MIT

DUAL AND COMPLEX FORMULATIONS OF THIN SHELL EQUATIONS

by

Z. M. Elias

DEPARTMENT OF CIVIL ENGINEERING

SCHOOL OF ENGINEERING

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Cambridge, Massachusetts 02139

Sponsored by

The National Aeronautics and Space Administration

Research Grant NGR-22-009-059 (251)

R68-20

March 1968

Dual and Complex Formulations of Thin Shell Equations

by

Z. M. Elias

Sponsored by

The National Aeronautics and Space Administration

Research Grant NGR-22-009-059 (251)

March 1968

1. Introduction

The general methods of solution of the basic equations of the linear theory of thin shells may be classified into displacement, force and mixed methods. The first method is well known and deals with equilibrium equations expressed in terms of displacements. The second and third methods take various forms. In (1)(2) a system of equations for "complex forces" is developed. In (3) the equations of the stress function method are obtained by expressing the strain compatibility equations in terms of stress functions. A well known mixed method for shallow shells deals with a system of two equations for a displacement component and a stress function (4)(5). A fundamental property of the basic equations of linear thin shell theory is the static geometric analogy (3)(6) or duality. In the following the duality is established for the non-homogenous shell problem and is used as a basis for establishing the equations of two types of formulations: a dual or mixed formulation in which the unknowns are displacements and stress functions and complex formulations in which the displacements and stress functions are combined into the real and imaginary parts of complex dependent variables.

2. Basic Equations

The principal lines of curvature of the middle surface are chosen as coordinate lines of a system of curvilinear coordinates ξ_1 and ξ_2. The first and second fundamental forms are written, respectively, in the form

$$d\bar{r} \cdot d\bar{r} = \alpha_1^2 \, d\xi_1^2 + \alpha_2^2 \, d\xi_2^2 \tag{1a}$$

$$d\bar{r} \cdot d\bar{t}_3 = \frac{\alpha_1^2}{R_1} d\xi_1^2 + \frac{\alpha_2^2}{R_2} \, d\xi_2^2 \tag{1b}$$

A local reference frame of unit vectors $\bar{t}_1$, $\bar{t}_2$ and $\bar{t}_3$ is defined through the relations

$$\bar{r}_{,1} = \alpha_1 \, \bar{t}_1 \tag{2a}$$

$$\bar{r}_{,2} = \alpha_2 \, \bar{t}_2 \tag{2b}$$

$$\bar{t}_3 = \bar{t}_1 \times \bar{t}_2 \tag{2c}$$

The vector equilibrium equations may be written in the form

$$(\alpha_2 \, \bar{N}_1)_{,1} + (\alpha_1 \, \bar{N}_2)_{,2} + \alpha_1 \, \alpha_2 \quad \bar{P} = 0 \tag{3a}$$

$$(\alpha_2 \, \bar{M}_1)_{,1} + (\alpha_1 \, \bar{M}_2)_{,2} + \alpha_1 \, \alpha_2 \, (\bar{t}_1 \times \bar{N}_1 + \bar{t}_2 \times \bar{N}_2) + \alpha_1 \, \alpha_2 \, \bar{Q} = 0 \tag{3b}$$

where $\bar{N}_i$ and $\bar{M}_i$ are stress resultant and stress couple vectors and $\bar{P}$ and $\bar{Q}$ are applied force and moment intensities.

let

$$(\bar{N}_i, \bar{M}_i) = (\bar{N}_i^* + \bar{N}_i^p, \bar{M}_i^* + \bar{M}_i^p), \quad i = 1, 2 \tag{4}$$

where $\bar{N}_i^p$ and $\bar{M}_i^p$ form a particular solution of Equations (3) and $\bar{N}_i^*$ and $\bar{M}_i^*$ the general solution of the homogenous equilibrium equations

$$(\alpha_2 \bar{N}_1^*)_{,1} + (\alpha_1 \bar{N}_2^*)_{,2} = 0 \tag{5a}$$

$$(\alpha_2 \bar{M}_1^*)_{,1} + (\alpha_1 \bar{M}_2^*)_{,2} + \alpha_1 \alpha_2 (\bar{t}_1 \times \bar{N}_1^* + \bar{t}_2 \times \bar{N}_2^*) = 0 \tag{5b}$$

Equations (5) are solved by means of two vector stress functions $\bar{F}$ and $\bar{G}$ in the form

$$\alpha_2 \bar{N}_1^* = \bar{F}_{,2} \tag{6a}$$

$$\alpha_1 \bar{N}_2^* = - \bar{F}_{,1} \tag{6b}$$

$$\alpha_2 \bar{M}_1^* = \bar{G}_{,2} + \alpha_2 \bar{t}_2 \times \bar{F} \tag{6c}$$

$$\alpha_1 \bar{M}_2^* = - \bar{G}_{,1} - \alpha_1 \bar{t}_1 \times \bar{F} \tag{6d}$$

The strain-displacement relations for the strain vectors, $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$, $\bar{\chi}_1$, and $\bar{\chi}_2$ corresponding, respectively, to $\bar{N}_1$, $\bar{N}_2$, $\bar{M}_1$, and $\bar{M}_2$ have the form

$$\alpha_1 \bar{\varepsilon}_1 = \bar{u}_{,1} + \alpha_1 \bar{t}_1 \times \bar{\omega} \tag{7a}$$

$$\alpha_2 \bar{\varepsilon}_2 = \bar{u}_{,2} + \alpha_2 \bar{t}_2 \times \bar{\omega} \tag{7b}$$

$$\alpha_1 \bar{\chi}_1 = \bar{\omega}_{,1} \tag{7c}$$

$$\alpha_2 \bar{\chi}_2 = \bar{\omega}_{,2} \tag{7d}$$

where $\bar{u}$ and $\bar{\omega}$ are the translation and rotation vectors, respectively, associated with a point of the middle surface and the normal thereto. The strain vectors satisfy the compatibility relations

$$(\alpha_2 \bar{\chi}_2)_{,1} - (\alpha_1 \bar{\chi}_1)_{,2} = 0 \tag{8a}$$

$$(\alpha_2 \bar{\varepsilon}_2)_{,1} - (\alpha_1 \bar{\varepsilon}_1)_{,2} + \alpha_1 \alpha_2 (\bar{t}_1 \times \bar{\chi}_2 - \bar{t}_2 \times \bar{\chi}_1) = 0 \tag{8b}$$

The component representation of $\overline{\chi}_1$, $\overline{\chi}_2$, $\overline{M}_1$ and $\overline{M}_2$ is taken in the form

$$(\overline{\ }) = - (\ )_2 \, \overline{t}_1 + (\ )_1 \, \overline{t}_2 + (\ )_3 \, \overline{t}_3 \tag{9a}$$

and that of all other vecotrs in the form

$$(\overline{\ }) = (\ )_1 \, \overline{t}_1 + (\ )_2 \, \overline{t}_2 + (\ )_3 \, \overline{t}_3 \tag{9b}$$

The stress-strain relations for an elastic shell are assumed obtainable from a strain energy density function W_e $(\varepsilon_{ij}, \chi_{ij})$ and, alternatively, from a complementary strain energy density function W_σ (N_{ij}, M_{ij}) in the form

$$(N_{ij}, M_{ij}) = \left(\frac{\partial W_e}{\partial \varepsilon_{ij}}, \frac{\partial W_e}{\partial \chi_{ij}}\right) \tag{10a}$$

$$(\varepsilon_{ij}, \chi_{ij}) = \left(\frac{\partial W_\sigma}{\partial N_{ij}}, \frac{\partial W_\sigma}{\partial M_{ij}}\right) \tag{10b}$$

We will consider a linearly elastic material without initial stresses so that W_e and W_σ are homogenous polynomials of second degree in their respective variables.

It will prove convenient for the purpose of presenting the static geometric analogy to express Equations (10) in terms of N^*_{ij} and M^*_{ij}. For this purpose let

$$W^*_\sigma = W_\sigma \, (N^*_{ij}, M^*_{ij}) \tag{11}$$

$$(\varepsilon^p_{ij}, \chi^p_{ij}) = \left(\frac{\partial W_\sigma}{\partial N_{ij}}, \frac{\partial W_\sigma}{\partial M_{ij}}\right)^P \tag{12}$$

where the partial derivatives are evaluated at N^p_{ij}, M^p_{ij}. Equations (10) may then be written in the form

$$(N^*_{ij} + N^p_{ij}, M^*_{ij} + M^p_{ij}) = \left(\frac{\partial W_e}{\partial \varepsilon_{ij}}, \frac{\partial W_e}{\partial \chi_{ij}}\right) \tag{13a}$$

$$(\varepsilon_{ij} - \varepsilon^p_{ij}, \chi_{ij} - \chi^p_{ij}) = \left(\frac{\partial W^*_\sigma}{\partial N^*_{ij}}, \frac{\partial W^*_\sigma}{\partial M^*_{ij}}\right) \tag{13b}$$

and the following relations hold

$$W_e = 1/2 \sum_{i,j} [(N^*_{ij} + N^p_{ij})\, \varepsilon_{ij} + (M^*_{ij} + M^p_{ij})\, \chi_{ij}] \tag{14a}$$

$$W^*_\sigma = 1/2 \sum_{i,j} [(\varepsilon_{ij} - \varepsilon^p_{ij})\, N^*_{ij} + (\chi_{ij} - \chi^p_{ij})\, M^*_{ij}] \tag{14b}$$

As for the stress resultants and stress couples the following notation will be adopted for the strain quantities.

$$(\ ) = (\ )^* + (\ )^p \tag{15}$$

If constraints are placed on the material of the shell they replace corresponding stress-strain relations which are then assumed excluded from Equations (13).

The constraints of zero transverse shear strains and zero couple-stress stress couples will be adopted i.e.,

$$\varepsilon_{i3} = 0 \tag{16a}$$

$$M_{i3} = 0 \tag{16b}$$

We let also

$$\varepsilon^p_{i3} = 0 \tag{16c}$$

$$M^{p}_{i3} = 0 \tag{16d}$$

$$\chi^{p}_{i3} = 0 \tag{16e}$$

3. Static-Geometric Analogy

The analogy between the forms of Equations (5) and (8), and (6) and (7) allows transforming one set of equations into the other by means of a correspondance between the statical and geometrical quantities. It will be convenient to establish this correspondance between pairs of quantities having the same physical dimensions. For this purpose all statical quantities are divided by an arbitrary factor k having the dimensions of a force as shown in Equation (17).

$$(\bar{n}_i, \bar{m}_i, \bar{f}, \bar{g}, \bar{p}, \bar{q}) = 1/k\ (\bar{N}_i, \bar{M}_i, \bar{F}, \bar{G}, \bar{P}, \bar{Q}) \tag{17}$$

$\bar{n}_i$ and $\bar{q}$ have the dimension of a curvature, $\bar{m}_i$ and $\bar{f}$ are non dimensional, $\bar{g}$ has the dimension of a length and $\bar{p}$ the dimension of the inverse of an area. Quantities $\bar{n}_i^*$, $\bar{m}_i^*$, $\bar{n}_i^p$ and $\bar{m}_i^p$ are defined similarly to $\bar{n}_i$ and $\bar{m}_i$. Letting

$$[w_\sigma\ (n_{ij}, m_{ij}), w_e\ (\varepsilon_{ij}, \chi_{ij})] = 1/k\ [W_\sigma\ (N_{ij}, M_{ij}), W_e\ (\varepsilon_{ij}, \chi_{ij})] \tag{18}$$

the stress-strain relations for the newly defined stress resultants and stress couples are obtained from w_σ and w_e through formulas similar to Equations (13).

The analogy between the homogenous equilibrium equations and the compatibility equations may be used to combine these two systems into one system of equations for complex dependent variables. These are defined through the relations

$$\tilde{\chi}_1 = -\ i\ \tilde{n}_2^* = \bar{\chi}_1 - i\ \bar{n}_2^* \tag{19a}$$

$$\tilde{\chi}_2 = i\ \tilde{n}_1^* = \bar{\chi}_2 + i\ \bar{n}_1^* \tag{19b}$$

$$\tilde{\varepsilon}_1 = - i \tilde{m}_2^* = \bar{\varepsilon}_1 - i \bar{m}_2^* \tag{19c}$$

$$\tilde{\varepsilon}_2 = i \tilde{m}_1^* = \bar{\varepsilon}_2 + i \bar{m}_1^* \tag{19d}$$

where $i = \sqrt{-1}$

The notation for a complex quantity associated with a real quantity ( ) or $(\bar{\ })$ is unambiguously $(\tilde{\ })$. No special notation is used to indicate complex vectors. The notation $(\bar{\tilde{\ }})$ indicates the complex conjugate of $(\tilde{\ })$.

Consistently with the notation

$$(\tilde{\ }) = (\tilde{\ })^* + (\tilde{\ })^p \tag{20}$$

we define the quantities

$$\tilde{\chi}_1^p = i \tilde{n}_2^p = \bar{\chi}_1^p + i \bar{n}_2^p \tag{21a}$$

$$\tilde{\chi}_2^p = - i \tilde{n}_1^p = \bar{\chi}_2^p - i \bar{n}_1^p \tag{21b}$$

$$\tilde{\varepsilon}_1^p = i \tilde{m}_2^p = \bar{\varepsilon}_1^p + i \bar{m}_2^p \tag{21c}$$

$$\tilde{\varepsilon}_2^p = - i \tilde{m}_1^p = \bar{\varepsilon}_2^p - i \bar{m}_1^p \tag{21d}$$

From Equations (20) and (21), Equations (19) may also be written in the form

$$\tilde{\chi}_1^* = - i \tilde{n}_2 = \bar{\chi}_1^* - i \bar{n}_2 \tag{22a}$$

$$\tilde{\chi}_2^* = i \tilde{n}_1 = \bar{\chi}_2^* + i \bar{n}_1 \tag{22b}$$

$$\tilde{\varepsilon}_1^* = - i \tilde{m}_2 = \bar{\varepsilon}_1^* - i \bar{m}_2 \tag{22c}$$

$$\tilde{\varepsilon}_2^* = i \tilde{m}_1 = \bar{\varepsilon}_2^* + i \bar{m}_1 \tag{22d}$$

$\bar{g}$, $\bar{f}$, $\bar{u}$ and $\bar{\omega}$ are combined in the forms

$$\tilde{u} = i\,\tilde{g} = \overline{u} + i\,\overline{g} \tag{23a}$$

$$\tilde{\omega} = i\,\tilde{f} = \overline{\omega} + i\,\overline{f} \tag{23b}$$

The complex quantities defined above will be referred to, as the notation suggests, complex strains, complex stress resultants, complex displacemnets etc.... The real and imaginary parts of a complex geometrical quantity as appearing in Equations (19), (21), (22), and (23) will be called dual. The same terminology is applied to the scalar components.

If dual quantities are interchanged the homogenous equilibrium and compatibility equations are interchanged. If a duality is to be defined in which the stress-strain relations remain invariant it is necessary to let the matrix of elastic constants, appropriately ordered, be the dual of its inverse.

We will consider an isotropic material having E as Young's modulus and ν as Poisson's ratio, for which, with

$$k = \frac{Eh^2}{12(1-\nu^2)} \tag{24a}$$

$$h_o = \frac{h}{12(1-\nu^2)} \tag{24b}$$

w_σ and w_e take the form

$$\begin{aligned} w_\sigma = {} & \frac{h_o}{2}\ [(n_{11} + n_{22})^2 - 2(1+\nu)(n_{11}\,n_{22} - n_{12}\,n_{21})] \\ & + \frac{1}{2(1-\nu^2)h_o}\ [(m_{11} + m_{22})^2 - 2(1+\nu)(m_{11}\,m_{22} - m_{12}\,m_{21})] \end{aligned} \tag{25a}$$

$$w_e = \frac{1}{2(1-\nu^2)h_o} [(\varepsilon_{11} + \varepsilon_{22})^2 - 2(1-\nu)(\varepsilon_{11}\,\varepsilon_{22} - \varepsilon_{12}\,\varepsilon_{21})]$$

$$+ \frac{h_o}{2} [(\chi_{11} + \chi_{22})^2 - 2(1-\nu)(\chi_{11}\,\chi_{22} - \chi_{12}\,\chi_{21})] \qquad (25b)$$

In Equations (25) h_o must be considered as an elastic constant. In the correspondance that leaves the stress strain relations unchanged h_o and ν correspond to - h_o and - ν, respectively, and w^*_σ corresponds to - w_e.

4. Equations for Complex Dependent Variables

Equations (5) and (8) may be combined into complex homogenous equilibrium equations in the form

$$(\alpha_2\, \tilde{n}_1^*)_{,1} + (\alpha_1\, \tilde{n}_2^*)_{,2} = 0 \tag{26a}$$

$$(\alpha_2\, \tilde{m}_1^*)_{,1} + (\alpha_1\, \tilde{m}_2^*)_{,2} + \alpha_1\, \alpha_2(\bar{t}_1 \times \tilde{n}_1^* + \bar{t}_2 \times \tilde{n}_2^*) = 0 \tag{26b}$$

It is also possible to write

$$(\alpha_2\, \tilde{n}_1)_{,1} + (\alpha_1\, \tilde{n}_2)_{,2} + \alpha_1\, \alpha_2\, \tilde{p} = 0 \tag{27a}$$

$$(\alpha_2\, \tilde{m}_1)_{,1} + (\alpha_1\, \tilde{m}_2)_{,2} + \alpha_1\, \alpha_2\, (\bar{t}_1 \times \tilde{n}_1 + \bar{t}_2 \times \tilde{n}_2) + \alpha_1\, \alpha_2\, \tilde{q} = 0 \tag{27b}$$

where

$$\tilde{p} = \bar{p} + i\, \frac{(\alpha_1\, \bar{\chi}_1^p)_{,2} - (\alpha_2\, \bar{\chi}_2^p)_{,1}}{\alpha_1\, \alpha_2} \tag{27c}$$

$$\tilde{q} = \bar{q} + i \left[\frac{(\alpha_1\, \bar{\varepsilon}_1^p)_{,2} - (\alpha_2\, \bar{\varepsilon}_2^p)_{,1}}{\alpha_1\, \alpha_2} + \bar{t}_2 \times \bar{\chi}_1^p - \bar{t}_1 \times \bar{\chi}_1^p \right] \tag{27d}$$

The stress-strain relations derived from w_σ and w_e and Equations (16a-d) may be combined in the complex form

$$\tilde{m}_{11}^* = - i\, h_o\, (\tilde{n}_{22} - \nu\, \bar{\tilde{n}}_{11}) \tag{28a}$$

$$\tilde{m}_{22}^* = - i\, h_o\, (\tilde{n}_{11} - \nu\, \bar{\tilde{n}}_{22}) \tag{28b}$$

$$\tilde{m}_{12}^* = i\, h_o\, (\tilde{n}_{12} + \nu\, \bar{\tilde{n}}_{12}) \tag{28c}$$

$$\tilde{m}_{21}^* = i\, h_o\, (\tilde{n}_{21} + \nu\, \bar{\tilde{n}}_{21}) \tag{28d}$$

$$\tilde{m}^*_{13} = 0 \tag{28e}$$

$$\tilde{m}^*_{23} = 0 \tag{28f}$$

where $\bar{\tilde{n}}_{ij}$ is the complex conjugate of $\tilde{n}_{ij}$. The relations between the quantities $\tilde{n}^p_{ij}$ and $\tilde{m}^p_{ij}$ associated with the particular solution of the equilibrium equations are obtained by changing the sign of one side of Equations (28). There follows that Equations (28) may alternatively be written with the sign (*) deleted from the left hand sides and attached to the quantities of the right hand sides.

Equations (26) are solved in terms of the two complex vector stress functions $\tilde{g}$ and $\tilde{f}$ in the form

$$\alpha_2 \, \tilde{n}^*_1 = \tilde{f}_{,2} \tag{29a}$$

$$\alpha_1 \, \tilde{n}^*_2 = - \tilde{f}_{,1} \tag{29b}$$

$$\alpha_2 \, \tilde{m}^*_1 = \tilde{g}_{,2} + \alpha_2 \, \bar{t}_2 \times \tilde{f} \tag{29c}$$

$$\alpha_1 \, \tilde{m}^*_2 = - \tilde{g}_{,1} - \alpha_1 \, \bar{t}_1 \times \tilde{f} \tag{29d}$$

The scalar equations obtained from Equations (27a, b) with $\tilde{m}_{13} = \tilde{m}_{23} = 0$ take the form

$$(\alpha_2 \, \tilde{n}_{11})_{,1} + (\alpha_1 \, \tilde{n}_{21})_{,2} - \alpha_{2,1} \, \tilde{n}_{22} + \alpha_{1,2} \, \tilde{n}_{12} + \frac{\alpha_1 \, \alpha_2}{R_1} \tilde{n}_{13} + \alpha_1 \, \alpha_2 \, \tilde{p}_1 = 0 \tag{30a}$$

$$(\alpha_1 \, \tilde{n}_{22})_{,2} + (\alpha_2 \, \tilde{n}_{12})_{,1} - \alpha_{1,2} \, \tilde{n}_{11} + \alpha_{2,1} \, \tilde{n}_{21} + \frac{\alpha_1 \, \alpha_2}{R_2} \tilde{n}_{23} + \alpha_1 \, \alpha_2 \, \tilde{p}_2 = 0 \tag{30b}$$

$$(\alpha_2 \, \tilde{n}_{13})_{,1} + (\alpha_1 \, \tilde{n}_{23})_{,2} - \alpha_1 \, \alpha_2 \left(\frac{\tilde{n}_{11}}{R_1} + \frac{\tilde{n}_{22}}{R_2} - \tilde{p}_3\right) = 0 \tag{30c}$$

$$(\alpha_2 \tilde{m}_{11})_{,1} + (\alpha_1 \tilde{m}_{21})_{,2} - \alpha_{2,1} \tilde{m}_{22} + \alpha_{1,2} \tilde{m}_{12} - \alpha_1 \alpha_2 \tilde{n}_{13} + \alpha_1 \alpha_2 \tilde{q}_2 = 0 \tag{30d}$$

$$(\alpha_1 \tilde{m}_{22})_{,2} + (\alpha_2 \tilde{m}_{12})_{,1} - \alpha_{1,2} \tilde{m}_{11} + \alpha_{2,1} \tilde{m}_{21} - \alpha_1 \alpha_2 \tilde{n}_{23} - \alpha_1 \alpha_2 \tilde{q}_1 = 0 \tag{30e}$$

$$\tilde{n}_{12} - \tilde{n}_{21} + \frac{\tilde{m}_{12}}{R_1} - \frac{\tilde{m}_{21}}{R_2} + \tilde{q}_3 = 0 \tag{30f}$$

and from Equations (29) there comes

$$\alpha_1 \alpha_2 \tilde{n}^*_{11} = \alpha_1 \tilde{f}_{1,2} - \alpha_{2,1} \tilde{f}_2 \tag{31a}$$

$$\alpha_1 \alpha_2 \tilde{n}^*_{22} = - \alpha_2 \tilde{f}_{2,1} + \alpha_{1,2} \tilde{f}_1 \tag{31b}$$

$$\alpha_1 \alpha_2 \tilde{n}^*_{12} = \alpha_1 \tilde{f}_{2,2} + \alpha_{2,1} \tilde{f}_1 + \frac{\alpha_1 \alpha_2}{R_2} \tilde{f}_3 \tag{31c}$$

$$\alpha_1 \alpha_2 \tilde{n}^*_{21} = - \alpha_2 \tilde{f}_{1,1} - \alpha_{1,2} \tilde{f}_2 - \frac{\alpha_1 \alpha_2}{R_1} \tilde{f}_3 \tag{31d}$$

$$\alpha_1 \alpha_2 \tilde{m}^*_{11} = \alpha_1 \tilde{g}_{2,2} + \alpha_{2,1} \tilde{g}_1 + \frac{\alpha_1 \alpha_2}{R_2} \tilde{g}_3 \tag{31e}$$

$$\alpha_1 \alpha_2 \tilde{m}^*_{22} = \alpha_2 \tilde{g}_{1,1} + \alpha_{1,2} \tilde{g}_2 + \frac{\alpha_1 \alpha_2}{R_1} \tilde{g}_3 \tag{31f}$$

$$\alpha_1 \alpha_2 \tilde{m}^*_{12} = - \alpha_1 \tilde{g}_{1,2} + \alpha_{2,1} \tilde{g}_2 - \alpha_1 \alpha_2 \tilde{f}_3 \tag{31g}$$

$$\alpha_1 \alpha_2 \tilde{m}^*_{21} = - \alpha_2 \tilde{g}_{2,1} + \alpha_{1,2} \tilde{g}_1 + \alpha_1 \alpha_2 \tilde{f}_3 \tag{31h}$$

$$\alpha_2 \tilde{n}^*_{13} = \tilde{f}_{3,2} - \frac{\alpha_2}{R_2} \tilde{f}_2 \tag{31i}$$

$$\alpha_1 \tilde{n}^*_{23} = - \tilde{f}_{3,1} + \frac{\alpha_1}{R_1} \tilde{f}_1 \tag{31j}$$

$$\alpha_2 \tilde{m}^*_{13} = \tilde{g}_{3,2} - \frac{\alpha_2}{R_2} \tilde{g}_2 - \alpha_2 \tilde{f}_1 \tag{31k}$$

$$\alpha_1 \tilde{m}^*_{23} = - \tilde{g}_{3,1} + \frac{\alpha_1}{R_1} \tilde{g}_1 - \alpha_1 \tilde{f}_2 \tag{31\ell}$$

letting $\tilde{m}^*_{13} = \tilde{m}^*_{23} = 0$ yields

$$\alpha_2 \tilde{f}_1 = \tilde{g}_{3,2} - \frac{\alpha_2}{R_2} \tilde{g}_2 \tag{32a}$$

$$\alpha_1 \tilde{f}_2 = - \tilde{g}_{3,1} + \frac{\alpha_1}{R_1} \tilde{g}_1 \tag{32b}$$

Equations (31 may then be expressed in terms of $\tilde{g}_1$, $\tilde{g}_2$, $\tilde{g}_3$ and $\tilde{f}_3$.

5. Mixed Formulation in Terms of Displacements and Stress Functions

Assuming that a particular solution of the equilibrium equations has been determined, the stress-strain relations when expressed in terms of the stress functions and displacements form a system of eight equations in the eight unknowns g_1, g_2, g_3, f_3, u_1, u_2, u_3, and ω_3. These equations are the real and imaginary parts of Equations (28a-d).

By eliminating $\tilde{f}_3 = f_3 - i\,\omega_3$ from Equations (28c-d) in which $\tilde{f}_3$ appears in non differential form a system of six real equations for g_1, g_2, g_3, u_1, u_2, and u_3 is obtained. From Equations (28c-d) written in the alternative form mentioned earlier with the asterisk on the right hand side and Equation (30f) used in the homogenous form there comes

$$\tilde{m}_{12} - \tilde{m}_{21} = h_o^2\,(1-\nu^2)\left(\frac{\tilde{n}_{12}}{R_1} - \frac{\tilde{n}_{21}}{R_2}\right) \tag{33a}$$

and from Equations (31g-h)

$$\tilde{m}_{12} - \tilde{m}_{21} = \frac{1}{\alpha_1\,\alpha_2}\,[(\alpha_2\,\tilde{g}_2),_1 - (\alpha_1\,\tilde{g}_1),_2] - 2\tilde{f}_3 + \tilde{m}^p_{12} - \tilde{m}^p_{21} \tag{33b}$$

It may be concluded from Equations (33) that a negligible error of order $\frac{h^2}{R^2}$ is made if in the expression (31c-d) of $\tilde{n}^*_{12}$ and $\tilde{n}^*_{21}$, $\tilde{f}_3$ is determined from Equation (33b) by letting $\tilde{m}_{12} - \tilde{m}_{21} = 0$, i.e.,

$$\tilde{f}_3 = \frac{1}{2\,\alpha_1\,\alpha_2}\,[(\alpha_2\,\tilde{g}_2),_1 - (\alpha_1\,\tilde{g}_1),_2] + 1/2\,(\tilde{m}^p_{12} - \tilde{m}^p_{21}) \tag{34}$$

Noting that $(m^*_{12} + m^*_{21})$ is independent of f_3 the three complex equations for determing u_1, u_2, u_3, g_1, g_2, and g_3 take the form

$$\tilde{m}^*_{11} = - i \, h_o \, (\tilde{n}_{22} - \nu \, \bar{\tilde{n}}_{11}) \tag{35a}$$

$$\tilde{m}^*_{22} = - i \, h_o \, (\tilde{n}_{11} - \nu \, \bar{\tilde{n}}_{22}) \tag{35b}$$

$$\tilde{m}^*_{12} + \tilde{m}^*_{21} = - i \, h_o \, [\tilde{n}_{12} + \tilde{n}_{21} + \nu \, (\bar{\tilde{n}}_{12} + \bar{\tilde{n}}_{21})] \tag{35c}$$

6. Complex Formulation in the Case $\nu = 0$

If $\nu = 0$ Equations (28a-d) do not contain the complex conjugates $\bar{\tilde{n}}_{11}$, $\bar{\tilde{n}}_{22}$, $\bar{\tilde{n}}_{12}$, and $\bar{\tilde{n}}_{21}$. They form, after use of Equation (31a-h) and (32), four equations for $\tilde{g}_1$, $\tilde{g}_2$, $\tilde{g}_3$, and $\tilde{f}_3$. As outlined in the preceding section a negligible error of order $\frac{h^2}{R^2}$ is made if Equations (35) with $\nu = 0$, are used as a system of three equations for determining $\tilde{g}_1$, $\tilde{g}_2$, and $\tilde{g}_3$.

This complex formulation is essentially the same as that of Equations 15.5 in Reference (1). In these equations however, displacements and stress functions are combined into complex quantities in the homogenous problem only and the particular solution is restricted to be a membrane solution. Another complex formulation in the case $\nu = 0$ will be obtained in section 8.

7. Complex Formulation in the Case $\nu \neq 0$. Expansion in Powers of ν

Let Equations (28a-d) be represented in matrix notation in the form

$$\{\tilde{m}^*\} - i\, h_0 \{\tilde{n}\} = i\, h_0\, \nu\, [J]\, \{\bar{\tilde{n}}\} \tag{36}$$

where

$$\{\tilde{m}^*\} = \{\tilde{m}^*_{11}\ \tilde{m}^*_{22}\ \tilde{m}^*_{12}\ \tilde{m}^*_{21}\} \tag{37a}$$

$$\{\tilde{n}\} = \{-\tilde{n}_{22}\ -\tilde{n}_{11}\ \tilde{n}_{12}\ \tilde{n}_{21}\} \tag{37b}$$

$$[J] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{37c}$$

The dependent variables may be sought as power series in ν in the form

$$(\tilde{\ }) = (\tilde{\ })_0 + \nu(\tilde{\ })_1 + \nu^2 (\tilde{\ })_2 + \ldots \tag{38}$$

Equating coefficients of equal powers of ν in Equation (36) obtain the system of equations

$$\{\tilde{m}^*\}_0 - i\, h_0 \{\tilde{n}\}_0 = 0 \tag{39a}$$

$$\{\tilde{m}^*\}_k - i\, h_0 \{\tilde{n}\}_k = i\, h_0 [J] \{\bar{\tilde{n}}\}_{k-1} \qquad k = 1, 2, 3, \ldots \tag{39b}$$

The solution of Equations (39) may proceed sequentially and, because Equations (39) have the same homogenous part, only particular solutions of (39b) need be determined.

The obtention of particular solutions of Equation (39b) presents in general no significant effort beyond the obtention of the general solution of the homogenous equation (39a). Practically the system of equations is truncated at an appropriate value of k.

8. Approximate Complex Formulation in the case $\nu \neq 0$. Vector Equations

The solution of the shell problem may be viewed as consisting in determining two complex vectors $\tilde{f}$ and $\tilde{g}$ such that the components of the left hand sides of Equations (29) satisfy the stress-strain relations Equations (28)

Equations (28) may be written in the vector form

$$\tilde{m}_1^* = i\, h_o\, (-\tilde{n}\, \bar{t}_2 + (1 + \nu(\bar{\ })) \, \bar{t}_3 \times \tilde{n}_1) \tag{40a}$$

$$\tilde{m}_2^* = i\, h_o\, (\tilde{n}\, \bar{t}_1 + (1 + \nu(\bar{\ })) \, \bar{t}_3 \times \tilde{n}_2) \tag{40b}$$

where

$$\tilde{n} = \tilde{n}_{11} + \tilde{n}_{22} \tag{41}$$

and $(\bar{\ })$ is the operator transforming a complex quantity into its complex conjugate. Considering now Equations (29c,d) as a system of equations for determining $\tilde{g}$, obtain with use of Equations (40)

$$\tilde{g},_1 = -\, i\, h_o\, (\tilde{n}\, \alpha_1\, \bar{t}_1 + (1 + \nu(\bar{\ }))\, \bar{t}_3 \times \alpha_1\, \tilde{n}_2) - \alpha_1\, \bar{t}_1 \times \tilde{f} \tag{42a}$$

$$\tilde{g},_2 = -\, i\, h_o\, (\tilde{n}\, \alpha_2\, \bar{t}_2 - (1 + \nu(\bar{\ }))\, \bar{t}_3 \times \alpha_2\, \tilde{n}_1) - \alpha_2\, \bar{t}_2 \times \tilde{f} \tag{42b}$$

From Equations (42) and (29a,b) the total differential of $\tilde{g}$ takes the form

$$d\tilde{g} = -\, i\, h_o\, [\tilde{n}\, d\bar{r} - (1 + \nu(\bar{\ })\, \bar{t}_3 \times d\tilde{f}] - d\bar{r} \times \tilde{f}$$

$$+\, i\, h_o\, (1 + \nu(\bar{\ })\, \bar{t}_3 \times (\alpha_2\, \tilde{n}_1^p\, d\xi_2 - \alpha_1\, \tilde{n}_2^p\, d\xi_1) \tag{43}$$

For $d\tilde{g}$ to be an exact differential the right hand sides of Equations (42) must satisfy the condition

$$\tilde{g},_{21} - \tilde{g},_{12} = 0 \tag{44}$$

which, after use of Equations (27a), (29a,b) and differentiation formulas for the unit normal vector $\bar{t}_3$, takes the form

$$\begin{aligned} & i\, h_o\, (\alpha_1\, \tilde{n}_{,2}\, \bar{t}_1 - \alpha_2\, \tilde{n}_{,1}\, \bar{t}_2) + \alpha_1\, \alpha_2\, (\bar{t}_1 \times \tilde{n}_1^* + \bar{t}_2 \times \tilde{n}_2^*) \\ & + \alpha_1\, \alpha_2\, i\, h_o\, (1 + \nu(\bar{\ }))(\frac{\bar{t}_1 \times \tilde{n}_1}{R_1} + \frac{\bar{t}_2 \times \tilde{n}_2}{R_2} - \bar{t}_3 \times \tilde{p}) = 0 \end{aligned} \tag{45}$$

Equation (45) may be expressed in terms of $\tilde{f}$ and $\bar{\tilde{f}}$ by means of Equations (29a,b). If $\nu = 0$, $\bar{\tilde{f}}$ does not occur and Equation (45) may be considered as a complex vector differential equation in one unknown $\tilde{f}$. For $\nu \neq 0$ an expansion of $\tilde{f}$ in powers of ν may be made following the method of section 7. It may be seen, however, that an equation for $\tilde{f}$ amy be obtained from Equation (45) after neglecting terms of order $\frac{h}{R}$. This is

$$\begin{aligned} & i\, h_o\, (\alpha_1\, \tilde{n}_{,2}\, \bar{t}_1 - \alpha_2\, \tilde{n}_{,1}\, \bar{t}_2) + \alpha_1\, \alpha_2\, (\bar{t}_1 \times \tilde{n}_1^* + \bar{t}_2 \times \tilde{n}_2^*) \\ & + \alpha_1\, \alpha_2\, i\, h_o\, (1 + \nu(\bar{\ }))\, (\frac{\bar{t}_1 \times \tilde{n}_1^p}{R_1} + \frac{\bar{t}_2 \times \tilde{n}_2^p}{R_2} - \bar{t}_3 \times \tilde{p}) = 0 \end{aligned} \tag{46}$$

Further simplifications are possible in Equation (46) if $\tilde{n}_1^p$ and $\tilde{n}_2^p$ are not of a larger order of magnitude than $\tilde{n}_1^*$ and $\tilde{n}_2^*$. Having solved Equation (46) for $\tilde{f}$, $\tilde{g}$ is determined through integration of $d\tilde{g}$ in Equation (43) where simplifications consistent with the obtention of Equation (46) and such that $d\tilde{g}$ remain an exact differential must be made. These simplifications may be seen to consist in letting in Equation (43)

$$\bar{t}_3 \times d\tilde{f} = d(\bar{t}_3 \times \tilde{f}) - d\bar{t}_3 \times \tilde{f} \tag{47}$$

then neglecting $i\, h_o\, d\bar{t}_3 \times \tilde{f}$ with regard to $d\bar{r} \times \tilde{f}$. The result is

$$\begin{aligned} & d\, (\tilde{g} - i\, h_o\, (1 + \nu(\bar{\ }))\bar{t}_3 \times \tilde{f}) = -\, i\, h_o\, \tilde{n}\, d\bar{r} + \tilde{f} \times d\bar{r} \\ & + i\, h_o\, (1 + \nu(\bar{\ }))\, \bar{t}_3 \times (\alpha_2\, \tilde{n}_1^p\, d\xi_2 - \alpha_1\, \tilde{n}_2^p\, d\xi_1) \end{aligned} \tag{48a}$$

letting

$$\tilde{y} = \tilde{g} - i\ h_o\ (1 + \nu(\tilde{\ })\ \bar{t}_3 \times \tilde{f} \tag{48b}$$

Equation (48a) yields

$$\tilde{y},_1 = -\ i\ h_o\ \alpha_1\ \tilde{n}\ \bar{t}_1 + \alpha_1\ \tilde{f} \times \bar{t}_1 - i\ h_o\ \alpha_1\ (1 + \nu(\tilde{\ }))\bar{t}_3 \times \tilde{n}_2^p \tag{48c}$$

$$\tilde{y},_2 = -\ i\ h_o\ \alpha_2\ \tilde{n}\ \bar{t}_2 + \alpha_2\ \tilde{f} \times \bar{t}_2 + i\ h_o\ \alpha_2\ (1 + \nu(\tilde{\ }))\bar{t}_3 \times \tilde{n}_1^p \tag{48d}$$

Scalar Equations

The scalar components of Equation (45) take the form

$$\tilde{n}^*_{13} + \frac{i\ h_o}{R_1}\ (\tilde{n}_{13} + \nu\ \bar{\tilde{n}}_{13}) = -\ i\ h_o\ (\frac{\tilde{n},_1}{\alpha_1} + \tilde{p}_1 + \nu\ \bar{\tilde{p}}_1) \tag{49a}$$

$$\tilde{n}^*_{23} + \frac{i\ h_o}{R_2}\ (\tilde{n}_{23} + \nu\ \bar{\tilde{n}}_{23}) = -\ i\ h_o\ (\frac{\tilde{n},_2}{\alpha_2} + \tilde{p}_2 + \nu\ \bar{\tilde{p}}_2) \tag{49b}$$

$$\tilde{n}^*_{12} - \tilde{n}^*_{21} = -\ i\ h_o\ (1 + \nu(\tilde{\ }))(\frac{\tilde{n}_{12}}{R_1} - \frac{\tilde{n}_{21}}{R_2}) \tag{49c}$$

The simplifications leading to Equation (46) transform Equations (49) into

$$\tilde{n}^*_{13} = -\ i\ h_o\ (\frac{\tilde{n},_1}{\alpha_1} + \tilde{p}'_1 + \nu\ \bar{\tilde{p}}'_1) \tag{50a}$$

$$\tilde{n}^*_{23} = -\ i\ h_o\ (\frac{\tilde{n},_2}{\alpha_2} + \tilde{p}'_2 + \nu\ \bar{\tilde{p}}'_2) \tag{50b}$$

$$\tilde{n}^*_{12} - \tilde{n}^*_{21} = -\ i\ h_o\ (1 + \nu(\tilde{\ }))\ (\frac{\tilde{n}^p_{12}}{R_1} - \frac{\tilde{n}^p_{21}}{R_2}) \tag{50c}$$

where

$$\tilde{p}_1' = \tilde{p}_1 + \frac{\tilde{n}^p_{13}}{R_1} \tag{51a}$$

$$\tilde{p}_2' = \tilde{p}_2 + \frac{\tilde{n}^p_{23}}{R_2} \tag{51b}$$

In terms of $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$ Equations (50) take the form

$$\tilde{f}_{3,2} - \frac{\alpha_2}{R_2}\tilde{f}_2 = - i\, h_o \frac{\alpha_2}{\alpha_1}\left[\frac{(\alpha_1 \tilde{f}_1)_{,2} - (\alpha_2 \tilde{f}_2)_{,1}}{\alpha_1 \alpha_2}\right]_{,1} - i\, h_o\, \alpha_2 \left(\frac{\tilde{n}^p_{,1}}{\alpha_1} + \tilde{p}_1' + \nu\, \bar{\tilde{p}}_1'\right) \tag{52a}$$

$$-\tilde{f}_{3,1} + \frac{\alpha_1}{R_1}\tilde{f}_1 = - i\, h_o \frac{\alpha_1}{\alpha_2}\left[\frac{(\alpha_1 \tilde{f}_1)_{,2} - (\alpha_2 \tilde{f}_2)_{,1}}{\alpha_1 \alpha_2}\right]_{,2} - i\, h_o\, \alpha_1 \left(\frac{\tilde{n}^p_{,2}}{\alpha_2} + \tilde{p}_2' + \nu\, \bar{\tilde{p}}_2'\right) \tag{52b}$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\tilde{f}_3 + \frac{(\alpha_1 \tilde{f}_2)_{,2} + (\alpha_2 \tilde{f}_1)_{,1}}{\alpha_1 \alpha_2} = - i\, h_o\, (1 + \nu(\bar{\ })) \left(\frac{\tilde{n}^p_{12}}{R_1} - \frac{\tilde{n}^p_{21}}{R_2}\right) \tag{52c}$$

With $\tilde{f}_3$ determined from Equation (52c), Equations (52a,b) form a system of two differential equations for $\tilde{f}_1$ and $\tilde{f}_2$. For shells of zero mean curvature the coefficient of $\tilde{f}_3$ in Equation (52c) vanishes. A system of two equations for $\tilde{f}_1$ and $\tilde{f}_2$ is formed then of Equation (52c) and of the equation obtained by eliminating $\tilde{f}_3$ from Equations (52a,b)

A particular solution of Equations (52) may be viewed as a correction to the particular solution of the equilibrium equations, in the sense that superposition of the two solutions yields a solution of the complete system of shell euqations. In particular, Equations (52) may be used to investigate the approximate character of a membrane solution.

The scalar components of Equations (48c,d) take the form

$$\tilde{y}_{1,1} + \frac{\alpha_{1,2}}{\alpha_2}\tilde{y}_2 + \frac{\alpha_1}{R_1}\tilde{y}_3 = - i\, h_o\, \alpha_1\, (\tilde{n} - (1 + \nu(\tilde{~}))\, \tilde{n}^p_{22}) \tag{53a}$$

$$\tilde{y}_{2,1} - \frac{\alpha_{1,2}}{\alpha_2}\tilde{y}_1 = \alpha_1\, \tilde{f}_3 - i\, h_o\, \alpha_1\, (1 + \nu(\tilde{~}))\, \tilde{n}^p_{21} \tag{53b}$$

$$\tilde{y}_{3,1} - \frac{\alpha_1}{R_1}\tilde{y}_1 = - \alpha_1\, \tilde{f}_2 \tag{53c}$$

$$\tilde{y}_{1,2} - \frac{\alpha_{2,1}}{\alpha_1}\tilde{y}_2 = - \alpha_2\, \tilde{f}_3 - i\, h_o\, \alpha_2\, (1 + \nu(\tilde{~}))\, \tilde{n}^p_{12} \tag{53d}$$

$$\tilde{y}_{2,2} + \frac{\alpha_{2,1}}{\alpha_1}\tilde{y}_1 + \frac{\alpha_2}{R_2}\tilde{y}_3 = - i\, h_o\, \alpha_2\, (\tilde{n} - (1 + \nu(\tilde{~}))\, \tilde{n}^p_{11}) \tag{53e}$$

$$\tilde{y}_{3,2} - \frac{\alpha_2}{R_2}\tilde{y}_2 = \alpha_2\, \tilde{f}_1 \tag{53f}$$

where $\tilde{n}$ may be expressed in terms of $\tilde{f}_1$ and $\tilde{f}_2$ in the form

$$\tilde{n} = \tilde{n}^p + \frac{(\alpha_1\, \tilde{f}_1)_{,2} - (\alpha_2\, \tilde{f}_2)_{,1}}{\alpha_1\, \alpha_2} \tag{54}$$

Novozhilov's Equations

Substituting for n_{13} and n_{23} from Equations (50a,b) into the complex equilibrium equations (30a-c) and assuming that a negligible error of order $\frac{h}{R}$ is made by letting in Equations (50a,b) $p_1' = p_1$ and $p_2' = p_2$ there comes

$$(\alpha_2\, \tilde{n}_{11})_{,1} + (\alpha_1\, \tilde{n}_{21})_{,2} - \alpha_{2,1}\, \tilde{n}_{22} + \alpha_{1,2}\, \tilde{n}_{12} - \frac{i\, h_o}{R_1}\alpha_2\, \tilde{n}_{,1} + \alpha_1\, \alpha_2\, \frac{n^p_{13}}{R_1}$$
$$+ \alpha_1\, \alpha_2\, (1 - \frac{i\, h_o}{R_1})\, \tilde{p}_1 - \nu\, \frac{i\, h_o}{R_1}\, \bar{\tilde{p}}_1 = 0 \tag{55a}$$

$$(\alpha_1 \tilde{n}_{22})_{,2} + (\alpha_2 \tilde{n}_{12})_{,2} - \alpha_{1,2} \tilde{n}_{11} + \alpha_{2,1} \tilde{n}_{21} - \frac{i h_o}{R_2} \alpha_1 \tilde{n}_{,2} + \alpha_1 \alpha_2 \frac{n^p_{23}}{R_2}$$

$$+ \alpha_1 \alpha_2 (1 - \frac{i h_o}{R_2}) \tilde{p}_2 - \nu \frac{i h_o}{R_2} \bar{\tilde{p}}_2 = 0 \qquad (55b)$$

$$\frac{\tilde{n}_{11}}{R_1} + \frac{\tilde{n}_{22}}{R_2} + i h_o \Delta \tilde{n} - \tilde{p}_3$$

$$- \frac{1}{\alpha_1 \alpha_2} \left\{ \left[\alpha_2 (n^p_{13} - i h_o (\tilde{p}_1 + \nu \bar{\tilde{p}}_1)) \right]_{,1} + \left[\alpha_1 (n^p_{23} - i h_o (\tilde{p}_2 + \nu \bar{\tilde{p}}_2)) \right]_{,2} \right\} = 0$$

(55c)

Equations (55) reduce to equations 16.10 in Reference 1 if there is no moment load $\bar{q}$ and if the particular solution is identified, withoug having actually to be determined, with a solution of the equilibrium equation of the membrane theory. In that case $n^p_{13} = n^p_{23} = 0$, $\tilde{\chi}^p_1 = \tilde{\chi}^p_2 = 0$ and $\tilde{p} = \bar{\tilde{p}} = \bar{p}$.

Expressing the complex stress resultants in terms of $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$ by means of Equations (31a-d), Equation (55c) becomes a consequence of Equations (55a,b) and these take the form of Equations (52a,b). Equations (52a,b) express then the conditions for the complex stress resultant - stress function relations to solve Equations (55).

9. Equations in Invariant Form I

From Equations (50a,b) or, equivalently, from Equations (52a,b) and for shells of non zero Gaussian curvature $\tilde{f}_1$ and $\tilde{f}_2$ may be expressed in terms of $\tilde{f}_3$ and $\tilde{n}$ in the form

$$\tilde{f}_1 = \frac{R_1}{\alpha_1} \tilde{f}_{3,1} - i\, h_o\, R_1 \left(\frac{\tilde{n}_{,2}}{\alpha_2} + \tilde{p}_2' + \nu\, \bar{\tilde{p}}_2'\right) \tag{56a}$$

$$\tilde{f}_2 = \frac{R_2}{\alpha_2} \tilde{f}_{3,2} + i\, h_o\, R_2 \left(\frac{\tilde{n}_{,1}}{\alpha_1} + \tilde{p}_1' + \nu\, \bar{\tilde{p}}_1'\right) \tag{56b}$$

where, as found in Equation (54)

$$\tilde{n} = \tilde{n}^p + \frac{(\alpha_1\, \tilde{f}_1)_{,2} - (\alpha_2\, \tilde{f}_2)_{,1}}{\alpha_1\, \alpha_2} \tag{57}$$

Substituting for $\tilde{f}_1$ and $\tilde{f}_2$ from Equations (56) into Equations (57) and (52c) there comes

$$\left(\frac{R_2\, \alpha_2}{\alpha_1} \tilde{n}_{,1}\right)_{,1} + \left(\frac{R_1\, \alpha_1}{\alpha_2} \tilde{n}_{,2}\right)_{,2} - i\, \frac{\alpha_1\, \alpha_2}{h_o} (\tilde{n} - \tilde{n}^p) + \frac{i}{h_o} [(R_1\, \tilde{f}_{3,1})_{,2} - (R_2\, \tilde{f}_{3,2})_{,1}]$$

$$+ (1 + \nu(\bar{\ }))\, [(R_1\, \alpha_1\, \tilde{p}_2')_{,2} + (R_2\, \alpha_2\, \tilde{p}_1')_{,1}] = 0 \tag{58a}$$

$$\left(\frac{R_1\, \alpha_2}{\alpha_1} \tilde{f}_{3,1}\right)_{,1} + \left(\frac{R_2\, \alpha_1}{\alpha_2} \tilde{f}_{3,2}\right)_{,2} + \alpha_1\, \alpha_2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tilde{f}_3 + i\, h_o\, [(R_2\, \tilde{n}_{,1})_{,2} - (R_1\, \tilde{n}_{,2})_{,1}]$$

$$+ i\, h_o\, (1 + \nu(\bar{\ }))[(R_2\, \alpha_1\, \tilde{p}_1')_{,2} - (R_1\, \alpha_2\, \tilde{p}_2')_{,1} + \alpha_1\, \alpha_2 \left(\frac{\tilde{n}^p_{12}}{R_1} - \frac{\tilde{n}^p_{21}}{R_2}\right)] = 0 \tag{58b}$$

Equations (58) from a system of two differential equations of fourth order for $\tilde{f}_3$ and $\tilde{n}$. They may be written in an arbitrary system of curvilinear coordinates on noting that $\tilde{f}_3$ and $\tilde{n}$ are invariants and that the differential expressions involved in Equations (58) may be expressed as divergence expressions.

It may be noted that the homogenous membrane solution and its dual inextensional bending solution are obtained through the solution for $\tilde{f}_3$ of the differential equation obtained from Equation (58b) by letting all terms other than $\tilde{f}_3$ be zero.

10. Equations in Invariant form II

Another way of expressing Equations (52) in terms of two invariant functions is to express $\tilde{f}_1$ and $\tilde{f}_2$ in the form

$$\tilde{f}_1 = \frac{\tilde{\Phi}_{,1}}{\alpha_1} + \frac{\tilde{\Psi}_{,2}}{\alpha_2} \tag{59a}$$

$$\tilde{f}_2 = \frac{\tilde{\Phi}_{,2}}{\alpha_2} - \frac{\tilde{\Psi}_{,1}}{\alpha_1} \tag{59b}$$

From Equations (59) there comes

$$(\alpha_1 \tilde{f}_1)_{,2} - (\alpha_2 \tilde{f}_2)_{,1} = \alpha_1 \alpha_2 \Delta \tilde{\Psi} \tag{60a}$$

$$(\alpha_1 \tilde{f}_2)_{,2} + (\alpha_2 \tilde{f}_1)_{,1} = \alpha_1 \alpha_2 \Delta \tilde{\Phi} \tag{60b}$$

where Δ is Laplace's operator in the middle surface. From Equations (60), (57) and (52c) it is seen that $\tilde{n}$ and $\tilde{f}_3$ are related to $\Delta\tilde{\Psi}$ and $\Delta\tilde{\Phi}$ through the relations

$$\tilde{n}^* = \Delta \tilde{\Psi} \tag{61a}$$

$$\tilde{f}_3 = -\frac{1}{2C} \Delta\tilde{\Phi} - \frac{i\, h_o}{2C} (1 + \nu(\tilde{\ }))(\frac{\tilde{n}^p_{12}}{R_1} - \frac{\tilde{n}^p_{21}}{R_2}) \tag{61b}$$

where it is assumed that

$$C = \frac{1}{2} (\frac{1}{R_1} + \frac{1}{R_2}) \neq 0 \tag{62}$$

Equations (52a,b) expressed in terms of $\tilde{\Phi}$ and $\tilde{\Psi}$ take the form

$$i\, h_o \frac{\alpha_2}{\alpha_1} (\Delta\tilde{\Psi})_{,1} - \left(\frac{\Delta\tilde{\Phi}}{2C}\right)_{,2} + \frac{\alpha_2\, \tilde{\Psi}_{,1}}{\alpha_1\, R_2} - \frac{\tilde{\Phi}_{,2}}{R_2} = -\, i\, h_o\, \alpha_2\, \tilde{\ell}_1 \tag{63a}$$

$$i\, h_o \frac{\alpha_1}{\alpha_2} (\Delta\tilde{\Psi})_{,2} + \left(\frac{\Delta\tilde{\Phi}}{2C}\right)_{,1} + \frac{\alpha_1\, \tilde{\Psi}_{,2}}{\alpha_2\, R_1} + \frac{\tilde{\Phi}_{,1}}{R_1} = -\, i\, h_o\, \alpha_1\, \tilde{\ell}_2 \tag{63b}$$

where

$$\tilde{\ell}_1 = \frac{\tilde{n}^p_{,1}}{\alpha_1} + \tilde{p}'_1 + \nu\, \bar{\tilde{p}}'_1 - \frac{1 + \nu(\tilde{\ })}{2\alpha_2} \left[\frac{1}{C} \left(\frac{\tilde{n}^p_{12}}{R_1} - \frac{\tilde{n}^p_{21}}{R_2} \right) \right]_{,2} \tag{64a}$$

$$\tilde{\ell}_2 = \frac{\tilde{n}^p_{,2}}{\alpha_2} + \tilde{p}'_2 + \nu\, \bar{\tilde{p}}'_2 + \frac{1 + \nu(\tilde{\ })}{2\alpha_1} \left[\frac{1}{C} \left(\frac{\tilde{n}^p_{12}}{R_1} - \frac{\tilde{n}^p_{21}}{R_2} \right) \right]_{,1} \tag{64b}$$

It may be noted that the homogenous equations (63) admit as solutions conjugate harmonic function $\tilde{\Phi}$ and $\tilde{\Psi}$ making $\tilde{f}_1$ and $\tilde{f}_3$ zero and contributing zero to $\tilde{f}_3$. Such harmonic functions are of no interest.

11. Mushtari-Vlassov Equation

For rapidly varying states of stress such that

$$\frac{\phi_{,i}}{R_i} \ll \left(\frac{\Delta\phi}{2C}\right)_{,i} \qquad i = 1, 2 \tag{65}$$

the terms $\frac{\phi_{,i}}{R_i}$ may be neglected in Equations (63). Then upon elimination of $\left(\frac{\Delta\phi}{2C}\right)$ there comes the differential equation for $\tilde{\psi}$

$$i\ h_o\ \Delta\Delta\tilde{\psi} + \Delta_R\ \tilde{\psi} = -\ i\ h_o\ \frac{(\alpha_2\ \tilde{\ell}_1)_{,1} + (\alpha_1\ \tilde{\ell}_2)_{,2}}{\alpha_1\ \alpha_2} \tag{66}$$

where

$$\Delta_R(\ ) = \frac{1}{\alpha_1\ \alpha_2}\left[\left(\frac{\alpha_2(\ )_{,1}}{\alpha_1\ R_2}\right)_{,1} + \left(\frac{\alpha_1(\ )_{,2}}{\alpha_2\ R_1}\right)_{,2}\right] \tag{67}$$

Consistently with the above approximations the terms in $\tilde{\phi}$ should be deleted from Equations (59).

If the simplifying assumptions associated with Vlassov's shallow shell equations (7) are introduced it becomes possible to write

$$\tilde{\psi} \approx g_3 - i\ u_3 \tag{68}$$

It is interesting however, that the derivation of Equation (66) may be based solely on the order of magnitude relation (65) which for a surface of smooth geometry may be replaced by

$$\phi \ll R^2\ \Delta\phi \tag{69}$$

Equation (66) may be identified except for its right hand side with Equation 17.16 of Reference 1. The discrepancy between the right hand sides is due to the assumption made in Reference 1 and similar derivations that $\overline{q} = 0$, $p_1 = p_2 = 0$ and to the introduction in the present derivation of a particular solution of the equilibrium equations.

12. Application to a Spherical Shell

The coordinates ξ_1 and ξ_2 are identified, respectively, with the meridional angle ξ and the circumferential angle θ. Letting a denote the radius of the sphere and r the radius of a parallel circle we have

$$\alpha_1 = R_1 = R_2 = a \tag{70a}$$

$$\alpha_2 = r = a \sin \xi \tag{70b}$$

It is convenient to introduce the independent variable

$$\eta = \log \tan \frac{\xi}{2} \tag{71a}$$

and to adopt the notation

$$(\ )' = (\ )_{,1} \tag{72b}$$

$$(\ )^{\cdot} = (\ )_{,2} \tag{72c}$$

$$(\ )^* = \frac{\partial(\ )}{\partial\eta} = (\ )' \sin \xi \tag{72d}$$

The notation $(\ )^*$ of Equation (72d) applies only to this section. It may be shown that

$$\Delta_o(\ ) = a^2\, \Delta(\ ) = \sin^{-2} \xi\, [(\ )^{**} + (\ )^{\cdot\cdot}\,] \tag{72e}$$

Assuming $\tilde{n}^p_{12} = \tilde{n}^p_{21}$ Equation (61b) yields

$$\tilde{f}_3 = - \frac{\Delta_o\, \tilde{\Phi}}{2a} \tag{73}$$

and Equations (63) take the form

$$2[\frac{i\,h_o}{a}\ \Delta_o\ \tilde{\Psi} + \tilde{\psi}]^* - [(\Delta_o + 2)\tilde{\phi}]^{\bullet} = -\ 2\ i\ h_o\ a^2 \sin\xi\ \tilde{\ell}_1 \qquad (74a)$$

$$2[\frac{i\,h_o}{a}\ \Delta_o\ \tilde{\Psi} + \tilde{\psi}]^{\bullet} + [(\Delta_o + 2)\tilde{\phi}]^* = -\ 2\ i\ h_o\ a^2 \sin\xi\ \tilde{\ell}_2 \qquad (74b)$$

It is noted that the homogénous Equations (74) are Cauchy-Reimann conditions with respect to η and θ. The quantities in the brackets, as far as the solution of the homogenous equations is concerned, are therefore complex conjugate harmonic functions. It may be readily verified that a particular determination of $\tilde{\Phi}$ and $\tilde{\Psi}$ corresponding to these conjugate harmonic functions may be made such that $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$ are zero. Disregarding this determination of $\tilde{\Phi}$ and $\tilde{\Psi}$ as being of no interest the general solution of Equations (74) may be written in the form

$$\tilde{\Psi} = \tilde{\Psi}_p + \tilde{\Psi}_h \qquad (75a)$$

$$\tilde{\Phi} = \tilde{\Phi}_p + \tilde{\Phi}_h \qquad (75b)$$

where $\tilde{\Psi}_p$ and $\tilde{\Phi}_p$ are a particular solution and $\tilde{\Psi}_h$ and $\tilde{\Phi}_h$ are the general solution of the equations

$$\Delta_o\ \tilde{\Psi}_h - i\,\frac{a}{h_o}\ \psi_h = 0 \qquad (76a)$$

$$(\Delta_o + 2)\ \tilde{\Phi}_h = 0 \qquad (76b)$$

$\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$ are obtained from Equations (59) and (73) in the form

$$\tilde{f}_1 = \frac{1}{a \sin\xi}\ (\phi^* - \tilde{\Psi}^{\bullet}) \qquad (77a)$$

$$\tilde{f}_2 = \frac{1}{a \sin\xi}\ (\tilde{\Phi}^{\bullet} - \tilde{\Psi}^*) \qquad (77b)$$

$$\tilde{f}_3 = - \frac{\Delta_o \tilde{\Phi}}{2a} = \frac{\tilde{\Phi}_h}{a} - \frac{\Delta_o \tilde{\Phi}_p}{2a} \tag{77c}$$

Equation (76b) is associated with the homogenous membrane and inextensional solutions and Equation (76a) is associated with the edge-zone solution. $\tilde{g}$ is now determined through $\tilde{y}$, Equation (48b), by integrating Equations (53). These after using Equations (61a), (76a) and (77) take the form

$$\tilde{y}_1' + \tilde{y}_3 = \tilde{\Psi}_h - i\, h_o\, a\, (\tilde{n}_{11}^p - \nu\, \bar{\tilde{n}}_{22}^p + a^{-2}\, \Delta_o\, \tilde{\Psi}_p) \tag{78a}$$

$$\tilde{y}_2' = \tilde{\Phi}_h - i\, h_o\, a\, (1 + \nu(\bar{\ }))\, \tilde{n}_{21}^p - \frac{1}{2}\, \Delta_o\, \tilde{\Phi}_p \tag{78b}$$

$$\tilde{y}_3' - \tilde{y}_1' = - \frac{\dot{\tilde{\Phi}}_h}{\sin\xi} + \tilde{\Psi}_h' - \frac{\dot{\tilde{\Phi}}_p}{\sin\xi} + \tilde{\Psi}_p' \tag{78c}$$

$$\dot{\tilde{y}}_1 - \cos\xi\, \tilde{y}_2 = - \sin\xi\, \tilde{\Phi}_h - \sin\xi\, (i\, h_o\, a\, (1 + \nu(\bar{\ }))\, \tilde{n}_{12}^p - \frac{1}{2}\, \Delta_o\, \tilde{\Phi}_p) \tag{78d}$$

$$\dot{\tilde{y}}_2 + \cos\xi\, \tilde{y}_1 + \sin\xi\, \tilde{y}_3 = \sin\xi\, \tilde{\Psi}_h - i\, h_o\, a\, \sin\xi (\tilde{n}_{22}^p - \nu\, \bar{\tilde{n}}_{11}^p + a^{-2}\, \Delta_o\, \tilde{\Psi}_p) \tag{78e}$$

$$\dot{\tilde{y}}_3 - \sin\xi\, \tilde{y}_2 = \sin\xi\, \tilde{\Phi}_h' + \dot{\tilde{\Psi}}_h + \sin\xi\, \tilde{\Phi}_p' + \dot{\tilde{\Psi}}_p \tag{78f}$$

Equations (78) form a compatible system of six equations in three unknowns. A particular solution of Equations (78) is of the form

$$\tilde{y} = \tilde{y}_h + \tilde{y}_p \tag{79}$$

where $\tilde{y}_p$ corresponds to the terms in $\tilde{n}_{ij}^p$, $\tilde{\Psi}_p$ and $\tilde{\Phi}_p$, and $\tilde{y}_h$ corresponds to $\tilde{\Psi}_h$ and $\tilde{\Phi}_h$. To obtain $\tilde{y}_h$ it is convenient to introduce two complex harmonic functions $\tilde{H}_1$ and $\tilde{H}_2$ conjugate in the Cauchy-Riemann sense, i.e., satisfying the relations

$$\tilde{H}_1^* = H_2^{\bullet} \tag{80a}$$

$$\tilde{H}_1^{\bullet} = - \tilde{H}_2^* \tag{80b}$$

and related to Φ_h through the relation

$$\Phi_h = (\sin \xi \, H_2)' \tag{81}$$

To show that Equations (80) and (81) are consistent with Equation (76b) the following identity may be verified

$$(\Delta_o + 2) \, (\sin \xi \, (\ \ ))' = \frac{1}{\sin^2 \xi} (\sin^3 \xi \, \Delta_o(\ \ ))' \tag{82}$$

There follows from Equations (76) and (82) that the most general $\tilde{H}_2$ as defined in Equation (81) satisfies

$$\Delta_o \, \tilde{H}_2 = \frac{A(\theta)}{\sin^3 \xi} \tag{83}$$

where $A(\theta)$ is an arbitrary function of θ. For an arbitrary $A(\theta)$ it is possible to determine a particular solution for $\tilde{H}_2$ of Equation (83) in the form $\frac{B(\theta)}{\sin \xi}$. Such a particular solution gives $\Phi_h = 0$. There is thus no loss of generality by letting

$$A(\theta) = 0 \tag{84a}$$

and

$$\Delta_o \, \tilde{H}_2 = 0 \tag{84b}$$

$\tilde{H}_2$ is therefore a general complex harmonic function.

It may be readily verified that a solution for $\tilde{y}_{h1}$, $\tilde{y}_{h2}$ and $\tilde{y}_{h3}$ may be written in the form

$$\tilde{y}_{h1} = \sin \xi \, \tilde{H}_1 \tag{85a}$$

$$\tilde{y}_{h2} = \sin \xi \, \tilde{H}_2 \tag{85b}$$

$$\tilde{y}_{h3} = - (\sin \xi \, \tilde{H}_1)' + \tilde{\Psi}_h \tag{85c}$$

Since the homogenous Equations (78) express in scalar form the equation $d\tilde{y} = 0$ the general solution of Equations (78) is obtained by adding a constant vector $\tilde{y}_o$ to the particular solution. This need not be done, however, because a constant vector is representable through Equations (85) by letting

$$\tilde{\Psi}_h = 0 \tag{86a}$$

$$\tilde{H}_1 = \tilde{a}_x \cot \xi \cos \theta + \tilde{a}_y \cot \xi \sin \theta - \tilde{a}_z \tag{86b}$$

$$\tilde{H}_2 = - \tilde{a}_x \frac{\sin \theta}{\sin \xi} + \tilde{a}_y \frac{\cos \theta}{\sin \xi} \tag{86c}$$

where $\tilde{a}_x$, $\tilde{a}_y$ and $\tilde{a}_z$ are cartesian components of the constant complex vector.

Finally, the general solution for $\tilde{g}$ is obtained from Equation (48b) in the form

$$\tilde{g}_1 = \sin \xi \, \tilde{H}_1 - i \, h_o \, (1 + \nu(\tilde{\ })) \, \tilde{f}_2 + \tilde{y}_{p1} \tag{87a}$$

$$\tilde{g}_2 = \sin \xi \, \tilde{H}_2 + i \, h_o \, (1 + \nu(\tilde{\ })) \, \tilde{f}_1 + \tilde{y}_{p2} \tag{87b}$$

$$\tilde{g}_3 = - (\sin \xi \, \tilde{H}_1)' + \tilde{\Psi}_h + \tilde{y}_{p3} \tag{87c}$$

The general solution of the spherical shell problem obtained here agrees except for a negligible discrepancy in Equation (76a) due to the approximations introduced in obtaining Equations (52) with an earlier solution of the homogenous problem in terms of stress functions (8).

Summary and Conclusion

The static geometric analogy is established for the non homogenous shell problem and is used as a basis for a mixed formulation of the shell problem in terms of displacements and stress functions and for complex formulations in terms of complex dependent variables. Using complex stress functions (or complex displacements) two formulations are obtained in each of which the basic system of differential equations is reduced to a system of two equations for two invariant complex functions. The relationship between these equations and Novozhilov's system of three equations for complex stress resultants is established. The approximations that make feasible the complex formulations are all contained in the single step of obtaining the vector Equation (46) from Equation (45). The mathematical inconsistencies that are caused by these approximations in the determination of displacements from strain quantities and of stress functions from stress quantities are identified and eliminated. The determination of the displacements and stress functions is thereby reduced to the integration of an exact differential. This should be advantageous for satisfying boundary conditions involving displacements or stress functions in analytical as well as numerical methods of solution.

The Mushtari-Vlasov formulation for rapidly varying states of stress and for shallow shells is obtained from the general Equations (63) by deleting negligible terms.

Application to the case of the spherical shell results in two complex uncoupled differential equations of second order, each associated with a particular type of shell behavior.

The complex formulations presented here may be written in an arbitrary system of surface curvilinear coordinates upon making use of the invariance of the differential operators involved and of the unknown functions.

REFERENCES

1. Novozhilov, v.v., "The Theory of Thin Shells", P. Noordhoff Ltd., Groningen, The Netherlands, 1959.

2. Naghdi, P. M., "On the Differential Equations of the Linear Theory of Elastic Shells", Proceedings of the 11th International Congress of Applied Mechanics, Munich, Germany, 1964.

3. Elias, Z. M., "A Dual Formulation of Thin Shell Theory", NASA CR-568, National Aeronautics and Space Administration, Washington, D.C., September 1966.

4. Marguerre, K., "Zur Theorie der Gekrümmten Platte Grosser Formanderung", Proceedings of the 5th International Congress of Applied Mechanics, 1938.

5. Vlasov, V. Z., "General Theory of Shells and Its Application in Engineering", NASA TTF-99, National Aeronautics and Space Administration, Washington, D.C., April 1964.

6. Goldenweiser, A. L., "Theory of Elastic Thin Shells", Pergammon Press, 1961.

7. Elias, Z. M., "On The Derivation of Vlasov's Shallow Shell Equations and Their Application to non Shallow Shells", Tech. Report R67-67, November 1967, Department of Civil Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts.

8. Elias, Z. M., "Analysis of Spherical Shells with Stress Functions", Journal of the Engineering Mechanics Division, ASCE, Vol. 90, No. EM6, Proc. Paper 4173, December 1964, pp 135-163.