UNIVERSITY OF CONNECTICUT

SCHOOL OF ENGINEERING

STORRS, CONNECTICUT

LYAPUNOV CONTROL-SYSTEM SYNTHESIS OF A HIGHLY RESONANT PLANT

D. W. Jorgensen, Jr.

D. P. Lindorff

Department of Electrical Engineering

April 1968

This work has been sponsored by the
National Aeronautics and Space Administration
Research Grant NsG-309

LIAPUNOV CONTROL-SYSTEM SYNTHESIS OF A HIGHLY RESONANT PLANT

Donald W. Jorgenson, Jr.

and

David P. Lindorff

I. INTRODUCTION

A number of papers have been written on the synthesis of control systems via the Second Method of Liapunov.[1,2,3] These papers have been theoretical in nature, at most containing simulation studies of systems employing the method. It is of interest therefore to determine some of the problem areas and possible solutions which arise when a Liapunov design is applied to a physical system of some complexity.

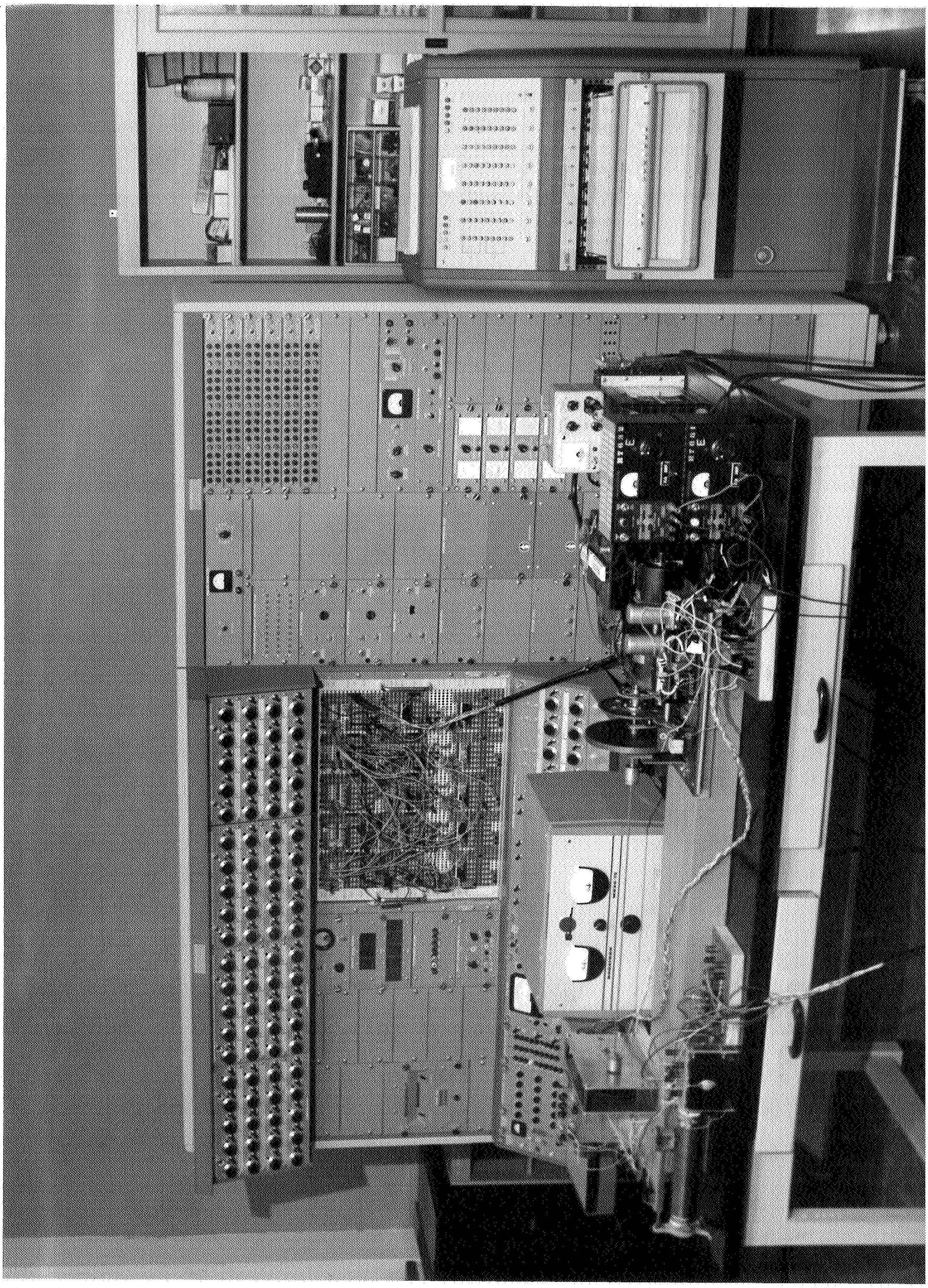
Since one of the principal advantages of Liapunov theory is a guarantee of stability within a region of the state space, a plant was selected in which stability problems might occur. This plant consisted of a motor-clutch combination driving a lightly damped resonant load, the objective being to position this load in some quasi-minimum-time manner and to control against disturbance torques. (See Figure 1.)

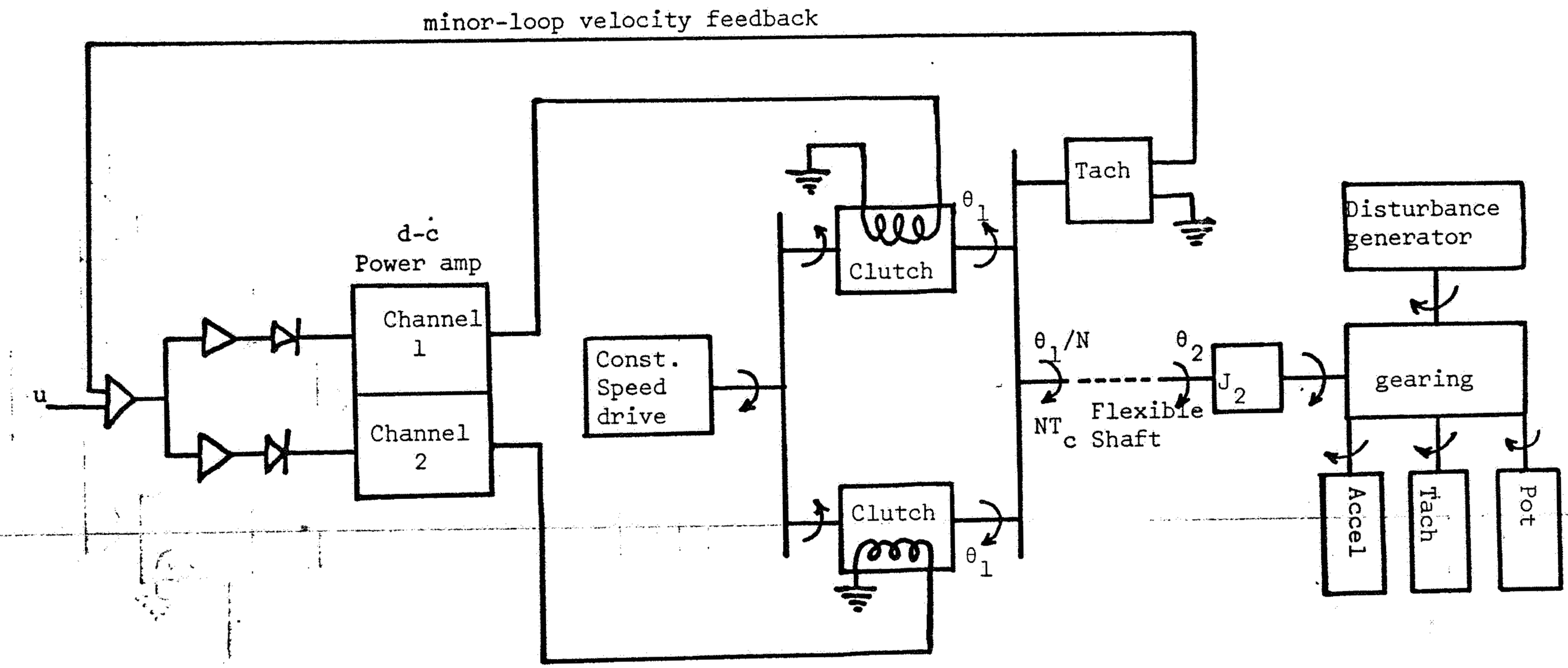
Certain problems are encountered before a final design can be arrived at. Noise is a potential problem, will large amounts of filtering be required or can the controller itself help in minimizing it? The disturbance problem in itself is interesting since in conventional design there is often a trade-off between good disturbance response and some other system specification, how will such a trade-off appear in the design being considered? Finally, can a reasonably simple synthesis technique be developed so that the engineer can design to a set of specifications without huge amounts of calculations or system studies?

The design objective was to control the plant as close to its resonant frequency as was possible, considering the physical limitations of the equipment. Studies were made of the step and disturbance response of the system with various values of controller parameters.

Various aspects of the engineering design problem have been considered, leading to an experimental method of arriving at a final design which yields a suitable response in the presence of instrument noise. This design procedure requires that a model, of the same order as the plant, be designed to meet the system specifications. The controller, which approximates a relay by a saturation function, is given the magnitude limits imposed by the physical system (for example, the magnitude of the voltage into a motor cannot exceed a certain value).

The system states are defined as the error between the states of the model and those of the plant. The summation of these states, weighted by controller parameters, form the argument of the saturation function. Within the contraints





SYSTEM SCHEMATIC

Figure 2

imposed by the design equations these controller parameters are chosen experimentally. Assuming an n^{th} order system whose m^{th} state is the derivative of its $m-1^{st}$ state it is shown that n+1 parameters must be specified. The procedure determines three of these n+1 controller parameters. These three parameters are primary factors in fixing the bounds on the error of the system and thus its input and disturbance responses, and of determining the quantity of noise which will appear in the control output. They are chosen to reduce the error bound, including the effect of transducer noise. The n-2 remaining parameters are then chosen to further refine the response of the system (subject to some restrictions imposed by Liapunov theory).

This synthesis procedure is attractive because it provides a method of designing to specifications, allows some uncertainty of plant parameters, is simple to instrument, and simplifies the computations required by Liapunov theory.

II. DESCRIPTION OF PLANT

The Plant was constructed to represent an approximation to a physical system such as a large radar antenna with structural resonance. As may be seen in Fig.2, the drive was obtained from two dry-partical, variable-slip clutches.

A limit of 24 volts on the clutch input voltage adds a saturation to the plant which must be included in the controller design.

The following symbols are to be used in development of equations describing the plant:

A	Feed forward gain of the clutches and their driving circuitry
B	Clutch internal feedback constant
D_1	Drag coefficient before flexible rod
D_2	Drag coefficient after flexible rod
J_1	Moment of inertia before flexible rod
J_2	Moment of inertia of rod, flywheel and Instrumentation
K	Spring constant of rod
K_f	Gain in ω_1 feedback path
N	Gear ratio from clutch to rod
T_c	Clutch output torque
T_d	Disturbance torque (applied at θ_2)
u	Control input
θ_1	Position of clutch output shaft
$\dot{\theta}_1, \omega_1$	Speed of clutch output shaft
θ_2, x_1	Position of flywheel (actual output)
$\dot{\theta}_2, \omega_2, x_2$	Speed of flywheel
$\ddot{\theta}_2, \dot{\omega}_2, x_3$	Acceleration of flywheel

Writing the equations of motion of the plant in the transform domain, we have

$$(2\text{-}1) \qquad T_c = J_1 S^2 \theta_1 + D_1 S \theta_1 + \frac{K}{N}(N\theta_1 - \theta_2)$$

$$(2\text{-}2) \qquad NT_d = J_2 S^2 \frac{\theta_2}{N} + D_2 S \frac{\theta_2}{N} + \frac{K}{N}(\theta_2 - N\theta_1).$$

Solving for θ_2 from (2-2) yields

$$(2\text{-}3) \quad \theta_2 = \frac{\frac{KN}{J_2}\theta_1 + \frac{N^2}{J_2}T_d}{S^2 + \frac{D_2}{J_2}S + \frac{K}{J_2}},$$

and substituting (2-3) into (2-1) yields

$$(2\text{-}4) \quad \theta_1 = \frac{\frac{1}{J_1}(S^2 + \frac{D_2}{J_2}S + \frac{K}{J_2})T_c + \frac{KN}{J_1J_2}T_d}{S\{S^3 + (\frac{D_1}{J_1} + \frac{D_2}{J_2})S_2 + (\frac{D_1D_2 + KJ_1 + KJ_2}{J_1J_2})S + K(\frac{D_1 + D_2}{J_1J_2})\}}.$$

If it is assumed that $|u|$ is constant,* then the torque from the clutches is given by

$$(2\text{-}5) \quad T_c = A(|u|)\,u - (B(|u|) + A(|u|)K_f)\,S\theta_1$$

where $A(|u|)$ decreases with increasing $|u|$, $B(|u|)$ increases with increasing $|u|$. If, however, various possible values of $A(|u|)$ and $B(|u|)$ are chosen and the resulting equation solved with $K_f = 1$, it is found that the resulting function $\theta_1 = f(u,T_d)$ exhibits poles and zeros which, except for a far-out dipole, are only very slightly affected by changes in A and B. The far-out dipole on the other hand, while remaining a dipole, varies its position widely but its closest approach to the origin is greater than 100, for the range of possible values of A and B. Furthermore, the gains of the function $\theta_1 = f(u,T_d)$ remain constant. Thus it was decided arbitrarily to choose some mid values for $A(|u|)$, $B(|u|)$ and treat the clutches as linear** The result is

$$(2\text{-}6) \quad T_c = Au - (B+AK_f)\,S\theta_1 = \frac{u}{6} - (9\times10)^{-4} + \frac{K_f}{6})\,S\theta_1$$

We can now solve (2-4) in terms of the control signal (u) and the disturbance torque (T_d). Thus

(2-7)

$$\theta_1 = \frac{\frac{A}{J_1}(S^2+\frac{D_2}{J_2}S+\frac{K}{J_2})u + \frac{KN}{J_1J_2}T_d}{S\{S^3+(\frac{D_1}{J_1}+\frac{D_2}{J_2}+\frac{B+AK_f}{J_1})S^2+(\frac{D_1D_2+J_1K+J_2K+D_2(B+AK_f)}{J_1J_2})S+K(\frac{D_1+D_2+B+AK_f}{J_1J_2})\}}$$

Substituting in the numerical values (see Appendix A), we obtain

* This assumption is not based on a rigorous argument. However, since in this study u(t) very nearly represents the output of an ideal relay, it follows that $|u(t)|$ is essentially constant.

**Unless otherwise stated, all units are in the m.k.s. system.

(2-8)

$$\theta_1 = \frac{341\,(S^2+.6S+47.5)\,U + 6440T_d}{S\{S^3 + (2.94+341K_f)S^2 + (60.42+170.6K_f)S + (121+16{,}100K_f)\}}$$

and using (2-3), it follows that

(2-)

$$\theta_2 = \frac{1070(S^2 + .6S + 57.5)U}{S(S^2+.6S+47.5)\{S^3+(2.94+341K_f)S^2+(60.42+170.6K_f)S+(121+16{,}100K_f)\}}$$

$$+\frac{38.8\{S^4+2.94+341K_f)S^3+(60.42+170.6K_f)S^2+121+16{,}100K_f)S+518\}T_d}{S(S^2+.6S+47.5)\{S^3+(2.94+341K_f)S^2+(60.42+170.6K_f)S+(121+16{,}100K_f)\}}.$$

It can be seen that the order of the polynomials in equation (2-9) is far too high for us to deal with, since Liapunov theory requires that all state variables be measured. Thus instrumentation would require derivatives of acceleration, resulting in extreme noise problems. It is for this reason that the feedback loop around the clutches is used to reduce the effective order of the system.

Reduction of Order of the Plant

Realizing that K_f consists of a gear ratio, a tachometer constant and the actual gain (G_f), we have $K_f = N_g K_T G_f$ where N_g = 40/13 and K_T = $.13\frac{\text{volt-sec.}}{\text{radian}}$. Thus $K_f = 0.4G_f$.

Since the polynomials of (2-8) exhibit the root contours seen in Figure 3 as a function of positive values of K_f, a K_f = 1 was chosen (G_f = 2.5) so as to form a set of dipoles and far-out poles. (2-8) and (2-9) then become

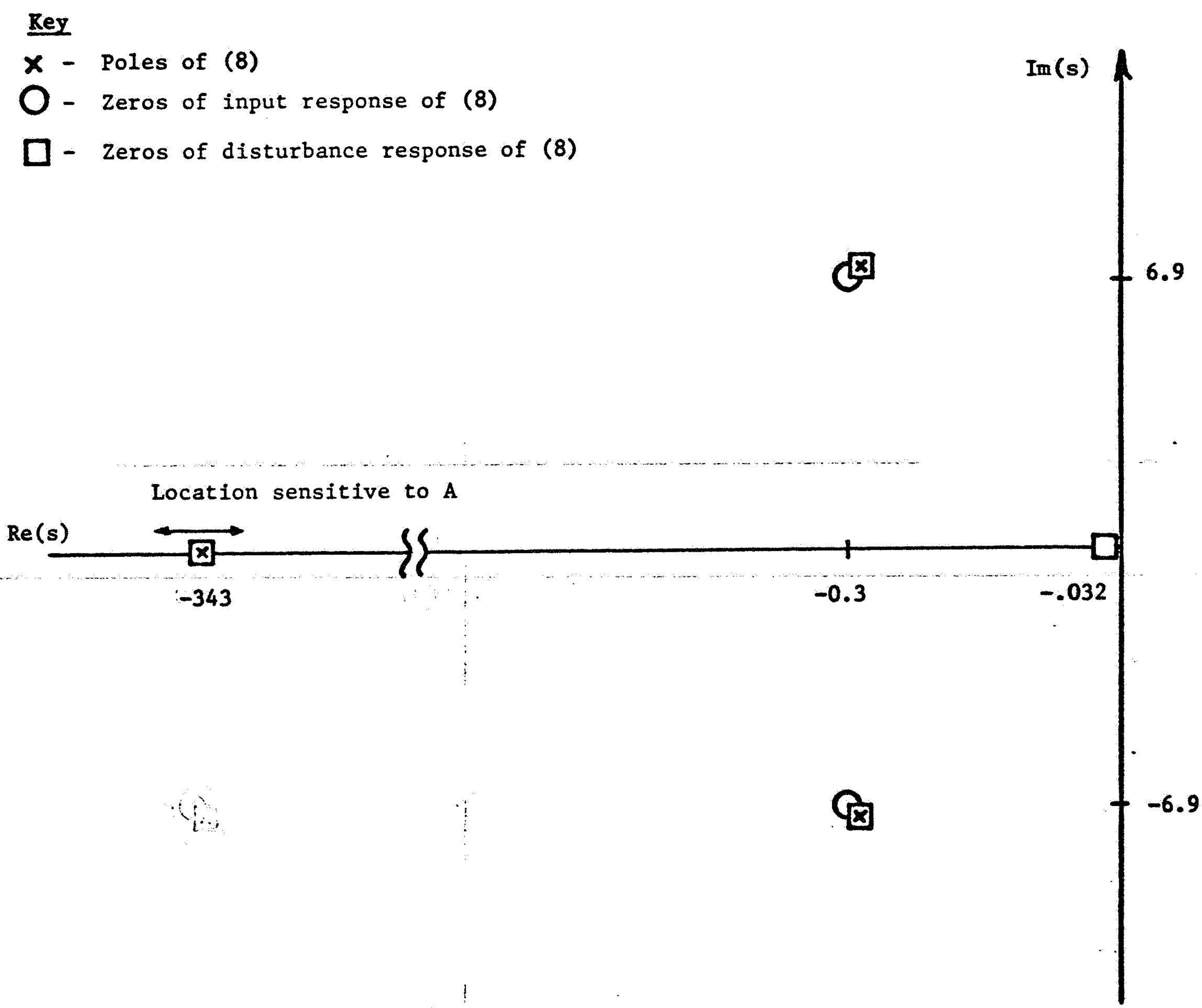
(2-10)

$$\theta_1 = \frac{341(S^2 + .6S + 47.5)U + 6440T_d}{S(S^3 + 343.94S^2 + 231.02S + 16{,}221)},$$

(2-11)

$$\theta_2 = \frac{1070(S^2+.6S+47.5)U+38.8(S^4+343.94S^3+231.02S^2+16{,}221S+518)T_d}{S(S^2+.6S+47.5)(S^3+343.94S^2+231.02S+16{,}221)}.$$

(2-11) can now be written in terms of factored polynomials so as to yield



POLES AND ZEROS OF (8) FOR $K_f=1$

Figure 3

(2-12)

$$\theta_2 + \frac{1070(S^2+.6S+47.5)\,U}{S(S^2+.6S+47.5)(S^2+.535S+47.236)(S+343.4)}$$

$$+\frac{38.8(S^2+.503S+47.215)(S+343.4)(S+.032)T_d}{S(S^2+.6S+47.5)(S^2+.535S+47.236)(S+343.4)}$$

The first expression in (2-12) reduces to a transfer function of lower order. Furthermore, the tern (S+343.4) is far enough out to be neglected. The second expression has a far-out dipole at (S+343.4). Thus this term can also be neglected and we are left with

(2-13)

$$\theta_2 = \frac{(1070/343.4)\,U}{S(S^2+.535+47.236)} + \frac{38.8(S^2+.503S+47.215)(S+.032)T_d}{S(S^2+.6S+47.5)(S^2+.535+47.236)}.$$

Now in the disturbance response of (2-13) we wish to neglect the term $\frac{S^2+.503S+47.215}{S^2+.6S+47.5}$. That this represents a reasonable assumption is borne out in the results which were obtained and therefore (2-13) becomes

(2-14)

$$\theta_2 = \frac{3.12\,U + 38.8(S + .032)T_d}{S(S^2 + .535S + 47.236)}$$

<u>State Space Equations of Reduced Order Plant</u>

From (2-14) the state equations of the reduced order plant can be written as

(2-15)

$$\dot{\underline{x}} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -47.5 & -.513 \end{vmatrix} \underline{x} + \begin{vmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ 3.13 & .515 & 38.8 \end{vmatrix} \begin{vmatrix} u \\ T_d \\ \dot{T}_d \end{vmatrix} .$$

Here the notation is adopted that $x_1 = \theta_2$.

Instrumentation

The necessary instrumentation was added to the system for measuring the states. This instrumentation consists of a wirewound potentiometer at θ_2, a d-c tachometers at θ_1 and θ_2, and a drag-cup servomoter which is connected to serve as an accelerometer for measuring $\ddot{\theta}_2$. (See Appendix B for calibration constants.) Another drag-up servomotor is used as a source of disturbance torque.

The potentiometer produced negligible noise output. At low speeds, however, the tachometer noise rose to 20 percent of the signal. The accelerometer noise was of two types, a low frequency component (about 10-20 cps) and large spikes. The low frequency component decreased as acceleration increased but the spike magnitudes increased, remaining over 50 percent of the signal magnitude at high acceleration levels. Although some filtering was used on these noisy signals, large amounts of noise were evident at the input to the clutches, due to the very large gains required in these loops.

III. DERIVATION OF THE LIAPUNOV CONTROLLER

Derivation of the Control Law

A Liapunov controller of the type described in [1] will be derived to control our system. A model will be chosen to meet the system specifications. The synthesis procedure insures that the plant states, $\underline{x}$, will follow the model states, $\underline{s}$, within some bound of error.

Using this technique, error coordinates are defined as

$$(3\text{-}1) \qquad \underline{e} = \underline{s} - \underline{x}.$$

Thus the differential equations of error are:

$$(3\text{-}2) \qquad \underline{\dot{e}} = -\underline{h}(\underline{x},t)$$

where

$$(3\text{-}3) \qquad \underline{h} = \underline{\dot{s}}(t) + \underline{g}(\underline{x},t) + \phi(\underline{x},t)\underline{u}(t)$$

or

$$(3\text{-}4) \qquad \underline{\dot{e}} = \underline{\dot{s}}-\underline{\dot{x}} = \begin{bmatrix} \dot{s}_1-\dot{x}_1 \\ \dot{s}_2-\dot{x}_2 \\ \dot{s}_3-\dot{x}_3 \end{bmatrix} = \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{s}_3 - g - \phi u \end{bmatrix} = -\underline{h} = -\underline{h}-A\underline{e}+A\underline{e}$$

where A is a stability matrix, i.e., all its eigenvalues have negative real parts. Now (3-4) can be rewritten as

$$(3\text{-}5) \qquad \underline{\dot{e}} = -\underline{h} - A\underline{e} + A\underline{e} = -\underline{f} + A\underline{e}$$

where

$$(3\text{-}6) \qquad A\underline{e} = \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} e_2 \\ e_3 \\ \pi(\underline{e}) \end{bmatrix}$$

where $\pi(\underline{e}) = -(a_1e_1 + a_2e_2 + a_3e_3)$, and

(3-7) $\quad \underline{f} = \underline{h}+A\underline{e} = \begin{bmatrix} -\dot{e}_1 + e_2 \\ -\dot{e}_2 + e_3 \\ -\dot{s}_3+g+\phi u + \pi(\underline{e}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\dot{s}_3+g+\phi u + \pi(\underline{e}) \end{bmatrix}$.

Now if a Liapunov function is chosen such that

(3-8) $\quad V = \underline{e}^t P \underline{e}$

is positive definate, then

(3-9) $\quad \dot{V} = -\underline{e}^t Q \underline{e} - 2\underline{e}^t P \underline{f}$ where $A^t P + PA = -Q$

and Q is positive definate.

Therefore, to assure stability $\dot{V}$ will be chosen to be negative definate for $\phi > 0$, $\underline{e}^t P \underline{f} \geq 0$. Thus we require that $|u| \geq \phi^{-1} |\dot{s}_3 - g - \pi(\underline{e})|$, and the sign of u be equal to the sign of γ where

(3-10) $\quad \gamma = p_{13}e_1 + p_{23}e_2 + p_{33}e_3$

and p_{ij} is the coefficient of the i^{th} row and j^{th} column of P.

In actuality, for the existance of a solution and due to noise considerations, a saturation function rather than a sign function will be used in u [2].

The control law which results for this system is thus

(3-11) $\quad |u| \geq \frac{1}{\phi} \dot{s}_3+\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3+\beta_1 T_d+\beta_2 \dot{T}_d+a_1 e_1+a_2 e_2+a_3 e_3|$

(3-12) sign u = sign γ = sign $(p_{13}e_1 + p_{23}e_3 + p_{33}e_3)$

where α_i, β_i are plant parameters in $g = \alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3+\beta_1 T_d+\beta_2 \dot{T}_d$.

Since the magnitude of u is limited by the motor-clutch combination to about 20 volts, then

(3-13) $\quad 20 \geq |u| \geq \frac{1}{\phi} \left| \dot{s}_3 - g - \pi(\underline{e}) \right|$

This limits $|\dot{s}_3|$, $|T_d|$ and $|\dot{T}_d|$.

The Model

It was decided to design a model which had as fast a step response as possible within the limitations imposed on $|\dot{s}_3|$.

It is worthwhile to note that, assuming the control law can be satisfied, system specifications can be met by using this technique if the model meets the specifications. The model configuration is limited only by the order of the plant and its physical limits (such as $|u| \geq L$). This allows the designer to employ almost any technique he wishes in order to meet his design specifications on the model. Therefore, if it is desired to control an n^{th} order plant and the n states are available, the designer can (within the physical limitations of the plant and its instrumentation) meet any set of specifications as long as a model can be designed to meet these specifications and the control law can be satisfied.

In this case the model was designed with a large gain preceding a saturation level on $\dot{s}_3$, thus approximating a bang-bang system (See Figure 4). A nonlinearity was introduced in the feedback path in order to realize a quasi-minimum time response.

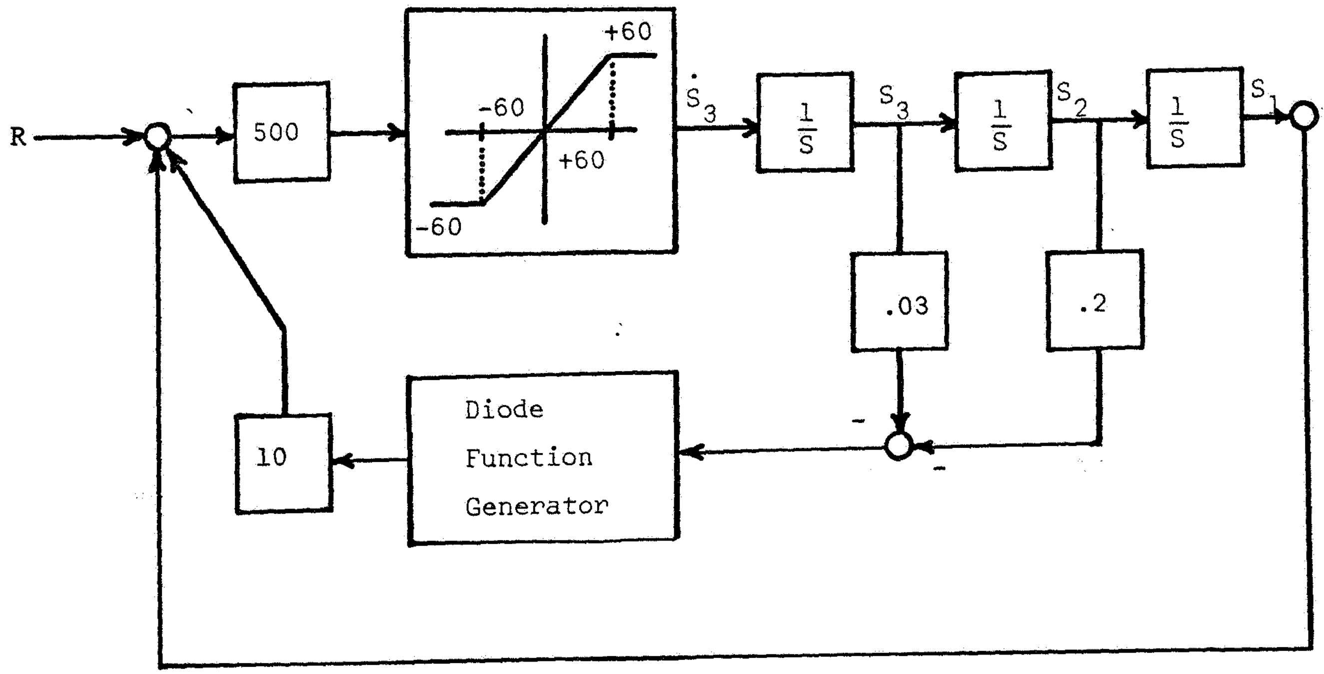
Since the input to the clutches consisted of two signals, u and $K_c\omega_1$, it was required that this feedback signal would not cause saturation of the clutches. Although it is the clutch input which is limited to 24 volts, it was stated in the previous section that there was a 20 volt limit imposed on u which in turn placed a limit on $|\dot{s}_3|$. This assumption was found to be justified in the final system design, and bears out the obvious fact that the speed of response of the system was limited by the motor-clutch combination. A larger motor and clutch would have to be employed in order to increase $|\dot{s}_3|$, thus increasing the speed of response. Now assuming $|u| \leq 20 \geq \frac{1}{\phi}|\dot{s}_3 - \bar{g} - \bar{\pi}(\underline{e})|$ where $\phi = 3.12$, $|\dot{s}_3| < 62.4$ volts. It was found, experimentally, that an $|\dot{s}_3|$ of 60 volts kept $|u|$ just under 20 volts. Experimentally determination was necessary since the magnitudes of x_2 and x_3 were unknown without knowing $|\dot{s}_3|$ and the terms in the summation $\sum a_i e_i$, i = 1 to 3, could only be known within a conservative bound [3].

The final model was observed to have a rise time, to 90 percent of its final value, of just under one second. It had a 10 percent overshoot and finally settled out in about 1.5 seconds.

Determination of Controller Parameters

Two possible methods of constructing u have been considered [1,2], i.e., with

$$(3\text{-}14) \quad u = 20\text{Sat } K(p_{13}e_3 + p_{23}e_2 + p_{33}e_3)$$



FLOW DIAGRAM OF MODEL

Figure 4

or

(3-15)

$$u = \frac{1}{3.12}\{|\dot{s}_3+.535x_3+47.23x_2+a_1e_1+a_2e_2+a_3e_3|+T_{dm}\}\mathrm{Sat}\{K(p_{13}e_1+p_{23}e_2+p_{33}e_3)\}$$

where T_{dm} is a constant which allows the system to control against disturbance torques and where K is a gain preceding the saturation function.

(3-15) can be rewritten as

(3-16)

$$u = \{|.321\dot{s}_3+.1715x_3+15.1x_2+.32|\sum a_ie_i|+T_{dm}\}\mathrm{Sat}\{K(p_{13}e_1+p_{23}e_2+p_{33}e_3)\}$$

It is now required to determine the coefficients a_i and p_{i3}, i=1 to 3.

Taylor [3] describes a method of determining a bound in the error space due to the use of an imperfect sign function.

Thus given a space $\Omega = \{\underline{e}: |\gamma_1(\underline{e})| \leq L \quad L \leq 1,$

where $\gamma_1(\underline{e}) = e_1 + \frac{p_{23}}{p_{13}} e_2 + \frac{p_{33}}{p_{13}} e_3$

then

(3-17) $$\Omega = \{\underline{e}: -L \leq e_1 + \frac{p_{23}}{p_{13}} e_2 + \frac{p_{33}}{p_{13}} e_3 \leq L\}$$

or

(3-18) $$\Omega = \{\underline{e}: \frac{p_{13}}{p_{33}}(-L-e_1-\frac{p_{23}}{p_{13}}e_2) \leq e_3 \leq \frac{p_{13}}{p_{33}}(L-e_1-\frac{p_{23}}{p_{33}}e_2)\}.$$

Now Ω can be expressed as

(3-19) $$-\alpha_1e_1-\alpha_2e_2-M \leq e_3 \leq -\alpha_1e_1-\alpha_2e_2+M$$

where

(3-20) $$M = \frac{p_{13}}{p_{33}}L, \quad \alpha_1 = \frac{p_{13}}{p_{33}}, \quad \alpha_2 = \frac{p_{23}}{p_{33}}$$

Equation (3-19) is certainly satisfied if $e_3 = \alpha_1e_1 - \alpha_2e_2+m(t)$ where $|m(t)| \leq M$. Thus it is true that

(3-21) $$\dot{e}_1 = e_2$$

(3-22) $\dot{e}_2 = e_3 = m(t) - \alpha_1 e_1 - \alpha_2 e_2$

or

(3-23) $H(s) = \frac{E_1}{M}(s) = \frac{1}{s^2 + \alpha_2 s + \alpha_1}$

Now if H(s) has an input of $M(s) = \frac{P_{13}L}{P_{33}s}$ then

(3-24) $|e_1(t)| < M \int_0^\infty |h(t)| dt.$

To evaluate this expression an H(s) was chosen which exhibits no sign changes, namely

(3-25) $H(s) = \frac{1}{(s+a)(s+b)}$

or

(3-26)

$$h(t) = \left(\frac{P_{33}}{\sqrt{P_{23} - 4P_{13}P_{33}}}\right) \exp\left(-\frac{P_{23}t}{2P_{33}}\right) \left[\exp\left(\frac{\sqrt{P_{23} - 4P_{13}P_{33}}}{2P_{33}}t\right) - \exp\left(-\frac{\sqrt{P_{23} - 4P_{13}P_{33}}}{2P_{33}}t\right)\right].$$

Thus, evaluating $M\int_0^\infty | h(t) | dt$ we obtain

(3-27) $|e_1(t)| \leq \frac{P_{33}L}{2P_{23}}$.

Similarly $|e_2(t)|$ can be found as

(3-28) $|e_2(t)| \leq M\int_0^\infty |h'(t)| dt.$

Ω can then be determined by solving

(3-29) $\left|e_1 + \frac{P_{23}}{P_{13}} e_2 + \frac{P_{33}}{P_{13}} e_3\right| \leq L$

for all e_1, e_2 less than or equal to their bounds.

Parameters for the controller can now be determined by using (3-27), (3-28) and the matrix equation $A^TP + PA = -Q$, realizing the P and Q must be positive definate and symmetric, and A must be a stability matrix.

Initially let us assume that u is instrumented according to (3-15). For this system it was decided to make $|e_1|$ very small for high positional accuracy. Since

$$(3\text{-}30) \quad u = \{|F(\dot{s}_3, \dot{x}_3, x_2) - \bar{\tau}(\underline{e})| + T'_{dM}\}\mathrm{Sat}(K\gamma) ,$$

in the steady state all terms in u could be zero except e_1, and $u = |a_1 e_1|\mathrm{Sat}(Kp_{13}e_1)$. Since this product is instrumented by a quarter-square multiplier which becomes inaccurate if signals are very small, a_1 must be large to enable the system to maintain a small value of $|e_1|$ in the presence of coulomb friction.

Assume $a_1 = 100$, $p_{13} = 1$, $|e_1|_{max} = .0357$, $L = 1$. Then using (3-27)

$$(3\text{-}31) \quad .0357 \le \frac{p_{33}}{2p_{23}}$$

from which we can say $p_{33} = .0714p_{23}$ and from $A^T P = PA = -Q$, where

$$(3\text{-}32) \quad Q = \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{23} \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} ,$$

the following equations result:

$$(3\text{-}33) \quad 2p_{13}a_1 = q_{11}$$

$$(3\text{-}34) \quad 2p_{33}a_3 - 2p_{23} = q_{33}$$

$$(3\text{-}35) \quad p_{13}a_3 + p_{33}a_1 = p_{23}a_2 - \frac{q_{22}}{2}$$

$$(3\text{-}36) \quad p_{12} = p_{33}a_1 + p_{13}a_3$$

$$(3\text{-}37) \quad p_{11} = p_{13}a_2 + p_{23}a_1$$

$$(3\text{-}38) \quad p_{22} = p_{23}a_3 + p_{33}a_2 - p_{13}$$

It is a tedious process to find the parameter set which will satisfy all the equations and stability conditions on the A matrix, but for this case such a set is

$p_{13} = 1 \quad p_{23} = .7 \quad p_{33} = .05 \quad p_{12} = 20 \quad p_{11} = 100 \quad p_{22} = 11$

$a_1 = 100 \quad a_2 = 30 \quad a_3 = 15$

$q_{11} = 200 \quad q_{22} = 2 \quad q_{33} = .1$

Taylor [3] has shown that with increasing K in Sat($K\gamma$), the error bound decreases proportionately [illegible]. Thus putting a K of 100 in the controller previously described would bring the bound on $|e_1|$ down to .001.

Final Determination of Controller

It was found that if the controller of (3-15) is used, noise limited the performance of the system considerably. The reason for this is that the noisy signals, x_2 and x_3, occur in both the magnitude of u and the saturation function. Due to the large gains involved, these components of u proved to be extremely noisy. For example, in the controller previously discussed the dominant term involving x_2 in $|u|$ is a_2e_2. To realize this term the output of the tachometer was multiplied by a gain of about 170. The output of the accelerometer in turn was multiplied by a gain of over 2500. When it is considered that the saturation function may have a large gain also, it is easy to see why this type of controller would be especially susceptible to noise problems, since a peak of noise in x_3 would occur in both $|u|$ and Sat($k\gamma$), thus multiplying its effect.

It was for these reasons that a controller of the type described in (3-14) was finally chosen. There are two main advantages in this choice aside from improving the noise situation. The instrumentation problem is greatly simplified and the disturbance constant, T_{dm}, is absorbed into the controller, allowing maximum effort against disturbances.

IV. EVALUATION OF SYSTEM PERFORMANCE

Transient Response to Inputs

Test runs were made with various values of controller parameters. The major factor limiting the system performance as determined from these tests were: 1) Size of the motor-clutch combination which limited response time, and 2) Existing instrumentation which exhibited poor resolution and high noise content. The results of these runs are shown in Table 1 and Figures 5 through 10.

This data suggests a number of conslusions. There is a minimum upper bound on error which can only be approached with the existing equipment. The error bound varies as N + L/K, where N is a bound on the noise magnitude. This sets a lower bound on the error, and as K decreases, the bound increases. As expected, an increase in P_{33} increases the magnitude of e_1 and e_2, and also increases the noise on u. For this reason, it is suggested that P_{33} be made as small as possible, still keeping the system stable. With P_{33} chosen, K should be increased so as to realize an acceptable error bound. A small P_{23} makes the system sluggish and overdamped, a large one makes it underdamped. The proper choice of p_{23} shapes the dynamic error for the chosen values of p_{13},p_{33} and K.

TABLE 1

RESPONSE OF SYSTEM TO 90° (2 volt) STEP INPUTS

RUN	p_{23}	p_{33}	K	e_1	e_2	e_3	N_1	N_2	N_3	u
1*	.6	.05	.1	1.5, .5	3.6	15	≈0	.2	15	5
2*	.6	.05	1.	.1, .01	1.6	10	.01	1.	20	14
3*	.1	.05	1.	.4, .04	2.	20	≈0	2.	15	5
4	.2	.05	1.	.32, .04	1.8	10	≈0	1.5	20	5
5	.3	.05	1.	.24, .03	1.6	10	≈0	1.5	15	5
6	.4	.05	1.	.16, .03	1.5	10	≈0	1.5	20	10
7	.5	.05	1.	.12, .02	1.5	10	.005	1.5	20	11.5
8	.7	.05	1.	.07, .025	1.5	10	.005	1.2	20	12.5
9	.8	.05	1.	.1, .02	1.5	10	.005	1.2	20	12.5
10	.9	.05	1.	.125, .025	1.5	12	.005	1.5	15	12.5
11	.6	.1	1.	.17, .02	1.	10	.005	1.	20	16

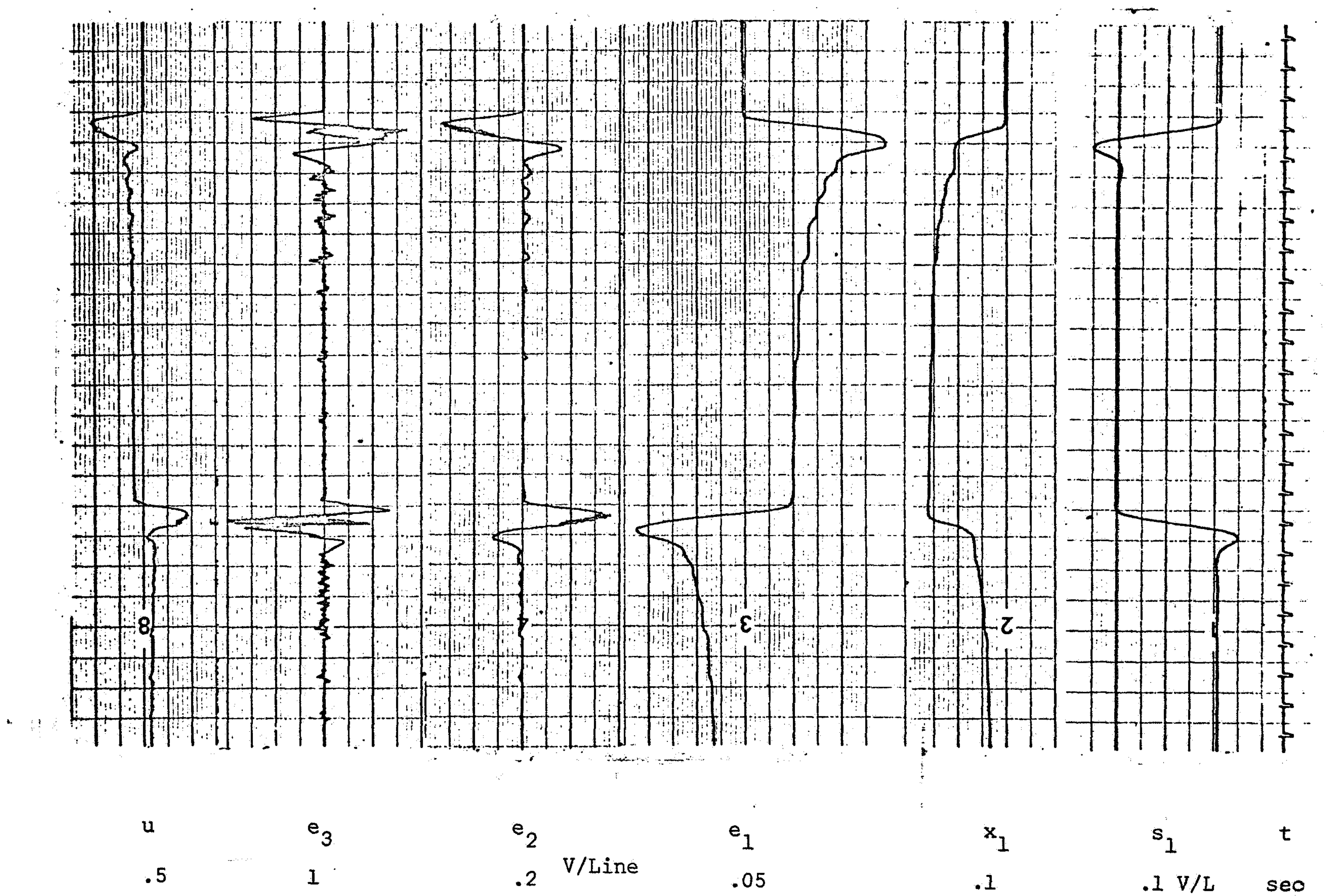
* denotes curves are shown for these runs

N_i is the peak to peak maximum of the noise on the e_i signal

K is the constant in $u = 20\mathrm{Sat}(K\gamma)$

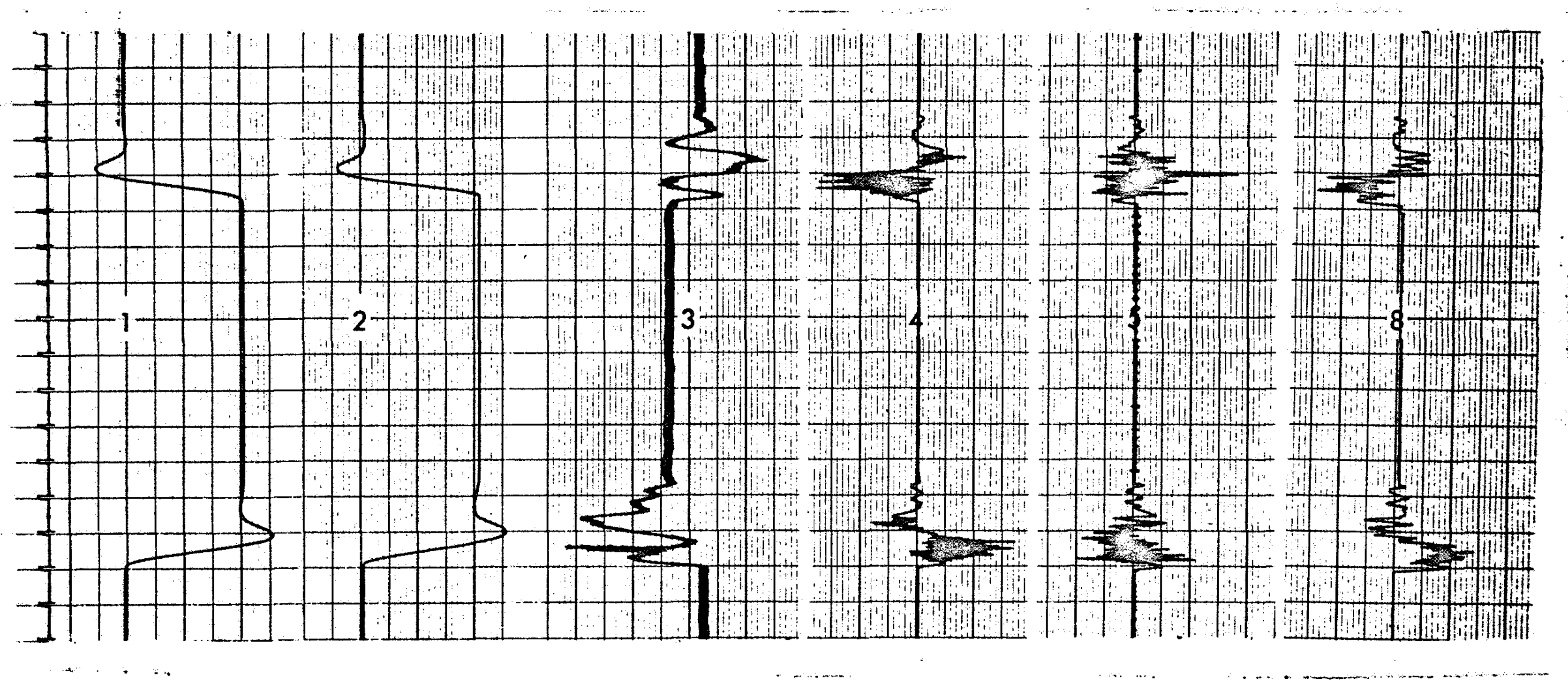
e_i is the maximum value of error

The second lower value given in the e_1 column is the steady state error position (attributed to coulomb friction).



RUN 1

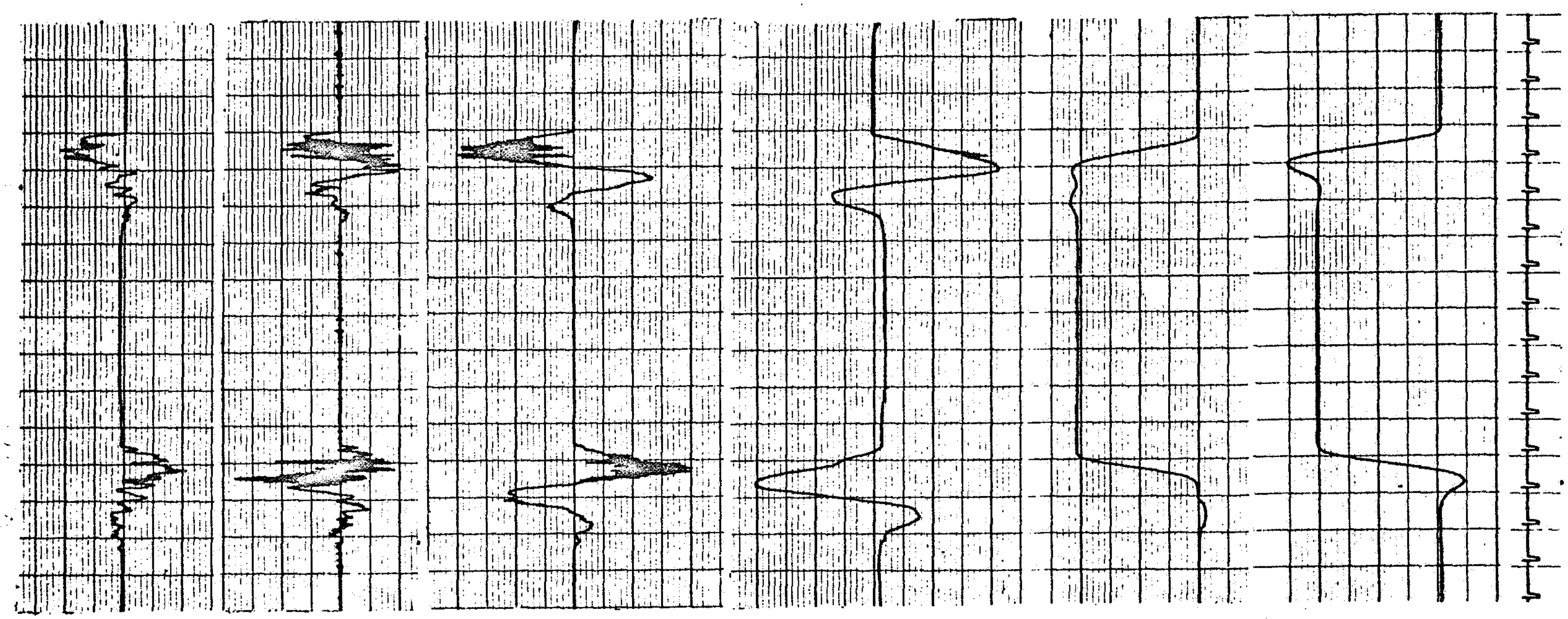
FIGURE 5



			V/Line				
u	e_3	e_2		e_1	x_1	s_1	t
1	1	.1		.005	.1	.1	sec.

RUN 2

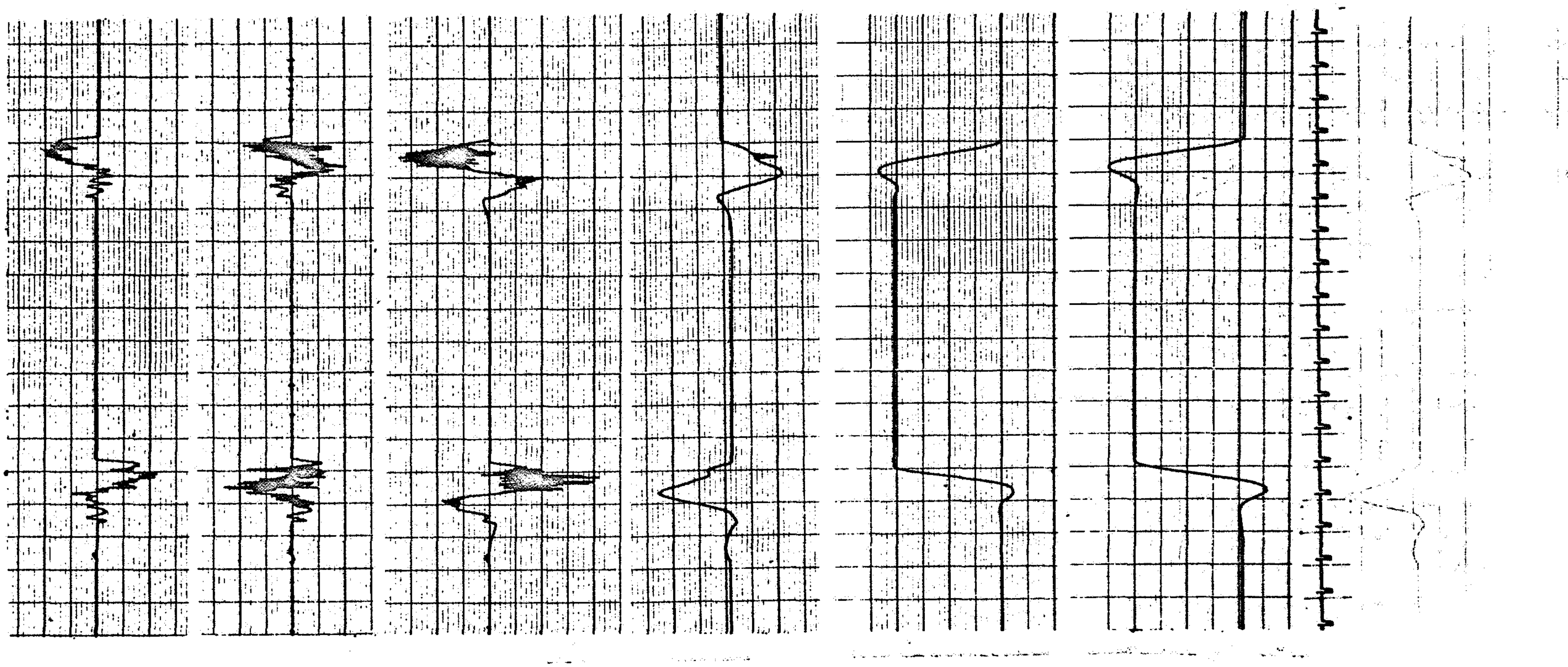
FIGURE 6



u	e_3	e_2	V/Line	e_1	x_1	s_1	t
1	1	.1		.02	.1	.1	sec.

RUN 3

FIGURE 7

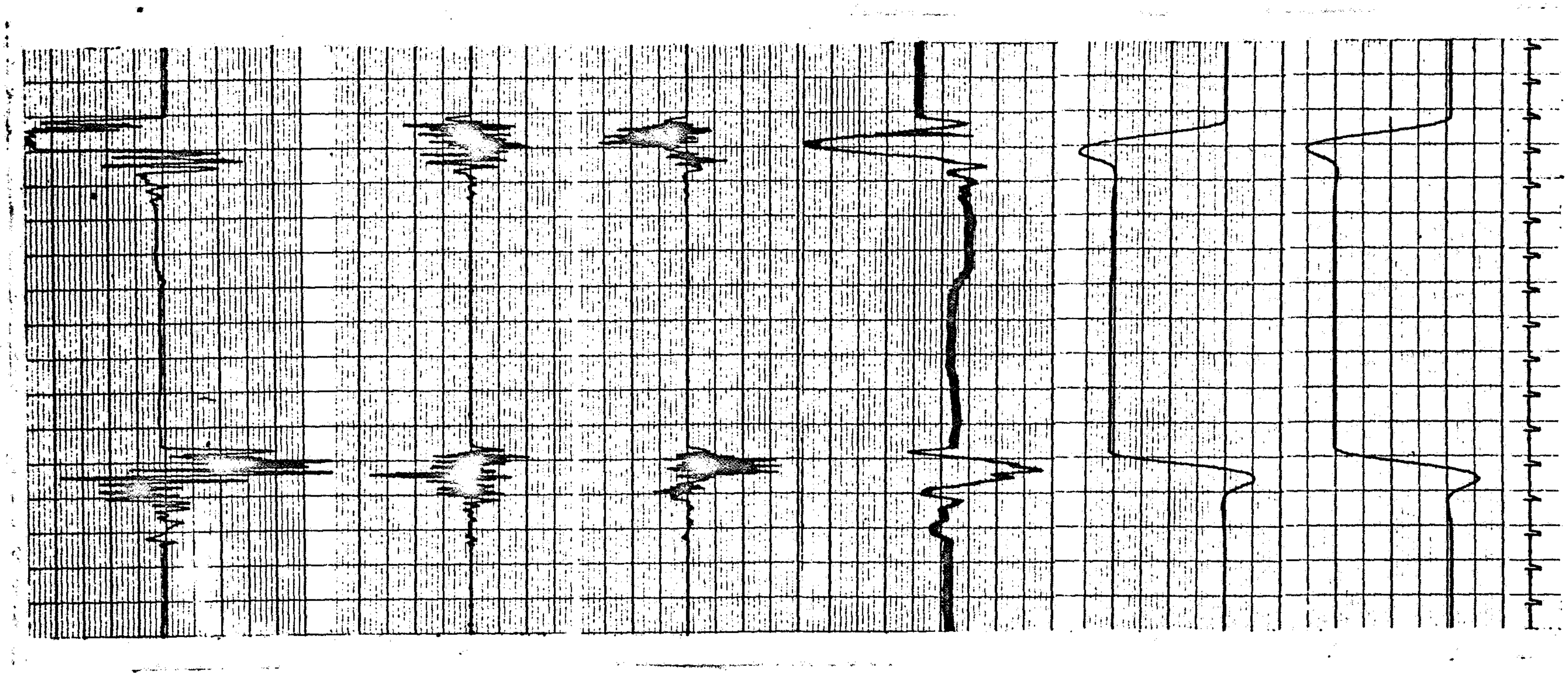


u	e_3	e_2	e_1	x_1	s_1	t
1	1	.1	.02	.1	.1	sec.

V/Line

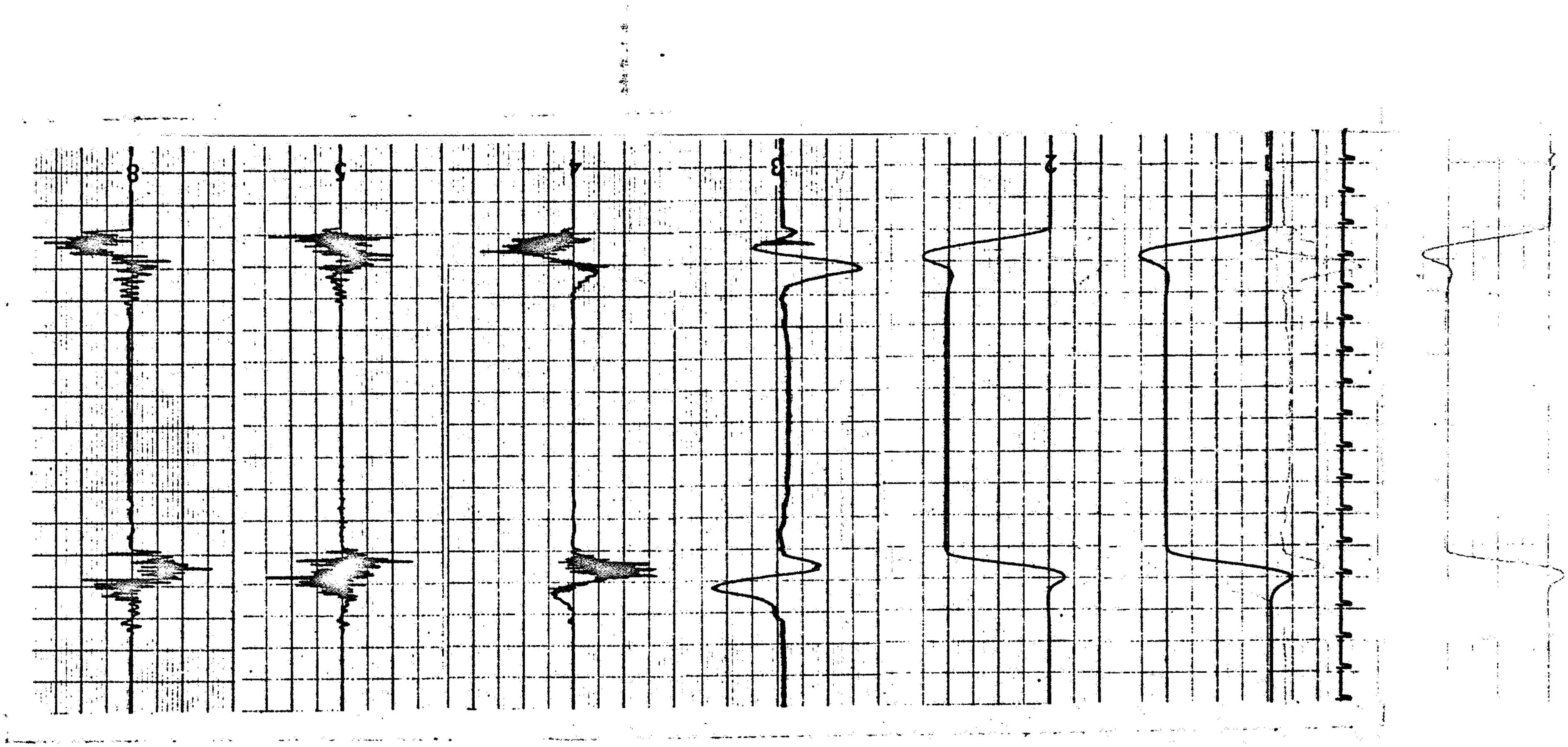
RUN 5

FIGURE 8



RUN 10

FIGURE 9



u	e_3	e_2	e_1	x_1	s_1	t
1	2	.1	.02	.1	.1	sec.

V/Line

RUN 11

FIGURE 10

Disturbance Response

The response of the system to disturbance torques of 7.5 in-lbs is shown in Table 2 and Figures 12-14. It was found that a controller of the type $u = 20\mathrm{Sat}(K\gamma)$ produces a much better response than if u is generated according to (3-15). When u is generated according to (3-14), the transient response of the system to step disturbances is very closely related to the error response to step inputs. The response of the system to step inputs while under the influence of a steady-state disturbance is only slightly different than that of the system without the disturbance, the difference mainly being in the magnitude of the steady-state error.

TABLE 2

SYSTEM RESPONSE TO STEP-TORQUE DISTURBANCE OF 7.5 IN-LBS

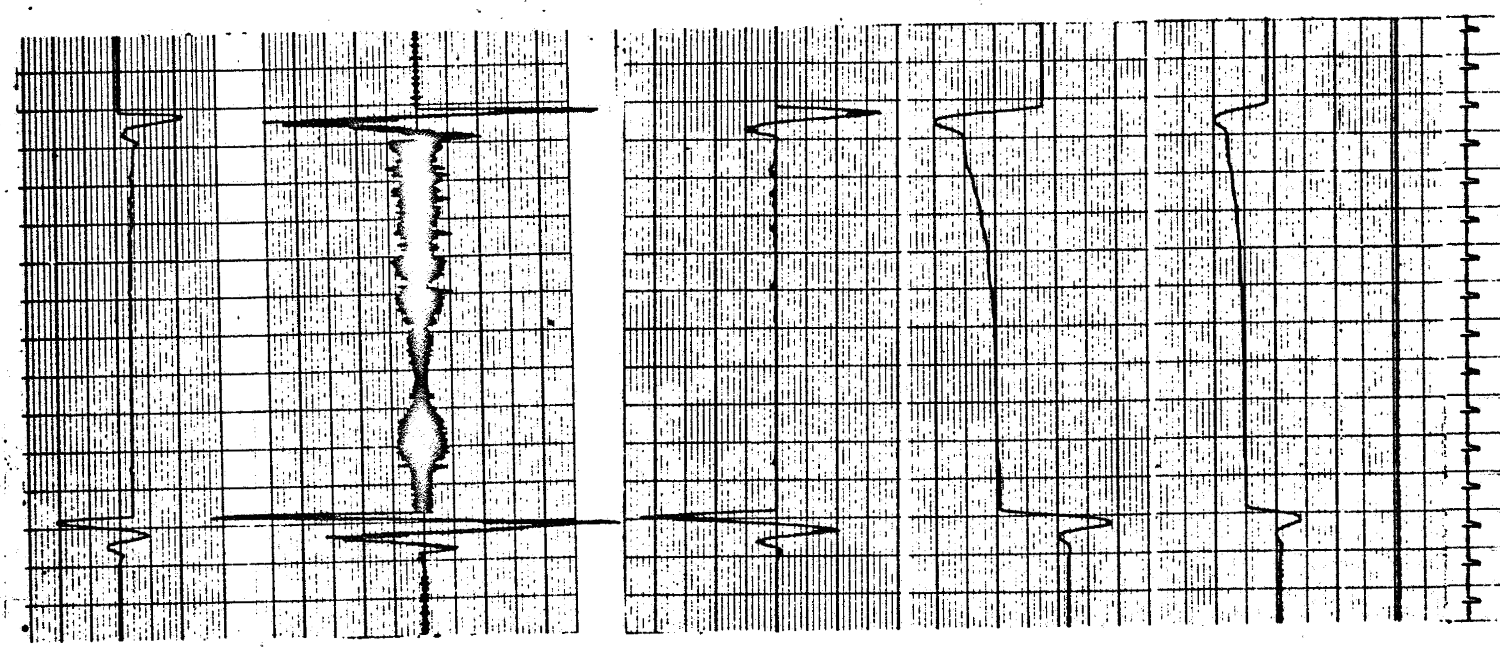
$P_{13}=1.0$

RUN	P_{23}	P_{33}	K	e_1	e_2	e_3	N_1	N_2	N_3	u
1A*	.6	.05	.1	1.,.5	3.6	30	0	.6	20	5.5
2A*	.6	.05	1.	.15,.03	2.	25	.01	.2	7	20
3A	.1	.05	1.	.26,.04	2.	30	0	.2	10	10
5A	.3	.05	1.	.24,.04	1.8	20	0	.2	7	10
10A	.9	.05	1.	.22,.03	2.	14	0	.2	28	20
11A	.6	.1	1.	.25,.02	1.8	30	.005	.2	10	20
1B*	.6	.05	.1	1.6,.5	3.6	20	0	.2	15	5.5
2B*	.6	.05	1.	.1,.05	1.6	20	.01	1.6	7	19
3B	.1	.05	1.	.4,.05	2.	15	0	2.	10	5
5B	.3	.05	1.	.24,.04	1.6	20	0	1.8	7	10
10B	.9	.05	1.	.14,.04	1.5	10	0	1.5	14	16
11B	.6	.1	1.	.17,.02	1.5	8	.005	.8	15	15

See note under Table 1.

The "A" runs are response to a step of disturbance torque.

The "B" runs are response to a 90° input step while under a steady-state disturbance of 7.5 in-lbs.

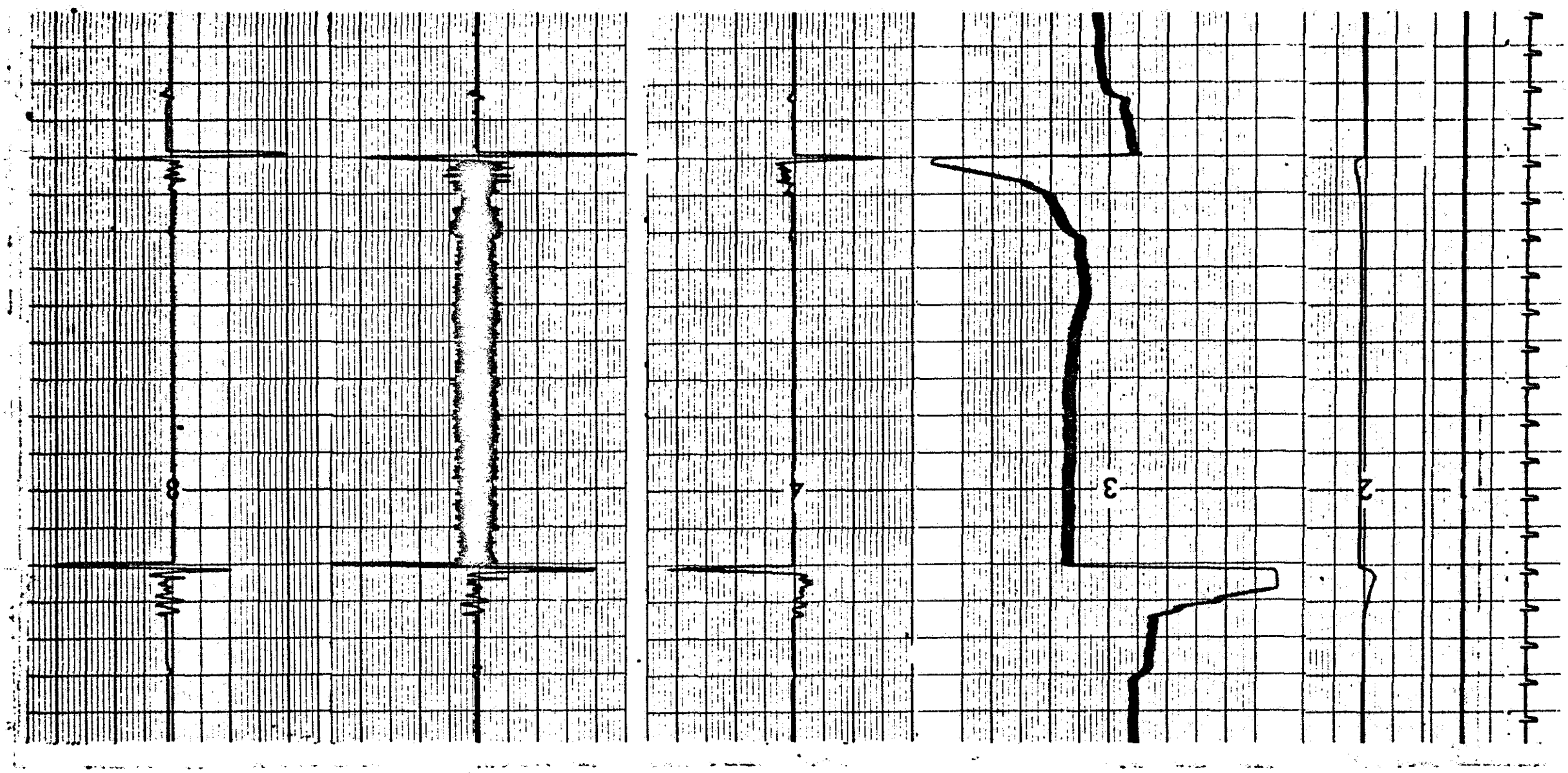


u	e_3	e_2	e_1	x_1	s_1	t
.5	1	.2	.05	.1	.1	sec.

V/Line

RUN 1A

FIGURE 11

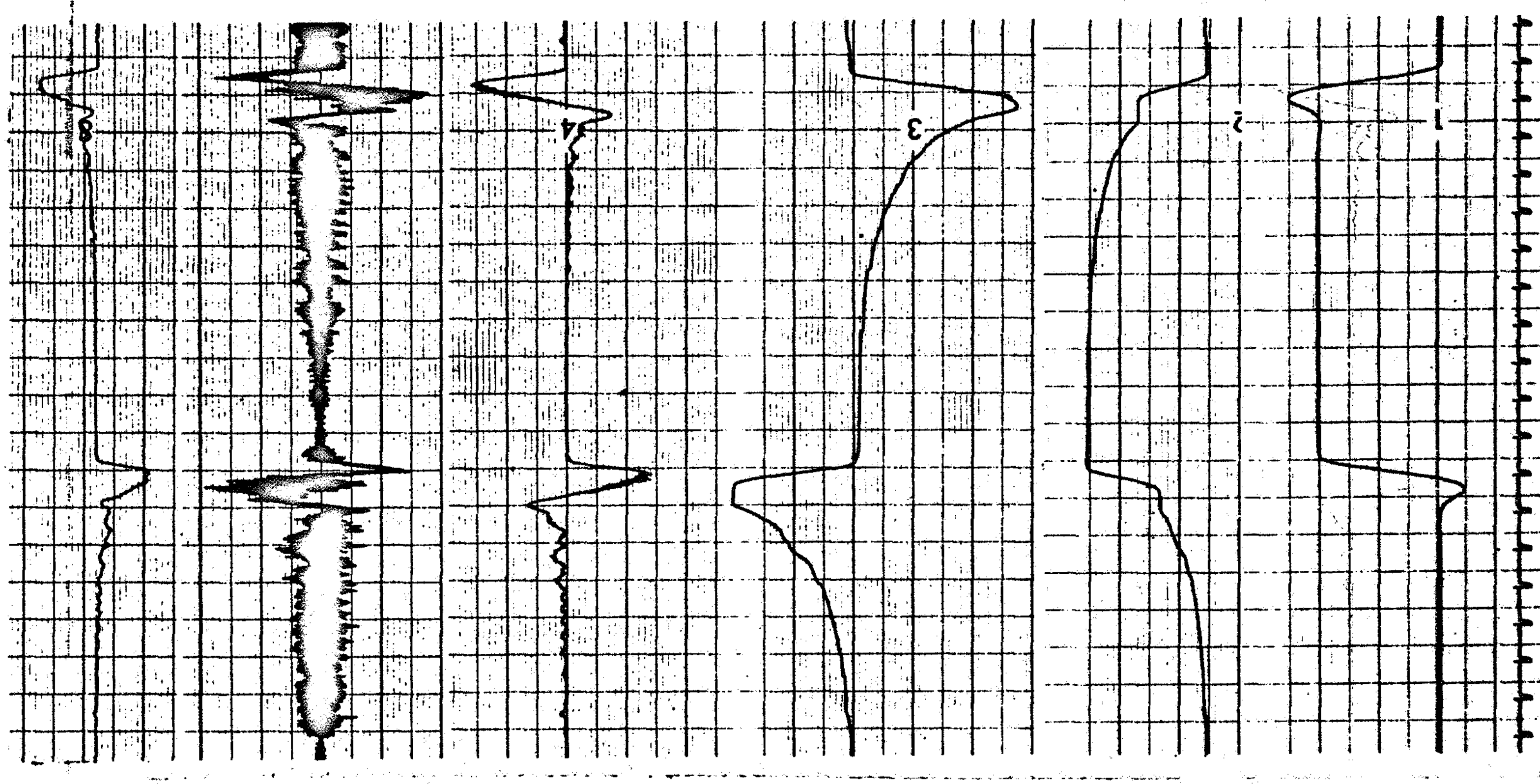


u	e_3	e_2	e_1	x_1	s_1	t
1	1	.1	.005	.1	.1	sec.

V/Line

RUN 2A

FIGURE 12

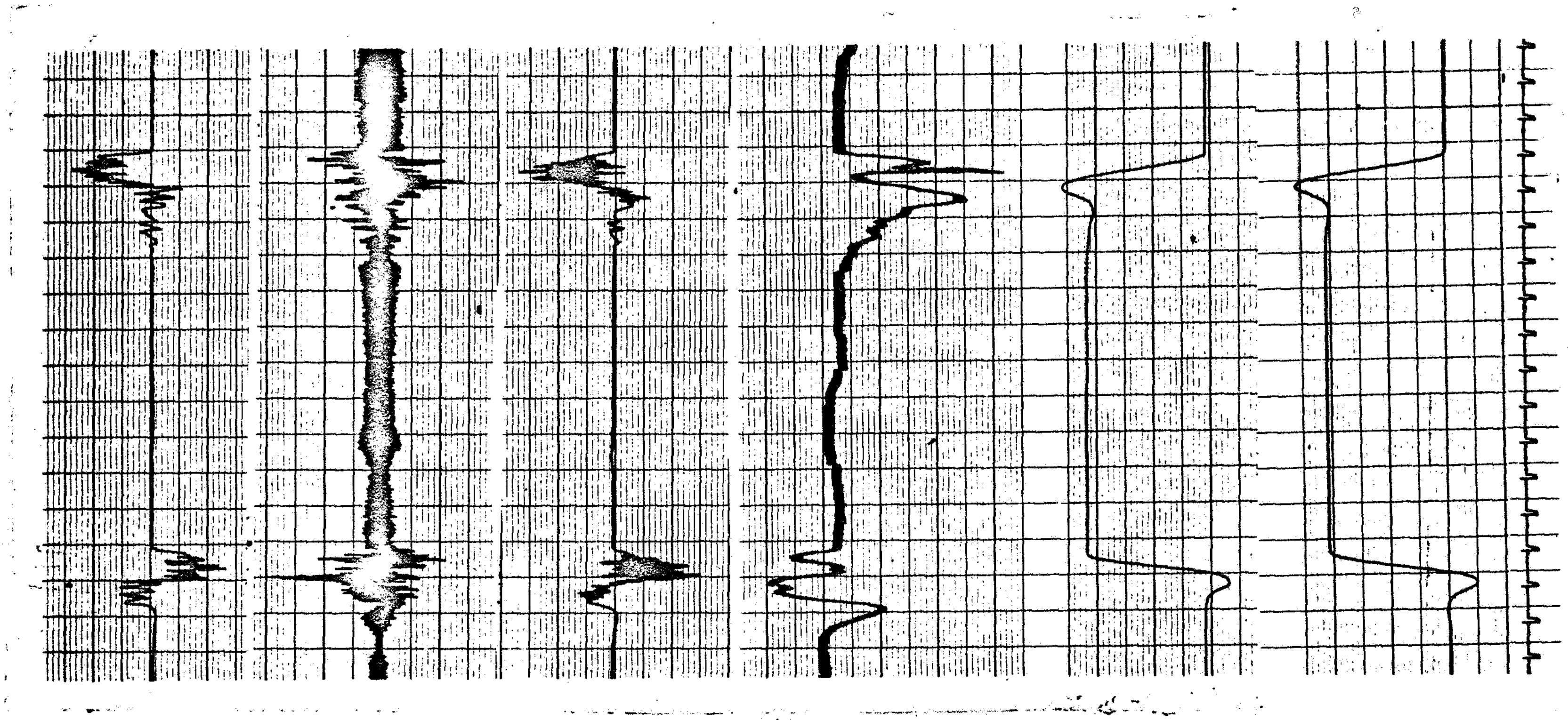


u	e_3	e_2	e_1	x_1	s_1	t
5	1	.2	.05	.1	.1	sec.

V/Line

RUN 1B

FIGURE 13



u	e_3	e_2	e_1	x_1	s_1	t
1	1	.1	.005	.1	.1	sec.

V/Line

RUN 2B

FIGURE 14

V. CONCLUSIONS

As a result of studies made on this system, a fairly systemized design procedure has been developed for applying the Liapunov synthesis technique when instrument noise is present. The procedure is as follows. First select a model whose order is the same as the plant to be controlled, and whose response essentially meets the specifications. Any technique may be employed to produce this model, even optimal techniques involving a digital computer, as long as all of the model states are available in real time. After the model has been chosen, a controller can be constructed of the following form:

$$u = L\,\mathrm{Sat}\{K(p_{1n}e_1 + p_{2n}e_2 + p_{3n}e_3 + \cdot\cdot\cdot + p_{nn}e_n)\}\ ,\ p_{1n}=1.0 \tag{5-1}$$

where L is the maximum permissible value of $|u|$, and the coefficients of the P matrix satisfy the conditions for positive definitness. A study should now be made to choose values of p_{nn} and K; it may be permissible to rely on a simulation study if noise in the plant is not extreme. In this study the minimum value of p_{nn} and the maximum value of K are determined by the following procedure: choosing the intermediate terms for p_{mn} between p_{1n} and p_{nn} as less than 1.0 and greater than p_{nn}, and decreasing in size as m increases, p_{nn} is made as small as possible without violating requirements for stability. Once P_{nn} is chosen, K is then increased in magnitude until it is as large as possible within saturation limits of the instrumentation. The remaining p_{mn} coefficients are selected to shape the response.

This design procedure seems to hold several advantages for the engineer. It appears relatively insensitive to parameter variations and offers a fairly simple method of designing to specifications. Its disturbance response can be predicted quite well with the result that the system will control against large disturbances with small error.

The parameter variation problem was studied only to the extent of determining that the existing parameter uncertainties (which in some cases were as large as 20%) did not introduce discrepancies between the theoretical and experimental results. A detailed study of this problem would constitute a worthwhile extension.

APPENDIX A

DETERMINATION OF SYSTEM PARAMETERS

Spring Constant K of Torsian Bar

$$k = 1.217 \frac{kg\text{-}M^2}{sec^2} \text{ at the flywheel}$$

$$K_c = N^2K = (4.4x10^{-3})(1.217) = 5.35x10^{-3} \frac{kg\text{-}M^2}{sec^2} \text{ at the clutches}$$

Evaluation of D_1, D_2, J_1, J_2

From transient and resonance studies ω = 1.1 cps = $\left(\frac{K}{J}\right)^{1/2}$ where ω is the resonant frequency of the rod and flywheel. From step responses of system without the rod

$$\frac{D_1}{J_1} = .5, \quad \frac{D_1}{J_1} = .6 \quad sec.^{-1}$$

and from steady state values of $\dot{\theta}_1$ under load and no load conditions

$$D_1 = 2.935x10^{-4} \quad \frac{.K_g\text{-}M^2}{sec}$$

Thus, since all equations are evaluated at the clutches,

$$J_1 = 4.89x10^{-4} \quad J_2 = 1.13x10^{-4} \quad Kg\text{-}M^2$$

$$D_1 = 2.935x10^{-4} \quad D_2 = 5.65x10^{-5} \quad \frac{Kg\text{-}M^2}{sec}$$

$$K = 5.35x10^{-3} \quad \frac{Kg\text{-}M^3}{sec^2}$$

$$N = 35/528$$

INSTRUMENTATION CONSTANTS

$\frac{\omega}{2\pi}$ (cps)	B volts	C volts	D volts	K_T	K_a	Peak to Peak volts of noise on Tach.	Accel.
.1	40	1.2	.025	.00755	.00025	.2	.5
.2	39	1.2	.05	.00387	.001285	.2	.75
.3	26	1.2	.1	.00387	.001715	.2	.75
.4	18	1.4	.1	.0049	.0014	.2	.75
.5	18	1.2	.1	.00335	.000089	.2	.6
.6	15.5	1.25	.2	.00338	.000144	.3	.75
.7	15	1.5	.25	.0036	.0001365	.2	.9
.8	14.5	1.6	.3	.00345	.0001288	.25	.75
.9	15.5	2.	.4	.00361	.0001275	.4	.75
1.0	18	2.45	.5	.00343	.000111	.4	.75
1.1	16	2.4	.6	.00344	.0001243	.4	.75
1.1	8.3	1.35	.3	.00372	.0001195	.4	.75
1.1	11.0	1.65	.375	.00342	.000113	.4	.75
1.1	12.0	1.8	.425	.00343	.000117	.4	.75
1.1	19.	2.7	.55	.00325	.0000955	.4	.75

CHART B-1

The values chosen for K_T and K_a were .0035 and .00012 respectively.

Large amounts of noise appeared in the outputs of these instruments. The accelerometer always had spikes of noise whose magnitude was comparable to its signal.

BIBLIOGRAPHY

1. D. P. Lindorff, "Control of Nonlinear Multivariable Systems," NASA, CR-716, Wash. D. C., Feb. 67; I.E.E.E. Trans. AC 12, N. 5, pp 506-515, Oct. 1967.

2. R. V. Monopoli, "Engineering Aspects of Control System Design via the 'Direct Method' of Liapunov," NASA, CR-654, Washington, D. C., Oct. 1966.

3. T. M. Taylor, "Determination of a Realistic Error Bound for a Class of Imperfect Nonlinear Controllers," to be published as a NASA Report, 1967; and JACC Preprints (1968), University of Michigan, Ann Arbor, Mich.

4. D. W. Jorgensen, "Liapunov Control Synthesis of a Highly Resonant Plant," M.S. Thesis, Department of Electrical Engineering, University of Connecticut, 1967.