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HIGH PERFORMANCE STRUCTURES

by Ralph L. Barnett and Paul C. Hermann

Prepared by
IIT RESEARCH INSTITUTE
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HIGH PERFORMANCE STRUCTURES

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IIT Research Institute

I. INTRODUCTION

A. Objectives

The traditional problem of structural design is to select a material and distribute it in such a way that the loads are equilibrated and the functional objectives are satisfied. There is a missing link in this formulation of the design process which relegates its practice to the realm of art and precludes its existence as a scientific discipline. The missing element is an efficiency yardstick which measures a design with respect to its weight or cost. By imposing the additional requirement that a structure be "optimum," the character of the traditional design problem is changed; indeed, the process of guessing at the required geometry is replaced by a systematic procedure which determines certain of the open dimensions of the structure.

Unfortunately, the available methods for synthesizing optimum geometries do not select the best possible structures, but only the best of a certain class of structures. We note, for example, that the optimum solid column is usually inferior to an ordinary thin wall column. It is at this juncture that the creative process is needed; needed to give us that "quantum jump" which brings us to a new class of possibilities. Having used our intuition to create a different type of structure, we must soon supplement it with various optimization tools which can penetrate more skillfully into the ramifications of our brainchildren. Even the most creative people are prejudiced by their experience which is often a good servant but a bad master. As an illustration it is almost inconceivable to a structural engineer that a bar in tension could buckle. In fact, however, Biezeno and Grammel describe precisely this problem in their famous books.

The objective of this program is to study the problems of high performance structures at both the creative level and the formal optimization level. Specifically, the program calls for the development of basic optimum components, the generation of new concepts, the extension of existing capabilities, the demonstration of feasibility, the collection and critical review of available works, the development of new theories, and the clarification of apparent anomolies. Since the investigation is visualized as a continuing effort, the various activities can be expected to be at different stages of progress. This necessitates a rather heterogeneous report treating, as it does, both completed studies and those currently in progress.

In a rather personal vein, the authors have to admit to a growing disposition towards a structural game which we call "zero mass structures." The rules of the game require simply that one adopt a structural behavior model in a given circumstance and squeeze it, so to speak, until it yields a zero weight component. This may, for example, require that some dimensions transcend the finite or that some physical and structural laws be broken which were originally excluded from the behavior model. The results of such diversions are usually instructive; they often expose the shortcomings of our assumptions or models; and they sometimes point to important new lines of inquiry. Several examples of zero mass structures appear in this report.

B. The Problem of Structural Design

For the purposes of this program, a structure is defined as a material object which must reliably maintain its geometry within certain limits when subjected to a loading environment. In the way of amplification, the loading will be taken as the mechanical, thermal, chemical, and electrical environments. Also, liquids and gases are added to the traditional solid structural materials.

The classical approach to structural design embraced only two functions; selection of a material from a finite number of candidates and development of the structural geometry. The introduction of prestressing greatly enlarged the designer's capabilities by enabling him to abandon the restriction that a zero state of stress correspond to an unloaded structure. Very recently the technique of proof testing was introduced in a special way which enables the designer to select the stronger elements from a set of nominally identical ones thereby capitalizing on the statistical nature of materials. During the course of the present program, it became apparent that energy must be added to the list of design parameters. Summarizing, then, the tools currently available to the designer are:

1. Selection of Materials from a Finite Number of Candidates
2. Development of the Structural Geometry
3. Prestressing
4. Proof Testing/Statistical Screening
5. System Energy

C. Summary of Results

Adopting the five general design tools, we have undertaken the study of optimum tension members, columns, and trusses. Some of our results are abstracted in this section.

1. Design Philosophy Based on Proof Testing or Statistical Screening. - A simple design philosophy is introduced which turns the characteristic scatter in fracture strengths and yield strengths into an asset. It does this by focusing its attention on the strong elements in a population rather than the weak ones. The problem, of course, is to identify the strong members and we discuss two distinct methods for doing this. The first of these considers the consequences of screening out all weak elements by means of a destructive proof test. The idea, which is applied to brittle

state materials, is that members which survive the proof test are stronger and more reliable than those in the original population. When nondestructive test methods are available, as they are for establishing yield strength, weak elements need not be eliminated; all members are monitored and labeled with their individual strength values. With this information the designer may select high strength members when he demands minimum weight; he may choose low strength bulky members for elastic stability problems; or he may utilize members in a degree proportional to their actual strength. In this latter case, for example, a collection of A7 structural steel tension members can be coordinated to provide an average resistance of 40 ksi as opposed to the deterministic minimum strength value of 30 ksi. Within this collection we will find a few members which provide nearly double the minimum stress level.

In the broadest terms, proof testing recasts the performance aspects of brittle design into an economic framework. When no regard is given to cost, it is readily apparent that unlimited strength distribution functions lead to components of infinite strength, 100 percent reliability, and zero weight. Even with limited distributions one can obtain exceptionally high strength and, therefore, very lightweight structures. Furthermore, application of the return period concept always leads to the highly desirable legal position of attaining components which are 100 percent reliable. The question, then, is, "What must we pay for these achievements?"

Once a proof testing capability is assumed, it is a straightforward matter to relate behavior to cost. It then becomes possible to select structural weight in a rational way. Furthermore, we are able, for the first time, to uniquely identify a best material within the framework of cost. Finally, an optimum structural geometry can be defined in terms of minimum cost; whereas, all geometries are equally good from the standpoint of weight and strength when the materials used have unlimited strength distributions.

2. <u>Size Effect in Ductile Tension Members</u>. - A statistical load-redistribution model for ductile tension members is proposed that reflects the influence of both length and cross-sectional area on member strength and reliability. The mathematical consequences of this model are explored in an exact and in an asymptotic framework. The variability of the hypothesized member is shown to decrease with increasing length or area; the most probable value of yield stress decreases with length and increases with area.

3. <u>Pressurized Column</u>. - The concept of prestressing is used to eliminate local buckling in a tubular column. Affecting the prestress by pressurizing the member, we obtain what is certainly a most remarkable column. In the first place, it represents the lightest column yet proposed for low structural indices; it can easily be several orders of magnitude lighter than the tension tied column which is also prestressed. Next, its weight is proportional to PL (load times length) which makes it unique in the column family where we normally find $P^{\alpha}L^{\beta}(\alpha < 1, \beta > 1)$. Because of this property we may, for the first time, design realistic Michell structures for low structural indices. Another provocative property of the prestressed column is that its weight is independent of the modulus of elasticity E of the column material. To be sure, higher E's lead to less bulky members. Several other special characteristics can be noted; much thinner gages may be used in the prestressed column compared with the unprestressed or conventional one; the prestressed column may be foldable; and finally the extremely simple design procedure avoids any dependance on the highly controversial classical buckling theory for thin walled circular cylinders.

4. <u>Classical Column Design</u>. - Ignoring the possibility of local buckling, as is the custom in the classical problems, two column designs were considered which required the optimum longitudinal distribution of a specified total mass to prevent Euler buckling.

The first of these treated the thickness distribution in a column with a bounded diameter. A simple approach, based on the constant bending stress property of optimum columns, replaced the usual variational calculus method normally used to attack this problem. Our second problem dealt with the classic simply supported solid column which is approached by means of dynamic programming. This effort is preparatory to the study of optimum columns constrained by yielding and local buckling criteria and by minimum thickness specifications.

5. Energy-Strength Tradeoff. - Sometimes energy may be converted into resisting force at very attractive force/weight ratios. Consider, for example, a hollow torus filled with a circulating fluid. The centripetal force developed by the fluid creates a tensile prestress in the walls of the torus. We observe that an increase in the fluid velocity increases the tensile prestress without increasing the weight of the system. Now, in the limit we can imagine an infinite fluid velocity producing any desired force with a vanishingly small quantity of fluid. The practical limitations which attend the application of these ideas to near perfect fluids such as liquid helium have not yet been explored; but, theoretical prestressed columns of vanishing weight can be forecast.

Certain technical problems have been dealt with in connection with both pressurized and "fluid flowing" columns. In particular, it has been shown that resistance to Euler buckling is not contributed by gas or liquid pressure or by the flow of fluids. That is, the Euler load of a column-fluid system is independent of the static pressure or the kinetic energy of the fluid. Buckling resistance is derived in the classical manner from the stiffness of the system.

6. Design of Trusses for Minimum Weight and Deflection. - The stiffness/weight ratios of statically determinant plane or space trusses are maximized by adjusting their bar areas and by optimizing their configurations. When minimum bar areas are

specified together with the outline of a truss, a simple non-linear programming problem is obtained which yields a global optimum. In the pure deflection problem where minimum bar areas are not assigned, three cases are encountered. In one, no physical solutions exist; in the second, a unique set of bar areas are obtained which represent the absolute minimum weight design for a specified deflection or conversely; and in the last, a degenerate case is obtained in which positive, negative, or zero deflection can be achieved at a node with an infinite number of truss designs of vanishing weight. Under very special circumstances the minimum deflection trusses display uniform stresses. Here, the optimum truss configuration corresponds to a Michell structure designed for equal tensile and compressive stresses. In general, however, the truss outline may be adjusted to produce the degenerate case in which any deflection is obtainable with structures of arbitrarily small weight.

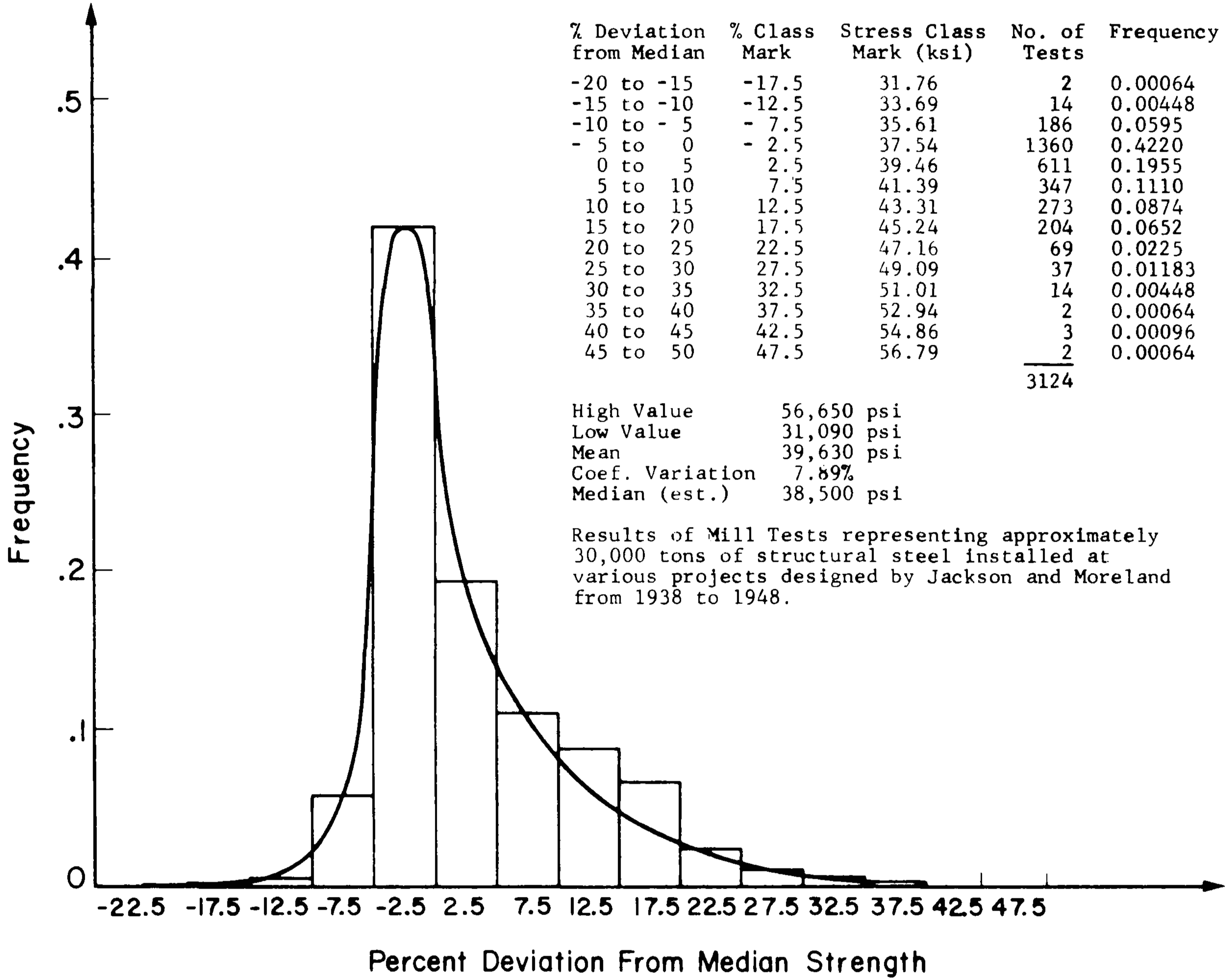
II. TENSION MEMBERS

A. Statistical Nature of Strength

A design philosophy dealing with fracture strength and yield strength is propounded in this chapter. Using the simple tension member for illustration, the proposed procedures lead to practical tension members of lighter weight and lower cost and to theoretical members of zero mass. The basis of these procedures is the recognition that strength is statistical in nature.

When nominally identical tension specimens are tested to yield or fracture, these strength values are characteristically spread over a wide range which appears to broaden as more and more elements are sampled. A typical strength distribution is illustrated in Figure 1 where a histogram and its associated probability density or frequency curve are shown for the yield strength of A7 structural steel specimens. These results represent 3124 mill test specimens that were drawn from 30,000 tons of structural steel over a period of ten years. It should be noted that two test results fell below the specified minimum of 33 ksi; the lowest value was 31,090 psi.

The familiar "bell shaped" frequency curve is not the most meaningful way of describing a strength distribution; the closely related cumulative distribution function is preferable since it provides the engineer with the very important tradeoff relationship between strength and reliability. The solid line in Figure 2 illustrates a typical distribution curve. For any stress value along the abscissa, the ordinate to the curve gives the probability of failure. The solid curve has been limited on the left and right at values of stress corresponding to the zero probability and the 100 percent probability strengths. For stress levels below the zero probability strength, no failures can occur; for stress levels above the 100 percent probability strength, there can be no survival. For many materials, it appears that the zero probability stress approaches its physical lower bound zero.



% Deviation from Median	% Class Mark	Stress Class Mark (ksi)	No. of Tests	Frequency
-20 to -15	-17.5	31.76	2	0.00064
-15 to -10	-12.5	33.69	14	0.00448
-10 to - 5	- 7.5	35.61	186	0.0595
- 5 to 0	- 2.5	37.54	1360	0.4220
0 to 5	2.5	39.46	611	0.1955
5 to 10	7.5	41.39	347	0.1110
10 to 15	12.5	43.31	273	0.0874
15 to 20	17.5	45.24	204	0.0652
20 to 25	22.5	47.16	69	0.0225
25 to 30	27.5	49.09	37	0.01183
30 to 35	32.5	51.01	14	0.00448
35 to 40	37.5	52.94	2	0.00064
40 to 45	42.5	54.86	3	0.00096
45 to 50	47.5	56.79	2	0.00064
			3124	

High Value	56,650 psi
Low Value	31,090 psi
Mean	39,630 psi
Coef. Variation	7.89%
Median (est.)	38,500 psi

Results of Mill Tests representing approximately 30,000 tons of structural steel installed at various projects designed by Jackson and Moreland from 1938 to 1948.

Fig. 1 YIELD STRENGTH DISTRIBUTION OF A7 STRUCTURAL STEEL

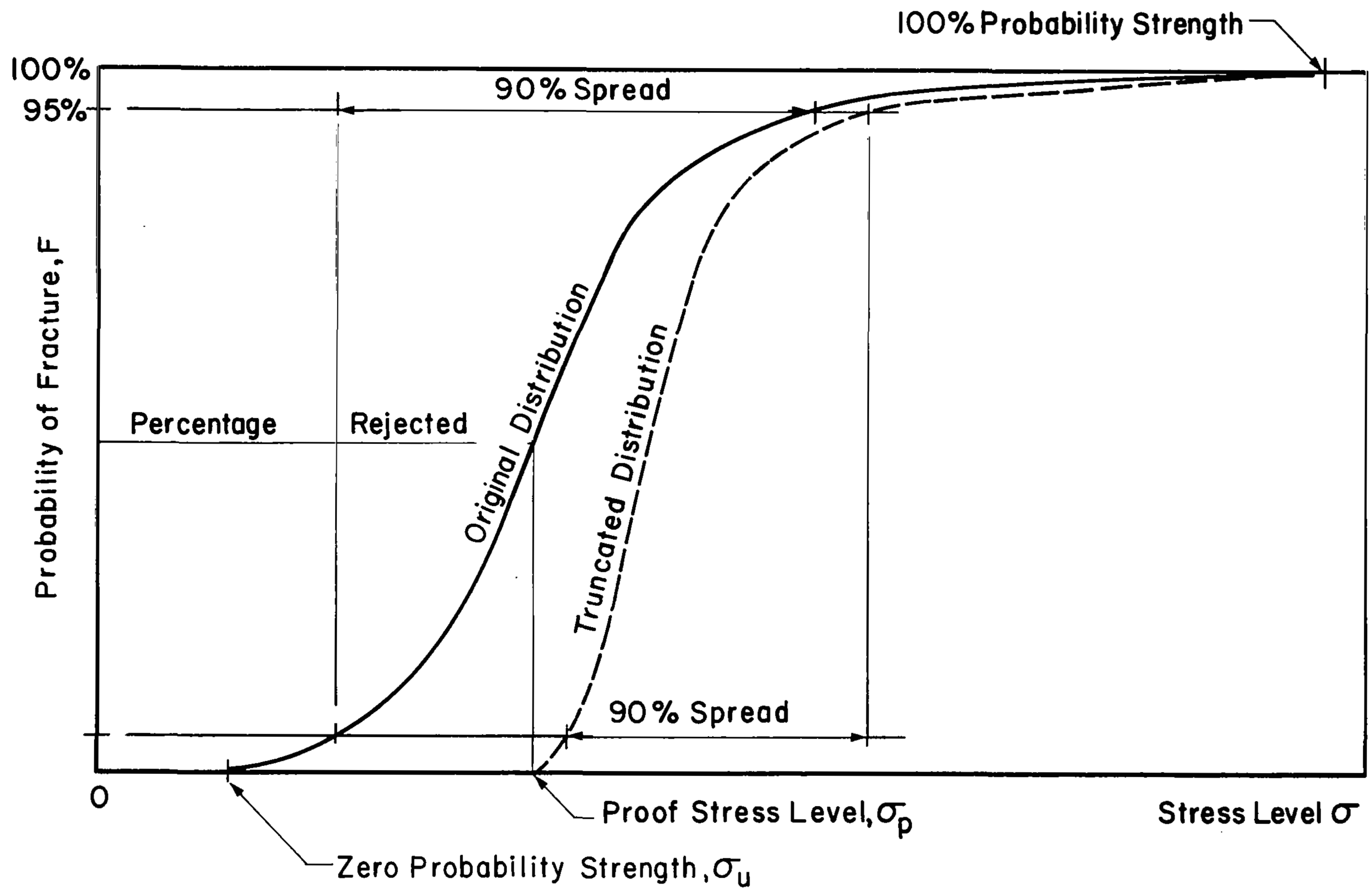


Fig. 2 TYPICAL CUMULATIVE PROBABILITY CURVE FOR THE STRENGTH OF AN ELEMENT

In the case of the 100 percent probability stress, there does not appear to be an obvious upper bound; however, it may be conjectured for brittle materials that strengths as high as the theoretical molecular strength are possible in view of recent evidence that single crystal filaments approach such stress levels.

B. Design Philosophy

The classical approach to structural design in ductile materials is based on a deterministic description of material response which either incorporates statistical behavior implicitly or ignores it altogether. During the last decade we have seen the growth of statistical design procedures which recognize the true nature of strength and account for it explicitly. This new point of view is almost indispensible for analyzing the integrity of brittle state materials.

In the authors' opinion, both the deterministic and recent statistical theories emphasize the negative aspects of strength, i.e., those strength values associated with low failure probabilities. A procedure is proposed in this section which looks to the higher strength values in a distribution and the possibilities of using them. We shall briefly outline each of the three design philosophies just identified.

1. Specified Minimum Strength. - The classical deterministic design theory is based on one of two concepts: either a strength value exists which leads to failure if exceeded and insures safety when unreached or a non-zero minimum strength value exists (zero probability strength). In either case such values are designated as the minimum specified strength for a material and they become the basis for design. Let us speculate for a moment on the yield strength distribution for A7 structural steel as reported in two test series; that described in Figure 1 and another involving 850 tests which may be found in Freudenthal's paper (ref. 1).

In both series the lowest observed value was 31.1 ksi which is below the specified minimum of 33 ksi. The value 33 ksi is clearly unconservative and will lead to one failure in a thousand specimens which is an intolerable reliability in most applications. As a matter of interest we note Freudenthal's remark, "Considering that the results of mill tests are consistently higher than those of standard tests as a result of the higher testing speeds, it appears that the specification of a minimum of 27,000 psi to 28,000 psi might be more reasonable if the probability of values falling below the minimum is to be lower than 10^{-3}."

The ductile metals usually display high values of the zero probability strength. Brittle materials, on the other hand, often show minimum strength values which are quite low even when their average strengths are considerable. When one adopts the deterministic design theory, it is quite clear that an accurate determination of the zero probability strength is desirable. Error in the determination is compensated, we hope, by the traditional safety factor; the allowable tensile strength of A7 structural steel is 20 ksi providing a safety factor of 1.65. Using methods described in the next section, we have estimated the zero probability strength of the data in Figure 1 as 30 ksi.

2. Reliability Design. - When materials display low "zero probability strengths," economy demands that some risk be assumed by designing with higher strength values. No alternative procedure exists, of course, when the zero probability strength is zero. Using the cumulative distribution curve, which by definition is strictly monotonic, the engineer can select the strength associated with any failure probability F; the lower the reliability (1-F) the higher the strength. The use of such a curve in design differs from the usual statistical applications which are concerned with values of the variate close to the mean value. Here, the

demand for structural components of high reliability forces the designer to deal with the low failure probabilities associated with the lower portion of the distribution curve.

Unfortunately, because of the rareness of extreme events, a suitable definition of the lower distribution tail will require an enormous amount of data. The conventional and more economical alternative to such a prospect is to find an analytic expression of a distribution function which closely describes the available data and then to extrapolate to find the stresses which correspond to low failure probabilities. Indeed, this is the only possible procedure for defining the zero probability stress. We hasten to point out, however, that regardless of the goodness of fit obtained for the existing data, the behavior at the lower distribution tail will always remain a mystery.

Considering this matter a little further, we must remember that a distribution function represents the behavior of an infinite amount of data. In spite of this, however, we must try to describe it, using only a finite number of tests. When various samples of strength data are drawn from the same infinite population, the distribution curves for each sample will be different. Consequently, the estimate of the zero probability strength from each sample will be different and the collection of such estimates will themselves have a distribution. Thus, we have not really escaped a statistical problem when we adopt the deterministic theory, we just sort of ignor it.

It is our opinion that the most perplexing problem in statistical strength theory is the description of the lower tail of the distribution curve. The estimated behavior in this range is quite sensitive to the precise form adopted for the trial distribution function. This, of course, must be of the type that is limited on the left. We note that Freudenthal fit the distribution of A7 steel with a log-normal distribution function that was limited on the left at zero. Even at stress levels as large as 33 ksi his probability of failure estimate of 0.01 is off by an order of magnitude from the observed data estimate of 0.001.

One of the most important implications of strength variability can be found in a "size effect" or dependence of strength upon volume. To see this let us consider a basic element or building block with the failure probability curve shown in Figure 2. We now visualize a structural component constructed from n such elements in which the i^{th} member sustains a stress σ_i. Clearly, to find the strength of a component we must relate the overall strength to that of the elements. As an illustration, in a perfectly brittle material failure at any point in a specimen necessarily constitutes overall failure of the specimen. We represent this behavior with the series or chain model shown in Figure 3a. Here, the probability that the chain will survive, $1-F_s$, is equal to the probability that the various links will simultaneously survive,

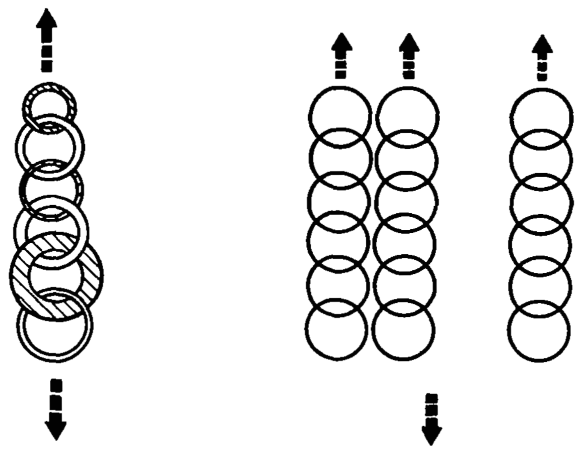
$$1 - F_s = (1-F_1)(1-F_2)\cdots(1-F_n) \tag{1}$$

For nominally identical links subjected to different stresses we obtain,

$$1 - F_s = \prod_{i=1}^{n} \left[1-F(\sigma_i)\right] \tag{2}$$

where F is the fracture probability of our basic element. The size effect may be inferred from the observation that the number of links n is proportional to the length of the chain.

When a basic element fails in most materials, alternative load paths are provided and component failure does not necessarily occur. A typical load redistribution model is shown in Figure 3b which indicates that failure of a link removes one strand from the cable; overall failure will not take place if the surviving strands can equilibrate the load. Clearly, the series model



A. Series Model B. Series-Parallel Model

FIG. 3 LOAD REDISTRIBUTION MODELS

provides no load redistribution. Consequently, if we adopt this model for real materials we will either be exact in our predictions or we will underestimate the strength of our components which are assumed to fail when only local failure has occurred. For this reason the series model can be used as the basis of a conservative analysis theory applicable to any material regardless of its actual load redistribution behavior. For ductile materials such a theory is too conservative; for brittle materials which act predominately in a series fashion it is more representative.

To illustrate some of the features of statistical design, we shall consider the problem of proportioning a simple tension member fabricated from a brittle (series) material whose fracture strength distribution is given by the Weibull distribution function (ref. 2)

$$F = 1 - \exp\left[- \frac{V}{v}\left(\frac{\sigma-\sigma_u}{\sigma_o}\right)^m\right] \qquad \sigma \geq \sigma_u$$

$$= 0 \qquad \sigma \leq \sigma_u \tag{3}$$

where V is the volume of a tension member, v is a unit volume, and σ_o, σ_u, and m are statistical parameters and constants of the material. By rewriting Equation (3) we can express the resistance σ as a function of the volume, the material constants, and a specified probability of fracture F, thus,

$$\sigma = \sigma_u + \sigma_o\left(\frac{v}{V}\right)^{1/m}\left[-\log(1-F)\right]^{1/m} \quad \text{...resistance} \tag{4}$$

where the term in the brackets is always non-negative since the survival probability (1-F) is always in the closed interval zero to one. Now, the applied stress on a tension member of length L and cross-sectional area A is

$$\sigma = \frac{P}{A} = \frac{PL}{V} \quad \cdots \quad \text{applied} \tag{5}$$

If we plot the member volume against the applied and resisting stresses as in Figure 4, we find that the smallest admissible volume is obtained when the two stresses are equal. This condition results in the following algebraic equation from which we obtain the minimum volume.

$$(W/\rho)\sigma_u + (W/\rho)^{(m-1)/m}\sigma_o\left[-v\ \log(1-F)\right]^{1/m} - PL = 0 \qquad (6)$$

where W is the member weight and ρ is the weight density.

From Equation (6) we find that the weight of a tension member designed for 100 percent reliability (F=0) is given by

$$W = \frac{PL}{(\sigma_u/\rho)} \qquad (7)$$

This, of course, is the deterministic design weight based on the zero probability strength σ_u. At the other extreme, Equation (6) gives us a zero weight member when we assume the greatest possible risk, F=1. This latter result is not unexpected since the Weibull distribution function requires an infinite stress to produce 100 percent fracture probability. For values of F between zero and unity, Equation (6) must in general be solved numerically for the tension member weight. Explicit solutions may be found, however, in the three special cases where $\sigma_u = 0$, m = 1, or m = 2. Taking $\sigma_u = 0$, for example, the weight becomes

$$W = \frac{1}{\left[x_o^{m/(m-1)}/\rho\right]} \frac{(PL)^{m/(m-1)}}{\left[-v\ \log(1-F)\right]^{1/(m-1)}} \qquad (8)$$

When the load, length, and reliability of a tension member are specified, one can readily evaluate various materials by comparing their associated W's. This procedure is illustrated in Table I using three materials. Because m/(m-1) is approximately unity for reasonably large values of m, the loading index term (PL) in Equation (1)does not greatly influence weight comparisons among materials. Furthermore, when $m \to \infty$, the weight approaches $W = PL/(\sigma_o/\rho)$ where σ_o/ρ is the specific tensile strength of a classical deterministic material.

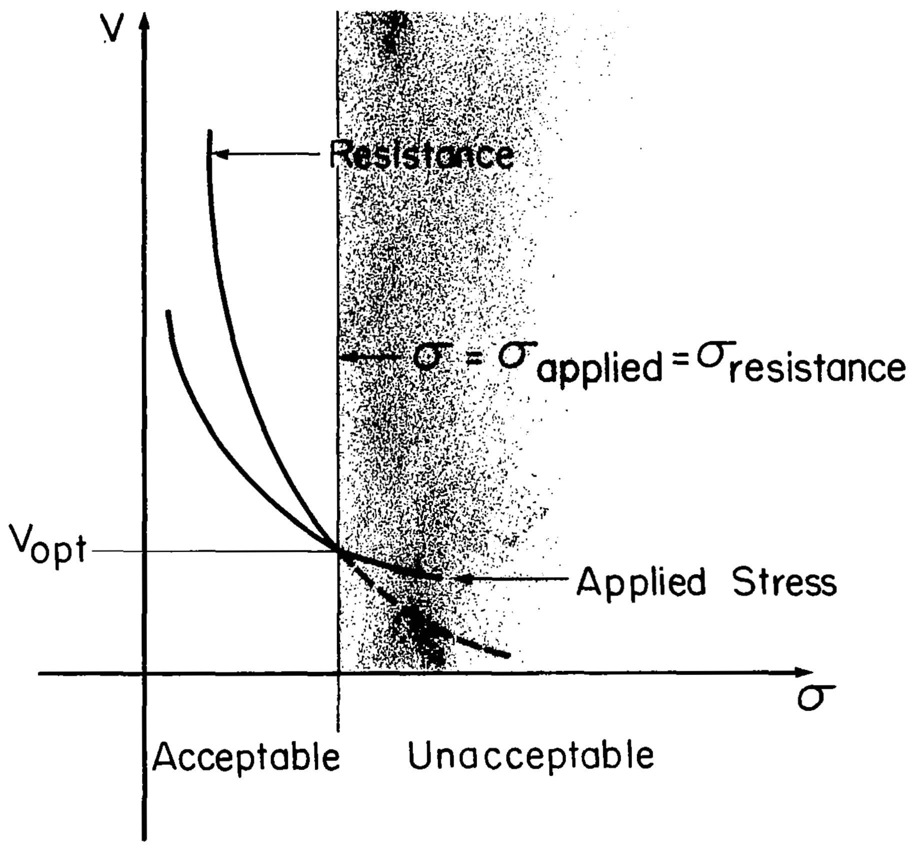


FIG. 4 MINIMUM VOLUME TENSION MEMBER

Table I

TENSION MEMBER WEIGHTS
(P = 1000 lb, L = 20 in., F = 0.01)

Material	m	σ_0 psi	W lb
Plaster	7.7	1,680	3.1
Beryllium oxide	7.25	7,800	0.66
Porcelain-white glazing	16.2	13,220	0.176

3. Statistical Screening or Grouping. - One of the most provocative characteristics of the distribution curve is that it indicates the existence of a few extraordinarily strong members in the population. Our problem then is to identify and utilize these strong members. This may be done in two ways depending on whether destructive or nondestructive methods are available for gaging the strength of an element. The destructive procedure eliminates the weak elements leaving a truncated distribution of strong members. When nondestructive test methods can be used one simply monitors every element and records its strength for future reference. Each procedure is discussed in the following sections.

C. Destructive Testing (Review)

Effective nondestructive methods are not presently available for determining the fracture strength of brittle state materials. In such circumstances the strong members in a population are indistinguishable from the weak ones which are also present and safety demands the assumption of low strength for all members. Thus, not only are weak elements intrinsically undesirable, but their presence precludes the full utilization of the strong ones. For this reason, it seems expedient to seek some method of screening out the weaker elements and a mechanical proof test appears to provide just such a screening operation.

The dashed line in Figure 2 is the distribution curve obtained by eliminating all of the elements which fail under a proof stress of σ_p. If the components which survive the proof test are unaffected by the test, that is, if no damage is accumulated, the truncated distribution function $F_p(\sigma)$ for these survivors can be related to the original distribution function $F(\sigma)$ in a manner originally described by Weibull. The probability of failure at a stress σ in the truncated population is equal to the ratio of the number of failures at stress σ in a sample of size N, $NF(\sigma) - NF(\sigma_p)$, and the total number of survivors, $N - NF(\sigma_p)$. Hence,

$$
\begin{aligned}
F_p(\sigma) &= \frac{F(\sigma) - F(\sigma_p)}{1 - F(\sigma_p)} \qquad && \sigma \geq \sigma_p \\
&= 0 && \sigma < \sigma_p
\end{aligned}
\tag{9}
$$

When structural elements are proof tested, the survivors show greater strength at every reliability level and less variability in strength. The shifting of the truncated curve to the right of the parent curve in Figure 2 illustrates the increase in strength; the constricted scatter between the 5 and 95 percent probability levels indicates the decreased variability. The improvement in strength and scatter with increasing proof stress follows immediately from Equation (9), i.e., $F_p(\sigma) \leq F(\sigma)$ and $F_p'(\sigma) \geq F'(\sigma)$.

The benefits of proof testing continue to increase until σ_p is taken as great as the 100 percent probability stress, at which point, all the survivors assume this strength. The advantages of high proof stress levels are, however, attended by economic disadvantages. Specifically, the number of weaker elements that are removed from the population increases with the proof stress level. Figure 2 indicates that the percentage of rejected elements $F(\sigma_p)$ is found from the parent population at a stress σ_p. Clearly then, when σ_p approaches the 100 percent probability stress, the number of specimens screened from the population approaches 100 percent.

Now then, proof testing of a specific structural component provides the designer with an additional parameter with which he may achieve higher strength and reliability at the expense of greater scrap. When no regard is given to cost, it is readily apparent that unlimited distribution functions lead to components of infinite strength, 100 percent reliability, and zero weight. Even with limited distributions one can obtain exceptionally high strength and, therefore, very lightweight structures. Furthermore, when circumstances permit the proof test to be matched identically with the actual loading, the surviving components are 100 percent reliable. The question, then, is: "What must we pay for these achievements?"

A rational approach to this question can be made using a concept from extreme value statistics called the return period. The return period predicts the number of prototypes required, on the average, to produce one prototype which is satisfactory (ref. 3). Its formula is simply the reciprocal of the survival probability, i.e., $1/\left[1-F(x)\right]$. To utilize the return period as a design tool, one must have some means of selecting the good parts from the bad, that is, a service or proof test must be performed on the various prototypes. If the proof test does not damage the one good prototype which is present in the group predicted by the return period, the resulting structure is 100 percent reliable. To illustrate the use of the return period we shall consider two fundamental problems; the cost-strength problem and the cost-weight problem.

a. Cost-Strength Tradeoff: In our first example we shall develop the relationship between the strength and cost of a perfectly brittle tension member of length L_2, volume V_2, and cross-sectional area A_2. We begin by establishing the member's strength distribution $F_2(f_2)$ where f_2 is the resisting force of the member. This may be done by fracture testing tensile specimens of length L_1, volume V_1, and area A_1. Recording their strengths f_1, we can plot the specimen strength distribution $F_1(f_1)$. Now, F_2 and F_1 may be related through Equation (2) where the volume V_2 is considered to contain V_2/V_1 volumes of V_1. Thus,

$$1 - F_2(f_2) = \left[1-F_1(f_1)\right]^{V_2/V_1} \tag{10}$$

where

$$f_2 = f_1(A_2/A_1) \tag{11}$$

Writing the return period in terms of F_1 and f_2 we obtain the desired cost-strength relationship,

$$\text{Cost} \sim \text{Return Period} = \frac{1}{\left[1-F_1(f_2A_1/A_2)\right]^{V_2/V_1}} \tag{12}$$

A typical plot of this equation is shown in Figure 5 where we observe that the region of rapid rise in cost for only a modest improvement in integrity is characteristic of the proof testing procedures. This figure enables one to select the greatest strength consistant with his budget.

When the return period approaches infinity, the strength f approaches its 100 percent reliability strength which is either unbounded or of the order of magnitude of the theoretical strength of the material. This is true of any design, and as a consequence, all designs can be made for all practical situations to have the same strength. Therefore, the real criterion of the best design is one that achieves a given strength at the least cost.

b. Cost-Weight Tradeoff: In those instances where we can reflect the relationship between the dimensions of a component and its reliability through a simple formula, the return period provides a powerful design tool. For example, the total volume of material G which is required to produce one acceptable component of volume V is given by the product of the return period and the component volume,

$$G = V/(1-F) \tag{13}$$

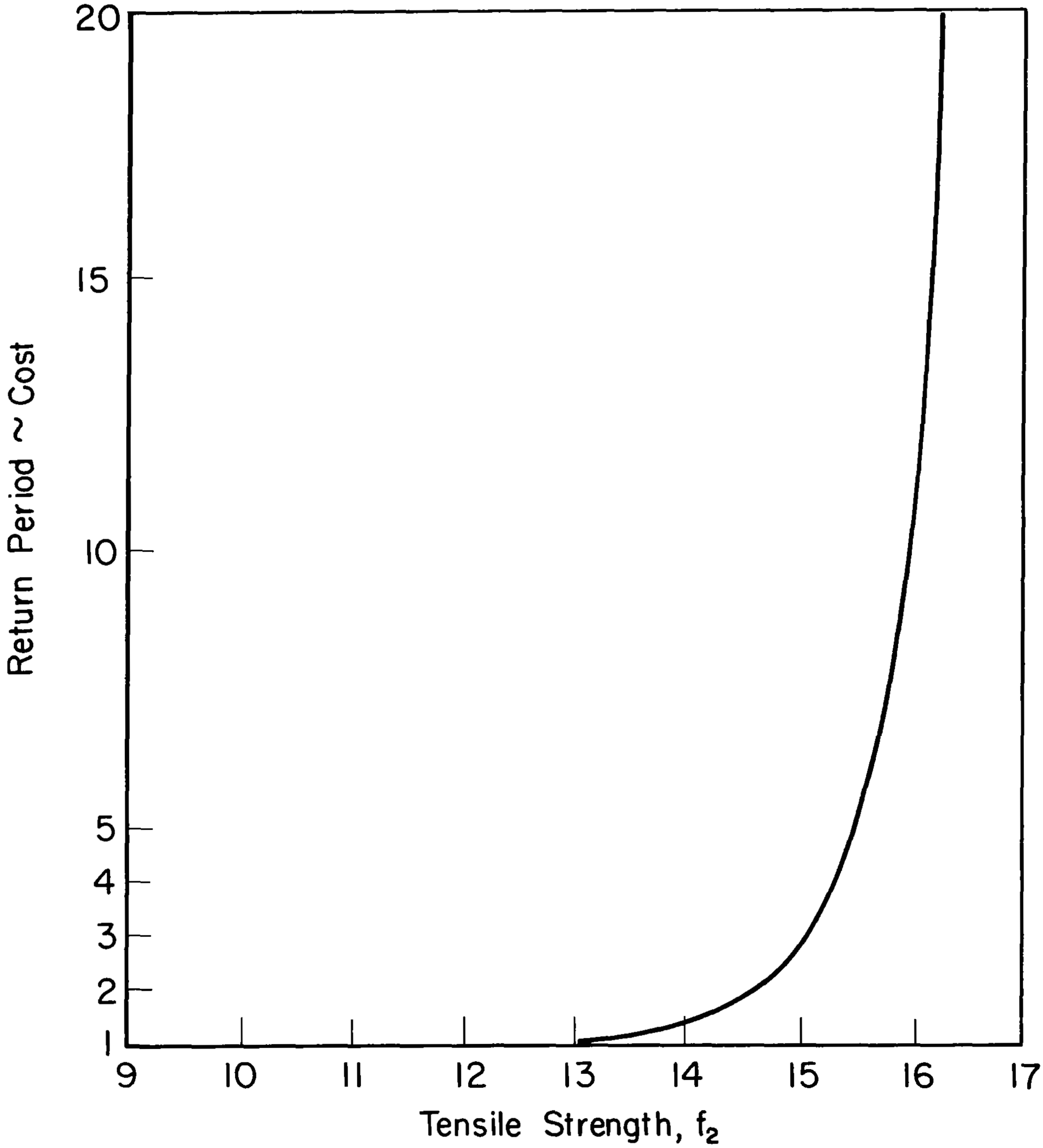


FIG. 5 VARIATION OF RETURN PERIOD WITH DESIGN STRENGTH

In those cases where cost is proportional to volume, a minimum cost design is obtained by minimizing G. In the case of a tension member of length L which is carrying a load P and which is constructed from a Weibull material, the total volume becomes

$$G = V \exp\left[\frac{V}{v}\left(\frac{x-x_u}{x_o}\right)^m\right] = V \exp\left[\frac{V}{v}\left(\frac{\frac{PL}{V} - x_u}{x_o}\right)^m\right] . \tag{14}$$

Setting the first derivative of G with respect to V equal to zero, we obtain

$$vx_o^m + V\left(\frac{PL}{V} - x_u\right)^m - mPL\left(\frac{PL}{V} - x_u\right)^{m-1} = 0. \tag{15}$$

In general, stationary values of Equation (15) must be found numerically; however, closed form solutions exist for m = 1, m = 2, and $x_u = 0$. For example, when $x_u = 0$ we obtain

$$V_{opt} = \left[\left(\frac{PL}{x_o}\right)^m \frac{(m-1)}{v}\right]^{1/(m-1)} \tag{16}$$

The associated minimum total volume and stress are respectively,

$$G_{opt} = \left[\left(\frac{PL}{x_o}\right)^m \frac{e(m-1)}{v}\right]^{1/m-1)} \tag{17}$$

$$x_{opt} = \left[\frac{x_o{}^m v}{PL(m-1)}\right]^{1/(m-1)} \tag{18}$$

The second derivative of G evaluated at V_{opt} is given by

$$\left.\frac{d^2G}{dV^2}\right|_{V=V_{opt}} = (m-1)\left[\frac{ev}{(m-1)(PL/x_o)^m}\right]^{1/(m-1)} > 0 \tag{19}$$

when m > 1.

For m greater than unity this quantity is positive and V_{opt} leads to a relative minimum of G.

It can also be shown that for m = 2 and $x_u \neq 0$ we achieve a relative minimum for G when

$$V_{opt} = \frac{\left[v^2 + 4\left(\frac{x_u}{x_o}\right)^2 \left(\frac{PL}{x_o}\right)^2\right]^{1/2} - v}{2\left(\frac{x_u}{x_o}\right)^2} \qquad (20)$$

For m = 1 and $x_u \neq 0$, Equation (15) leads to a relative maximum at $V = vx_o/x_u$. Equation (15) will probably lead to minima for all $m > 1$ and $x_u > 0$.

Taking typical values for the parameters in Equation (14) we have plotted the cost G against the component volume V in Figure 6. We observe that when V_{opt} is too large for a particular application, a smaller component volume can be obtained at the expense of greater total volume. Equation (13) can be solved numerically for the smallest V associated with any specified G which is greater than G_{opt}. We hasten to point out that the general approach is applicable for cost criteria that are more sophisticated than Equation (13).

As a final remark on proof testing we shall comment on the extrapolation-interpolation aspects of the cumulative distribution function. The problem of describing the lower tail of the distribution curve takes on an entirely different character when the population is truncated. Here, the proof stress σ_p becomes the zero probability strength and, therefore, values of stress corresponding to low-failure probabilities can be found by interpolation. Furthermore, the density of data in the neighborhood of the proof stress becomes relatively high for even moderate levels of σ_p, and, consequently, the lower tail can be accurately defined with a limited amount of data.

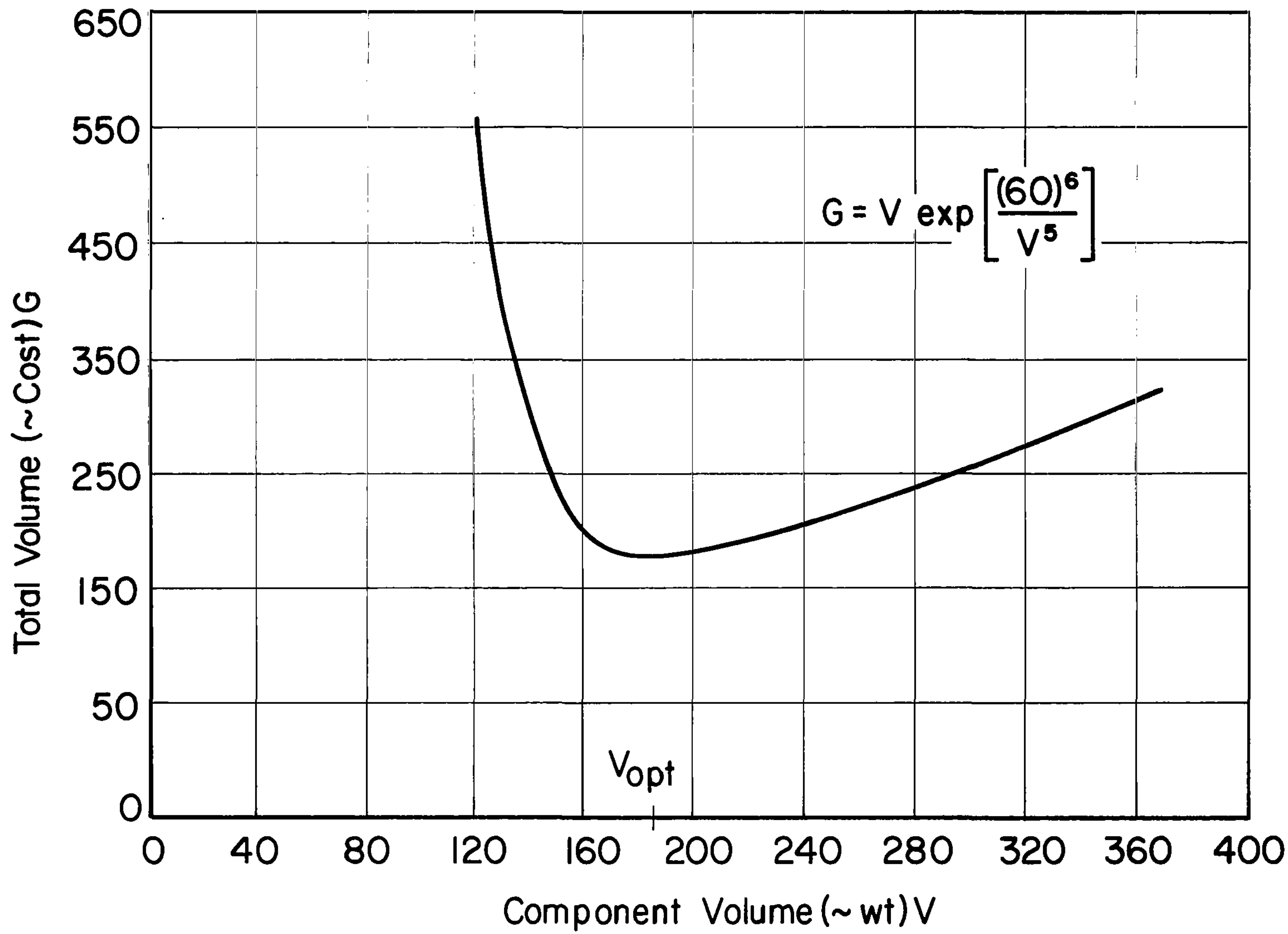


FIG. 6 VARIATION IN TOTAL VOLUME WITH COMPONENT VOLUME

D. Nondestructive Testing

The ductile tension member lends itself naturally to a nondestructive determination of its yield strength. In most materials, bringing a ductile component to incipient yield by mechanical means produces almost no degradation in its static or fatigue properties. Proceeding on this basis we shall assume that all members considered by a designer have been monitored and labeled with their individual strength values.

Two possibilities arise concerning the behavior of a collection of tension members. Either there is a member-to-member variation in strength or there are point-to-point variations in each member. We shall briefly consider each of these cases.

1. Member-to-Member Variation. - There is a possibility that all metal elements poured from a single heat have the same yield strength or at worst display a very small scatter. In such circumstances either a full scale test or a coupon test can be used to establish the strength of a member. Granting this state of affairs, why then do we find the variability shown in Figure 1 for structural steel? One explanation may be that the scatter is caused by heat-to-heat variations which occur when steel is produced in different plants and at different times.

An interesting implication arises from the possibility of experiencing heat-to-heat differences. The strength distribution expressed by the frequency curve of Figure 1 may not exist.* That is, the next 3000 tests may produce a radically different curve. Since the entire foundation of statistics rests on the assumption that the frequency curve exists, the reliability design procedures described in section II-B-2 will break down immediately in the face of this inconsistant manufacturing capability. Unless the distribution curve is substantially reproducible, predictions of the low probability strengths are impossible.

*The distribution curve of each infinite subset of the original population must be identical if the distribution in fact exists.

In the light of the nondestructive testing procedures assumed in our discussions, it is not vital that the distribution curve exist. The strength of any individual member is never in question; however, to forecast precisely the number of members falling between say 40 ksi and 45 ksi will in fact require a proper distribution curve. The distinction between the reliability and the proof testing philosophies is enormous. In reliability design any "funny business" with the distribution curve may adversely affect the strength of a component. On the other hand, a false distribution curve cannot affect integrity when members are pretested. In this latter case, however, any planning which involves group predictions such as economic strategy or logistics may be seriously jeapordized.

The concepts of minimum weight and minimum cost design can readily be explored by referring to Figure 1. This collection of data provides no information regarding an upper bound on the yield strength. As a matter of fact, the data can be effectively portrayed by distribution functions which are unlimited on the right. Indeed, there is no reason to believe that any extremely high strength value cannot be obtained if patience and funding enable one to sample a sufficiently large number of specimens. This possibility rules out minimum weight design as a goal to be pursued without consideration of cost.

Even the most cursory consideration of Figure 1 reveals the economic potential of the NDT philosophy. The rational deterministic design of a 1000 tension members would be based on a working stress of 30 ksi (zero probability strength) in each member. The actual strength of these 1000 members would correspond to the mean strength, 39,630 psi, recorded in Figure 1. Here, of course, each of the 1000 nominally identical members would operate at a different stress level and correspondingly carry a different load.

Clearly, if the cost of determining the individual yield strengths is less than the cost of the potential weight savings, 33 percent, we have achieved a net advantage.Furthermore, that portion of a safety factor which covers the uncertainty surrounding material strength can be reduced.

Because no variability is assumed within a member, we have no strength scaling problems and the foregoing observations are applicable to any type of structural steel component that is yield strength limited in its performance. The potential cost advantage when NDT is used is at least $(\sigma_m - \sigma_u)/\sigma_u$ over the deterministic design where σ_m is the mean strength and σ_u the zero probability or specified minimum strength. This is, of course, the simplest view of the problem since it weights or evaluates the value of a member in proportion to its yield strength. A more sophisticated approach might place a premium on high stress members because of their lighter weight. Further, the value of low stress members may be greater than expected because of their bulk and the important effects this may have on moment of inertia in beams and columns.

The most provocative implication of the proposed design procedure lies in the prospect of basing designs on pounds not psi. For statically determinate trusses this clearly poses no problem; but, for hyperstatic structures the cost of a member will depend upon both its strength and bulk. Minimum cost design in this framework must employ a very broad-based approach involving the entire design, fabrication, and inventory process. Our future studies will consider optimum tradeoff relationships between component weight and cost.

2. Point-to-Point Variation. - In this section we shall explore the implications of a point-to-point yield strength variation in a ductile tension member. As previously pointed out for brittle materials, such scatter in strength implies a size effect. To deal with this problem we must again relate the overall behavior of a tension member to its local behavior. To do this we shall imagine that the tension member is a collection of tension specimens for which the yield strength distribution has been obtained.

Now, we observe that yielding of a tension member requires that yielding occur throughout one or more transverse sections. We reflect this behavior in the model shown in Figure 7.

Referring to this figure, each spring-weight combination represents a tension specimen. We note the following characteristics:

1. The coefficients of static and dynamic friction are both equal to μ.

2. The maximum resistance of each slider is equal to μ times its total weight.

3. Each spring-weight combination produces an elastic-perfectly plastic type of load-deflection diagram.

4. The total weights of the individual sliders are statistically distributed. Their distribution is proportional to the specimen yield strength distribution.

5. The minimum "maximum resistance" of a slider is $W\mu$. This reflects the possibility of having a nonzero zero probability yield strength.

6. There are three links shown in Figure 7. A link cannot yield unless all of its spring-weight combinations are at incipient sliding.

7. Yielding of a four element link causes overall yielding. The link strength is equal to the sum of the corresponding four slider resistances.

8. The overall strength is equal to the strength of the weakest link.

Based on the foregoing observations, we shall discuss the ductile tension member model in its mathematical setting. First, the length of the tension member L shall be taken as the multiple n times the gage length of the specimen L_g. Also, the member area A will be taken as k times the specimen gage area A_g.

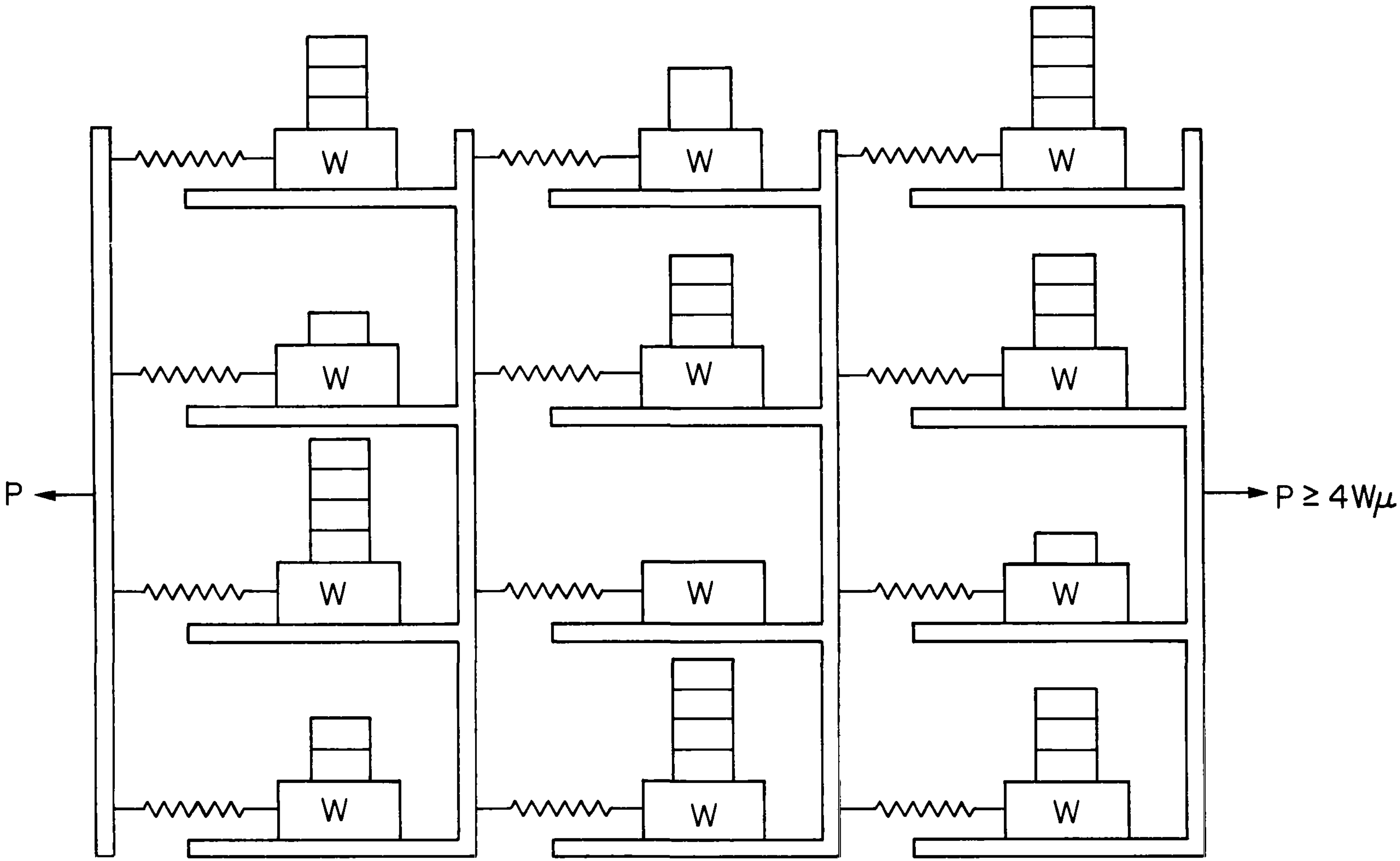


Fig.7 STRENGTH MODEL FOR A DUCTILE TENSION MEMBER

Now, focusing our attention on the behavior of a link, we take into account that it is made up of k tension specimens whose strength distribution we shall designate by $F(\sigma)$ where the statistical variate σ is the specimen yield strength. Since the link strength is the sum of the associated specimen strengths, i.e.,

$$A_g \sum_{i=1}^{k} \sigma_i ,$$

we find that the resisting link stress σ_r is

$$\sigma_r = \frac{A_g \sum_{i=1}^{k} \sigma_i}{kA_g} = \frac{\sigma_1 + \sigma_2 + \sigma_3 + \ldots + \sigma_k}{k} \qquad (21)$$

where the subscript i refers to the i^{th} tension specimen. Clearly, the resisting link stress σ_r is the average yield stress of the specimens comprising the link.

Using the known specimen strength distribution $F(\sigma)$, our job is to establish the link strength distribution $Q(\sigma_r)$. This problem is treated differently for k small and large; both cases give rise to well known statistical problems.

When a link is equivalent to only a few specimens (k... small), its distribution is studied under the topic heading "summation of chance variables." In broad terms, the distribution function $Q(\sigma_r)$ is found by integration of its corresponding frequency function $q(\sigma_r)$; this in turn is obtained by inversion of its characteristic function. This characteristic function is formed by the k^{th} power of the specimen characteristic function which is associated with the yield strength distribution function $F(\sigma)$. This standard procedure is considered in detail in reference 4; its application leads to the following complete and explicit solution:

$$Q_k(\sigma_r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-e^{-iu\sigma_r}}{iu}\left[\int_{-\infty}^{\infty} e^{i\beta u}\ f(\beta)d\beta\right]^k du + \text{constant} \qquad (22)$$

where $f(\sigma)$ is the frequency function associated with $F(\sigma)$ and where the constant is determined by the condition that $\lim_{\sigma\to-\infty} F(\sigma) = 0$. Evaluation of this integral is usually very difficult especially when k is large. In problems involving reasonable large k's an approximate asymptotic expression is available for $Q_k(\sigma_r)$.

Appealing to the central limit theorem in the case of large k, we can state:

If σ has a distribution $F(\sigma)$ with mean $\bar{\sigma}$ and variance r^2, then the distribution $Q_k(\sigma_r)$ or $(\sigma_1 + \sigma_2 + \ldots + \sigma_k)/k = \sigma_r$ is approximately normal with mean value $\bar{\sigma}$ and variance r^2/k. To show how effectively the approximation works, Figure 8 shows the exact frequency distributions $q_k(x)$ for a rectangular distribution. We note that even $q_3(x)$ looks similar to a Gauss density. It should be observed, however, that the rectangular distribution which is limited on the right and left produces "distributions of sums" which are also limited on the right and left. Clearly then, the normal distribution which is unlimited in both directions cannot represent a truncated distribution at the thresholds regardless of the size of k (approximation improves with k). Consequently, the use of the proposed model for <u>high reliability analysis</u> demands that one utilize the exact distribution of sums given by Equation (22). On the other hand the more powerful asymptotic result may be employed in design problems in which proof testing is used since these problems deal with strength values away from the zero probability strength.

Having studied the properties of a link, we now consider the behavior of the chain indicated in Figure 7. Once again we can distinguish between two problems on the basis of size; long chains (large n) and short ones (small n).

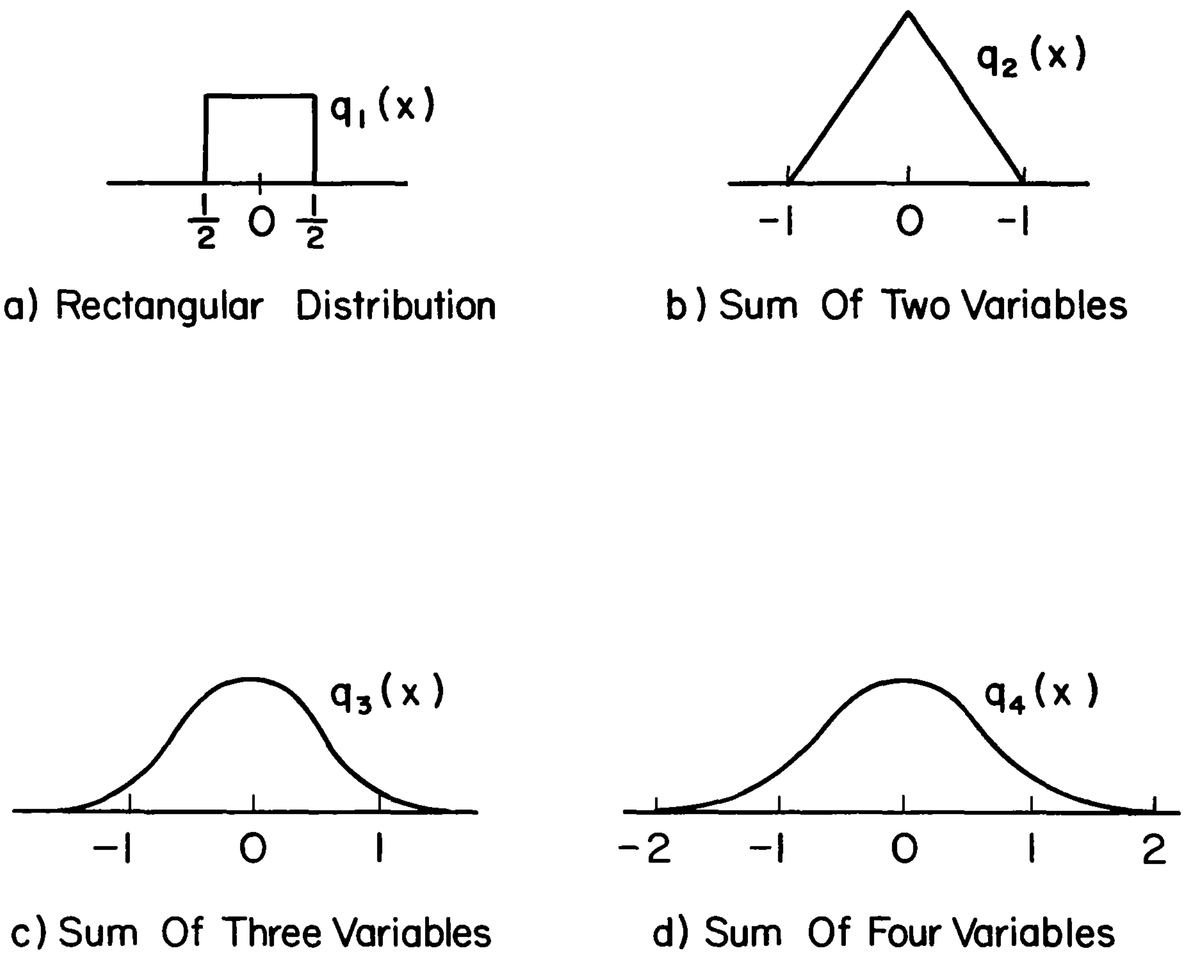


Fig. 8 FREQUENCY DISTRIBUTIONS FOR THE SUMS OF CHANCE VARIABLES DRAWN FROM A RECTANGULAR DISTRIBUTION

The statistical theory of chain behavior is treated extensively in works on Extreme Value Statistics. These works, among other things, concern themselves with the distribution of smallest values in samples of size n. Since a chain is as strong as the weakest of its n links, the application of extreme value statistics is obvious.

We shall first direct our attention to the exact description of the chain strength distribution $P(\sigma_c)$ where σ_c is the chain strength. Referring to Equation (2) and using the link strength distribution $G_k(\sigma_c)$, we obtain

$$P(\sigma_c) = 1 - \left[1-G(\sigma_c)\right]^n \tag{23}$$

Since $P(\sigma_c)$ is the distribution function for a prototype ductile tension member, we are vitally interested in its properties and how they may be affected by scaling up the length or the area of the tension element. Recalling that $G(\sigma_r)$ is approximately normal, it is quite clear that we are faced with the study of normal extremes. Since such an important problem has surely received considerable attention we might anticipate a rather elegant treatment; however, the formulas for the normal extremes are very complicated and require extensive numerical investigation. None of the fundamental theorems concerning extremes are related in a simple way to the normal distribution. To illustrate one of the difficulties, we observe that G is the cumulative distribution function for the normal distribution and that as such it is given in integral form; the values of G are tabulated in every statistics book. Consequently, to use Equation (23) we must raise tabulated values to the power n. Such a task has been performed for the largest normal extreme (ref. 5); the author has not come across a similar effort for the smallest values in a normal population.

In situations where the prototype member is considerably larger than the tension specimen, n large, it is possible to gain considerable insight into the behavior of $P(\sigma_c)$. In 1947 Epstein (ref. 6) adopted the asymptotic theory of extremes to study the smallest value in a sample of size n where n was assumed large.

Using our notation, we shall present his results for the smallest extreme normal:

Probability density function of link (k...large):

$$g(\sigma_c) = \frac{1}{\sqrt{2\pi r^2/k}} \exp - \left[\frac{\sigma_c - \bar{\sigma}}{2r^2/k}\right] \qquad (24)$$

Mode of tension member strength distribution:

$$\bar{\sigma} - \sqrt{2(r^2/k)\log n} + \sqrt{r^2/k}\,\frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}} \qquad (25)$$

Yield strength distribution of a tension member:

$$\bar{\sigma}_c = \bar{\sigma} - \sqrt{2(r^2/k)\log n} + \sqrt{r^2/k}\,\frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}} + \sqrt{\frac{r^2/k}{2 \log n}} \log \xi \qquad (26)$$

where ξ is distributed with probability density function

$$h(\xi) = e^{-\xi} \qquad \xi \geq 0.$$

Remarks on Yield Strength Distribution of a Tension Member (n...large)

1. The most probable strength value (mode) decreases as a multiple of $\sqrt{\log L/L_g}$.
2. The variance decreases as L/L_g increases and is given by

   $$\frac{\pi^2 r^2}{12\ k\ \log(L/L_g)}$$

3. The asymptotic form for normal extremes is approached with extreme slowness as n becomes large.
4. The most probable strength value increases as A/A_g increases; the mode is given approximately in the form $\bar{\sigma} - \sqrt{r^2/(A/A_g)}$ (constant)
5. The variance decreases as $1/(A/A_g)$

6. For small area members (k...small), the input to the behavior model consists of the distribution curve of a tension specimen. With large areas (k...large), the average yield strength of a tension specimen $\bar{\sigma}$ together with its standard deviation r provide the complete input data. For the A7 structural steel, for example, the asymptotic theory uses $\bar{\sigma}$ = 39,630 psi and r = 3127 psi. Theoretical studies of the ductile behavior model may require that the mean and variance be given in terms of the statistical parameters describing the tensile specimen distribution. In the case of a Weibull fit, we find that

$$\bar{\sigma} = \sigma_u + \sigma_o V^{-1/m} \Gamma(1 + \frac{1}{m}) \tag{27}$$

$$r^2 = \sigma_o^2 V^{-2/m} \left[\Gamma(1 + \frac{2}{m}) - \Gamma^2(1 + \frac{1}{m}) \right] \tag{28}$$

7. If we consider the behavior of the model as the member area approaches infinity, we find that the link strength becomes $\bar{\sigma}$ with no scatter. The member then becomes a chain with deterministic links; it behaves elastically at stresses below $\bar{\sigma}$ and it flows at stresses equal to or greater than $\bar{\sigma}$.

8. To study the behavior of long tension members we can examine Equation (23) as $n \rightarrow \infty$. With an infinitely long member we see that P = 0 when G = 0 and that P = 1 when $G \neq 0$. Hence, survival is possible only when $G(\sigma) = 0$. This implies that an infinitely long member can be realized only when the link distribution is limited on the left at a positive value (this includes a deterministic

link in the limit), i.e., if $G(\sigma) = 0$ leads to $\sigma = \sigma_u > 0$ or $\sigma = \sigma_p$ where proof testing is employed. We see then that the exact theory predicts that an infinitely long chain is deterministic with a strength equal to the zero probability strength. We note that the asymptotic theory of long chains also predicts zero variability (deterministic behavior); however, the strength prediction from Equation (25) is negative infinity as we would expect from a link distribution which is unlimited on the left.

There was not enough time during the current effort to examine the literature on size effects in ductile elements. We did find, however, a paper by Richards (ref. 7) which showed a 13 percent drop in the upper yield-point stress when the volume was increased by three orders of magnitude. Our future investigations will undertake a review of the available data on yield strength.

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III. COLUMNS

A. Prestressed Columns

Perhaps the most exasperating problem in structural design is the proportioning of simple columns for very low structural indices (P/L^2) where L is the length of a concentrically loaded column under compressive forces P acting at either end. Problems of this kind occur frequently in space structures where small compressive loads must be transmitted over large distances. Stability demands that relatively massive columns be used to carry these loads and the resulting structures characteristically entertain low stress levels throughout most of their volume. Beginning in 1773 with Lagrange, a really significant amount of work has been directed toward optimum column design. Almost all of this has been done within the classical framework, i.e., development of geometry and selection of material. In the work presented here, we shall reap a considerable advantage over classical approaches by utilizing the principal of prestressing to eliminate the possibility of local buckling.

We shall study the optimum design of a circular thin walled tubular column which is uniform throughout its length. A prestress shall be applied by pressurizing the tube to a level which is just sufficient to assure that the net axial stress in the tube walls will not become compressive upon application of the load P. This implies that the pressure acting on the cross-sectional area must provide a resisting force P, i.e.,

$$P = \pi r^2 p \tag{1}$$

where r is the tube radius and p is the absolute pressure level. The weight of the gas will now be determined.

1. <u>Properties of Gas</u>. - We begin by computing the pressure/weight ratio of a gas from the equation of state for an ideal gas,

$$pV = nRT \tag{2}$$

where p is the absolute pressure in dynes/cm^2, V is the gas volume in cm^3, n is the number of moles in gram-moles, T is the temperature in degrees Kelvin, and R = 8.31×10^7 ergs/mole K°. Noting that n is the ratio of the mass of the gas m and its gram molecular weight M, Equation (2) yields

$$p/\gamma = \frac{RT}{Mg} \tag{3}$$

where

$$\gamma \equiv mg/V \text{ ... weight density} \tag{4}$$

and g is the gravitational constant. We observe that p/γ = constant for a given material at a specified temperature. The following pressure/weight ratios are computed for T = 20° C(293°K):

Table I

PRESSURE/WEIGHT RATIOS OF SEVERAL GASES (20°C)

Gas	Formula	Gram Molecular Weight M...grams	Pressure/Weight Ratio inches $\frac{p}{\gamma} = \frac{RT}{Mg} = \frac{8.31 \times 10^7 (293)}{M(980)(2.54)}$
Air	--	28.97	3.38×10^5
Oxygen	O_2	32.00	3.06×10^5
Helium	He	4.003	2.44×10^6
Hydrogen	H_2	2.016	4.86×10^6

The efficiency of gas as a compression carrying medium can be inferred by comparing the pressure/weight ratios to the specific compressive strengths of mild steel ($\sigma_c/\rho = 1.41 \times 10^5$ in.) and beryllium ($\sigma_c/\rho = 1.35 \times 10^6$ in.).

Using the fact that p/γ = constant for a given gas and temperature, we can easily show that the weight of the gas W_g required to prestress the tubular column is independent of the pressure level or radius employed. Thus

$$W_g = \gamma V = \gamma \pi r^2 L = \pi r^2 p \frac{L}{p/\gamma} \tag{5}$$

or using Equation (1),

$$W_g = \frac{PL}{p/\gamma} \tag{6}$$

where we note the linear dependence on P and L.

2. Weight of Tube. - We shall ignore for the moment any consideration of Euler buckling and proceed to the calculation of tube weight. Two configurations will be examined: an open ended tube and a completely closed one. In each, sufficient material must be used to contain the gas pressure.

a. Open Tube: There are circumstances in which the end pressure and the gas pressure always remain equal, for example, when a column is inflated and reacted against end plates as in a jacking operation. Here, we shall assume that the closure of an open ended cylinder is provided by the device used to transmit the load to the column; we may visualize an infinitely thin membrane covering the ends. Under these conditions we must proportion a cylinder to resist only hoop stresses given by

$$\sigma = pr/t \tag{7}$$

where t is the wall thickness. For a hoop resistance σ_t and wall density ρ, the required weight of the cylinder W_c is simply

$$W_c = 2\pi r t L \rho = \pi r^2 p \; 2L/(\sigma_t/\rho) \tag{8}$$

or using Equation (1)

$$W_c = \frac{2PL}{\sigma_t/\rho} \tag{9}$$

As in the case of the gas weight we find that the tube weight is independent of the pressure level or radius and that it depends linearly on P and L. The total weight of the pressurized open end column is simply the sum $W_g + W_c$; thus,

$$W = \frac{PL}{\sigma_t/\rho}\left[2 + \frac{\sigma_t/\rho}{p/\gamma}\right] \text{... Open End Column} \tag{10}$$

b. Closed Tube: For most conventional column applications, a closed tube must be used to contain the gas under both loaded and unloaded conditions. Equation (6) gives us the weight of gas W_g required to prevent local buckling when the column is under a load P. In the unloaded state the column is in effect a pressure vessel which must contain W_g. We shall adopt an isotensoid structure for our cylinder (ref. 1); these are formed by winding fine filaments in the shape of a vessel in such a way that under load the filament stress is a constant tensile value σ.

Assuming a linear material, an isotensoid vessel under an internal pressure p will entertain a constant strain e throughout its volume given by $e = \sigma/E$ where E is the modulus of elasticity of the filaments. Consequently, the walls of the pressurized vessel will absorb a total strain energy U given by the product of the volume of the walls V_c and the constant strain energy per unit volume $\sigma e/2$; thus,

$$U = \frac{\sigma e V_c}{2} \tag{11}$$

The column will expand under internal pressure and all its dimensions will increase by a factor (1 + e). The associated change in the enclosed volume V is

$$\Delta V = V(1 + e)^3 - V \doteq 3eV \tag{12}$$

where products of e have been neglected. The work done by the expanding gas is simply $p\Delta V/2$; thus,

$$\text{Work} = p\Delta V/2 = 3epV/2 \tag{13}$$

Equating the internal and external work given by Equations (11) and (13) we obtain

$$V_c = \frac{3pV}{\sigma} \tag{14}$$

or

$$W_c = \frac{3(p/\gamma)(W_g)}{(\sigma/\rho)} \tag{15}$$

We observe that the weight of an isotensoid vessel does not depend on its size or shape; but only on the weight of gas it contains.

Substituting for W_g from Equation (6) and using the largest permissible filament stress $\sigma = \sigma_t$, the cylinder weight becomes

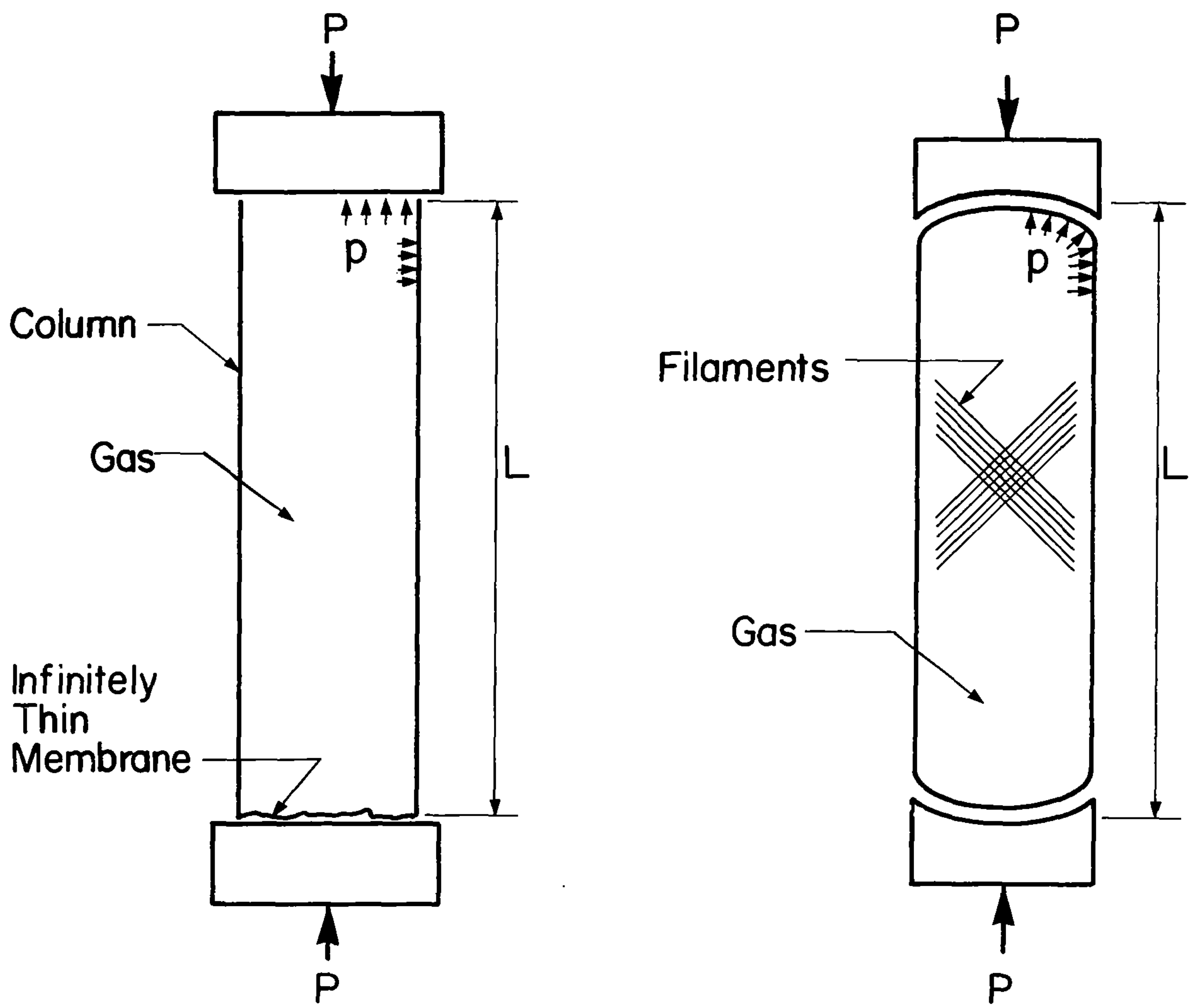
$$W_c = \frac{3PL}{\sigma_t/\rho} \tag{16}$$

Again we find that the shell weight varies linearly with P and L. Adding this weight to that of the gas, the total column weight becomes

$$W = \frac{PL}{\sigma_t/\rho}\left[3 + \frac{\sigma_t/\rho}{p/\gamma}\right] \text{... Closed Tube} \tag{17}$$

The open tube and the isotensoid columns are shown schematically in Figure 1.

3. Euler Buckling. - We have shown that the weight of a column which is pressurized to preclude local buckling is independent of its radius. To take advantage of this fact, we observe that an infinite moment of inertia I can be obtained by increasing the column radius indefinitely. Specifically, the product rt for the cylindrical column remains constant and $I = (\pi rt)r^2$ increases as the square of the radius. Since the Euler buckling resistance, $\pi^2\ EI/L^2$, is directly proportional to I, any desired resistance can be achieved without changing the column weight.



$$W = \frac{PL}{\sigma_t/\rho}\left[2 + \frac{\sigma_t/\rho}{p/\gamma}\right]$$

a) Open Tube

$$W = \frac{PL}{\sigma_t/\rho}\left[3 + \frac{\sigma_t/\rho}{p/\gamma}\right]$$

b) Closed Tube

FIG. 1 PRESSURIZED TUBULAR COLUMNS

Taking the open tube column as an example, we can easily establish a value of the radius above which all radii provide ample Euler resistance. Euler's formula for a thin walled circular tube may be written as

$$\frac{\pi^2 E(\pi r^2)rt}{L^2} \geq P \tag{18}$$

Noting, from Equations (8) and (9), that

$$rt = \frac{P}{\pi\sigma_t} , \tag{19}$$

we obtain

$$r \geq L\sqrt{\frac{\sigma_t}{\pi^2 E}} \tag{20}$$

to prevent Euler buckling. The minimum radius and the associated thickness and pressure are immediately

$$r_{min} = L\sqrt{\frac{\sigma_t}{\pi^2 E}} \quad ; \quad t = \frac{P}{L}\sqrt{\frac{E}{\sigma_t^3}} \; ; \; p = (P/L^2)(E/\sigma_t)\pi \tag{21}$$

It is remarkable that the minimum radius does not depend on the loading; for a pressurized structural steel tube, the slenderness ratio is approximately $L/r_{min} = 95$. Although the column weights were found to be uninfluenced by the modulus of elasticity of the shell, Equation (21) indicate that very low E's give rise to large radius columns with small wall thicknesses and small pressures.

4. Strength of Stable Tubes. - Disregarding the possibility of either local or Euler buckling, it is instructive to investigate the potential weight and cost savings of a pressurized tube compared to a conventional short solid column. The open ended thin

walled cylinder will be studied for several different combined stress theories. The biaxial stress state which occurs in the cylinder walls will ultimately result in a maximum hoop resistance σ_t and a maximum axial compressive resistance σ_c. On this basis the shall resistance P_s is given by

$$P_s = 2\pi rt\sigma_c \qquad \sigma_c \geq 0 \tag{22}$$

The maximum pressure attainable in the tube is

$$p = \sigma_t \, t/r \tag{23}$$

and the corresponding axial resistance of the pressure P_p is

$$P_p = \pi r^2 p = \pi rt\sigma_t \tag{24}$$

Now, the total resistance P becomes

$$P = P_s + P_p = \pi rt(2\sigma_c + \sigma_t) \tag{25}$$

Writing the shell weight W_s in terms of P we obtain

$$W_s = 2\pi rtL\rho = \frac{2PL\rho}{2\sigma_c + \sigma_t} \tag{26}$$

a. Maximum Stress Theory: Assuming a material with a square yield diagram, we can take $\sigma_t = \sigma_c = \sigma_o$. Then, Equation (26) yields

$$W_s = \frac{2}{3}\left[\frac{PL}{\sigma_o/\rho}\right] \tag{27}$$

Since $PL/(\sigma_o/\rho)$ is the weight of a solid stable strut, the pressurization leads to a 33.3 percent savings in the weight of the shell. In minimum cost applications where either air or water are used as the pressurization medium, the cost is proportional to W_s and it follows that the potential cost savings is also one third.

To find the total weight of the pressurized column, we observe from Equations (24) and (25) that in this case the pressure supports P/3. With this information, Equation (6) gives the gas weight as $W_g = PL/3(p/\gamma)$. This is added to W_s to give

$$W = \frac{PL}{\sigma_o/\rho}\left[\frac{2}{3} + \frac{(\sigma_o/\rho)}{3(p/\gamma)}\right] \tag{28}$$

It follows readily from this equation that a weight advantage over the solid stable strut can be realized only if $p/\gamma > \sigma_o/\rho$.

b. Distortion Energy Theory: The Mises yield criterion for the pressurized tube can be written

$$\sigma_t^2 + \sigma_t\sigma_c - \sigma_c^2 = \sigma_o^2 \tag{29}$$

where σ_o is the yield strength in pure tension. Eliminating σ_t between Equations (26) and (29), we obtain

$$W_s = \frac{4PL\rho}{3\sigma_c - \sqrt{4\sigma_o^2 - 3\sigma_c^2}} \tag{30}$$

To minimize this quantity, we set

$$\frac{dW_s}{d\sigma_c} = 0.$$

This leads to $\sigma_c = \sigma_o$ and $\sigma_t = 0$. Here, there is no advantage gained by pressurizing. The Tresca yield criterion leads to the identical result.

c. Modified Tresca Failure Condition: To study a material whose tensile strength is higher than its compressive strength, we assume a failure criterion given by

$$\sigma_t + k\sigma_c = k\sigma_o \qquad \sigma_c \geq 0 \tag{31}$$

where the ultimate compressive strength is σ_o and the ultimate tensile strength is $k\sigma_o$. Eliminating σ_t between Equations (31) and (26), we obtain

$$W_s = \frac{2PL\rho}{k\sigma_o + \sigma_c(2-k)} \tag{32}$$

When $k \leq 2$ it is clear that W_s is minimized by taking the largest possible σ_c, i.e., $\sigma_c = \sigma_o$. This implies that $\sigma_t = 0$ and that $W_s = PL/(\sigma_o/\rho)$ which is the conventional design. When $k > 2$ the smallest possible σ_c is chosen to minimize W_s, $\sigma_c = 0$. Here, $\sigma_t = k\sigma_o$ and there is a weight advantage over the solid stable strut; hence, $W_s = (2/k)\left[PL/(\sigma_o/\rho)\right]$. Thus, when the tensile strength of a material is over twice its compressive strength there is a weight and cost saving in the column shell.

The situation is only slightly more complicated when the total column weight is involved. Here, the gas weight must be added to W_s. Using Equations (24) and (25) we find that the pressure resistance $P_p = P\sigma_t/(2\sigma_c + \sigma_t)$. It then follows from Equation (6) that

$$W_g = PL\,\frac{\sigma_t/(p/\gamma)}{2\sigma_c + \sigma_t} \tag{33}$$

Now, the total weight becomes

$$W = W_s + W_g = \frac{PL\left[2\rho + \sigma_t/(p/\gamma)\right]}{2\sigma_c + \sigma_t} \tag{34}$$

Substituting for σ_t from Equation (31), W becomes

$$W = PL\,\frac{\left[2\rho + k(\sigma_o - \sigma_c)/(p/\gamma)\right]}{\left[k\sigma_o + \sigma_c(2-k)\right]} \tag{35}$$

When $k \leq 2$ it is readily apparent from this equation that W is minimized by taking the largest possible value for σ_c; thus, $\sigma_c = \sigma_o$, $\sigma_t = 0$ and W is equal to the conventional $PL(\sigma_o/\rho)$. To examine the weight when $k > 2$, we shall rewrite W taking $\sigma_c = \sigma_o - \alpha\sigma_o$ where $0 \leq \alpha \leq 1$.

$$W = \frac{PL}{\sigma_o/\rho}\left\{\frac{2 + \left[k\,\frac{\sigma_o/\rho}{p/\gamma}\right]\alpha}{2 + (k-2)\,\alpha}\right\} \tag{36}$$

To minimize W we observe that when the quantity in the square brackets exceeds the quantity in parentheses, α must be taken as small as possible, i.e., $\alpha = 0$. When the bracketed quantity is smaller than that in parentheses, we choose the largest possible α, $\alpha = 1$. Only in this latter case do we find an advantage over the solid stable strut. Summarizing the minimum weight problem, we find,

$$\left.\begin{array}{l} k \leq 2 \\ \\ k > 2 \text{ and } p/\gamma \leq \frac{k}{k-2}\,(\sigma_o/\rho) \end{array}\right\} \begin{array}{l} \sigma_c = \sigma_o \\ \sigma_t = 0 \\ W = PL/(\sigma_o/\rho) \end{array}$$

$$k > 2 \text{ and } p/\gamma \geq \frac{k}{k-2}\,(\sigma_o/\rho) \left\} \begin{array}{l} \sigma_c = 0 \\ \sigma_t = k\sigma_o \\ W = \frac{PL}{\sigma_o/\rho}\left[(2/k) + (\sigma_o/\rho)/(p/\gamma)\right] \end{array}\right. \tag{37}$$

where we recall that k is the ratio of the ultimate tensile to compressive strength.

5. Discussion of Results. -

a. Efficiency of Pressurized Columns: The efficiency of our pressurized columns can be assessed by comparing their weights to that of the classical optimum prismatic circular cylinder, (ref. 2), i.e.,

$$W^o = \frac{P^{2/3}L^{5/3}}{(E^{2/3}/\rho)(\pi/16)^{1/3}} \tag{38}$$

The radius and wall thickness of this column are determined in such a way that the local and Euler buckling modes occur simultaneously. For purposes of comparison it is advantageous to rewrite Equation (38) in terms of the structural indes P/L^2. Thus

$$(W^o/L^3) = \frac{(16/\pi)^{1/3}}{(E^{2/3}/\rho)} (P/L^2)^{2/3} \qquad P/L^2 \leq \frac{16\ \sigma_c^3}{\pi\ E^2}$$

$$(W^o/L^3) = \frac{1}{\sigma_c/\rho} (P/L^2) \qquad P/L^2 \geq \frac{16\ \sigma_c^3}{\pi\ E^2} \tag{39}$$

The associated optimum thickness is

$$(t^o/L) = \sqrt{\frac{2}{\pi\ E}(P/L^2)}^{\,1/2} \tag{40}$$

Referring to Equations (10) and (21), the weight and thickness for the open pressurized tube may be written as

$$(W/L^3) = \left[\frac{2}{\sigma_t/\rho} + \frac{1}{p/\gamma}\right](P/L^2) \tag{41}$$

$$(t/L) = \sqrt{E/\sigma_t^3}\ (P/L^2) \tag{42}$$

Equations (39) through (40) are plotted in Figure 2 for a beryllium tube filled with hydrogen at 20°C. The constants entering

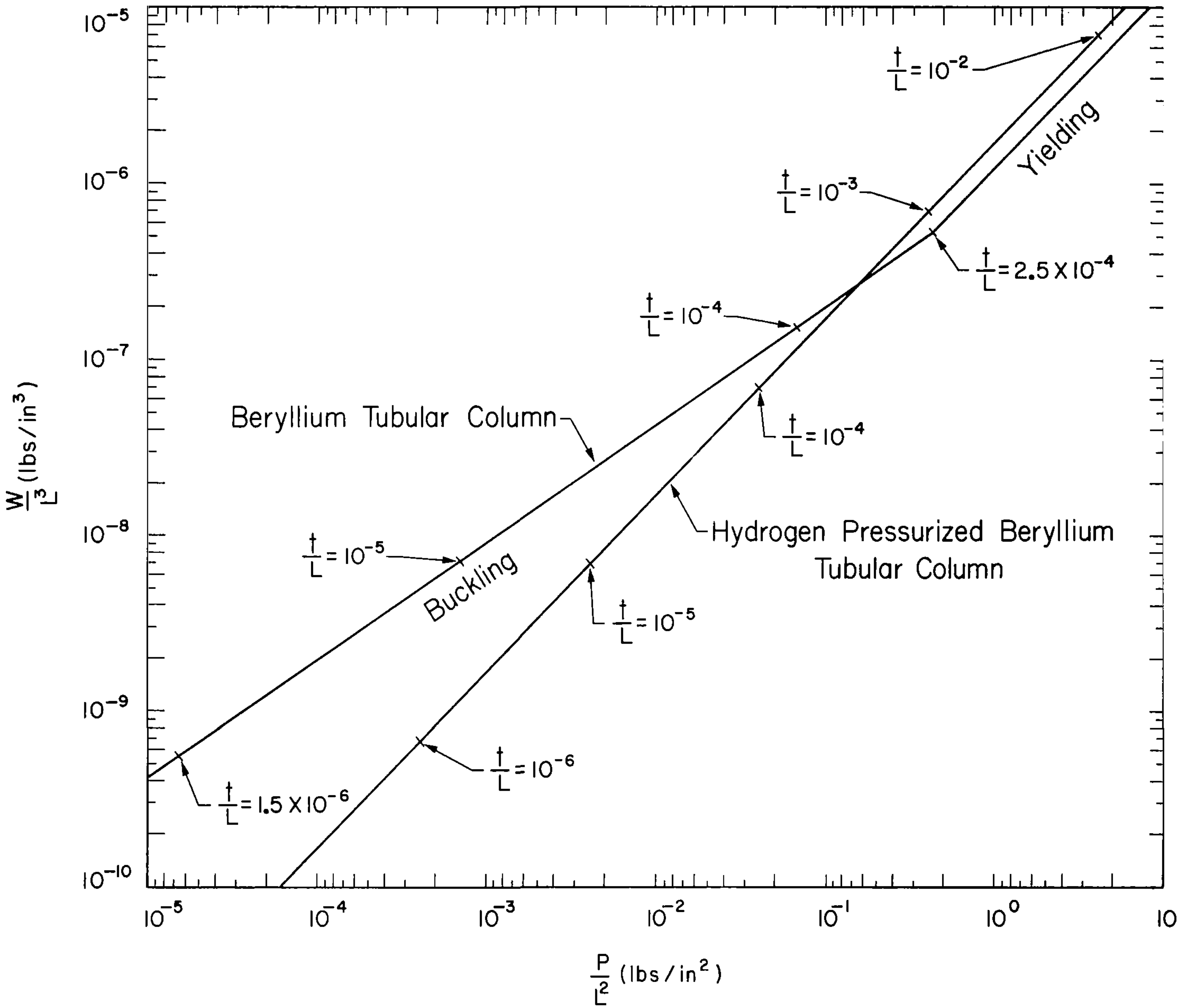


Fig. 2 COLUMN WEIGHT vs. STRUCTURAL INDEX

these equations are listed below:

$$\sigma_c = 55 \times 10^3 \text{ psi}$$

$$\sigma_t = 90 \times 10^3 \text{ psi}$$

$$E = 44 \times 10^6 \text{ psi}$$

$$\rho = 0.0667 \text{ lbs/in}^3$$

$$p/\gamma = 4.86 \times 10^6 \text{ inches (hydrogen at 20°C)}$$

It is clear from Figure 2 that the pressurized column favors low values of the structural index and that the optimum conventional column favors high values. In a recent paper, Mauch and Felton (ref. 9) consider the optimum design of a tension tied column. They also compare the results of this prestressed member to the optimum prismatic tube and they conclude that it is superior in the range of low structural indices. Furthermore, their results show a maximum theoretical weight savings of about 60 percent over the optimum tube. In the case of the pressurized tube, the potential weight savings is 100 percent.

b. Special Properties of Pressurized Columns: Certainly, the most provocative properties of the pressurized column stem from the fact that its weight varies linearly with P and L and is independent of E. These unique characteristics are not found in conventional columns whose weight formulas resemble Equation (38) for the optimum tube.

Because the modulus of elasticity does not enter the weight expression, almost any "gas tight" material becomes a candidate for the pressurized column. The higher E's provide more compact members. Some materials lead to columns which are foldable and inflatable.

We recall that the central feature of both the Maxwell (ref. 4) and Michell (ref. 5) structures was that the weight of their constituent members was proportional to their load and span. Because conventional compression elements do not meet this requirement, most of the authors in this field have viewed the Michell structure as an important academic development. The pressurized column could provide the practical realization of such structures. The design must be based on an allowable tensile stress σ_t and an allowable compressive stress given by $\sigma_c = \sigma_t / \left[N + (\sigma_t/\rho)/(p/\gamma)\right]$ where $N = 2$ or 3 depending upon whether the column is open or closed respectively.

Because of the linearity in P, n columns each carrying a load P/n will weigh the same as a single column under P. The reliability of a column cluster is greater than that of a single pressurized column and may preclude catastrophic failure under accidental conditions. Furthermore, the cluster concept may lead to certain standardizations in the cylinder sizes which may have important logistic and cost implications.

c. Practical Considerations: The development of high performance columns for low structural indices (P/L^2) invariably results in impractically thin sections or gages. The limitations imposed on real columns because of minimum thickness specifications favor the pressurized column. Extremely thin gages are possible in the "balloon type" structures considered, and indeed, the minimum thickness may be established on the basis of gas diffusion through the column walls. The behavior of these columns is quite insensitive to their precise geometry because they never experience compressive stresses.

d. Comments on Analysis: The local buckling of very thin-walled circular cylinders has been the subject of intensive investigation during the last several decades; however, our understanding of the buckling process is yet very incomplete. There are several characteristics of this problem that are worth noting in the present context:

1. The experimental data on the buckling load falls between one-half and one-fifth of the classical linear predictions.

2. The axial resistance of the cylinder characteristically drops off abruptly after the buckling load has been reached.

3. Initial imperfections have large effects on the buckling load.

4. As the walls of a cylinder become thinner, there is an increasing tendency for it to buckle in a developable diamond pattern formed by a regular arrangement of flat triangular surfaces. Under ideal conditions, this buckling mode can occur without membrane extension; in real cylinders, bending is concentrated around the ridges which results in a small amount of stretching.

This last observation may be relevant to the construction of inflatable columns. The first three represent shortcomings present in the conventional minimum weight prismatic column. In our development of the pressurized column, we have elected to ignore any local buckling strength which might be provided by the walls. When this is included, a smaller weight of gas is required; but, we introduce uncertainties in our analysis arising from the three shortcomings noted. For low structural indices, the very small wall thicknesses preclude any significant weight savings.

We shall now comment on several topics omitted from our discussion of Euler buckling. First, it was tacitly assumed that the pressure would not affect the Euler buckling load, and indeed, this is demonstrated to be the case in Section III-C-3. Next, the role of tapering was not discussed. In classical columns, one can always improve on the uniform column by using an optimum axial distribution of material. This is not true in the pressurized column where the weight is already the minimum required to contain the pressure and prevent local buckling. Finally, we did not consider the reduction in the buckling load due to shear deflection. Owing to the action of shearing forces, the critical load is diminished by the factor (ref. 6)

$$\frac{1}{1 + \frac{\beta P_{cr}}{AG}} \tag{43}$$

where A is the cross-sectional area, G is the modulus of rigidity, P_{cr} is the Euler load, and β is a numerical factor that depends on the shape of the cross section. For rectangular sections $\beta = 1.2$; for circular sections $\beta = 1.11$; and for H sections β is approximately 2. The area of the pressurized column is found from Equation (19) to be $A = 2P_{cr}/\sigma_t$; hence, Equation (43) becomes

$$\frac{1}{1 + \frac{\beta \sigma_t}{2G}} \tag{44}$$

For most materials this factor will be very close to unity.

B. Optimum Column Geometry

1. Dynamic Programming. - The first attempt to determine the optimum distribution of material along the length of a column was made unsuccessfully by Lagrange in 1773 who approached the problem using variational calculus. A further investigation of the problem was made by Clausen (ref. 7) in 1851 who determined that the most efficient tapered solid column has a volume $\sqrt{3}/2$ times the volume of a cylindrical column of the same strength. This highly academic column design tapers from a maximum diameter in the center to zero diameter at the ends. Variational calculus has also been used to establish the optimum diameter distribution of a constant thickness tubular column.

In both the solid and the constant thickness tubular columns, an increase in circumference is accompanied by an increase in weight. This guarantees a well set problem which yields a unique area distribution. Unfortunately, the formulation is restricted to a class of relatively inefficient cross sections which are extremely difficult to fabricate. Furthermore, the formulations do not consider yielding or local buckling. These failure criteria may be formally introduced into the calculus of variations as inequality constraints; however, they represent difficulties of both a fundamental and computational nature. These difficulties may be overcome by reformulating the problem in the setting of dynamic programming. Here, the introduction of constraints actually simplifies the computations.

As a preliminary exercise in the study of minimum weight column design, the classical unconstrained problem of the solid strut was attacked using dynamic programming and excellent agreement was obtained with the exact solution. Specifically, the optimum area distribution was developed for a simply supported solid column of length L and under the axial loads P. The failure criterion was taken as Euler buckling.

In order to facilitate comparison with the dynamic programming method of solution, the solution obtained using calculus of variations will be presented first. If we assume that all the cross sections of the strut are similar and similarly situated, we can express the moment of inertia as $I = kA^2$ where k is a constant depending upon the shape of the cross section and A is its area. Using the differential equation of bending

$$E\,k\,A^2\,\ddot{y} + Py = 0 \qquad (45)$$

we can solve for A and, hence, we can express the column weight as

$$W = 2\rho\left(\frac{P}{Ek}\right)^{1/2}\int_0^{L/2}\left(-\frac{y}{\ddot{y}}\right)^{1/2} dx \qquad (46)$$

where x is measured from the center and symmetry has been employed. Application of variational calculus to Equation (46) leads to the equation

$$\dot{y}\,\ddot{y} + 3y\,\dddot{y} = 0 \qquad (47)$$

If we assume that $\ddot{y} \not\equiv 0$, then Equation (47) can be integrated twice to yield

$$\dot{y}^2 = -\,c_1\,y^{2/3} + c_2 \qquad (48)$$

where c_1 and c_2 are constants of integration. By making the change in variable $y = a^3 \cos^3\theta$ in Equation (48) and using the boundary conditions $\dot{y}(0) = 0$ and $y(L/2) = 0$ the following parameterized solution is obtained

$$y = \beta \cos^3\theta \qquad (49)$$

$$x = \left(\frac{L}{\pi}\right)\left(\theta + \frac{1}{2}\sin 2\theta\right) \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

where β is a constant.

Using the solution, Equation (49) and Equation (45), the expression for the cross-sectional area is

$$(A)_{opt} = \left(\frac{4PL^2}{3\pi^2 Ek}\right)^{1/2} \cos^2\theta \tag{50}$$

The weight of the optimum strut is found, using Equation (46), to be

$$(W)_{opt} = \rho\left(\frac{3PL^4}{\pi^2 Ek}\right)^{1/2} \tag{51}$$

Since the sections are similar, we can write the maximum distance from the centroid to the outer fibers as $c = \alpha\sqrt{A}$. Hence, the maximum bending stress at any station along the column becomes

$$\sigma = \frac{(Py)c}{I} = \frac{P(\beta\cos^3\theta)\alpha\sqrt{A}}{kA^2} = \frac{P(\beta\cos^3\theta)\alpha}{k\left[\left(\frac{4PL^2}{3\pi^2 Ek}\right)^{1/2}\cos^2\theta\right]^{3/2}}$$

$$\sigma = \alpha\beta(P/k)^{1/4}\left(\frac{3\pi^2 E}{4L^2}\right)^{3/4} = \text{constant} \tag{52}$$

We see then that the most efficient column is a strut of uniform bending stress.

Bellman and Dreyfus (ref. 8) considered the problem of minimizing the functional

$$J(y) = \int_a^b F(x,y,\dot{y})\ dx \qquad (53)$$

subject to the initial condition y(a) = c. Using dynamic programming, they introduce the function

$$f(a,c) = \min_y J(y) \qquad (54)$$

and consider a and c to be parameters ($-\infty < a < b$ and $-\infty < c < \infty$). Then, by noting the additive property of the integral

$$\int_a^b = \int_a^{a+\Delta} + \int_{a+\Delta}^b \qquad (55)$$

and using the "principle of optimality," they deduce the following functional equation

$$f(a,c) = \min_{\substack{y \\ (a<x<a+\Delta)}} \left\{ \int_a^{a+\Delta} F(x,y,\dot{y})dx + f\left[a+\Delta, c(y)\right] \right\} \qquad (56)$$

By making the restriction that y(x) will be determined at only the points $a = k\Delta, (k+1)\Delta, \ldots, n\Delta = b$ and by approximating the derivative by $(y_{i+1} - y_i)/\Delta$, the following recursion relationship is obtained

$$f(i\Delta, y_i) = \min_{y_{i+1}} \left\{ F\left[i\Delta, y_i, (y_{i+1} - y_i)/\Delta\right] + f\left[(i+1)\Delta, y_{i+1}\right] \right\} \qquad (57)$$

Recall that the expression for the weight of the strut was

$$W = 2\rho\left(\frac{P}{Ek}\right)^{1/2}\int_0^{L/2}\left(-\frac{y}{\ddot{y}}\right)^{1/2}dx \tag{58}$$

In order to get this expression into the form of Equation (53), we make the change in variable $y = (\frac{L}{2} - x)e^{v}$; thus

$$W = 2\rho\left(\frac{P}{Ek}\right)^{1/2}\int_0^{L/2}\left[\frac{(\frac{L}{2} - x)}{2\dot{v} - (\frac{L}{2} - x)(\dot{v}^2 + \ddot{v})}\right]^{1/2} dx \tag{59}$$

By introducing the nondimensional variables $\xi = 2x/L$ and $w = \dot{v}L/2$ we can write

$$\overline{W} \equiv \frac{W}{\rho\left(\frac{PL^4}{4Ek}\right)^{1/2}} = \int_0^1 G(\xi,w,\dot{w})\ d\xi \tag{60}$$

where

$$G(\xi,w,\dot{w}) \equiv \left[\frac{(1-\xi)}{2w - (1-\xi)(w^2 + \dot{w})}\right]^{1/2} \tag{61}$$

Similarly, the expression for the cross-sectional area becomes

$$\overline{A} \equiv \frac{A}{\left(\frac{PL^2}{4Ek}\right)^{1/2}} = G(\xi,w,\dot{w}) \tag{62}$$

The recursion relationship, Equation (57), is now applicable and thus we have

$$f(i\Delta,w_i) = \min_{w_{i+1}}\left\{G\left[i\Delta,w_i,(w_{i+1}-w_i)/\Delta\right]\Delta + f\left[(i+1)\Delta,w_{i+1}\right]\right\} \tag{63}$$

where $i = 0, 1, 2, \ldots n$ and $\Delta = 1/n$. By introducing the restriction that w can take on only the values of $w = 0, \delta, 2\delta, \ldots, p\delta$ where p is an integer, Equation (63) can be written as

$$r(i,j_i) = \min_{j_{i+1}} \left[g(i,j_i,j_{i+1})\Delta + r(i+1,j_{i+1}) \right] \quad (64)$$

where we have defined $w_i \equiv j_i\delta$ and

$$g(i,j_i,j_{i+1}) \equiv G\left[i\Delta, j_i\delta, (j_{i+1}-j_i)\delta/\Delta\right] \quad (65)$$

$$r(i,j_i) \equiv f(i\Delta, j_i\delta) \quad (66)$$

As the recursion relationship is used to go from $\xi = (i+1)\Delta$ to $\xi = i\Delta$, it is desirable to keep track of the value of j_{i+1} that minimized the right hand side of Equation (64) because it represents the optimum policy for getting from $i\Delta$ to $(i+1)\Delta$ starting with j_i. Thus, we define

$$\ell(i,j_i) \equiv (j_{i+1})_{min} \quad (67)$$

The desired solution $(w_i)_{opt} = (j_i)_{opt}\delta$ will be obtained after the tables $\ell(i,j_i)$ have been computed using

$$(j_i)_{opt} = \ell\left[i-1,(j_i)_{opt}\right] \quad (68)$$

The boundary conditions on the strut can now be used to determine conditions on w_i (or equivalently j_i). The condition $y(L/2) = 0$ implies through the differential equation (since A is arbitrary) that $\ddot{y}(L/2) = 0$ and, consequently, $w(1) = 0$ which in turn implies that $(j_n)_{opt} = 0$. The condition $\dot{y}(0) = 0$ becomes simply $w(0) = 1$ and thus $(j_0)_{opt} = m$ and $\delta = 1/m$ $(m \lesssim p)$.

Thus, in the strut problem we no not initially know the values of either j_0 and j_1 or j_{n-1} and j_n which would have made applying the recursion equation straightforward. What we do know are the starting and ending values j_0 and j_n. Inspection of Equations (61) and (65) reveals that $g(n,j_n,j_{n+1}) = 0$ unless $j_{n+1} = j_n = 0$. This property makes it possible to start at $i = n$. The computational scheme may now be outlined:

(0) Set $r(n,j_n) = 0$ for all j_n

(1) Since the first boundary condition is $j_n = 0$, compute for each j_{n-1}:

$$r(n-1,j_{n-1}) = \Big[g(n-1,j_{n-1},j_n)\Delta + r(n,j_n)\Big]_{j_n = 0}$$

and set $\ell(n-1,j_{n-1}) = 0$

(2) Compute for each j_{n-2}:

$$r(n-2,j_{n-2}) = \min_{j_{n-1}}\Big[g(n-2,j_{n-2},j_{n-1})\Delta + r(n-1,j_{n-1})\Big]$$

$$\ell(n-2,j_{n-2}) = (j_{n-1})_{min}$$

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(n) Finally, compute for each j_0:

$$r(0,j_0) = \min_{j_1}\Big[g(0,j_0,j_1)\Delta + r(1,j_1)\Big]$$

$$\ell(0,j_0) = (j_1)_{min}$$

When the second condition $j_0 = m$ is used, we note that the minimum weight is determined,

$$(\overline{W})_{opt} = r(0,m),$$

and we can proceed with the calculation of $(j_i)_{opt}$ using Equation (68):

$$(j_0)_{opt} = m$$

$$(j_1)_{opt} = \ell\left[0,(j_0)_{opt}\right]$$

$$\vdots$$

$$(j_{n-1})_{opt} = \ell\left[n-2,(j_{n-2})_{opt}\right]$$

$$(j_n)_{opt} = \ell\left[n-1,(j_{n-1})_{opt}\right] = 0$$

Finally, with the set of values $(j_i)_{opt}$ it is possible to calculate the optimum area distribution

$$(\overline{A}_i)_{opt} = g\left[i,(j_i)_{opt},(j_{i+1})_{opt}\right]$$

With $n = 50$ and $m = 50$, $(\overline{W})_{opt}$ and $(\overline{A}_0)_{opt}$ were computed to be 0.5648 and 0.1985 respectively while the exact solution yielded $\sqrt{3}/\pi = 0.5513$ and $2/\pi\sqrt{3\pi} = 0.2074$ respectively. The values obtained via dynamic programming should approach those from the exact solution when either n and m become large or when an interpolation scheme is introduced which allows optimum values to occur between the tabulated values.

2. Columns with Fixed Diameters. - The approach used to determine the diameter distribution for a solid column can be adopted for the more efficient hollow cross sections. When local buckling can be ignored, it is good practice to spread out the cross sections as much as possible. Using the largest permissible width, we shall determine the optimum section thicknesses along the column length. This may be done very simply in the present circumstances by using the uniform stress property described in the previous subsection.

Consider a hollow prismatic column in which we will adjust the wall thickness (or any single open parameter) in such a way that the bending stress will be uniform in the buckled form. In such a case the moment of inertia must be related to the applied moment through the bending stress formula,

$$\sigma = \frac{Mc}{I(x)} = \text{constant} \tag{69}$$

Hence, the differential equation of bending becomes

$$EI \frac{d^2y}{dx^2} = - M = - \frac{\sigma I}{c} \tag{70}$$

We obtain immediately by quadrature,

$$y = \frac{\sigma L^2}{2Ec} \left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2 \right] \tag{71}$$

where we have imposed the boundary conditions

$$\begin{aligned} y &= 0, \text{ at } x = 0 \\ \frac{dy}{dx} &= 0, \text{ at } x = L/2 \end{aligned} \tag{72}$$

To find an open parameter corresponding to y we simply set $P_y = \sigma I(x)/c$. Thus, for a thin-walled circular section with

$I = \pi R^3 t(x)$, we obtain the parabolic thickness distribution t(x).

$$t(x) = \frac{PL^2}{2\pi R^3 E}\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right] \qquad (73)$$

where R is the radius and the associated volume of the column is given by

$$V_{opt} = PL^3/6R^2E \qquad (74)$$

The corresponding optimum constant radius and constant thickness column has

$$t = PL^2/\pi^3R^3E \qquad (75)$$

and

$$V = 2PL^3/\pi^2R^2E \qquad (76)$$

Therefore, when the length, radius, and load are equivalent,

$$\frac{V_{opt}}{V} = \frac{\pi^2}{12} = 0.822 \qquad (77)$$

The weight saving is therefore 17.7 percent. When the thickness is constant and the radius varies in an optimum fashion, the most efficient hollow strut shows a weight saving of 10.9 percent over the comparable prismatic strut. An examination of either Equations (74) or (76) indicates the possibility of arbitrarily increasing R with an attendant reduction in volume and decrease in wall thickness. It is abundantly clear that local buckling will occur at the ends of the column and, consequently, a real column will have to satisfy a condition which precludes this mode of failure.

C. Energized Columns*

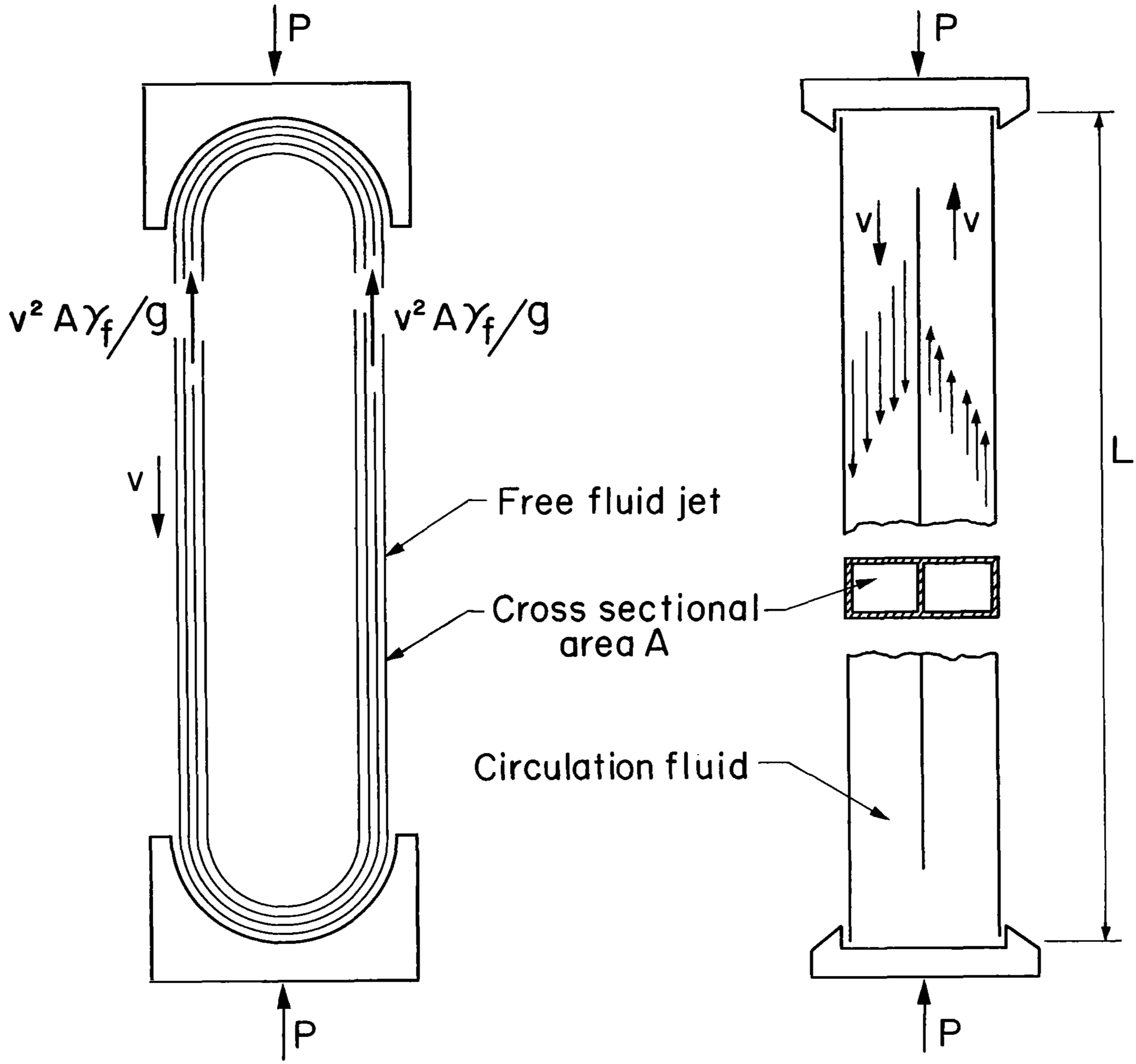
Traditionally the structural engineer equilibrates forces by placing objects in their paths which develop equal and opposite resisting forces. There are, of course, many other ways of providing these equal and opposite forces; but, they are seldom employed in structural design. For example, we might use magnetic forces, centripetal forces, electrostatic forces, nuclear forces, or forces created by chemical explosions. In other words, some energized system may supply the resistance normally provided by a structure. The question is, can an energized system develop this resistance at lower cost or lower weight than the conventional structure?

The system/energy concept admits many exciting possibilities. For instance, by using thermal energy, a bimetallic strip, and a simple servo mechanism, an infinitely stiff beam can be constructed. We have, unfortunately, only touched on this subject in the present section which utilizes a circulating fluid to prestress a column. Nevertheless, certain first order features can be illustrated even in this preliminary inquiry.

1. Weight of Fluid. - To preclude local buckling, we shall employ the steady flow of a perfect fluid to achieve a prestress level equal to the column load P. Referring to Figure 3a, the force due to a fluid jet impinging on a fixed vane will be computed in the usual way from the momentum theorem, i.e., force on a mass is equal to its time rate of change in momentum. Application of this theorem gives rise to the two forces of magnitude $v^2 A\gamma_f/g$ acting on the upper vane in Figure 3a where v is the steady mean speed of the fluid, A is the cross-sectional area of the jet stream, γ_f is the weight density of the fluid and g is the gravitational acceleration. We therefore require that

$$P = 2v^2 A\gamma_f/g \tag{78}$$

*Dr. Theodore Liber of IIT Research Institute served as a consultant in the stability investigations.



(a) Free Jet Impinging on Fixed Vanes

(b) Simple Open End Column

Fig. 3 COLUMNS USING CIRCULATING FLUIDS

Using this fluid momentum in the simple column shown in Figure 3b, we will never experience axial compression in the column walls. The weight of the fluid W_f used in this column is

$$W_f = 2AL\gamma_f \tag{79}$$

or using Equation (78),

$$W_f = \frac{PL}{v^2/g} \tag{80}$$

We note that the fluid weight varies linearly with P and L; but, the more important result lies in the observation that the fluid weight goes to zero as the fluid speed approaches infinity. It is clear from Equation (78) that any resistance may be obtained by increasing v without changing the amount of fluid.

It is of some interest to compare the gas and fluid weights given by Equations (6) and (80). For a fluid to weigh less than a gas, the fluid velocity must be taken greater than the square root of the pressure-mass density ratio, i.e.,

$$v > \sqrt{g(p/\gamma)} \tag{81}$$

To surpass hydrogen at 20°C, the velocity must be greater than 3610 ft/sec.

2. Tube Weight Based on Strength.-

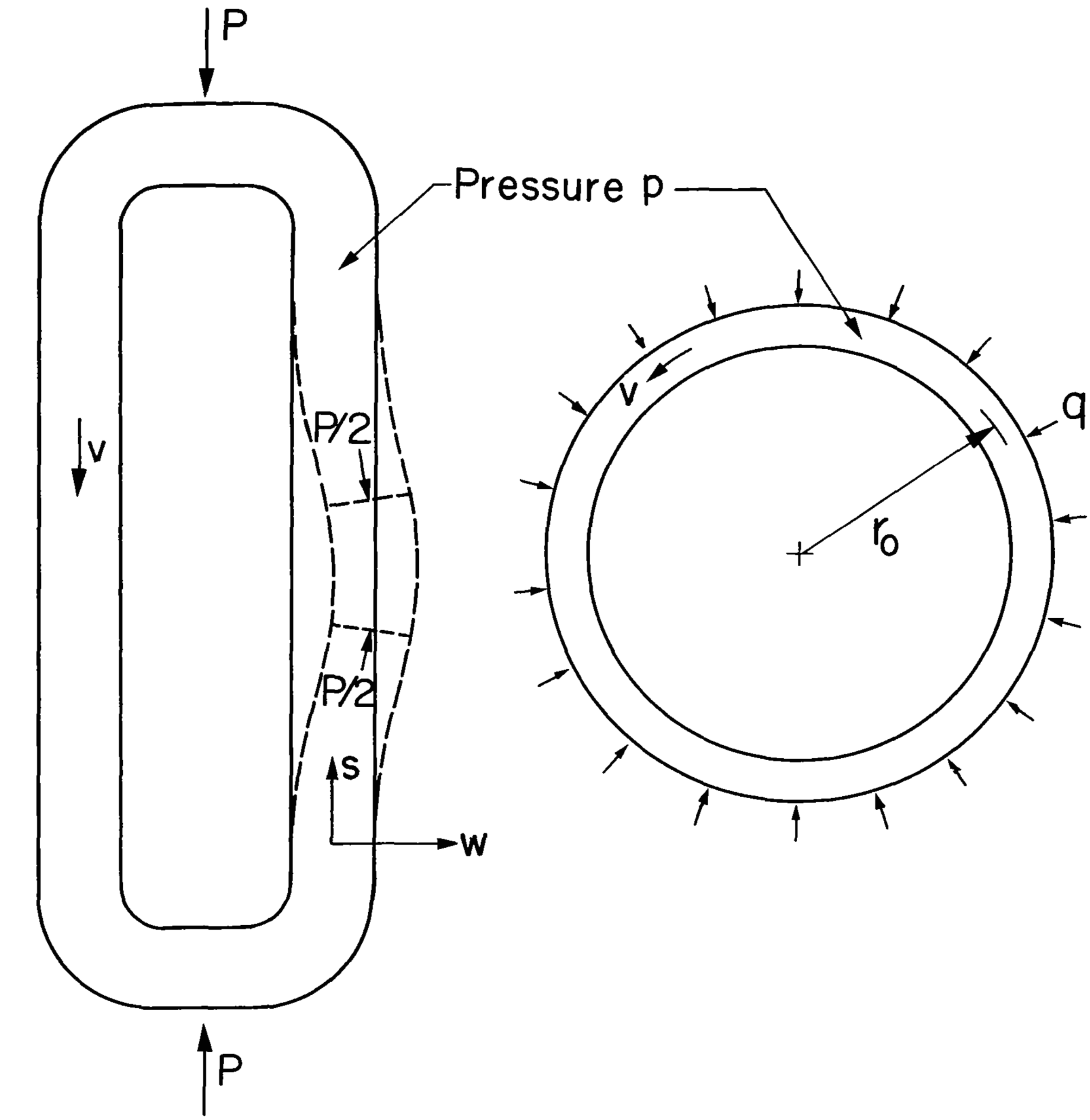
a. Open End Tube: We refer to the column shown in Figure 3b as an open end column. The end closure and seal is furnished by the loading mechanism or contiguous members and it is assumed that the fluid velocity is adjusted so that its reaction just balances the column load. This precludes any axial stress in the tube walls. It is possible to accelerate a perfect fluid to high velocities by using a very small pressure gradient over a long time span. Theoretically, when the gradient is removed the fluid will continue to circulate indefinitely at the terminal velocity. In such a situation, the hoop stresses in the tube walls are negligible, and indeed, no walls are required at all. This limiting situation is illustrated in Figure 3a where two fixed vanes are shown with a free jet impinging on them.

b. Closed Tube: We shall assume that a perfect fluid under zero pressure is circulated in the circuit shown in Figure 4a . To prevent local buckling under the column load P, the fluid velocity is selected to produce an axial preforce of this magnitude. In the unloaded condition, both the horizontal and vertical members must resist an axial force P/2. In the loaded condition, only the horizontal axial force persists. Thus, the total length of tubing must be designed as a tension member supporting a force P/2. Minimizing the total length by placing the two vertical tubes next to each other, as in Figure 3b , the effective tube length is approximately 2L and the required weight is simply

$$W_c = \frac{PL}{\sigma_t/\rho} \tag{82}$$

where again σ_t is the tensile strength and ρ is the weight density of the tube material.

3. <u>Stability</u>. - With the proper fluid velocity, it seems clear that the system shown in Figure 3a can equilibrate the loading; however, our intuition suggests that this liquid column is unstable. We can easily show that the walls of a tubular column must provide the required Euler stability; neither a moving liquid nor an internal pressure can effect the column buckling resistance. To see this we shall examine the behavior of the tubular column shown in Figure 4a which contains a perfect fluid circulating with a mean speed v under a mean pressure **p**. When the tubes are undistorted, the unloaded column sustains an axial tension created by the pressure and inertia forces, i.e., $P = 2A(p + v^2\gamma_f/g)$ where A is cross sectional area of the tube.



(a) Tubular Column (b) Hollow Torus

Fig. 4 COMPRESSION MEMBERS FILLED WITH A PRESSURIZED CIRCULATING PERFECT FLUID

If a tube undergoes a distortion w measured normal to the original centroid, the effective lateral restoring force is simply

$$\frac{P}{2}\left[\frac{d^2w}{ds^2}\right]$$

where s is measured along the tube. Now, referring to the dashed element in Figure 4a , compressive forces acting on the fluid at the end cross sections are caused by the pressure, pA, and by the rate of change in the momentum, $v^2A\gamma_f/g$. But, these axial forces are equal to -P/2; they exactly cancel out the force in the tube walls. Consequently, no effective lateral force is generated by the flow or pressure. The resistance to Euler buckling proceeds as if there were no fluid; local buckling is prevented by the flow and pressure.

The same basic argument can be used to describe the behavior of a hollow torus which is filled with a pressurized and circulating fluid. When such an element entertains a uniform circumferential force S, any radial deviation of its centroid w from the circular equilibrium position results in a radial restoring force per unit arc length given by (ref. 6, section 40)

$$S\left[\frac{d^2w}{ds^2} + \frac{w}{r_o^2}\right]$$

where r_o is the initial radius of the centroid and w is positive when $w > r_o$. Now, $S = pA + v^2A\gamma_f/g$ for a pressurized circulating fluid. On the other hand, a compressive force acts on the fluid at any cross section and is made up of a pressure contribution pA and a change in momentum contribution $v^2A\gamma_f/g$. For a radial element of unit arc length, the change in the curvature due to w is

$$\left[\frac{d^2w}{ds^2} + \frac{w}{r_o^2}\right]$$

Therefore, the forces on the fluid produce a radial outward force per unit arc length of

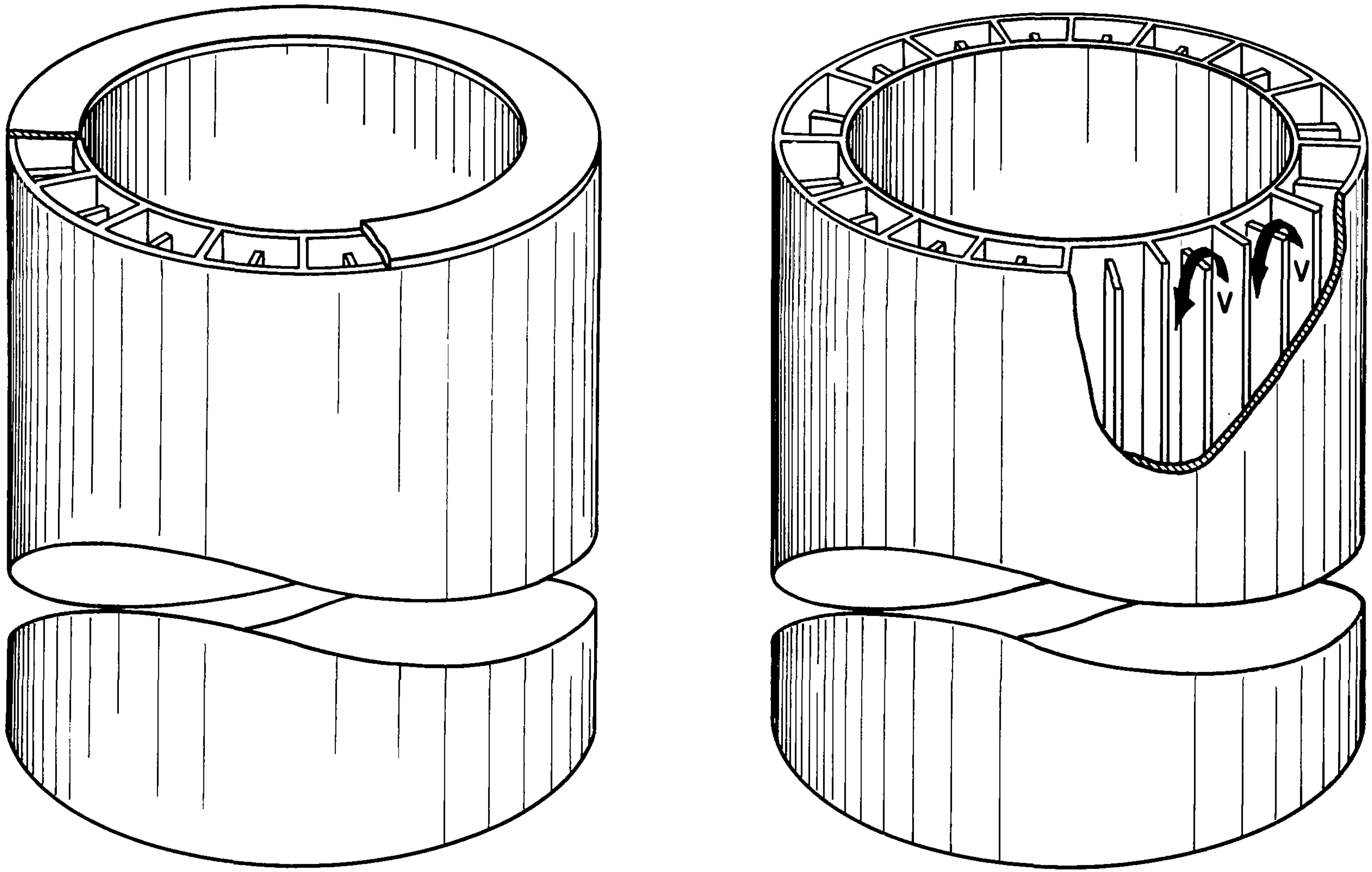
$$-S\left[\frac{d^2w}{ds^2} + \frac{w}{r_o^2}\right]$$

Since the two radial forces arising from the distortion w cancel out, no secondary bending is produced by the fluid. The overall stability of the torus under a uniform radial compressive loading is the same for the empty and the fluid filled member. The problem of rotating the torus is the same as that of the circulating fluid, and consequently, this method of prestressing also has no effect on the overall stability.

We now turn our attention to the problem of designing a stable "fluid column." Beginning with the closed tube column, we recall that both the tube weight and the fluid weight vary linearly with P and L. Consequently, n columns of the type shown in Figure 4a may each support P/n without compromising the total weight. We now treat each of these n columns as elements comprising the periphery of a single circular column in the fashion illustrated in Figure 5a. Using the fixed weight of material W_c given by Equation (82), we can construct our circular column with any radius desired by adjusting n and the wall thickness of the closed tube elements. It follows that any moment of inertia and, therefore, any Euler buckling resistance may be achieved by adopting a sufficiently large radius.

Clearly then, we can always construct columns which preclude Euler buckling. The total weight of such columns is found from Equations (80) and (82); thus,

$$W = W_c + W_f = \frac{PL}{\sigma_t/\rho}\left[1 + \frac{\sigma_t/\rho}{v^2/g}\right] \qquad (83)$$



(a) CLOSED COLUMN (b) OPEN END COLUMN

Fig. 5 OPTIMUM FLUID COLUMNS

This weight is less than that of the pressurized closed cylindrical column when

$$v > \sqrt{\frac{g}{\frac{2}{\sigma_t/\rho} + \frac{1}{p/\gamma}}} \tag{84}$$

In the limiting case, v approaches infinity, the v^2 term drops out of Equation 83, and the fluid column has the same weight as a tension member of length L which is under a load P. The weight of the "infinite speed" fluid column is about one third to one fourth that of the pressurized column.

The open tube problem is similar to that of the closed tube in the sense that we can construct a circular column of any radius from n open tube elements as shown in Figure 5b. Here, an infinite fluid speed eliminates the fluid weight and an infinite radius eliminates Euler buckling. We recall that a tube of vanishingly small weight was required to contain the fluid; consequently, for the "open conditions," the limiting column has zero mass.

4. Energy Losses. - The assumption of an ideal fluid was adopted throughout our previous discussions. On this basis, any velocity could be imparted to a fluid and the motion would be maintained indefinitely. One can introduce the required energy into a system prior to its mission; the energy-force relationship can be modified without changing the system mass. Although some real fluids, such as liquid helium, closely approximate ideal behavior, we must generally take energy losses into account in an explicit fashion.

Real fluids always have some viscosity and this frictional phenomenon accounts for the energy dissipated when the fluids are set in motion. To maintain a desired fluid velocity, one must continually replace the irrecoverable energy. This may be done by drawing energy from the environment or from a source which is part of the system. In either case, additional mass is required for its storage, distribution, and conversion.

It is clear that the efficiency of a real fluid column must be judged on the weight of the entire system. We also observe that extended missions require more massive storage hardware when there is no external energy source, and as a consequence, mission time may become an important parameter. Further exploitation of the energized system concept will certainly embrace considerations of laminar and turbulent flow, vapor pressure, and shock waves in addition to solar collectors, batteries, and pumps.

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IV. DESIGN OF STATICALLY DETERMINATE TRUSSES FOR MINIMUM WEIGHT AND DEFLECTION

A. Introduction

As materials of ever increasing strength are made available, the proportioning of structural components will be governed more and more by stiffness and stability rather than strength. This section addresses itself to the problem of designing a statically determinate plane or space truss under a single load system so that its stiffness/weight ratio is maximum. This may be accomplished by optimizing the location of the truss nodes and by optimizing the distribution of bar areas; both procedures are treated.

Specifying the truss outline together with certain member sizes, our first studies required that the "open" bar areas in the truss be varied to produce a given node deflection with a minimum volume of material. Depending on the loading and the specified deflection, three situations were encountered. In one, no physical bar areas exist which will satisfy the deflection requirement. In another, one finds an infinite number of bar area distributions which produce not only the specified deflection, but also, negative deflections or zero deflection at the given node. For this case, when strength and stability are disregarded, any deflection can be achieved with trusses of vanishing weight. In the final case, a unique bar area distribution is obtained which represents the absolute minimum weight design of the truss. One of the characteristics of this truss is that the product of the actual stress and the virtual stress is the same for all bars; the virtual stresses arise from a unit load placed in the direction and at the node of the desired deflection. Designs of minimum weight and uniform stress beams and trusses are compared for equal constant depth members. The beam is found to be superior to the truss on a strictly stiffness/weight basis.

In our second study, we again consider a truss of fixed outline with certain member sizes specified; only here, the open bar areas are required to be no smaller than certain specified minimum areas established from perhaps code, strength, or stability requirements. If the deflections of such a truss are excessive when the bar areas are taken as their minimum specified values, a method is presented for stiffening the truss with a minimum increase in the weight. The associated mathematical problem may be formulated as a nonlinear programming problem with a nonlinear objective function and linear constraints. A very rapid procedure suitable for a desk calculator is described for finding the exact solution to this problem in a finite number of iterations. The resulting truss is unique and represents the absolute minimum weight design producing a specified node deflection.

In our final minimum weight design problem, both the location of the truss nodes and the bar areas are allowed to vary. For this problem, previous investigators have concluded that the optimum stiffness/weight truss is a Michell structure. We show that this uniformly stressed structure is optimum only when the actual and virtual loadings are proportional. When this is not the case, an infinite number of truss configurations can be found which produce any desired node deflection, including zero deflection, with a structure of vanishing weight.

B. Trusses with Given Configurations

The deflection of any joint of a pin-connected truss is given by the virtual work expression

$$\Delta = \sum \frac{SuL}{A\,E} \tag{1}$$

where, for any member, S is the direct stress resulting from the applied loading, u is the direct stress resulting from a unit load applied in the specified direction at the joint where the deflection is desired, L is the length, A is the area, and E is the modulus of elasticity. The summation extends over all bars in the structure.

Consider the bars in a statically determinate truss to be divided into two groups. In the first, the members will be completely described and denoted by the subscript c (closed). In the second group, everything except the member areas will be specified and these will be treated as open parameters. This group will be denoted by the subscript o. Equation (1) may be rewritten as

$$\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c} = \sum_o \frac{S_o u_o L_o}{A_o E_o} \tag{2}$$

where the symbols $\sum_c$ and $\sum_o$ mean the summation over the members of group c and summation over group o respectively. Since we are considering a design problem as opposed to an analysis problem, the deflection Δ will be specified and the areas A_o will be sought. It is then meaningful to distinguish four cases and these will be treated in the following sections.

Case 1: The sign of the product $S_o u_o$ is either nonpositive or non-negative for all members and the left side of Equation (2) is zero; or analytically,

$$\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c} = 0$$

and either $S_o u_o \geq 0$ or $S_o u_o \leq 0$ for all members o.

For this case, Equation (2) cannot be satisfied using only finite values for A_o. Thus, no physical solution exists.

Case 2: The sign of the product $S_o u_o$ is different from that of the left side of Equation (2) for all members; or analytically,

$$\frac{S_o u_o}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c}} \leq 0$$

for all members o.

For this condition, the signs of the right and left side of Equation (2) cannot be made the same unless negative values of the areas A_o are admitted. Again, no physical solution exists.

Case 3: The product $S_o u_o$ is positive for some truss members and negative for others, $S_o u_o < 0$ and $S_o u_o > 0$ for the members o.

We shall show that in this case any specified deflection value can be obtained using a truss of arbitrarily small weight. Let each open member with a negative product $S_o u_o$ have an area A_1, and let the members with a positive product $S_o u_o$ have an area A_2. Then, Equation (2) may be written as

$$\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c} = \frac{1}{A_2} \sum_{+o} \frac{S_o u_o L_o}{E_o} - \frac{1}{A_1} \sum_{-o} \left| \frac{S_o u_o L_o}{E_o} \right| \qquad (3)$$

where the symbols $\sum\limits_{+o}$ and $\sum\limits_{-o}$ are the sums over the members with positive and negative products $S_o u_o$ respectively. Solving Equation (3) for A_1 we find that any finite deflection Δ can be achieved with non-negative areas A_1 and A_2 when A_1 is given by

$$A_1 = \frac{\sum\limits_{-o} \left| \frac{S_o u_o L_o}{E_o} \right|}{\frac{1}{A_2} \sum\limits_{+o} \frac{S_o u_o L_o}{E_o} - \left(\Delta - \sum\limits_{c} \frac{S_c u_c L_c}{A_c E_c} \right)} \tag{4}$$

and A_2 is taken sufficiently small. It is clear that

$$\lim_{A_2 \to 0} A_1 = 0 \tag{5}$$

Physically, we note that the stresses and deflections at joints other than the one with the specified deflection approach infinity as the two areas A_1 and A_2 approach zero.

It can be seen from Equation (3) that by "beefing up" members with negative products $S_o u_o$ (increasing A_1), the deflection is increased. On the other hand, by making such members more flexible we produce unusual effects such as upward deflections of simply supported trusses under downward acting loads. This situation is illustrated in the photograph shown in Figure 1. If the flexibility of members with $S_o u_o < 0$ are adjusted so that zero deflection is obtained at the specified node, this condition will persist as the loading is increased or decreased proportionally.

Case 4: The sign of the product $S_o u_o$ is the same as that of the left side of Equation (2) for all members; or analytically,

$$\frac{S_o u_o}{\Delta - \sum\limits_{c} \frac{S_c u_c L_c}{A_c E_c}} \geq 0$$

for all members o.

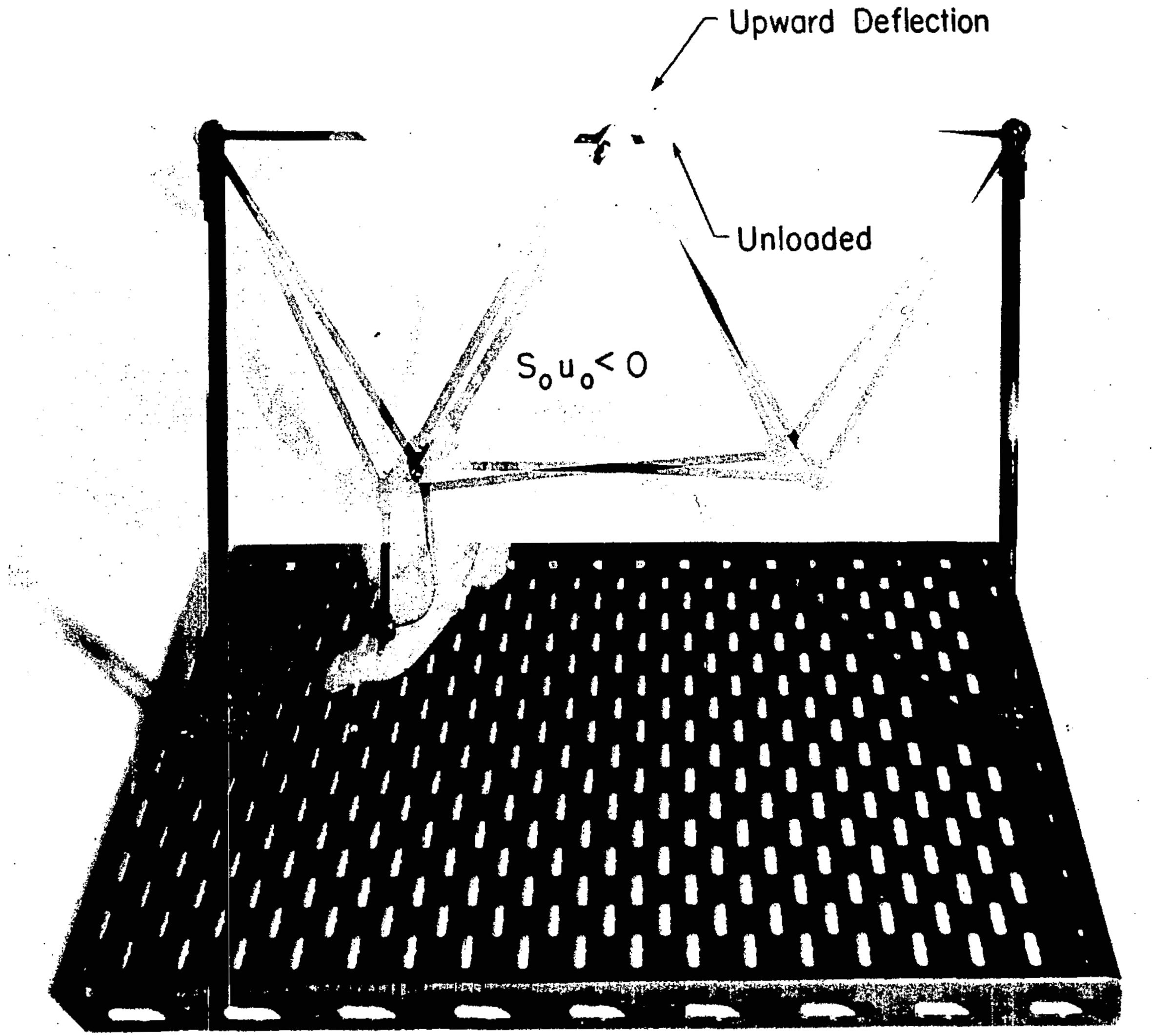


Fig. 1 TRUSS EXHIBITING UPWARD DEFLECTION DUE TO A DOWNWARD ACTING LOAD

The numerator and denominator in the above fraction will be considered, without loss in generality, as non-negative quantities. For this case, we shall find a set of bar areas A_o^* which minimize the truss weight subject to the condition that Δ is a specified constant. The weight of a truss may be written

$$W = \sum_c \rho_c A_c L_c + \sum_o \rho_o A_o L_o \tag{6}$$

where ρ is the weight density of a bar. Repeating Equation (2), the deflection is

$$\Delta = \sum_c \frac{S_c u_c L_c}{A_c E_c} + \sum_o \frac{S_o u_o L_o}{A_o E_o} = \text{specified constant.} \tag{7}$$

Using Lagrange's method of undetermined multipliers, a set of areas A_o^* may be found from Equations (6) and (7) which render W stationary; hence,

$$\frac{\partial}{\partial A_o}\left[\sum_c \rho_c A_c L_c + \sum_o \rho_o A_o L_o + \lambda\left(\sum_c \frac{S_c u_c L_c}{A_c E_c} + \sum_o \frac{S_o u_o L_o}{A_o E_o}\right)\right] = 0 \tag{8}$$

where λ is the Lagrangian multiplier. Performing the operations in Equation (8), solving for A_o, and eliminating λ by Equation (7), A_o^* becomes

$$A_o^* = \frac{\sqrt{\frac{S_o u_o}{\rho_o E_o}} \sum_o \left(\frac{S_o u_o \rho_o}{E_o}\right)^{1/2} L_o}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c}} \tag{9}$$

The weight W^* associated with A_o^* may be found by substituting Equation (9) into (6):

$$W^* = \sum_c \rho_c A_c L_c + \frac{\left[\sum_o \left(\frac{S_o u_o}{E_o/\rho_o}\right)^{1/2} L_o\right]^2}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c}} \tag{10}$$

We shall now show that the stationary value W^* is an absolute minimum; i.e., for any set of areas $A_o \geq 0$ satisfying Δ = specified constant, $W^* \leq W$.

Define $F \equiv W - W^*$

Using Equations (6) and (10), F becomes

$$F = \sum_c \rho_c A_c L_c + \sum_o \rho_o A_o L_o - \left\{\sum_c \rho_c A_c L_c + \frac{\left[\sum_o \left(\frac{S_o u_o \rho_o}{E_o}\right)^{1/2} L_o\right]^2}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c}}\right\} \tag{11}$$

Substituting for Δ from Equation (7), one obtains

$$F = \frac{1}{\sum_o \frac{S_o u_o L_o}{A_o E_o}} \left\{\left(\sum_o \frac{S_o u_o L_o}{A_o E_o}\right)\left(\sum_o \rho_o A_o L_o\right) - \left[\sum_o \left(\frac{S_o u_o L_o}{A_o E_o}\right)^{1/2} \left(\rho_o A_o L_o\right)^{1/2}\right]^2\right\} \tag{12}$$

The quantity in braces is non-negative by Schwarz's inequality. Since the quantity

$$\sum_o \frac{S_o u_o L_o}{A_o E_o}$$

is also non-negative, $F \geq 0$. Q.E.D.

It should be noted that the weight of the "open" members described by the second term in Equation (10) is inversely proportional to the specific stiffness E/ρ. This ratio is approximately equal to 10^8 in. for most of the common metals; for ceramics we find specific stiffnesses as high as 10^9 in.

If a truss is designed using one material, Equation (9) indicates that in the optimum stiffness/weight truss the product of the actual stress and the virtual stress is constant over all the open members, i.e.,

$$\left(\frac{S_o}{A}\right)\left(\frac{u_o}{A}\right) = \left[\frac{E\Delta - \sum_c \frac{S_c u_c L_c}{A_c}}{\sum_o \left(S_o u_o\right)^{1/2} L_o}\right]^2 = \text{constant} \qquad (13)$$

When the actual and virtual loadings are proportional, $S = ku$ where k is a constant. For such cases it is evident from Equation (13) that the optimum deflection design is a uniformly stressed truss.

In Figure 2 weight comparisons are made among minimum weight beams and trusses and uniform stress trusses when the designs are based on deflection. The detailed weight relationships for these members are developed in Appendix III. The design of optimum stiffness/weight beams is outlined in Appendix I; the weight of a uniform stress truss designed on the basis of deflection is given in Appendix II.

For low values of L/d where shear deformations are significant, a beam is found to be far superior to a truss when the designs are based on stiffness. For large values of L/d, most of the truss weight is concentrated in the chords to resist bending deformation. The resulting trusses are quite similar to webless I-beams and the comparisons drawn for this "ideal" member in Table 1 and Figure 4 of ref. 1 hold exactly for the trusses when their L/d approach infinity.

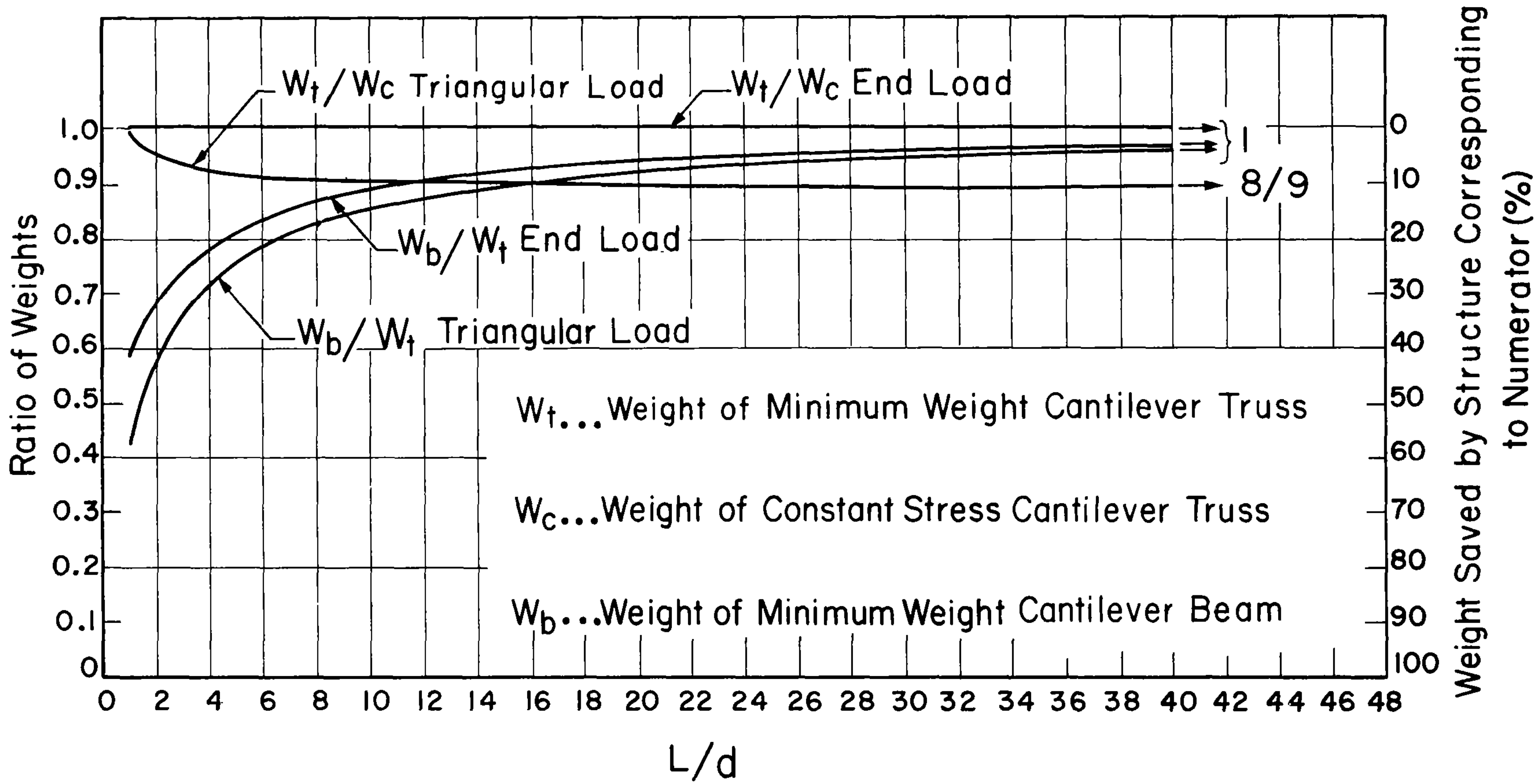


Fig. 2 COMPARISONS AMONG MINIMUM WEIGHT BEAMS AND TRUSSES AND UNIFORM STRESS TRUSSES

C. Trusses with Given Configurations and Minimum Bar Sizes

1. Design Algorithm. - Of the four cases discussed in section B, only the third and fourth gave rise to physically attainable solutions. In both of these cases the optimum areas for the open members were established without considering the consequences of strength, stability, corrosion, or code requirements. In this section we shall again examine cases three and four; but this time, the open members will be assigned minimum areas based on such criteria.

Consider a truss loaded in such a manner that some of the open members have positive products $S_o u_o$ and some negative. Suppose that the deflection at some joint of this truss is excessive when minimum bar areas A_m are used in the members. Since the specified deflection Δ can always be taken as non-negative, the foregoing situation can be expressed as

$$0 \leq \Delta < \sum_{+o} \frac{S_o u_o L_o}{E_o A_m} - \sum_{-o} \left| \frac{S_o u_o L_o}{E_o A_m} \right| + \sum_{c} \frac{S_c u_c L_c}{E_c A_c} \qquad (14)$$

With only the areas of the open members at our disposal, we must try to reduce the magnitude of the right side of this inequality to the specified value Δ. Clearly, the magnitude is reduced when the areas of the (+ o) group are increased and when the areas of the (-o) group are decreased. The latter course is preferred since it is accompanied by a decrease in the truss weight. However, in Equation (14) the members in the (-o) group have already the smallest admissible areas. The remaining possibility is to increase the areas of the (+ o) group with a corresponding increase in truss weight.

There are circumstances in which the flexibility of the closed members are so great that the specified deflection cannot physically be achieved. The condition for the existence of a solution is found from Equation (14) when the areas of the (+ o) group are allowed to approach infinity; hence, Case 5:

$$\Delta - \sum_{c} \frac{S_c u_c L_c}{E_c A_c} + \sum_{-o} \left| \frac{S_o u_o L_o}{E_o A_m} \right| \leq 0 \qquad (15)$$

physical solution is impossible.

Case 6:

$$\Delta - \sum_{c} \frac{S_c u_c L_c}{E_c A_c} + \sum_{-o} \left| \frac{S_o u_o L_o}{E_o A_m} \right| > 0 \qquad (16)$$

physical solutions are possible.

Infinitely many solutions exist when the inequality of Equation (16) holds; however, only one minimizes the truss weight. The following procedure stiffens a truss with a minimum increase in weight.

(1) Let the minimum areas be used for those members in which the product $S_o u_o$ is negative. These members should now be included in group c.

(2) Treat the remaining areas as open parameters (group o) and determine their magnitudes from Equation (9).

(3) If any of the areas A_o^* assume values lower than their minimum values, increase their magnitudes to their minimum values and transfer them to group c.

(4) Return to Step 2 and repeat the process until the areas determined by Equation (9) are all greater than or equal to their minimum values.

After demonstrating this design procedure by the following example, we shall return to Step 3 and comment on its validity.

2. Example. - We shall design the truss shown in Figure 3a so that the downward deflection at joint G is equal to 55a/E and the weight is a minimum.

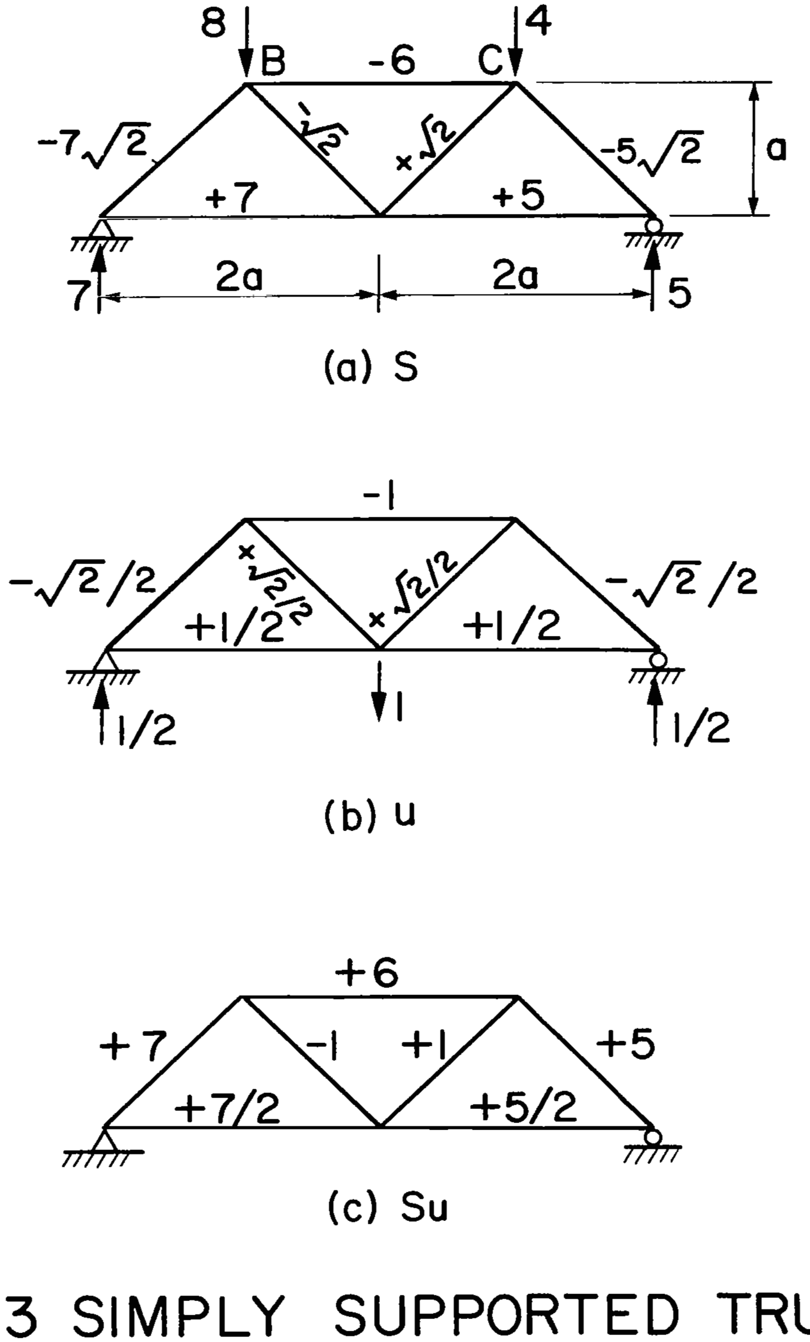


Fig.3 SIMPLY SUPPORTED TRUSS

Some of the properties of the truss shown in Figure 3 are given in Table 1.

TABLE 1

TRUSS PROPERTIES

Member	Designation	Specified Area	Specified Minimum Area	Su	L	$\sqrt{Su}$	$L\sqrt{Su}$	SuL/A_m
AB	1	0.5		7	$\sqrt{2}$ a	2.646	3.742a	$14\sqrt{2}$ a
AG	2		0.5	7/2	2a	1.871	3.742a	14a
BC	3		1.0	6	2a	2.449	4.899a	12a
BG	4		0.2	-1	$\sqrt{2}$ a	-----	------	$-5\sqrt{2}$ a
CD	5		0.5	5	$\sqrt{2}$ a	2.236	3.162a	$10\sqrt{2}$ a
CG	6		0.2	1	$\sqrt{2}$ a	1.000	1.414a	$5\sqrt{2}$ a
DG	7		0.5	5/2	2a	1.581	3.162a	10a

The truss deflection, when specified and minimum bar areas are used, is given by the sum of the righthand column; thus,

$$\text{Deflection} = \frac{69.936\ a}{E}$$

However, the specified deflection is

$$\Delta = \frac{55.000\ a}{E}$$

Compute Equation (16):

$$\Delta - \frac{S_1 u_1 L_1}{E\ A_1} + \left|\frac{S_4 u_4 L_4}{E\ A_4}\right| = \frac{55a}{E} - \frac{14\sqrt{2}\ a}{E} + \left|\frac{-5\sqrt{2}\ a}{E}\right| > 0,$$

therefore, solutions exist.

Step 1:

Set $A_4 = 0.2$ and transfer it to group c. Then

$$\sum_c \frac{S_c u_c L_c}{E\ A_c} = \frac{14\sqrt{2}\ a}{E} - \frac{5\sqrt{2}\ a}{E} = \frac{12.728\ a}{E}$$

Step 2:

$$A_o^* = \frac{\sqrt{S_o u_o}\sum_o (S_o u_o)^{1/2} L_o}{E\Delta - \sum_c \frac{S_c u_c L_c}{A_c}}$$

$$A_o^* = \frac{\sqrt{S_o u_o}(16.379)}{55.000 - 12.728} = 0.3875\ (S_o u_o)^{1/2}$$

Member	Minimum Area	A_o^*
1	0.5	------
2	0.5	0.7250
3	1.0	0.9490 ← $A_3^* < A_{3m}$
4	0.2	------
5	0.5	0.86645
6	0.2	0.3875
7	0.5	0.6126

Step 3:

Set $A_3 = 1.0$ and transfer to group c. Then

$$\sum_c \frac{S_c u_c L_c}{E_c A_c} = \frac{12.728\ a}{E} + \frac{12.000\ a}{E} = \frac{24.728\ a}{E}$$

Step 4:

Return to Step 2.

$$A_o^* = \frac{\sqrt{S_o u_o}(11.480)}{55.000 - 24.728} = 0.37923\ (S_o u_o)^{1/2}$$

Member	Minimum Area	A_o^*
1	0.5	-------
2	0.5	0.70954
3	1.0	-------
4	0.2	-------
5	0.5	0.84796
6	0.2	0.37923
7	0.5	0.59956

Note that $A_o^* > A_m$ and the procedure ends.

Summary

(1) Member	(2) Area (A_m or A_o^*)	(3) SuL/A	(4) AL	(5) $A_m L$
1	0.5000	19.79900a	0.7071a	0.7071a
2	0.7095	9.86554a	1.4190a	1.0000a
3	1.0000	12.00000a	2.0000a	2.0000a
4	0.2000	-7.07105a	0.2828a	0.2828a
5	0.8480	8.33889a	1.1991a	0.7071a
6	0.3792	3.72916a	0.5362a	0.2828a
7	0.5996	8.33944a	1.1992a	1.0000a
		55.00098a	7.3434a	5.9798a

Column (2) lists the optimum areas. Using these areas the deflection is computed in column (3) as a check on the computations. The volume of material in the optimum truss is given in column (4), and the volume of material based on minimum areas is given in column (5).

Step 3 of the design procedure is the only step which requires elaboration. When an optimum area A_j^* is increased to its minimum value, the optimum values of all the other member areas are affected. If such an increase can cause some of the optimum values of the open members to increase, our design procedure breaks down. There would always be the possibility that we had assigned a minimum value of area to a member which might have required a larger area after other member areas had been increased to their minimum values in accordance with Step 3. We must, therefore, show that an increase in any optimum area value will decrease all of the other optimum area values.

Assume that Steps 1 and 2 have been performed. Now consider any open member, say the ith. If we fix that area of this member at A_i, the expression for the remaining optimum areas is found by appropriately modifying Equation (9); thus,

$$A_o^* = \frac{\sqrt{\frac{S_o u_o}{\rho_o E_o}} \sum_{o-i} \left(\frac{S_o u_o \rho_o}{E_o}\right)^{1/2} L_o}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c} - \frac{S_i u_i L_i}{A_i E_i}} \tag{17}$$

where the symbol $\sum_{o-i}$ denotes the summation over all the open members except the ith. When the area A_i is fixed at its optimum value A_i^*, Equation (17) reduces to Equation (9). When the area A_i is fixed at a value greater than A_i^*, we shall denote the values given by Equation (17) as A_o^{**}. We must show that $A_o^{**} \leq A_o^*$ or that

$$\frac{\sqrt{\frac{S_o u_o}{\rho_o E_o}} \sum_{o-i} \left(\frac{S_o u_o \rho_o}{E_o}\right)^{1/2} L_o}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c} - \frac{S_i u_i L_i}{A_i E_i}} \leq \frac{\sqrt{\frac{S_o u_o}{\rho_o E_o}} \sum_{o-i} \left(\frac{S_o u_o \rho_o}{E_o}\right)^{1/2} L_o}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c} - \frac{S_i u_i L_i}{A_i^* E_i}} \tag{18}$$

Since $A_i \geq A_i^*$ the denominator on the left is greater than or equal to the denominator on the right and the inequality obviously holds. Hence Step 3 is justified.

D. Optimum Truss Configurations

The equivalence of maximum rigidity with minimum total strain energy or uniform stress for a given volume of material has been suggested by a number of authors (ref. 2,3,4,5,6) beginning with H. R. Cox in 1936 and continuing with Richards and Chan in 1966. Saelman (ref.7) demonstrated in 1958 that these conditions do not result in maximum stiffness for the torsion problem; Barnett (ref. 1,8) proved a similar result for beams and established the circumstances under which minimum deflection designs display uniform maximum fiber stresses. We shall begin this section by re-emphasizing the previously established relationship between optimum stiffness/weight trusses and uniform stress trusses.

For statically determinate trusses proportioned entirely on the basis of stiffness, the minimum weight W^* (Case 4) is given by Equation (10). The corresponding weight of the uniform stress truss W_c is given by Equation (31) in Appendix II. When all bar areas are open and only one material is used, we wish to show that $W^* \leq W_c$. Thus,

$$W^* - W_c = \left[\sum_o \sqrt{\frac{S_o u_o L_o}{\Delta |S_o|}} \sqrt{\frac{L_o |S_o|}{E/\rho}} \right]^2 - \sum_o \frac{S_o u_o L_o}{\Delta |S_o|} \sum_o \frac{L_o |S_o|}{E/\rho} \leq 0 \tag{19}$$

From Schwarz's inequality we conclude that $W^* \leq W_c$ and that the equality holds if and only if u_o is proportional to S_o, i.e., $bu_o + cS_o = 0$ where b and c are constants.

We shall first consider the case where the actual and virtual loadings are proportional. Here, S_o is proportional to u_o and the minimum deflection truss is uniformly stressed.

Its weight can be found from either Equation (10) or (31) when we take $S_o/u_o = k$ and only one material is used.

$$W^* = W_c = \frac{(k/\Delta)}{(E/\rho)}\left[\sum_o |S_o| L_o\right]^2 \qquad (20)$$

This equation represents the lowest possible weight for a statically determinate truss of given configuration which is designed for a specified deflection Δ (or stiffness k/Δ). If we wish to select the optimum truss configurations from all possible minimum weight candidates, we must choose those which minimize the quantity shown in the brackets of Equation (20).

In 1904, Michell (ref.9) developed the conditions for minimizing the quantity $\sum_o |S_o| L_o$. The associated minimum weight structures are usually found to be statically determinate; however, hyperstatic Michell structures may sometimes occur. If this should happen, it is always possible to find an equal weight statically determinate Michell structure. Referring to the literature written in English, this is guaranteed by the theorems of Sved (ref. 10) and Barta (ref. 11) which state that in pin-jointed plane or space structures of n bars involving r redundancies, it is possible to obtain a statically determinate structure which yields the least possible weight by removing r properly chosen redundant bars from the given network. The theorem holds for fixed, not necessarily equal, permissible stresses in tension and compression.

To summarize, when the actual and virtual loadings on a truss are proportional, the optimum stiffness/weight truss is given by a Michell structure, either statically determinate or indeterminate, designed for equal magnitude tensile and compressive stresses. Such a structure minimizes the total strain energy as shown by Richards and Chan (ref.5) who propose this condition arbitrarily as a general stiffness criterion.

The authors H. L. Cox (ref. 3) and Hemp (ref. 4) also adopt this minimum strain energy argument; but, they incorrectly propose the general Michell structure without requiring that all stresses have the same magnitude.

Very few situations arise where the actual and virtual loadings are proportional. Such cases are encountered when a single concentrated load acts on a truss and the deflection under the load is minimized. Examples of this case are furnished by the Michell structures shown in Figures 4a and 4b which minimize, respectively, the central and tip deflections. We note that the length to depth ratio for these optimum members may be impractically large; for the optimum simply supported beam $L/d = \sqrt{2}$. For such problems, the work done by the single force F acting through the deflection δ must equal the strain energy U; thus,

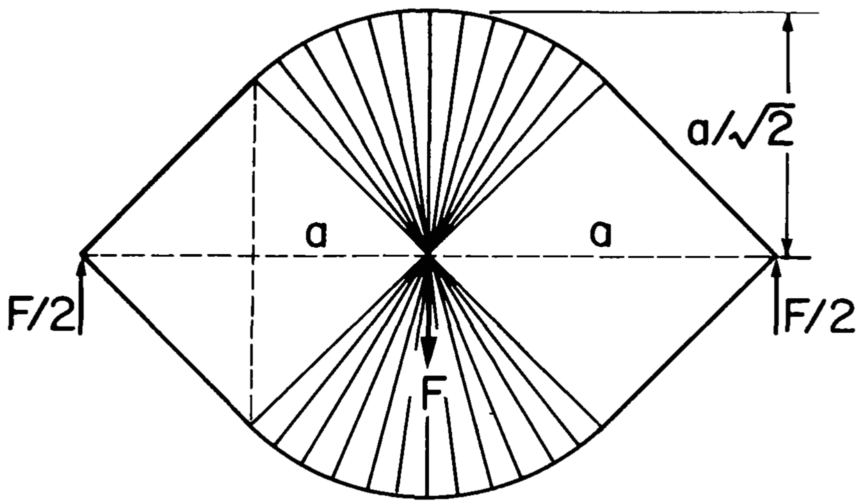
$$\frac{F\,\delta}{2} = U$$

In this simple situation it is clear that the structure minimizing U will also minimize δ.

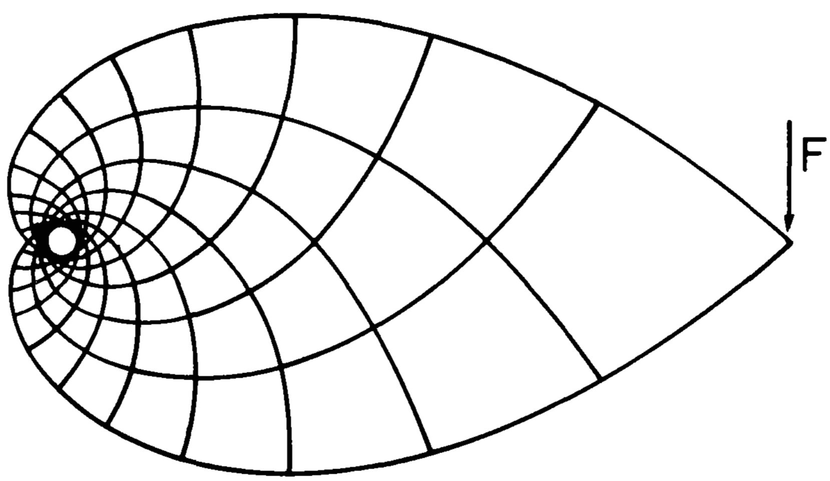
If several loads of the same magnitude F act on a truss and δ_i represents the deflection under the i th load in the direction of its action, the optimum may again lead to a Michell structure. Equating the strain energy U to the work done by the forces F we obtain

$$F \sum_i \delta_i = U$$

Consequently, a structure which minimizes the strain energy will also minimize the sum of the displacements under the forces F. To formulate this problem using virtual loads, we note that the deflection formula, Equation (1), used in the unit load method when



(a) Simply Supported Beam



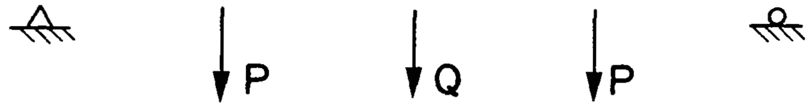
(b) Cantilever Beam

Fig.4 OPTIMUM STIFFNESS / WEIGHT MEMBERS
(MICHELL STRUCTURES)

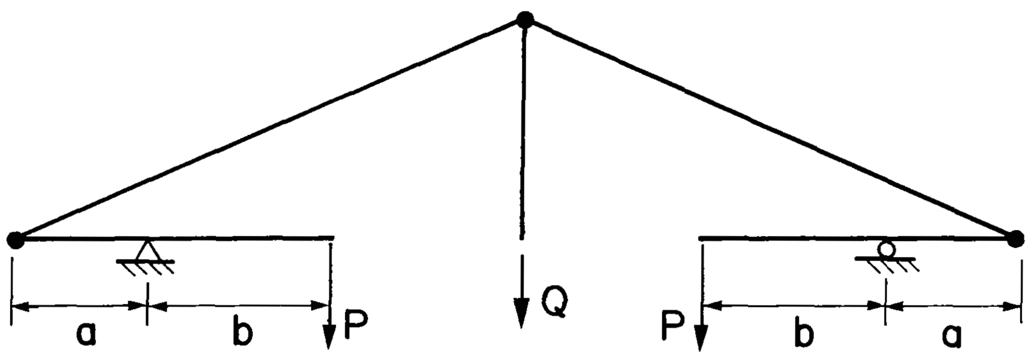
several unit loads are placed on a structure predicts the sum of the deflections which occur under the unit loads. Thus, in this case, we would place a unit load to correspond to each load F. The resulting virtual loading would be proportional to the actual loading, Δ would be interpreted as $\sum_i \delta_i$, and the optimum truss would be a uniformly stressed Michell structure. Now, if we wish to minimize the sum of the deflections under a set of loads of unequal magnitudes, we observe that the actual and virtual loadings are no longer proportional and that the optimum truss cannot be a Michell structure.

The two cases leading to Michell structures which we have just examined are probably the only situations where S_o and u_o are proportional. In all other problems minimum strain energy structures will not provide optimum stiffness/weight trusses. For such problems it does not appear to be difficult to choose bar arrangements which lead to the degenerate case described under Case 3 in Section B. Here, we recall that positive, negative, or zero deflections can be obtained with an infinite number of trusses of vanishingly small weight. The structure shown in Figure 1 provides an example of a degenerate truss design. for a single concentrated load which does not act at the node where we are interested in the deflection.

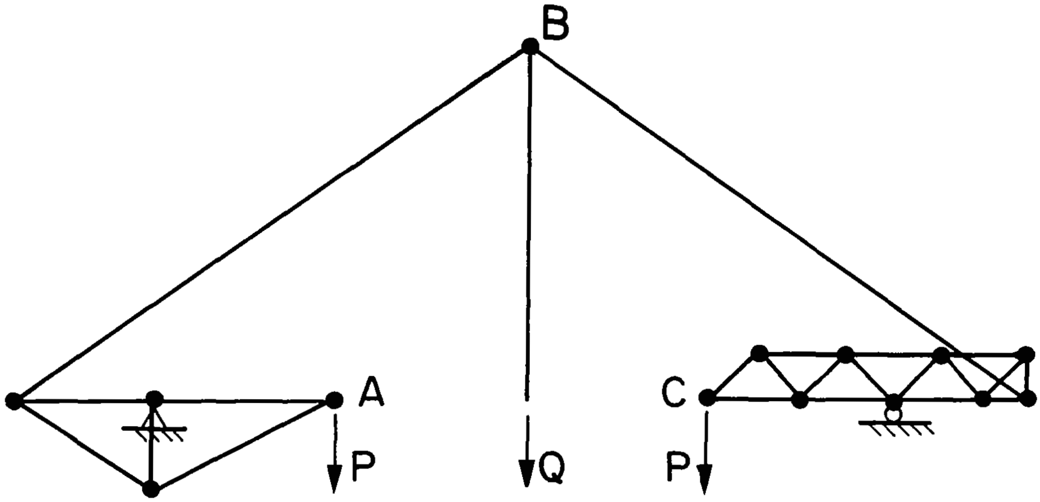
Another example is illustrated in Figure 5 where it is required that we minimize the central deflection in a truss under three symmetrically located forces. Examination of Figure 5a makes it clear that a virtual unit load placed in the center of the span cannot be proportional to the three forces shown; consequently, the optimum truss will not be a Michell structure. Observe that the linkage shown in Figure 5b will provide an upward force and movement at the center node to counteract the load Q. Adding bars to this mechanism, we can obtain the statically determinate truss shown in Figure 5c.



(a) Specified Loading



(b) Linkage



(c) Degenerate Truss

Fig. 5 OPTIMUM TRUSS DESIGN
(Virtual and Actual Loadings not Proportional)

A simple static analysis indicates that the members AB and BC in this truss will provide negative values of the product Su when the dimension a is adjusted so that

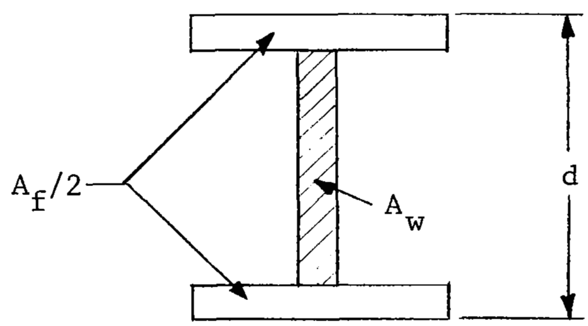
$$Q < \frac{2b}{a} P \qquad a \neq 0$$

Hence, the conditions of Case 3 are realized; namely, both positive and negative values of $S_o u_o$ occur in the same truss. The bar areas may be adjusted in an infinite number of ways to obtain any central deflection desired. We note in closing that when the actual and virtual loadings are proportional, a negative deflection would violate the conservation of energy.

APPENDIX I*

OPTIMUM STIFFNESS/WEIGHT BEAMS

A detailed development of minimum weight beams designed for deflection may be found in ref. 1. The relatively exact treatment given there for I-beams, unfortunately, masks the influence of some of the important beam parameters on optimum beam weight. Here, we obtain a very simple formula for minimum beam weight by using approximate expressions for the moment of inertia and shape factor.



I-BEAM SECTION

The moment of inertia and shape factor for the section shown in the above figure are,

$$I \doteq \frac{d^2}{12}(3A_f + A_w)$$

$$\alpha/A \doteq 1/A_w \; ... \text{ (see ref. 13).}$$

---

*The material in this appendix was taken from the paper by Barnett (ref. 12) and is included here for the sake of completeness.

The deflection of any statically determinate beam is given by

$$\Delta = \int_S \left(\frac{Mm}{EI} + \frac{\alpha Vv}{GA}\right) dx = \int_S \left[\frac{12Mm}{Ed^2(3A_f + A_w)} + \frac{Vv}{GA_w}\right] dx \tag{21}$$

where M is the bending moment, V is the shear, m is the virtual moment and v is the virtual shear caused by a unit load placed where the deflection is desired, G is the shear modulus and the integral is taken over the span S. The weight of the beam is given by,

$$W = \int_S \rho(A_f + A_w)\, dx \tag{22}$$

Using variational calculus, the conditions for minimizing the weight W subject to the requirement that Δ is equal to a specified constant are

$$\frac{\partial}{\partial A_F}\left\{\rho(A_f + A_w) + \gamma\left[\frac{12Mm}{Ed^2(3A_f + A_w)} + \frac{Vv}{GA_w}\right]\right\} = 0 \tag{23}$$

$$\frac{\partial}{\partial A_w}\left\{\rho(A_f + A_w) + \gamma\left[\frac{12Mm}{Ed^2(3A_f + A_w)} + \frac{Vv}{GA_w}\right]\right\} = 0 \tag{24}$$

where γ is a constant multiplier. Performing the operations in Equation (23) and (24), and eliminating γ with Equation (21), the optimum area distributions become

$$A_w = \sqrt{\frac{3Vv}{2G}}\,\frac{1}{\Delta}\int_S\left(\sqrt{\frac{4Mm}{Ed^2}} + \sqrt{\frac{2Vv}{3G}}\right) dx \tag{25}$$

$$A_f = \left[\sqrt{\frac{4Mm}{Ed^2}} - \sqrt{\frac{Vv}{6G}}\right]\frac{1}{\Delta}\int_S\left(\sqrt{\frac{4Mm}{Ed^2}} + \sqrt{\frac{2Vv}{3G}}\right) dx \tag{26}$$

Substituting these areas into Equation (22), we obtain an expression for the minimum weight beam.

$$W_b = \frac{1}{\Delta(E/\rho)} \left[\int_S \left(\sqrt{\frac{4Mm}{d^2}} + \sqrt{\frac{2Vv\beta^2}{3}} \right) dx \right]^2 \tag{27}$$

where

$$\beta^2 \equiv E/G \tag{28}$$

The approximation represented by Equation (27) is useful for sensibly proportioned beams; however, for very small length-to-depth ratios, the weight may be greatly underestimated by virtue of the negative flange width described by Equation (26). If we desire to study an infinitely deep beam, for example, we should abandon this development and reformulate the problem. An infinitely deep flat plate of finite cross-sectional area A_w has an infinite moment of inertia and, consequently, will not entertain any bending deflection. The shear deflection, on the other hand, depends only on the lengthwise distribution of A_w. The optimum area distribution can be shown to be

$$A_w = \frac{6}{5\Delta G} \sqrt{Vv} \int_S \sqrt{Vv}\, dx \text{....infinitely deep plate} \tag{29}$$

The corresponding beam weight is

$$W = \frac{6}{5\Delta(G/\rho)} \left[\int_S \sqrt{Vv}\, dx \right]^2 \text{....infinitely deep plate} \tag{30}$$

APPENDIX II

CONSTANT STRESS TRUSSES

The weight of a uniform stress truss designed for deflection often approximates and sometimes equals the weight of a minimum weight truss. The bar areas of such a truss are given by

$$A_o = \frac{|S_o|}{\sigma} \tag{31}$$

where the stress σ is a constant. By substituting A_o into Equation (1), the value of σ may be found for any specified deflection; thus

$$\sigma = \frac{\Delta}{\sum_o \frac{S_o}{|S_o|} \frac{u_o L_o}{E_o}} \tag{32}$$

Using the expressions for A_o and σ, the weight of a constant stress truss W_c becomes,

$$W_c = \frac{1}{\Delta} \left(\sum_o \rho_o L_o |S_o| \right) \left(\sum_o \frac{S_o}{|S_o|} \frac{u_o L_o}{E_o} \right) \tag{33}$$

APPENDIX III

EXAMPLES

1. <u>End Loaded Cantilevers</u>. - In this appendix, we shall compute the weight of a minimum weight, constant depth, cantilever truss and beam subjected to a concentrated end load. The truss geometry is defined in Figure 6. Expressions are given below for the actual and virtual bar forces and the bar lengths.

$$\left.\begin{aligned} S_n &= (-1)^{\frac{n+3}{2}} P\sqrt{1+\alpha^2} \\ u_n &= (-1)^{\frac{n+3}{2}} \sqrt{1+\alpha^2} \end{aligned}\right\} \quad n = 1, 3, 5, \ldots N-1$$

$$\left.\begin{aligned} S_n &= (-1)^{\frac{n}{2}} \frac{P\alpha n}{2} \\ u_n &= (-1)^{\frac{n}{2}} \frac{\alpha n}{2} \end{aligned}\right\} \quad n = 2, 4, 6, \ldots N$$

$$L_n = d\sqrt{1+\alpha^2} \qquad n = 1, 3, 5, \ldots N-1$$

$$L_n = 2a \qquad n = 2, 4, 6, \ldots N-2$$

$$L_n = a \qquad n = N$$

where

$$\alpha \equiv a/d$$

$$N = 2(L/a)$$

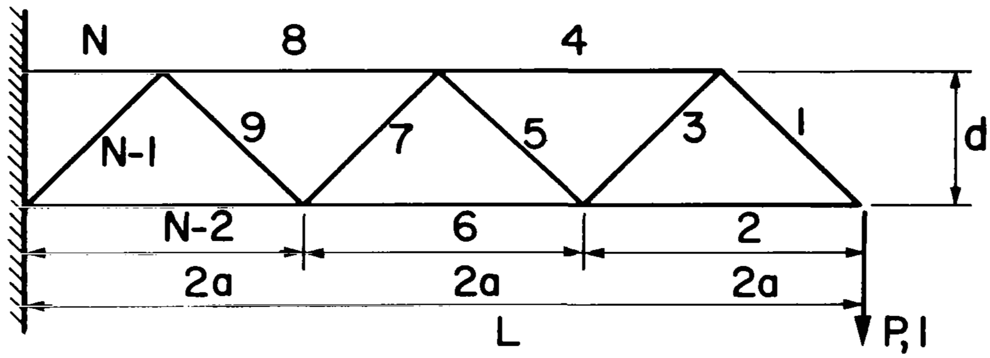


Fig. 6 END LOADED CANTILEVER TRUSS

It is noted that the ratio $S/u = P$ for each of the truss members. Since this ratio is constant throughout the truss, the minimum weight design and the uniform stress design are equivalent.

Using the expressions for S_n, u_n, L_n, the truss weight may be computed from Equation (10).

$$W^* = \frac{1}{\Delta}\left[\sum_o \left(\frac{S_o u_o}{E_o/\rho_o}\right)^{1/2} L_o\right]^2$$

$$W^* = \frac{1}{\Delta}\left[\sum_{n=1,3}^{N-1} \sqrt{\frac{P}{E/\rho}}(1 + \alpha^2)d + \sum_{n=2,4}^{N-2} \sqrt{\frac{P}{E/\rho}}\,\frac{n\alpha}{2}\,2a + \sqrt{\frac{P}{E/\rho}}\,\frac{N\alpha}{2}\,a\right]^2$$

$$W^* = \frac{PL^2}{\Delta(E/\rho)}\left(\frac{1 + \alpha^2}{\alpha} + L/d\right)^2$$

The minimum of this expression occurs when the web members are placed at 45 degree angles, i.e., $\alpha = 1$. The variation of W^* for angles near 45 degrees is quite small as can be seen from Figure 7. For $\alpha = 1$, W^* becomes

$$W^* = \frac{PL^2}{\Delta(E/\rho)}(2 + L/d)^2 \qquad (34)$$

If an I-beam is used instead of a truss, its weight is computed from Equation (27) when

$$M = Px$$
$$m = x$$
$$V = P$$
$$v = 1$$

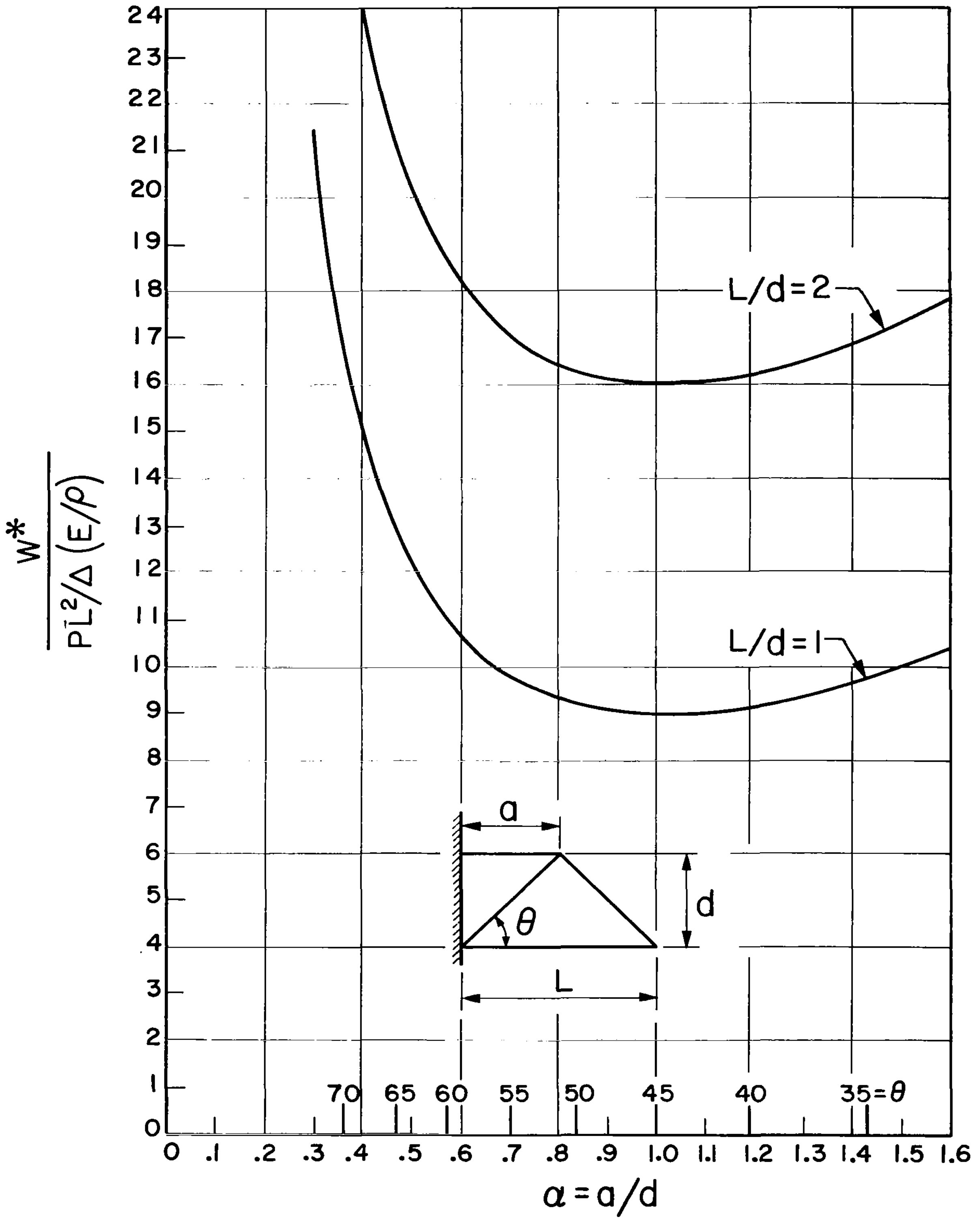


Fig. 7 VARIATIONS OF TRUSS WEIGHT WITH WEB ANGLE

and x is measured from the free end. Thus,

$$W_b = \frac{1}{\Delta(E/\rho)}\left[\int_o^L \sqrt{\frac{4Px^2}{d^2}} + \sqrt{\frac{2P\beta^2}{3}}\, dx\right]^2$$

$$W_b = \frac{1}{\Delta(E/\rho)}\left[(L/d)^2 + \sqrt{\frac{8}{3}}\,\beta(L/d) + \frac{2}{3}\beta^2\right] \tag{35}$$

The ratio of beam to truss weight is found from Equation (34) and (35).

$$\frac{W_{beam}}{W_{truss}} = \frac{(L/d)^2 + \sqrt{\frac{8}{3}}\,\beta(L/d) + \frac{2}{3}\beta^2}{(2 + L/d)^2} = \frac{(L/d)^2 + 2.580\ (L/d) + 1.667}{(2 + L/d)^2} \tag{36}$$

for $\beta^2 = 2.5$. This ratio is plotted against L/d in Figure 2.

2. Cantilever Under a Triangular Loading. - The truss defined in Figure 8 shall be designed for minimum weight and uniform stress. The resulting deflection designs will be compared to a minimum weight constant depth I-beam of equivalent stiffness.

The actual and virtual bar stresses and the member lengths for the truss are given by the following expressions:

$$\left.\begin{aligned} S_n &= (-1)^{\frac{n+3}{2}} \frac{Ra^2}{L^2}\sqrt{1+\alpha^2}\left(\frac{n^2}{4} + \frac{1}{12}\right) \\ u_n &= (-1)^{\frac{n+3}{2}} \sqrt{1+\alpha^2} \end{aligned}\right\} \quad n = 1, 3, 5, \ldots N-1$$

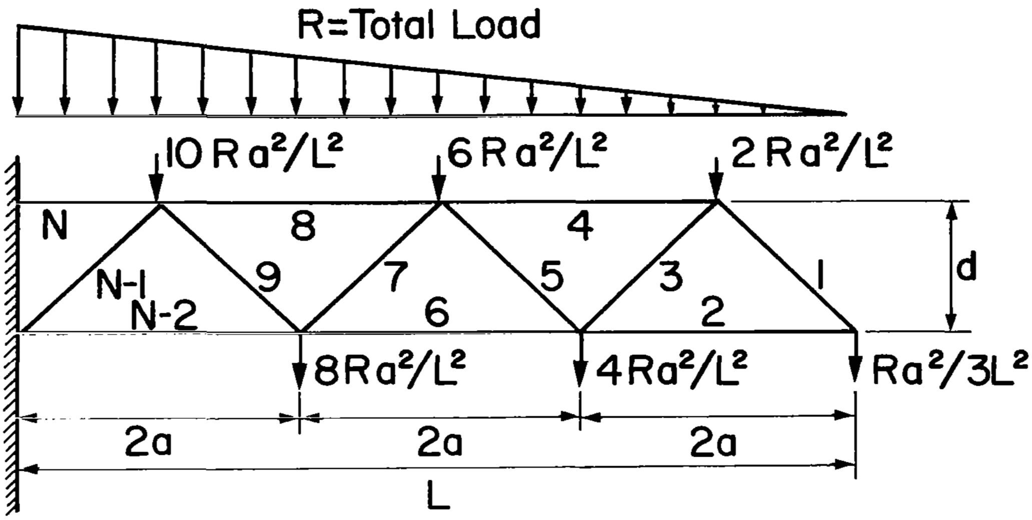


Fig. 8 TRUSS UNDER A TRIANGULAR LOADING

$$\left.\begin{aligned} S_n &= (-1)^{n/2} \frac{Ra^2\alpha}{3L^2} \left(\frac{n}{2}\right)^3 \\ u_n &= (-1)^{n/2} \alpha\left(\frac{n}{2}\right) \end{aligned}\right\} \quad n = 2, 4, 6, \ldots N$$

$$L_n = d\sqrt{1 + \alpha^2} \qquad n = 1, 3, 5, \ldots N-1$$

$$L_n = 2a \qquad n = 2, 4, 6, \ldots N-2$$

$$L_n = a \qquad n = N$$

where

$$\alpha \equiv a/d$$

$$N = 2(L/a)$$

Substituting these into Equation (10), we obtain the weight of a minimum weight truss.

$$W^* = \frac{1}{\Delta}\left[\sqrt{\frac{R}{E/\rho}} \frac{ad}{L} (1 + \alpha^2) \sum_{n=1,3}^{N-1} \sqrt{\left(\frac{n}{2}\right)^2 + \frac{1}{12}} + \sqrt{\frac{R}{E/\rho}} \frac{2\, a^2\, \alpha}{\sqrt{3}\, d} \sum_{n=2,4}^{N-2} \left(\frac{n}{2}\right)^2 + \sqrt{\frac{R}{E/\rho}} \frac{a^2\, \alpha}{\sqrt{3}\, d} \left(\frac{N}{2}\right)^2\right]^2$$

Introducing the approximation

$$\sum_{n=1,3}^{N-1} \sqrt{\left(\frac{n}{2}\right)^2 + \frac{1}{12}} \doteq \sum_{n=1,3}^{N-1} \left(\frac{n}{2}\right) = \frac{L^2}{2a^2} ,$$

W^* becomes

$$W^* = \frac{RL^2}{27\Delta(E/\rho)} \left[\frac{\alpha^2}{(L/d)} + \frac{3\sqrt{3}}{2\alpha}(1+\alpha^2) + 2(L/d)\right]^2 \qquad (37)$$

For this loading, it is clear that the optimum web angle is not 45 degrees. For L/d = 1, $\alpha = 0.79$; as $L/d \to \infty$, $\alpha \to 1$. The weight of a uniform or constant stress truss is found from Equation (33). Using the expressions for S_n, u_n, and L_n, W_c becomes

$$W_c = \frac{1}{\Delta(E/\rho)} \left[\sum_{n=1,3}^{N-1} \frac{Ra^2 d}{12L^2}(1+\alpha^2)(3n^2+1) + \sum_{n=2,4}^{N-2} \frac{Ra^2\alpha}{3L^2}\left(\frac{n}{2}\right)^2 2a + \frac{Ra^2\alpha}{3L^2}\left(\frac{N}{2}\right)^2 a\right] \cdot \left[\sum_{n=1,3}^{N-1} d(1+\alpha^2) + \sum_{n=2,4}^{N-2} \frac{n\alpha}{2} 2a + \frac{N\alpha}{2} a\right]$$

$$W_c = \frac{RL^2}{6\Delta(E/\rho)} \left[\frac{\alpha^3}{L/d} + 2\alpha^2 + \frac{\alpha}{L/d} + 2\alpha(L/d) + (L/d)^2 + 3 + \frac{3}{\alpha}(L/d) + \frac{2}{\alpha^2}\right] \qquad (38)$$

The ratio of the weights of the minimum weight truss and the constant stress truss is found from Equation (37) and (38). For $\alpha = 1$ it becomes

$$\frac{W_{min.}}{W_{const.}} = \frac{\frac{1}{27}\left[3\sqrt{3} + 2(L/d) + \frac{1}{(L/d)}\right]^2}{\frac{1}{6}\left[(L/d)^2 + 5(L/d) + 7 + \frac{2}{(L/d)}\right]} \qquad (39)$$

This ratio is plotted against L/d in Figure 2.

The following expressions give the actual and virtual bending moments and shear acting on an I-beam subjected to a triangular loading.

$$M = \frac{R\ x^3}{3L^2}$$

$$m = x$$

$$V = \frac{R\ x^2}{L^2}$$

$$v = 1$$

where x is measured from the free end of the cantilever. Using these relationships, the weight of a minimum weight I-beam can be computed from Equation (27); thus,

$$W_b = \frac{1}{\Delta(E/\rho)}\left[\int_o^L\left(\sqrt{\frac{4\ Rx^4}{d^2 3L^2}} + \sqrt{\frac{2Rx^2\beta^2}{3L^2}}\right)dx\right]^2$$

$$W_b = \frac{4RL^2}{27\Delta(E/\rho)}\left(\frac{3\sqrt{2}}{4}\beta + \frac{L}{d}\right)^2 \qquad (40)$$

The ratio of the weights of the minimum weight beam and the minimum weight truss can be found from Equation (37) and (40). For $\alpha = 1$ and $\beta^2 = 2.5$, the ratio becomes

$$\frac{W_{beam}}{W_{truss}} = \frac{4(1.667 + L/d)^2}{\left[5.196 + 2(L/d) + \frac{1}{(L/d)}\right]^2} \qquad (41)$$

This ratio is plotted against L/d in Figure 2.

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