FINAL REPORT

EXTENSION OF MODERN CONTROL SYSTEM THEORY

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I. INTRODUCTION

This report describes the results obtained on DRI Project No. 3638 which extended from 1 July 1966 to 31 December 1967. The objective of the research was to develop analysis and synthesis techniques for general linear systems with emphasis upon the development of methods to minimize the sensitivity of optimal control systems to large parameter variations.

During the period of study the principal investigators supervised the 3 dissertations and 4 publications listed in Section III of this report. These publications contain the most significant results obtained in this study.

II. SUMMARY OF RESULTS OF THE RESEARCH EFFORT

During the course of the research five distinct results were obtained.

A. A method has been developed for analytically determining the optimal control for a linear system with respect to a performance functional that includes trajectory sensitivity. Specifically, the control law $\underline{u}$ has been found for the system described by the state-variable differential equation

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

so that the performance index J is minimized:

$$J = \int_0^{\infty} (\underline{x} \cdot Q\underline{x} + \underline{u} \cdot R\,\underline{u} + \nu \cdot S\,\nu)\,dt$$

where Q, R and S are positive definite matrices and the trajectory sensitivity vector ν is defined by

$$\underline{\nu} = \frac{\partial \underline{x}}{\partial \alpha}$$

where α is a system parameter appearing only in A. Numerical determination of the optimal-control law is achieved by the solution of a nonlinear algebraic matrix equation. Implementation of the optimal control requires generation of the $\underline{\nu}$ vector which is done in a straight-forward manner: for an n-th order system, the trajectory sensitivity vector $\underline{\nu}$ requires n integrators for its simulation[1*].

B. A design algorithm was developed for the minimization of a cost functional which includes, in addition to state and control variables, a measure of sensitivity. This algorithm was used to solve a problem of practical interest: a flexible ballistic missile in powered flight. Comparisons were made with optimal designs which include sensitivity and those that do not. The efficacy of using a measure of

*Numbers refer to publications listed in Section III.

sensitivity in the performance index was demonstrated[2,4].

C. A technique was developed for compensator design which includes constraints on peak time, overshoot, settling time, peak value of the controller output, and velocity error constant. A basic feature of the design technique is the formation of a figure-of-merit, which is a linear combination of the design objectives, and a minimization by adjustment and addition of compensator gain, poles and zeros under the control of a multilevel decision procedure. Examples show a favorable comparison to several other synthesis techniques[3].

D. A method was developed for obtaining the optimal control law for plants containing a randomly slowly-varying parameter whose probability density function is known. The necessary conditions for the optimal control law were derived using the calculus of variations, and a method advanced for approximating the optimal control law in closed-loop fashion. Numerical examples have been worked and the results compared with both optimal control systems designed about the nominal value of the parameter, and an adaptive system whose control law changes to the optimal for the existing parameter value[5,7].

E. A design procedure was developed which assumed the plant input to be a combination of plant states and approximate sensitivity functions. Necessary conditions were developed and examples worked to illustrate the method[6].

III. PUBLICATIONS RELATED TO THE RESEARCH EFFORT

The following publications were prepared by or under the direction of the principal investigators during the period of study.

1. D'Angelo, H., Moe, M.L., and Hendricks, T.C., "Trajectory Sensitivity of an Optimal Control System", Proceedings of the Fourth Annual Allerton Conference, pp. 489-498, University of Illinois, October 1966.

2. Hendricks, T.C., "Trajectory Sensitivity of an Optimal Control System of Fixed Structure". Ph.D. Dissertation, University of Denver, May 1967.

3. Hauser, F.D. and Moe, M.L., "A Computer-Aided Design Technique for Sampled-Data Control Systems", Digest of the First Annual IEEE Computer Conference, pp. 69-72, Chicago, Illinois, September 1967.

4. Hendricks, T.C. and D'Angelo, H., "An Optimal Fixed Control Structure Design with Minimal Sensitivity for a Large Elastic Booster", Proceedings of the Allerton Conference on Circuit and System Theory, pp. 142-151, University of Illinois, October 1967.

5. D'Angelo, H. and Patrick, L.B., "Optimal Control of Plants with Random Slowly-Varying Parameters", Proceedings of the Allerton Conference on Circuit and System Theory, pp. 711-720, University of Illinois, October 1967.

6. Bradt, A.J., "The Design of Optimal Controllers to Minimize a Performance Index Containing Trajectory Sensitivity Functions", Ph.D. Dissertation, University of Denver, October 1967.

7. Patrick, L.B., "Optimization of Random Systems by Minimization of the Expected Value of a Quadratic Performance Index", Ph.D. Dissertation, University of Denver, March 1968.

IV. COPIES OF PUBLISHED PAPERS

A detailed description of the results is best provided by including copies of the following four papers[1,3,4,5].

MORGAN-VOYCE POLYNOMIALS IN RECURRENT LADDERS

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ABSTRACT

Ladder networks have been previously analysed by a number of methods such as continued fractions, Tchebycheff polynomials, iteration, signal flow graphs, continuants, etc. Morgan-Voyce demonstrated in the special case of a resistive ladder network, an interesting mathematical relationship between some of the network functions and the Fibbonacci series of numbers and defined a set of polynomials by

$$b_n(w) = wB_{n-1}(w) + b_{n-1}(w) \qquad (n\geq 1)$$

$$B_n(w) = (w+1)\ B_{n-1}(w) + b_{n-1}(w) \qquad (n\geq 1)$$

with

$$b_o(w) = B_o(w) = 1$$

In subsequent papers, Swamy studied in detail, the properties of these polynomials and defined the polynomial $c_n(w)$ as

$$c_n(w) = \frac{1}{2}\ (b_n + b_{n-1}) = \frac{1}{2}\ (B_n - B_{n-2})$$

In the present paper these polynomials b_n, B_n and c_n are used to make a detailed study of recurrent ladders with a general series impedance z_1 and a shunt impedance z_2. The [ABCD] parameters have been expressed in terms of these polynomials.

Since the polynomials are factorisable, the matrix parameters can be conveniently utilised to yield network response to any arbitrary excitation. Also, the resonant and anti-resonant frequencies of the network may be found in terms of the zeros of these polynomials.

For purposes of illustration, responses to a step and a square wave input have been worked out in detail in the case of an RCG ladder. The steady state response of the RCG ladder for the periodic excitation has also been obtained from the total response.

The matrix parameters of a uniform line have also been obtained by considering it to be the limiting case of a recurrent ladder. The chief merit of the analysis lies in its simplicity and compactness.

TRAJECTORY SENSITIVITY OF AN OPTIMAL CONTROL SYSTEM

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ABSTRACT

A method is introduced for analytically determining the optimal control for a linear system with respect to a performance functional that includes trajectory sensitivity. Specifically, the control law u is found for the system described by the state-variable differential equation

$$\dot{x} = Ax + Bu$$

so that the performance index J is minimized:

$$J = \int_0^\infty \left(x^T Qx + u^T\ Ru + \sum_{i=1}^{m}\ v_i^T\ S_i v_i \right) dt$$

where Q, R and the S_i are positive definite matrices and the trajectory sensitivity vector v_i is defined by

$$v_i = \frac{\partial^i x}{\partial \alpha^i}$$

where α is a system parameter. Numerical determination of the optimal-control law is achieved by the solution of a nonlinear algebraic matrix equation. Implementation of the optimal control requires generation of the v_i vector which is done in a straight-forward manner: for an n-th order system, the trajectory sensitivity vector v_i requires n integrators for its simulation.

1. The Main Results

Consider the plant

$$\dot{x} = Ax + Bu \qquad (1)$$

where

$A = A(\alpha)$, $B = B(\alpha)$ and

$$x = x(t,\alpha) = \begin{bmatrix} x_1(t,\alpha) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t,\alpha) \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_r \end{bmatrix}$$

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Assuming a nominal a_0 the Taylor series expansion can be written about this value as

$$x(t, a) = x(t, a_0) + \frac{\partial x(t, a_0)}{\partial a} (a - a_0) + \frac{\partial^2 x(t, a_0)}{\partial a^2} \frac{(a - a_0)^2}{2!} + \ldots$$
$$= x(t, a_0) + v_1(t, a_0)(a - a_0) + v_2(t, a_0) \frac{(a - a_0)^2}{2!} + \ldots \tag{2}$$

where

$$v_i(t, a) = \frac{\partial^i x(t, a)}{\partial a^i} \tag{3}$$

Clearly if the trajectory $x(t, a)$ is to be independent of a it is necessary that

$$v_i(t, a) = 0 \text{ for } i = 1, 2, 3, \ldots \tag{4}$$

Generally, for any significant system parameter a, making the $v_i(t, a) = 0$ will conflict with the system's intended function. A performance index which allows one to weight trajectory deviations relative to the control effort and the state transfer in the "regulator" problem is

$$J = \int_0^\infty \left(x^T Qx + u^T Ru + \sum_{i=1}^{\infty} v_i^T S_i v_i \right) dt = \text{minimum} \tag{5}$$

For practical purposes it is assumed that $x(t, a)$ is well approximated by a finite number of terms. Thus equation (2) can be replaced by

$$x(t, a) = x(t, a_0) + \sum_{i=1}^{m} v_i(t, a_0) \frac{(a - a_0)^i}{i!} \tag{6}$$

Thus the performance index J becomes

$$J = \int_0^\infty \left(x^T Qx + u^T Ru + \sum_{i=1}^{m} v_i^T S_i v_i \right) dt \tag{7}$$

Taking derivatives repeatedly with respect to a of equation (6) results in

$$v_j(t, a) = 0, \quad j = m+1, m+2, \ldots \tag{8}$$

A new state vector X is formed by adjoining the system state variable vector x with the trajectory sensitivity vectors $v_1, \ldots v_m$. Thus

$$X = [x^T, v_1^T, \ldots, v_m^T]^T \tag{9}$$

and has dimension $(m+1)n$. In terms of this new state vector X, the performance index J, given by equation (7), can be written as

$$J = \int_0^\infty (X^T \hat{Q} X + u^T Ru) \, dt \tag{10}$$

where

$$\hat{Q} = \begin{bmatrix} Q & 0 & \ldots & 0 \\ 0 & S_1 & \ldots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \ldots & S_m \end{bmatrix} \tag{11}$$

For the present, let it be assumed that this new state vector X satifies the differential equation

$$\dot{X} = \hat{A}X + \hat{B}u \tag{12}$$

If this assumption is true, then the optimal control u minimizing the performance index J of equation (10) is known to be

$$u = -R^{-1} \hat{B}^T KX \tag{13}$$

where K satisfies the Riccati equation

$$K\hat{A} + \hat{A}^T K - K\hat{B}R^{-1}\hat{B}^T K + \hat{Q} = 0 \tag{14}$$

Substituting the optimal control u of equation (13) into the original plant equation (1) gives

$$\dot{x} = Ax - BR^{-1}\hat{B}^T KX \tag{15}$$

A more convenient form of the above equation is obtained by partitioning the K matrix:

$$K = \begin{bmatrix} K_{00} & K_{01} & K_{02} & \ldots & K_{0m} \\ K_{01}^T & K_{11} & K_{12} & \ldots & K_{1m} \\ K_{02}^T & K_{12}^T & K_{22} & \ldots & K_{2m} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ k_{0m}^T & K_{1m}^T & K_{2m}^T & \ldots & K_{mm} \end{bmatrix} \tag{16}$$

where the K_{ii} are symmetric for $i = 0, \ldots, m$. Also

$$K = [K_0, K_1, K_2, \ldots, K_m] \tag{17}$$

where

$$K_i = [K_{0i}^T, K_{1i}^T, \ldots, K_{i-1,i}^T K_{ii}, K_{i,i+1}, \ldots, K_{i,m-1}, K_{im}]^T,$$
$$i = 0, 1, 2, \ldots, m \tag{18}$$

Equation (15) can now be written

$$\dot{x} = Ax - BR^{-1}\hat{B}^T [K_0, K_1, \ldots, K_m] [x^T, v_1^T, \ldots, v_m^T]^T \tag{19}$$

or

$$\dot{x} = (A - BR^{-1}\hat{B}^T K_0) x - BR^{-1}\hat{B}^T \sum_{i=1}^{m} K_i v_i \tag{20}$$

Equation (20) is written in the compact form

$$\dot{x} = A_{00}x + \sum_{i=1}^{m} A_{0i}\, v_i \tag{21}$$

where

$$A_{00} = A - BR^{-1}\hat{B}^T K_0 \tag{22}$$

and

$$A_{0i} = -BR^{-1}\hat{B}^T K_i, \quad i = 1, \ldots, m \tag{23}$$

Equations (22) and (23) can be combined into the single equation

$$A_{0i} = \delta_{0i} A - BR^{-1}\hat{B}^T K_i, \quad i = 0, 1, \ldots, m \tag{24}$$

where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases} \tag{25}$$

Taking the derivative of equation (21) with respect to α gives a differential equation for v_1:

$$\dot{v}_1 = A_{10}x + \sum_{i=1}^{m} A_{1i}\, v_i + A_{1,m+1}\, v_{m+1} \tag{26}$$

where the coefficients A_{1i}, $i = 0, 1, \ldots, m$ are discussed in the sequel. In accordance with Equation (8)

$$v_{m+1} = 0 \tag{27}$$

Therefore equation (26) reduces to

$$\dot{v}_1 = A_{10}x + \sum_{i=1}^{m} A_{1i}\, v_i \tag{28}$$

Proceeding in a similar manner results in the general form

$$\dot{v}_j = A_{j0}x + \sum_{i=1}^{m} A_{ji}\, v_i, \quad j = 1, \ldots, m \tag{29}$$

Adjoining equations (1) and (29) results in the partitioned matrix equation

$$\begin{bmatrix} \dot{x} \\ \dot{v}_1 \\ \cdot \\ \cdot \\ \cdot \\ \dot{v}_m \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ A_{10} & A_{11} & A_{12} & \cdots & A_{1m} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ A_{m0} & A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix} \begin{bmatrix} x \\ v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_m \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} u \tag{30}$$

or simply

$$\dot{X} = \hat{A}X + \hat{B}u \tag{31}$$

where

$$\hat{A} = \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ A_{10} & A_{11} & A_{12} & & A_{1m} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ A_{m0} & A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix} \tag{32}$$

and

$$\hat{B} = [B^T, 0, \ldots, 0]^T \tag{33}$$

It is now possible to determine $\hat{A}$ and $\hat{B}$ in terms of known matrices and partitions of K. Substituting from equations (18) and (33) in equation (22) gives

$$A_{00} = A - BR^{-1}B^T K_{00} \tag{34}$$

Similarly, substituting from equations (18) and (33) in equation (23) gives

$$A_{0i} = -BR^{-1}B^T K_{0i}, \quad i = 1, \ldots, m \tag{35}$$

Equations (34) and (35) can be combined into the single equation

$$A_{0i} = \delta_{0i} A - BR^{-1}B^T K_{0i}, \quad i = 0, 1, \ldots, m \tag{36}$$

where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i=j \end{cases} \tag{37}$$

To simplify the ensuing formulations the following notation is introduced:

$$A^{(k)} = \frac{\partial^k A}{\partial \alpha^k} \tag{38}$$

$$\beta^{(k)} = \frac{\partial^k (-BR^{-1}B^T)}{\partial \alpha^k} \tag{39}$$

Thus equations (21) and (38) can be written as

$$\dot{x} = A_{00}x + \sum_{i=1}^{m} A_{0i}\, v_i \tag{40}$$

where

$$A_{0i} = \delta_{0i} A^{(0)} + \beta^{(0)} K_{0i}, \quad i = 0, \ldots, m \tag{41}$$

ng the derivative of equation (40) with respect to α results in

$$\dot{v}_1 = A_{10}x + \sum_{i=1}^{m} A_{1i} v_i \qquad (42)$$

where

$$A_{1i} = \theta(i-1) A_{0,i-1} + \frac{\partial A_{0i}}{\partial \alpha}$$

$$= \theta(i-1)\,[\delta_{0,i-1} A^{(0)} + \beta^{(0)} K_{0,i-1}] + [\delta_{0i} A^{(1)} + \beta^{(1)} K_{0i}],\ i = 0, \ldots, m \qquad (43)$$

where $\theta(t)$ is the unit-step function defined by

$$\theta(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad (44)$$

Taking the derivative of equation (42) with respect to α results in

$$\dot{v}_2 = A_{20}x + \sum_{i=1}^{m} A_{2i} v_i \qquad (45)$$

where

$$A_{2i} = \theta(i-2)\,[\delta_{0,i-2} A^{(0)} + \beta^{(0)} K_{0,i-2}] + 2\,\theta(i-1)\,[\delta_{0,i-1} A^{(1)} + \beta^{(1)} K_{0,i-1}] + [\delta_{0i} A^{(2)} + \beta(2) K_{0i}],\ i = 0, \ldots, m \qquad (46)$$

Proceeding in this manner results in

$$\dot{v}_j = A_{j0}x + \sum_{i=1}^{m} A_{ji} v_i \qquad (47)$$

where

$$A_{ji} = \sum_{k=0}^{j} \binom{j}{k} \theta(i-k)\,[\delta_{0,i-k} A^{(j-k)} + \beta^{(j-k)} K_{0,i-k}] \qquad (48)$$

$$i = 0, \ldots, m$$

where $\binom{j}{k}$ is the binomial coefficient.

In accordance with equation (39) it is easily shown that

$$\beta^{(k)} = \sum_{j=1}^{k} \binom{k}{j} \frac{\partial^j B}{\partial \alpha^j} R^{-1} \frac{\partial^{k-j} B^T}{\partial \alpha^{k-j}} \qquad (49)$$

2. Simulation

Equation (12) can be used as the basis for simulation. By writing $\hat{A}$ as

$$\hat{A} = [\hat{A}_0^T\ \hat{A}_1^T\ \ldots,\ \hat{A}_m^T]^T \qquad (50)$$

where

$$\hat{A}_i = [A_{i0},\ A_{i1}, \ldots, A_{im}],\ i = 0, 1, \ldots, m \qquad (51)$$

one can write equation (12) as the m+1 equations

$$\dot{x} = Ax + Bu$$

$$\dot{v}_i = \hat{A}_i \mathbf{X},\ i = 1, 2, \ldots, m \qquad (52)$$

Further, from equations (13), (16) and (33) one can write

$$u = -R^{-1}B^T K_{00}x - R^{-1}B^T \sum_{i=1}^{m} K_{0i} v_i \qquad (53)$$

Equations (52) and (53) lead to the simulation shown in Figure 1.

3. Illustrative Example

Consider a first-order system characterized by

$$\dot{x} = -\alpha^2 x + u$$

It is desired to find the control u that minimizes the performance index

$$J = \int_0^{\sigma} (x^2 + u^2 + s v^2)\, dt$$

where $v = \partial x / \partial \alpha$.

For this example, the optimal control is given by $u = -k_{00}x - k_{01}v$. The feedback coefficients k_{00} and k_{01} are found by solving equation (14) for K. It is noted that B=1, R=1, Q=1, S=s and

$$\hat{A} = \begin{bmatrix} -\alpha^2 & 0 \\ -2\alpha & \alpha^2 - k_{00} \end{bmatrix}, \quad K = \begin{bmatrix} k_{00} & k_{01} \\ k_{01} & k_{11} \end{bmatrix}$$

For the case that the nominal parameter value $\alpha = \alpha_0 = 1$ we obtain: for s = 1, u = -0.65x + 0.178v; for s = 10, u = -1.33x + 0.855v; for s = 100, u = -2.81x + 3.13v.

Figure 2 shows the response, for various values of α, of the optimal reference system; trajectory sensitivity is not considered, i.e., s = 0. Figures 3, 4 and 5 show the responses of the optimal system for s = 1, s = 10 and s = 100 respectively; each shows the responses for various values of α. These figures also show the trajectory sensitivity generated by the nominal optimal system compared the computed values of trajectory sensitivity (dotted curve); the computation involves the taking the difference of two responses of the optimal system with plus and minus deviations of the parameter nominal value. As expected, the system responses deviate less from the nominal trajectory (note the smaller trajectory sensitivity) for larger values of s.

4. Acknowledgements

The authors gratefully acknowledge the support of the National Science Foundation (Grant No. GK 274) and NASA (Grant No. NsG-518). The authors also wish to express their thanks to Mr. Dan Curran who did all the digital and analog computer programming for the many numerical problems worked.

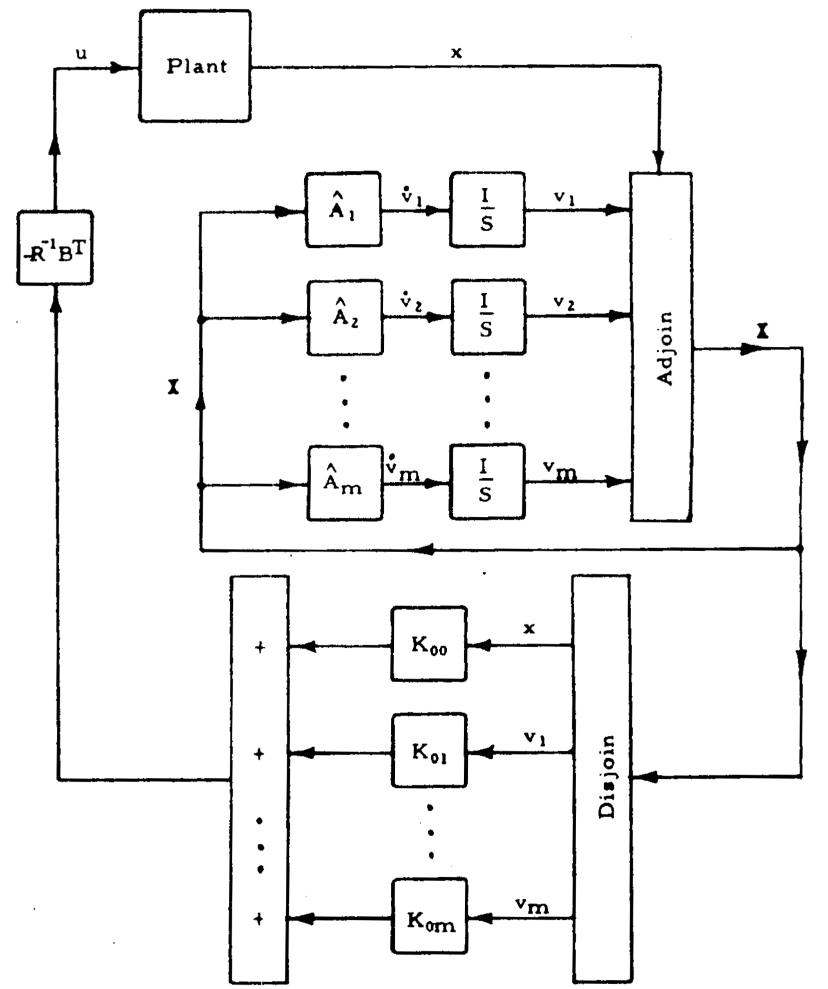

Figure 1. Simulation of Optimal Control

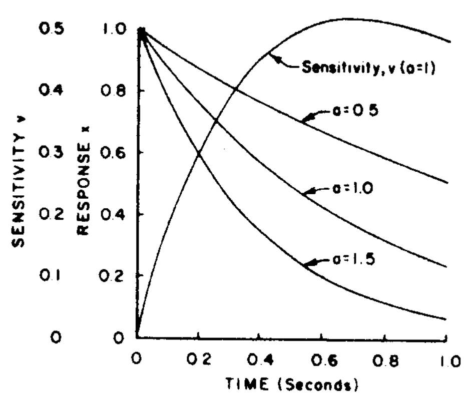

Figure 2. Response for s 0

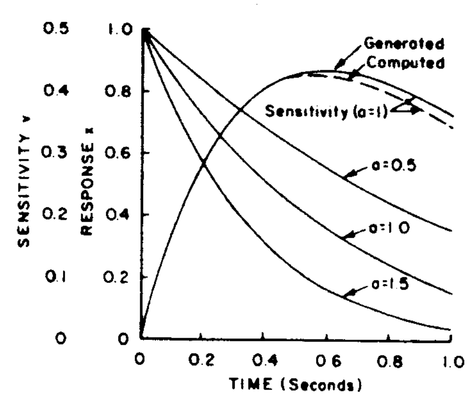

Figure 3. Response for s = 1

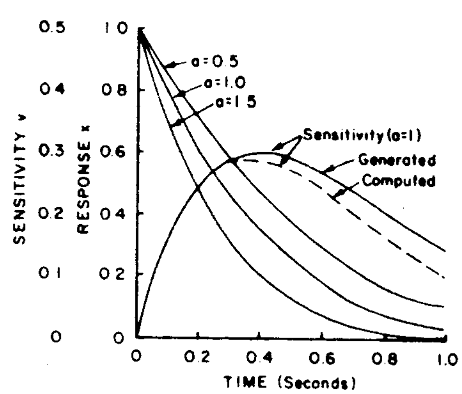

Figure 4. Response for s 10

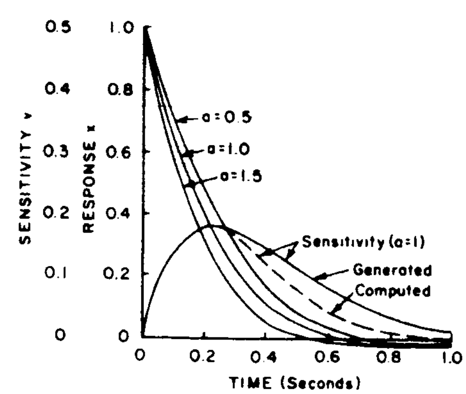

Figure 5. Response for s = 100

A COMPUTER-AIDED DESIGN TECHNIQUE FOR SAMPLED-DATA CONTROL SYSTEMS

N 68-23563

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ABSTRACT. A technique is presented for compensator design which includes constraints on peak time, overshoot, settling time, peak value of the controller output, and velocity error constant. A basic feature of the design technique is the formation of a figure-of-merit, which is a linear combination of the design objectives, and a minimization by adjustment and addition of compensator gain, poles, and zeros under the control of a multilevel decision procedure. Examples are given showing favorable comparisons to several other synthesis techniques.

INTRODUCTION

Several design techniques for sampled-data compensators have been proposed [2 - 8]. However, each allows only a limited number of design specifications and generally results in cancellation of poles and zeros. The computer-aided design technique described, does not result in pole and zero cancellation and allows a variety of specifications to be used. The particular specifications used for the development of the procedure include peak time, overshoot, settling time, peak signal input to the controlled system, and the velocity error constant. But the general technique need not be limited to these. The design technique is a search procedure as opposed to analytical synthesis. A figure-of-merit, which is a linear combination of the design objectives, is generated and then minimized by adjusting the compensator gain, poles, and zeros under the control of a multilevel decision procedure.

The procedure starts by attempting gain compensation, and if this fails it gradually adds poles and zeros until an acceptable design is achieved. Both real and complex poles and zeros may be used. Since the technique uses only the response of the fixed portion of the system, the approach does not become more involved as the order of the system increases. The complexity of the compensator is related to the difficulty of achieving the design objectives rather than the complexity of the system.

THE PROCEDURE

The design procedure synthesizes compensators of arbitrary complexity using both real and complex poles and zeros; however, for simplicity, the procedure will be described for a compensator consisting of a gain, one pole and one zero.

A figure-of-merit, FM, is defined

$$FM = C_1 T_p + C_2 E_s + C_3/K_v + C_4 M_s$$

where C_1, C_2, C_3, and C_4 are arbitrary weighting factors, E_s is the maximum error after the specified settling time, T_p is the time to the peak, K_v the velocity error constant, and M_s the peak signal into the fixed portion of the system. When comparing two compensators, the one with the smaller value for FM is considered the better. When a particular specification is satisfied for both compensators, it is deleted from the figure-of-merit calculation. This allows the search to concentrate on those specifications that are not yet met.

The design begins with what is termed a brute-force search (BFS) where combinations of pole and zero locations on the real axis inside the unit circle are considered. Large increments can be used in this search. For each pole and zero combination, the gain is adjusted to bring the overshoot within the specifications, if possible. If it is not possible, the particular pole and zero combination is eliminated from further consideration. For each pole and zero combination where the overshoot requirement can be satisfied, the value of FM is computed. As the search progresses, the lowest value of FM and the compensator which produced it are recorded and updated as better compensators are encountered.

If a suitable compensator has not been found at the end of the BFS, the program continues with a vernier search about the best pole and zero locations found in the BFS. In this search the pole and zero are alternately shifted with decreasing step size until FM is minimized, or a suitable compensator found. The gain is used to control the overshoot during the vernier search also. The use of gain to control overshoot created a two level search procedure which was much more efficient than one which considered gain as a parameter, on the same level with a pole or zero, and included overshoot in the figure-of-merit.

If the design is not completed at the end of the vernier search, the best compensator found is kept as a fixed compensator. Then an additional pole and zero is introduced and the procedure is repeated starting with the BFS again. This process may continue until a desired solution is found, the compensator has exceeded the allowed complexity, or the program execution time exceeds the specified limit.

If complex poles and zeros are allowed, the program first attempts to use a real pole and zero. If this fails, complex poles and zeros are introduced using a BFS and then a vernier search as with real poles and zeros. Of course, the search procedure is more complex and time consuming since four, rather than two, parameters are used.

The design procedure has been compared with seven design techniques on five different problems with very favorable results [1]. The following example will illustrate the results obtained.

Example

The effectiveness of the technique may be illustrated by comparing its solution with that obtained by Kuo on an example problem [5]. The fixed parts of the sampled-data system with zero-order hold are described by the transfer function

$$P(s) = \frac{(1-e^{-Ts})}{s^2(s+1)}$$

where the sampling period, T, is 0.1 seconds. The Z-transform is

$$P(z) = \frac{(.005)(z-.9)}{(z-1)(z-.905)}.$$

The specifications to be realized are:

1) The damping ratio, $\xi = .707$
2) The peak time, $T_p \leq 0.3$ seconds
3) The overshoot, $M_p \leq 10$ per cent
4) The velocity error constant $K_v \geq 5.$ seconds^{-1}.

The peak time, overshoot, and damping ratio are related, and since the program does not include provisions for damping ratio it was not explicitly used.

Kuo's method for discrete compensation of sampled-data systems is an extension of Truxal's synthesis technique developed for continuous systems [8]. Kuo's design does not consider the settling time, T_s, or the peak value of the controller output, M_s. For the computer-aided design, the values used for T_s and M_s were the actual values obtained from Kuo's solution. This, of course, puts more severe restrictions on the computer-aided solution than necessary.

Table 1 contains a summary of the results of Kuo's solution, as well as two separate solutions by the computer-aided design technique. The compensators resulting from the various techniques were as follows:

Kuo's Solution:

$$D(z) = 138\,\frac{(z-0.0737)(z-0.905)}{(z-0.23)(z-0.9)}$$

Computer Solution #1:

$$D(z) = 127.48\,\frac{z-0.1}{z-0.25}$$

Computer Solution #2:

$$D(z) = 200.11$$

Solution #1 has very similar properties to that of Kuo's. The response, as shown in Fig. 1, indicates that solution #1 has slightly more damping than Kuo's. However the computer-aided design used only a single pole and zero while Kuo's technique required a second order compensator.

The second solution illustrates that a simple design tends to result whenever possible. In this case, when the peak output of the controller was not restricted, only a gain was needed to obtain approximate deadbeat response. The solution as shown in Fig. 1, is within 0.5 per cent of the final value from the first sample onward. The closed-loop transfer function with compensator #2 is given by

$$\frac{C(z)}{R(z)} = \frac{(1.00056)(z-.9)}{(z-.005)(z-.899441)}$$

and indicates that the solution is well behaved between samples. The suitability of this solution is dependent upon the reason for the requirement of a damping ratio of 0.707. If this figure was chosen to allow a reasonable overshoot and fast response, then the requirement could be changed to $\xi \geq .707$ and solution #2 would certainly be the best.

CONCLUSIONS

A technique for computer-aided compensator design is presented which is not dependent on the complexity of the fixed portion of the system. The procedure is general and could be easily extended to include additional design specifications, e.g., mean-squared-error, bandwidth, gain margin, phase margin, etc. Although not tested as yet, it would appear that the procedure can also be extended to nonlinear and continuous systems, and because of its organization, could make effective use of facilities for hybrid computation.

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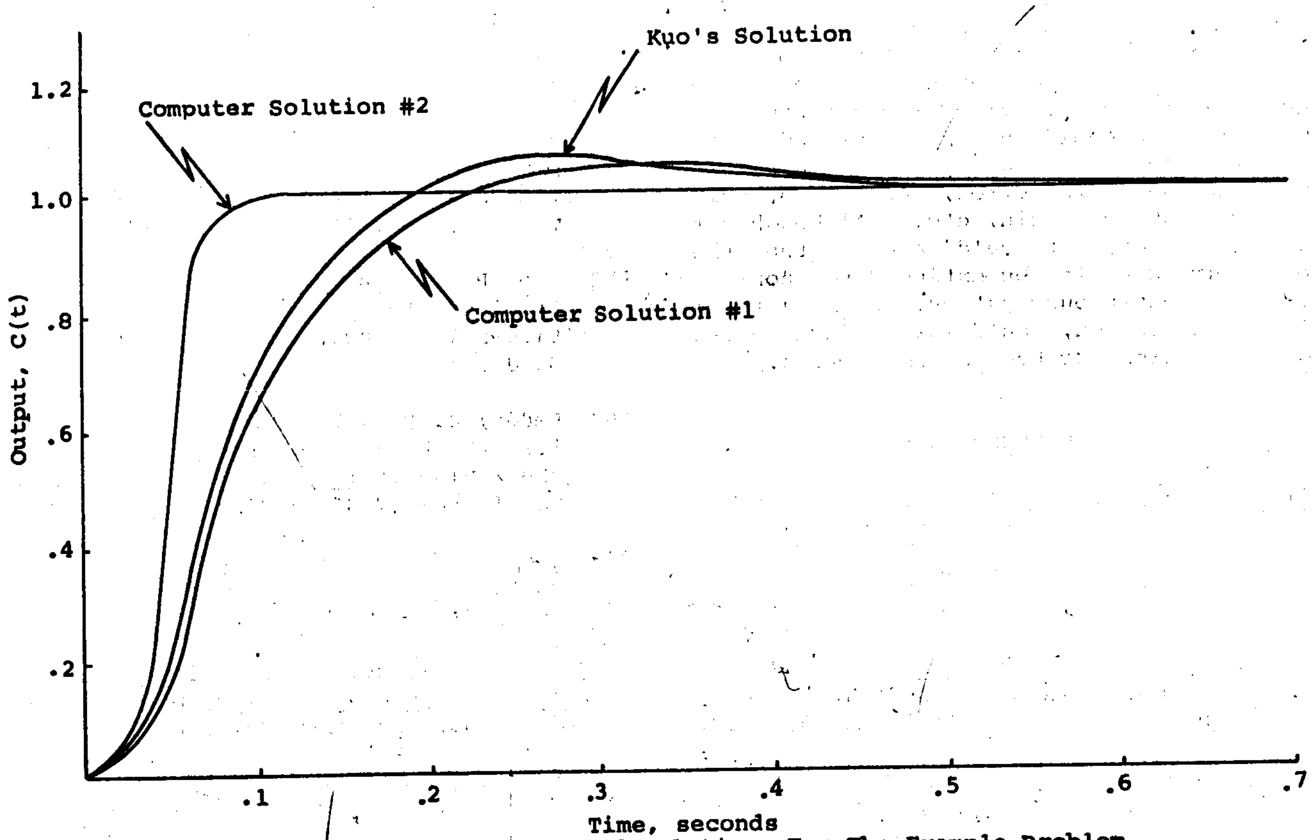
TABLE 1

Properties Of The Solutions For The Design Problem

	T_p	M_p	T_s*	K_v	M_s
1) Design Specifications	.3	.10	.5	5.	138.
2) Kuo's Solution	.3	.062	.5	8.301	138.
3) Computer Solution #1	.3	.051	.5	8.052	127.48
4) Computer Solution #2	.2	.005	.2	10.532	200.11

*Output within ±2% of its final value.



Transient Response Plots of Solutions For The Example Problem

Figure 1

AN OPTIMAL FIXED CONTROL STRUCTURE DESIGN WITH MINIMAL SENSITIVITY FOR A LARGE ELASTIC BOOSTER

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ABSTRACT

A design algorithm developed for the minimization of a cost functional which includes, in addition to state and control variables, a measure of sensitivity, is used to solve an example of practical interest: a flexible ballistic missile in powered flight. Comparisons are made between optimal designs which include sensitivity in the performance index and those that do not. The efficacy of using a measure of sensitivity in the performance index is demonstrated.

I. INTRODUCTION

Trajectory sensitivity, defined as $v = \partial x/\partial \alpha$ where x is the system state, α a system parameter, was introduced and used by Miller and Murray [1] in a mathematical error analysis of differential analyzers. Meissinger [2] used trajectory sensitivity functions in control system analysis in 1960. And in 1964 Kokotovic [3] demonstrated how sensitivity functions can be simulated in a straight forward manner on an analog computer. An early use of trajectory sensitivity functions in an optimal design was formulated, but not solved, by Siljak and Dorf [4]. Methods for the minimization of a performance index containing sensitivity terms have been advanced by numerous authors [5, 6, 7]. In the minimization of a performance index containing sensitivity functions the optimal control law, if feedback control is used, will in general be a function of these sensitivity variables. In this case, it is known [8] that the sensitivity functions can only be approximately realized. To avoid approximating sensitivity functions, the approach taken here is to optimize under the constraint that the control be a linear function of only the states, i.e., a constant feedback structure is assumed. The constant feedback structure is easiest to both implement and handle mathematically. Obviously, other feedback structures could also be assumed. However, for each case it would be necessary to determine if an optimal solution is independent of initial conditions. The minimization of a performance index containing sensitivity with the added constraint of fixed feedback structure is illustrated by solving a significant problem of practical interest: a fifth order flexible booster in powered flight. It is shown that if the trajectory dispersions, as system parameters are varied, are used to evaluate two optimal designs (one where sensitivity is used and one where sensitivity is not) that the design with sensitivity performs in a superior manner.

II. STATEMENT OF THE PROBLEM

The plant to be controlled is a large Titan III flexible booster. It is assumed that the states of the system are exactly known, i.e. the sensors are noiseless and contain no dynamics. Further, it is desired to compare designs by the optimum regulator theory with designs in which the performance index contains sensitivity terms. The booster is described by a set of nonlinear differential equations linearized about some nominal value. Control is maintained by gimballing the two engines which produce the thrust for the vehicle. Gimballing dynamics

are neglected. Rigid and lateral elastic bending geometry is shown in Figure 1 and 2 respectively. Two frames of reference are used in describing the motion of the center of gravity of the vehicle: the inertial reference fixed on earth and the vehicle body coordinates. The relationship between the two coordinate systems is given in Figure 1. The nonlinear equations describing the motion of the center of gravity (CG) of the vehicle are written in terms of θ, α, and η (attitude, angle of attack, and bending mode deflection).

$$\ddot{\theta} I = -TL_G \sin\delta + L_n q S_f C_{m_\alpha} \alpha + \varphi_{aft} \ddot{\eta}$$

$$Mv_o (\dot{\theta} - \dot{\alpha}) = T \sin\delta + qS_f C_{n_\alpha} \alpha \qquad (1)$$

$$\ddot{\eta} = -2\zeta\omega\dot{\eta} - \omega^2\eta + Th_g \delta$$

By defining coefficients a_i by

$$a_1 = T/M \qquad a_2 = (qS_f C_{n_\alpha}/M) \qquad a_3 = v_o$$

$$a_4 = (qS_f C_{m_\alpha} L_n/I) \qquad a_5 = (TL_G/I)$$

with T = Thrust, M = Mass, $q = 1/2\,\rho v_o^2$, S_f = Reference vehicle area, L_n = moment arm, C_{n_α} = normal force coefficient C_{m_α} = aerodynamic moment coefficient, L_G = normal coordinate to normal force moment arm, g = gravity, v_o = velocity, φ_{aft} = maximum angular deflection from rigid body center line, δ = engine angular deflection, ω = bending mode natural frequency, ζ = bending mode damping ratio, h_g = bending moment arm, ρ = air density; and using the small angle approximation $\delta \cong \sin\delta$, enables Eqn. (1) to be rewritten in state variable form.

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \dot{\theta}\\ \alpha\\ \eta\\ \dot{\eta}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 0 & 0 & a_4 & -\omega^2\varphi_{aft} & -2\zeta\omega\varphi_{aft}\\ 0 & 1 & -a_2/a_3 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -\omega^2 & -2\zeta\omega\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ \alpha\\ \eta\\ \dot{\eta}\end{bmatrix} + \begin{bmatrix}0\\ a_5 + \varphi_{aft} Th_g\\ -a_1/a_3\\ 0\\ Th_g\end{bmatrix}\delta \qquad (2)$$

Choosing δ (the engine deflection) to minimize the performance index

$$\Phi = \frac{1}{2}\int_0^\infty (x \cdot Qx + \delta^2)dt \qquad (3)$$

results in a linear constant feedback law $\delta = +g^*x$ (*denotes matrix transposition and $x \cdot Qx \equiv x^*Qx$) where g is obtained as a function of the solution to the nonlinear matrix Riccati equation

$$A^*K + KA - KBR^{-1}B^*K + Q = 0 \tag{4}$$

namely

$$g^* = -R^{-1}B^*K \tag{5}$$

The selection of δ to minimize a performance index

$$\Phi = \frac{1}{2}\int_0^\infty (x \cdot Qx + v \cdot Sv + \delta^2)dt \tag{6}$$

containing, in addition to "state" and "control", a measure of trajectory sensitivity is effected by an iterative algebraic algorithm developed in [9] under the additional constraint that the feedback law is a linear function of the state variables. The algorithm presented in [9] is a generalization, and a computational improvement, of the work of Schoenberger [10].

For notational simplicity the system to be controlled is written in the following form

$$\dot{x} = Ax + b\delta \tag{7}$$

The control is assumed to be a constant function of the state variables

$$\delta = g^*x \tag{8}$$

Combining equations (7) and (8) yields the compensated system

$$\dot{x} = (A + bg^*)x \triangleq \tilde{A}x \tag{9}$$

and the sensitivity equations, with respect to α, with equation (9) is

$$\dot{v} = \tilde{A}_\alpha x + \tilde{A}v \qquad v(0) = 0 \quad , \tilde{A}_\alpha \triangleq \partial\tilde{A}/\partial\alpha \tag{10}$$

In block form equations (9) and (10) become

$$\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} = \begin{bmatrix}\tilde{A} & 0\\ \tilde{A}_\alpha & \tilde{A}\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix} \tag{11}$$

To simplify equation (11) we define

$$y = \begin{bmatrix}x\\ v\end{bmatrix} \qquad \bar{A} = \begin{bmatrix}\tilde{A} & 0\\ \tilde{A}_\alpha & \tilde{A}\end{bmatrix} \qquad W = \begin{bmatrix}Q + gg^* & 0\\ 0 & S\end{bmatrix}$$

Then using the above nomenclature equation (11) and the performance index equation (6) can be rewritten

$$\dot{y} = \bar{A}y \tag{12}$$

$$\Phi = \frac{1}{2}\int_0^\infty (y \cdot Wy)dt \tag{13}$$

The minimization of equation (6) requires iteratively computing the gain vector g by the formula

$$g^{\nu+1} = \{P^\nu\}^{-1}\{d^\nu + e^\nu\} \tag{14}$$

which is terminated when

$$\left\|g^{\nu+1} - g^\nu\right\| \le \epsilon \tag{15}$$

where ϵ is a small positive number. To solve for P, d, and g of Eq. (7) requires, at each iteration, solving two $2n^{th}$ order algebraic equations.

$$\bar{A}^*H + H\bar{A} + W = 0 \tag{16}$$

and

$$\hat{H}\bar{A}^* + \bar{A}\hat{H} + \begin{bmatrix}x_o x_o^* & 0\\ 0 & 0\end{bmatrix} = 0 \tag{17}$$

Matrices H and $\hat{H}$ are symmetric and positive definite. From the partitions of H and $\hat{H}$, i.e.

$$H = \begin{bmatrix}H_1 & H_2\\ H_2^* & H_3\end{bmatrix} \tag{18}$$

$$\hat{H} = \begin{bmatrix}\hat{H}_1 & \hat{H}_2\\ \hat{H}_2^* & \hat{H}_3\end{bmatrix} \tag{19}$$

it is possible to solve for the matrices of the algorithm from the relationships

$$P = \hat{H}_1$$

$$d = (\hat{H}_1 H_1 + \hat{H}_2 H_2^*)b \tag{20}$$

$$e = (\hat{H}_2^* H_2 + \hat{H}_3 H_3)b$$

Since Eqn. (10) definitely depends upon the initial conditions it is natural to assume that the control law $\delta = g^*x$ likewise is a function of the initial conditions. It can be shown that this is not the case; numerous numerical examples with varying initial conditions demonstrate the nondependence of the control law upon initial conditions.

III. SOLUTION OF THE PROBLEM

As a numerical example consider the parameters for a Titan III - M configuration at 60 seconds after liftoff. The model given in equation (2) is used; the a_i are

$a_1 = 904.8$ $\quad a_2 = 213.5$ $\quad a_3 = 218$

$a_4 = .3358$ $\quad a_5 = -2.942$

Using the numerical values given above in equation (2) yields

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \dot\theta\\ \alpha\\ \eta\\ \dot\eta\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 0 & 0 & -1.5x10^{-5} & -17.5x10^{-4} & -4.4x10^{-6}\\ 0 & 1 & -9.76x10^{-3} & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -40.2 & -0.102\end{bmatrix}\begin{bmatrix}\theta\\ \dot\theta\\ \alpha\\ \eta\\ \dot\eta\end{bmatrix} + \begin{bmatrix}0\\ -1.67\\ -0.0414\\ 0\\ -2.9x10^{4}\end{bmatrix}\delta \quad (21)$$

Of all the parameters defined, experience has indicated that the parameter ϕ_{aft} (shown in equation (2)) is not accurately known and also greatly affects system performance. Using the algorithm shown in Section II the system performance is compared for an optimum system with and without sensitivity added to the performance index.

For the optimum regulator problem the control variable is chosen to minimize the performance index

$$\Phi = \frac{1}{2}\int_0^\infty (x^* Qx + \delta^2)dt \quad (22)$$

where x is defined as $x = [\theta , \dot\theta , \alpha , \eta , \dot\eta]^*$ and

$$Q = \begin{bmatrix}124 & 9 & 0 & 0 & 0\\ 9 & 3 & 0 & 0 & 0\\ 0 & 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} \quad (23)$$

The optimal control is found to be

$$\delta = -11.1\theta - 7.2\dot\theta - 0.848\alpha + 2.49 \times 10^{-4}\eta + 2.17 \times 10^{-4}\dot\eta \quad (24)$$

To include a measure of sensitivity in the performance index define the sensitivity variable as

$$v = \frac{\partial x}{\partial \varphi_{aft}} \quad (25)$$

Thus the optimal control problem then is to pick the control function δ, constrained to be a constant linear function of the state variables, to minimize the performance index

$$\Phi = \frac{1}{2}\int_0^\infty (x\cdot Qx + v\cdot Sv + \delta^2)dt \quad (26)$$

where Q has the same numerical value as before and
$S = \text{Diag.}\,[1x10^{-16}, 1x10^{-16}, 1x10^{-16}, 1x10^{-16}, 1x10^{-16}]$

The algorithm of Section II was programmed on a CDC6400 computer and the control function was found to be

$$\delta = -13.07\theta - 10.04\dot\theta - 1.95\alpha - 1.4x10^{-3}\eta - 2.4x10^{-5}\dot\eta \quad (27)$$

Figure 3 shows the state space trajectories of the optimum system without sensitivity, whereas Figure 4 depicts the state space trajectories of the optimum system with sensitivity. Each figure displays three trajectories obtained by varying φ_{aft} from its nominal value + 20%. The input into the system was a step function of negative unit height on the system attitude (θ). During an actual flight this might correspond to a wind gust on wind shear input. The magnitude of the input, however, was used only for purposes illustration. All other state variables are assumed to have zero initial conditions. It is obvious from Figure 3 that the system for $\varphi_{aft} = \varphi_{aft(low)}$ is unstable and the system's trajectories diverge with increasing time. Figure 4 depicts trajectories of the optimum system with sensitivity for the same variations in φ_{aft}. In all cases the system remains stable. Also, the dispersion of the system's trajectories is not great and the utility of minimizing a functional containing sensitivity is demonstrated. It should be mentioned that this method of comparing system performance for parameter variations is not without flaws. To obtain the insensitive performance requires larger feedback gains and more control effort. It is theoretically possible, in the optimum regulator problem, to choose the weighting matrix Q so that the feedback gains will match those shown in equation (27). However, the utility of including sensitivity in the performance is one of having analytical control over the degree of insensitivity required, and achieving insensitivity in an optimum manner.

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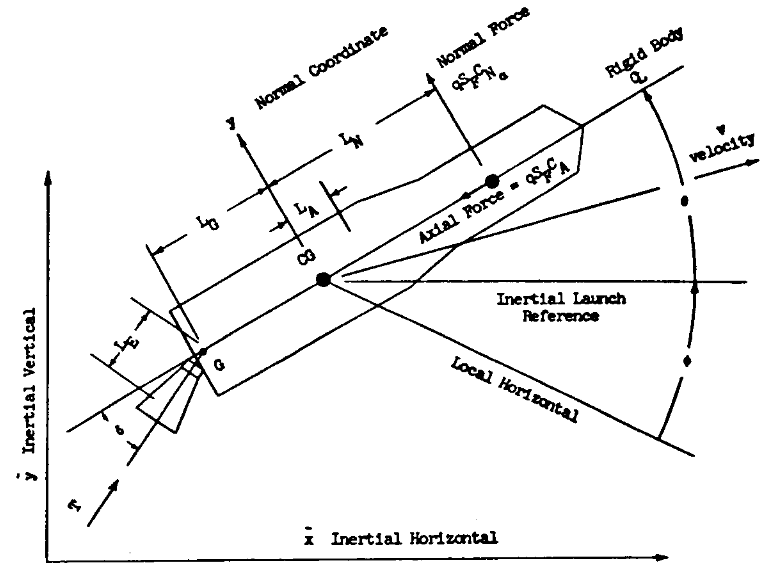

Figure 1. Rigid Body Missile Geometry

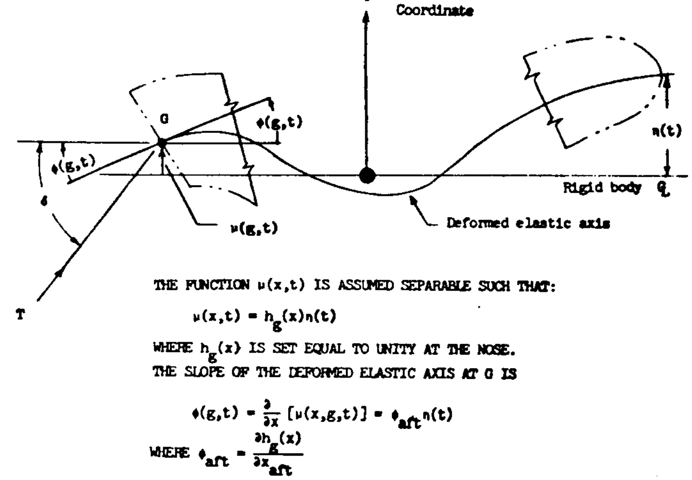

Figure 2. Lateral Elastic Bending Geometry

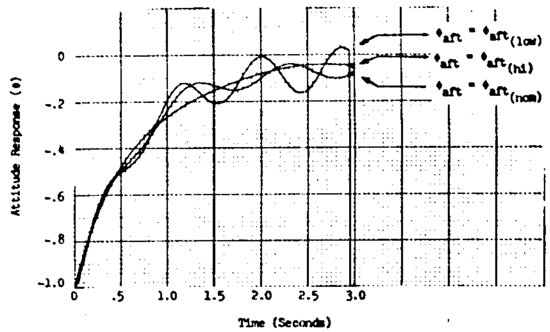

Figure 3a. Missile Attitude Response for S = 0

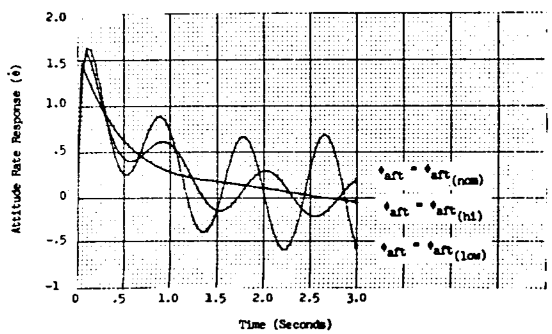

Figure 3b. Missile Attitude Rate Response for S = 0

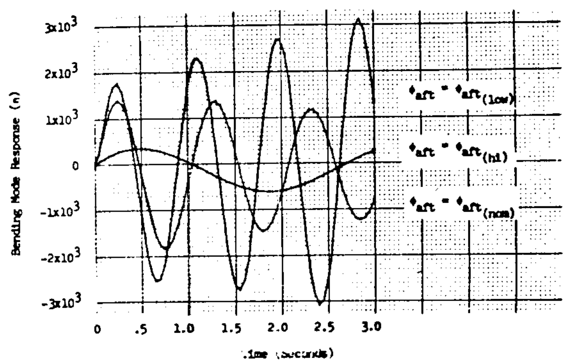

Figure 3c. Missile Bending Mode Response for S = 0

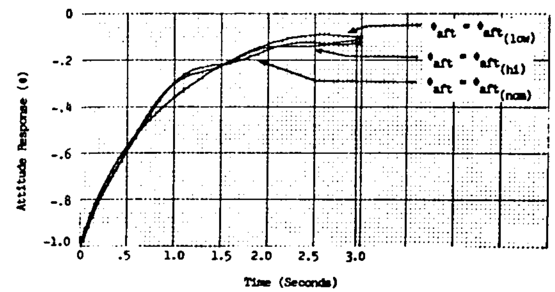

Figure 4a. Missile Attitude Response for $S = 1x10^{-16} I$

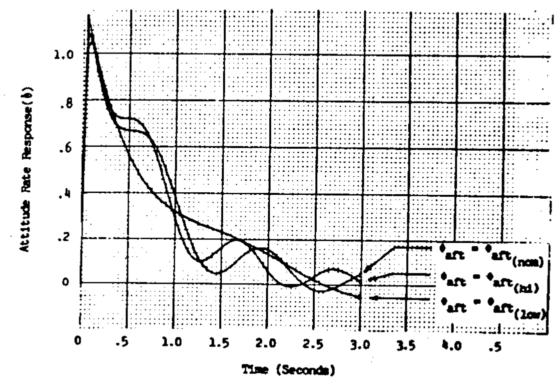

Figure 4b. Missile Attitude Rate Response for $S = 1x10^{-16} I$

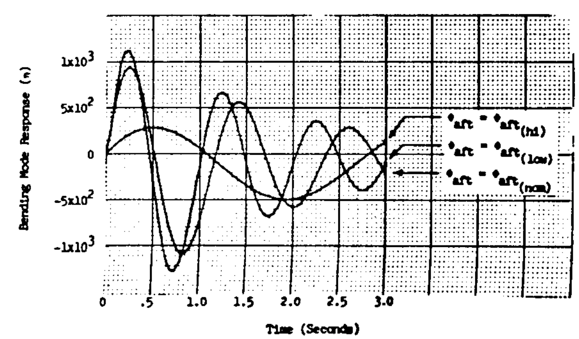

Figure 4c. Missile Bending Mode Response for $S = 1x10^{-16} I$

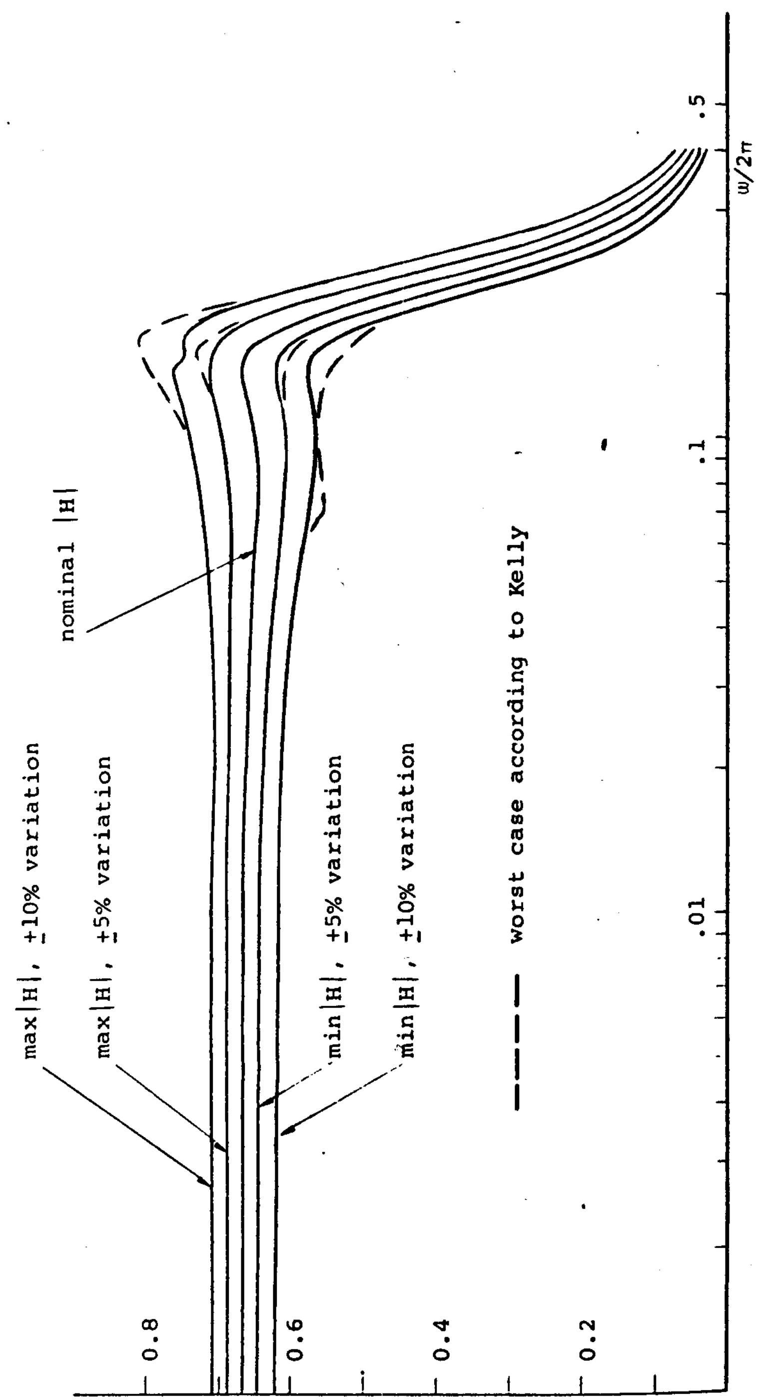

Fig. 5

OPTIMAL CONTROL OF PLANTS WITH RANDOM SLOWLY-VARYING PARAMETERS

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ABSTRACT

A method is given for obtaining the optimal control law for plants containing a random slowly-varying parameter whose probability density function is known. The necessary conditions for the optimal control law are derived using the calculus of variations, and a method is advanced for approximating the optimal control law in closed-loop fashion. A numerical illustrative example is worked and the results are compared with both an optimal system designed about the nominal value of the parameter, and an adaptive system whose control law changes to the optimal for the existing parameter value.

MAIN RESULTS

1. Introduction

Many methods have been proposed for reducing optimal system sensitivity to small parameter variations in which the resultant design is a function of the nominal parameter value. Relatively little has been done in utilizing a-priori knowledge that may be available of the parameter's probability density function. Assuming a-priori knowledge of the parameter's probability density function, and that the random parameter is slowly time-varying, an optimal control law, which considers the total parameter variation, is derived. The resulting optimal design thus takes into account both small and large parameter variations.

The problem is formulated as follows: Given the plant

$$\dot{x}(t,\alpha) = f[x(t,\alpha), u(t); t,\alpha] \quad (1)$$

with initial conditions

$$x(t_o,\alpha) = x_o \quad (2)$$

find the optimal control law u(t) which minimizes the performance index

$$J = \int_{\alpha_\ell}^{\alpha_h} \omega(\alpha)\{K[x(t_f,\alpha)] + \int_{t_o}^{t_f} L[x(t,\alpha), u(t); t,\alpha]dt\}d\alpha \quad (3)$$

where $\omega(\alpha)$ is the probability density function of the plant parameter α, $x(t,\alpha)$ is an n-vector, $u(t)$ is an m-vector with $m \leq n$, and α is a scalar. In addition, it is desired, when possible, to obtain the optimal control law $u(t)$ as a function of the state $x(t,\alpha)$ and thus allow a closed-loop synthesis of the optimal system.

2. Necessary conditions for optimal control law [1]

Denoting an optimal variable with a superscript o, any control u(t) and state $x(t,\alpha)$ can be expressed by the perturbations

$$u(t) = u^o(t) + \epsilon\, \eta(t) \quad (4)$$

$$x(t,\alpha) = x^o(t,\alpha) + \epsilon\, \phi(t,\alpha) \quad (5)$$

Defining the Hamiltonian by

$$H[x(t,\alpha), u(t), p(t,\alpha); t,\alpha] \equiv L[x(t,\alpha), u(t); t,\alpha] + \langle p(t,\alpha), f[x(t,\alpha), u(t); t,\alpha]\rangle \quad (6)$$

it is determined that

$$J(u) - J(u^o) = \int_{\alpha_\ell}^{\alpha_h} \omega(\alpha)\{K[x(t_f,\alpha)] - K[x^o(t_f,\alpha)]\}d\alpha$$
$$+ \int_{t_o}^{t_f}\int_{\alpha_\ell}^{\alpha_h} \omega(\alpha)\{H[x(t,\alpha),\ u(t),\ p(t,\alpha);\ t,\alpha]$$
$$- H[x^o(t,\alpha),\ u^o(t),\ p^o(t,\alpha);\ t,\alpha]\}d\alpha dt$$
$$+ \int_{t_o}^{t_f}\int_{\alpha_\ell}^{\alpha_h} \omega(\alpha)\ \langle p(t,\alpha),\ [x^o(t,\alpha) - \dot{x}(t,\alpha)]\rangle\ d\alpha dt \quad (7)$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product. Letting ϵ approach zero results in

$$\int_{\alpha_\ell}^{\alpha_h} \omega(\alpha)\langle\frac{\partial K}{\partial x}\Big|_{o,p} - p(t_f,\alpha),\ \phi(t_f,\alpha)\rangle d\alpha + \int_{t_o}^{t_f}\int_{\alpha_\ell}^{\alpha_h} \omega(\alpha)\ \langle[\frac{\partial H}{\partial x}\Big|_{o,p}$$
$$+ \dot{p}(t,\alpha)],\ \phi(t,\alpha)\rangle\ d\alpha dt + \int_{t_o}^{t_f}\int_{\alpha_\ell}^{\alpha_h} \omega(\alpha)\ \langle\frac{\partial H}{\partial u}\Big|_{o,p},\ \eta(t)\rangle\ d\alpha dt \to 0 \quad (8)$$

from which the following necessary conditions are deduced:

It is necessary that the differential equations

$$\dot{x}^o(t,\alpha) = \frac{\partial H}{\partial p} \quad (9)$$

$$\dot{p}^o(t,\alpha) = \frac{-\partial H}{\partial x} \quad (10)$$

with the boundary conditions

$$x^o(t_o,\alpha) = x_o \quad (11)$$

$$p^o(t_f,\alpha) = \frac{\partial K[x(t_f,\alpha)]}{\partial x} \quad (12)$$

and the integral equation

$$\int_{\alpha_\ell}^{\alpha_h} \omega(\xi)\ \frac{\partial H}{\partial u}\ d\xi = 0 \quad (13)$$

be satisfied.

3. The linear regulator problem

The previous results are applied to the state regulator problem in which the plant is given by

$$\dot{x}(t,\alpha) = A(\alpha)x(t,\alpha) + B(\alpha)u(t) \quad (14)$$

$$x(t_o,\alpha) = x_o \quad (15)$$

and the performance index to be minimized is

$$J = \int_{\alpha_\ell}^{\alpha_h} \omega(\alpha)\{x^*(t_f,\alpha)F\ x(t_f,\alpha) + \int_{t_o}^{t_f} [x^*(t,\alpha)Q\ x(t,\alpha)$$
$$+ u^*(t)R\ u(t)]dt\}d\alpha \quad (16)$$

(* denotes matrix transposition). The necessary conditions of equations (9) through (13) result in the following state and costate equations

$$\dot{x}(t,\alpha) = A(\alpha)\ x(t,\alpha) + B(\alpha)\ u(t) \quad (17)$$

$$\dot{p}(t,\alpha) = -Q\ x(t,\alpha) - A^*(\alpha)\ p(t,\alpha) \quad (18)$$

with boundary conditions

$$x(t_o,\alpha) = x_o \quad (19)$$

$$p(t_f,\alpha) = F\ x(t_f,\alpha) \quad (20)$$

The optimal control law $u^o(t)$, derived from equation (13), is

$$u^o(t) = -R^{-1}\int_{\alpha_\ell}^{\alpha_h} \omega(\xi)B^*(\xi)p(t,\xi)d\xi \quad (21)$$

4. Closed-loop optimal control

A method is now advanced which allows the control law to be approximated as a function of the state and thus provide closed-loop control. Expanding the costate $p(t,\xi)$ in a Taylor series [2] about α_o results in

$$p(t,\xi) = \sum_{i=0}^{\infty} p_i(t,\alpha_o)\ \frac{(\xi - \alpha_o)^i}{i!} \quad (22)$$

where

$$p_i(t,\alpha_o) \equiv \frac{\partial^i p(t,\xi)}{\partial \xi^i}\Big|_{\xi=\alpha_o} \quad (23)$$

Thus substituting from equation (22) in equation (21) gives

$$u(t) = -R^{-1}\int_{\alpha_\ell}^{\alpha_h} \omega(\xi)B^*(\xi)[\sum_{i=0}^{\infty} p_i(t,\alpha_o)\ \frac{(\xi - \alpha_o)^i}{i!}]d\xi \quad (24)$$

For notational simplicity c_i is defined as

$$c_i \equiv \int_{\alpha_\ell}^{\alpha_h} \omega(\xi)B^*(\xi)\ \frac{(\xi - \alpha_o)^i}{i!}\ d\xi \quad (25)$$

Thus

$$u(t) = -R^{-1}\sum_{i=0}^{\infty} c_i p_i(t,\alpha_o)$$

$$= -R^{-1}\ \hat{c}^*\hat{p}(t,\alpha_o) \quad (26)$$

where $\hat{c}$ and $\hat{p}(t,\alpha_o)$ are the appropriate infinite vectors. Differentiating equation (18) with respect to α gives

$$\dot{p}_i(t,\alpha_o) = -Q\ v_i(t,\alpha_o) - \frac{\partial^i[A^*(\xi)p(t,\xi)]}{\partial \xi^i}\Big|_{\xi=\alpha_o} \quad (27)$$

where $v_i(t,\alpha_o)$ denotes the i^{th} trajectory sensitivity [3] defined by

$$v_i(t,\alpha_o) \equiv \frac{\partial^i x(t,\xi)}{\partial \xi^i}\Big|_{\xi=\alpha_o}$$

Hence, the following matrix equation can be formed:

$$\begin{vmatrix} \dot{p}(t,\alpha_o) \\ \dot{p}_1(t,\alpha_o) \\ \dot{p}_2(t,\alpha_o) \\ \vdots \\ \dot{p}_r(t,\alpha_o) \\ \vdots \end{vmatrix} = - \begin{vmatrix} A^*(\alpha_o) & 0 & 0 & \cdots & 0 & \cdots \\ A_1^*(\alpha_o) & A^*(\alpha_o) & 0 & \cdots & 0 & \cdots \\ A_2^*(\alpha_o) & 2A_1^*(\alpha_o) & A^*(\alpha_o) & \cdots & 0 & \cdots \\ \vdots & \vdots & & & \vdots & \\ A_r^*(\alpha_o) & rA_{r-1}^*(\alpha_o) & \sum_{i=1}^{r-1} iA_{r-2}^*(\alpha_o) & \cdots & A^*(\alpha_o) & \cdots \\ \vdots & & \vdots & & \vdots & \end{vmatrix} \begin{vmatrix} p(t,\alpha_o) \\ p_1(t,\alpha_o) \\ p_2(t,\alpha_o) \\ \vdots \\ p_r(t,\alpha_o) \\ \vdots \end{vmatrix}$$

$$- \begin{vmatrix} Q & 0 & 0 & \cdots & 0 & \cdots \\ 0 & Q & 0 & \cdots & 0 & \cdots \\ 0 & 0 & Q & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & Q & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{vmatrix} \begin{vmatrix} x(t,\alpha_o) \\ v_1(t,\alpha_o) \\ v_2(t,\alpha_o) \\ \vdots \\ v_r(t,\alpha_o) \\ \vdots \end{vmatrix} \tag{29}$$

where

$$A_i(\alpha_o) = \left. \frac{\partial^i A(\alpha)}{\partial \alpha^i} \right|_{\alpha=\alpha_o} \tag{30}$$

with boundary conditions derived from equation (20):

$$\begin{vmatrix} p(t_f,\alpha_o) \\ p_1(t_f,\alpha_o) \\ p_2(t_f,\alpha_o) \\ \vdots \\ p_r(t_f,\alpha_o) \\ \vdots \end{vmatrix} = \begin{vmatrix} F & 0 & 0 & \cdots & 0 & \cdots \\ 0 & F & 0 & \cdots & 0 & \cdots \\ 0 & 0 & F & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & F & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{vmatrix} \begin{vmatrix} x(t_f,\alpha_o) \\ v_1(t_f,\alpha_o) \\ v_2(t_f,\alpha_o) \\ \vdots \\ v_r(t_f,\alpha_o) \\ \vdots \end{vmatrix} \tag{31}$$

Using matrix notation equations (29) and (31) are rewritten:

$$\dot{\hat{p}}(t,\alpha_o) = \tilde{A}(\alpha_o)\hat{p}(t,\alpha_o) - \hat{Q}\hat{x}(t,\alpha_o) \tag{32}$$

$$\hat{p}(t_f,\alpha_o) = \hat{F}\,\hat{x}(t_f,\alpha_o) \tag{33}$$

where $\tilde{A}(\alpha_o)$, $\hat{Q}$, $\hat{F}$ and $\hat{x}(t,\alpha_o)$ are appropriately defined. Similarly, differentiating equation (17) with respect to α results in

$$\begin{vmatrix} \dot{x}(t,\alpha_o) \\ \dot{v}_1(t,\alpha_o) \\ \dot{v}_2(t,\alpha_o) \\ \vdots \\ \dot{v}_r(t,\alpha_o) \\ \vdots \end{vmatrix} = \begin{vmatrix} A(\alpha_o) & 0 & 0 & \cdots & 0 & \cdots \\ A_1(\alpha_o) & A(\alpha_o) & 0 & \cdots & 0 & \cdots \\ A_2(\alpha_o) & 2A_1(\alpha_o) & A(\alpha_o) & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ A_r(\alpha_o) & rA_{r-1}(\alpha_o) & \sum_{i=1}^{r-1} iA_{r-2}(\alpha_o) & \cdots & A(\alpha_o) & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{vmatrix} \begin{vmatrix} x(t,\alpha_o) \\ v_1(t,\alpha_o) \\ v_2(t,\alpha_o) \\ \vdots \\ v_r(t,\alpha_o) \\ \vdots \end{vmatrix}$$

$$+ \begin{vmatrix} B(\alpha_o) \\ B_1(\alpha_o) \\ B_2(\alpha_o) \\ \vdots \\ B_r(\alpha_o) \\ \vdots \end{vmatrix} u(t) \tag{34}$$

where

$$B_i(\alpha_o) = \left. \frac{\partial^i B(\alpha)}{\partial \alpha} \right|_{\alpha=\alpha_o} \tag{35}$$

with boundary conditions derived from equation (19):

$$\begin{vmatrix} x(t_o,\alpha_o) \\ v_1(t_o,\alpha_o) \\ v_2(t_o,\alpha_o) \\ \vdots \\ v_r(t_o,\alpha_o) \\ \vdots \end{vmatrix} = \begin{vmatrix} x_o \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{vmatrix} \tag{36}$$

Using matrix notation equations (34) and (36) are rewritten:

$$\dot{\hat{x}}(t,\alpha_o) = \hat{A}(\alpha_o)\hat{x}(t,\alpha_o) + \hat{B}(\alpha_o)u(t) \tag{37}$$

$$\hat{x}(t_o,\alpha_o) = \hat{x}_o$$

where $\hat{A}(\alpha_o)$, $\hat{B}(\alpha_o)$ and $\hat{x}_o$ are appropriately defined. Substituting from equation (26) in equation (37) gives

$$\dot{\hat{x}}(t,\alpha_o) = \hat{A}(\alpha_o)\hat{x}(t,\alpha_o) - \hat{B}(\alpha_o)R^{-1}\hat{c}^*p(t,\alpha_o) \tag{39}$$

Thus equations (32) and (39) provide simultaneous equations for $x(t,\alpha_o)$ and $p(t,\alpha_o)$:

$$\dot{\hat{x}}(t,\alpha_o) = \hat{A}(\alpha_o)\hat{x}(t,\alpha_o) - \hat{B}(\alpha_o)R^{-1}\hat{c}^*p(t,\alpha_o) \tag{40a}$$

$$\dot{\hat{p}}(t,\alpha_o) = -\hat{Q}\,\hat{x}(t,\alpha_o) - \tilde{A}(\alpha_o)\hat{p}(t,\alpha_o) \tag{40b}$$

with the boundary conditions

$$\hat{x}(t_o,\alpha_o) = \hat{x}_o \tag{41a}$$

$$\hat{p}(t_f,\alpha_o) = \hat{F}\,\hat{x}(t_f,\alpha_o) \tag{41b}$$

Equations (40) and (41) can be utilized [4, 5] to show that

$$\hat{p}(t,\alpha_o) = \hat{K}(t,\alpha_o)\hat{x}(t,\alpha_o) \tag{42}$$

where $\hat{K}(t,\alpha_o)$ is the solution to a matrix Riccati equation:

$$\dot{\hat{K}}(t,\alpha_o) + \hat{K}(t,\alpha_o)\hat{A}(\alpha_o) + \tilde{A}(\alpha_o)\hat{K}(t,\alpha_o)$$

$$- \hat{K}(t,\alpha_o)\hat{B}(\alpha_o)R^{-1}\hat{c}^*\hat{K}(t,\alpha_o) + \hat{Q} = 0 \tag{43}$$

Substitution of equation (42) in equation (26) provides a relationship for $u(t)$ in terms of $\hat{x}(t,\alpha_o)$, i.e., in terms of the state $x(t,\alpha_o)$ and all the trajectory sensitivity vectors $v_i(t,\alpha_o)$ $i = 1, 2, \ldots$ In particular

$$u(t) = -R^{-1}\hat{c}^{*}\hat{K}(t,\alpha_o)\hat{x}(t,\alpha_o) \tag{44}$$

The trajectory sensitivity vectors are easily generated [6].

A simulation of such a closed-loop optimal control system is shown in Figure 1. It should be noted that in generating the trajectory sensitivity vectors $v_i(t,\alpha_o)$ knowledge of the plant parameter α is implied. Thus if gains are not changed as the plant parameter α changes, the control law generated in this fashion will be only an approximation to the optimal control law. Experience shows that this is generally a good approximation. Further, in any practical situation only a finite number of trajectory sensitivity vectors will be generated. Thus all the infinite-order equations are approximated by finite-order equations.

5. Illustrative numerical example

Consider the first-order system characterized by

$$\dot{x} = -\alpha x + u \tag{45}$$

Knowing that the parameter α is uniformly distributed on the interval (0,2), it is desired to obtain the control u that minimizes the performance index

$$J = \int_0^2 \frac{1}{2} \int_0^\infty \frac{1}{2}(x^2 + u^2)dt\, d\alpha \tag{46}$$

Since this is an infinite time regulator problem, the matrix Riccati differential equation reduces to an algebraic equation. Truncating the Taylor expansion of $p(t,\xi)$ given in equation (22) after 3 terms results in the Riccati equation (43) being

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} -\alpha_o & 0 & 0 \\ -1 & -\alpha_o & 0 \\ 0 & -2 & -\alpha_o \end{bmatrix} + \begin{bmatrix} -\alpha_o & 0 & 0 \\ -1 & -\alpha_o & 0 \\ 0 & -2 & -\alpha_o \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

$$- \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (1) \left[1,\ 1-\alpha_o,\ \frac{2}{3} - \alpha_o + \frac{1}{2}\alpha_o^2\right] \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The approximate control, in accordance with equation (44), is

$$u(t) = (-1)\left[1,\ 1-\alpha_o,\ \frac{2}{3} - \alpha_o + \frac{1}{2}\alpha_o^2\right] \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} x(t,\alpha_o) \\ v_1(t,\alpha_o) \\ v_2(t,\alpha_o) \end{bmatrix} \tag{48}$$

A block diagram showing the simulation of this system is shown in Figure 2.

Figure 3a shows a comparison of the cost C defined by

$$C = \int_0^\infty (x^2 + u^2)dt \tag{49}$$

over the range of α, (0,2), between a system designed to minimize J (the expected value of C), using such a 3-term approximation $\alpha_o = 1$ (Figure 3b shows similar results for a 5-term approximation), and the optimal regulator minimizing C at a nominal $\alpha = 1$. Also shown in this figure is the cost that would be obtained by an optimally adaptive system, i.e., a system with a control law u(t) that is adjusted, on line, to be optimal for the particular value of α that the plant is noted to have. Clearly, this is the minimal cost possible. It is noted that the expected value of the cost using this simple approximation to the control minimizing J is significantly decreased from that not taking into account variations in the parameter α.

Figure 4 shows that the trajectories of a system so designed are less sensitive to variations in the parameter α.

6. Acknowledgements

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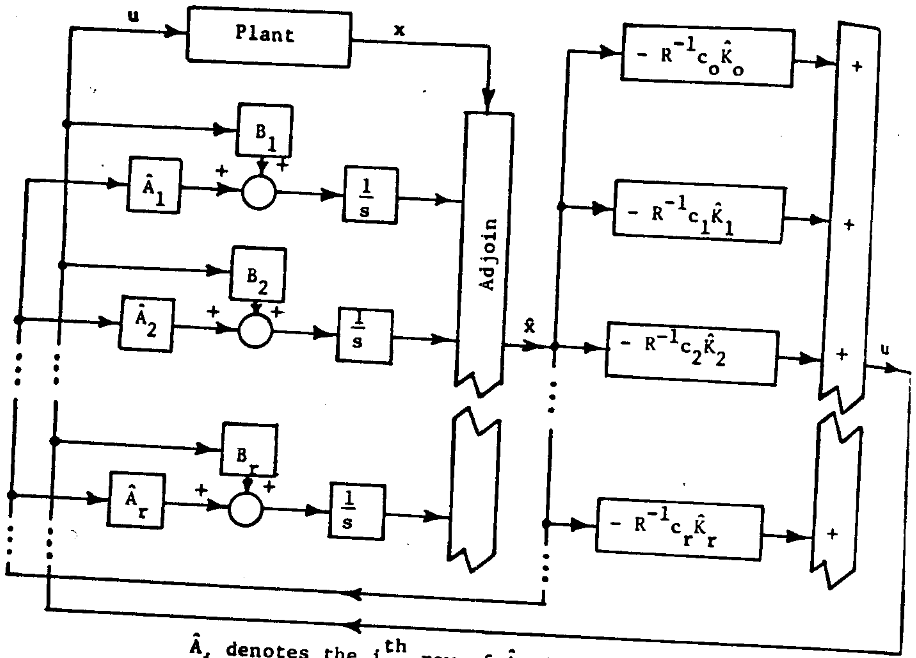
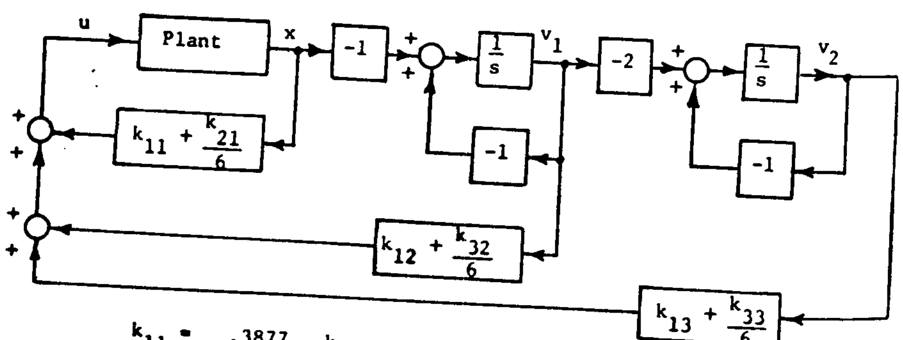


$\hat{A}_i$ denotes the i^{th} row of $\hat{A}$, $i = 0, \ldots, \infty$

$\hat{K}_i$ denotes the i^{th} row of $\hat{K}$, $i = 0, \ldots, \infty$

Figure 1. Simulation of Optimal System Characterized by Equations 40 and 41



$k_{11} = .3877 \quad k_{12} = .0342 \quad k_{13} = -.0125$

$k_{21} = -.3351 \quad k_{22} = .4471 \quad k_{23} = .0171$

$k_{31} = .6205 \quad k_{32} = -.8753 \quad k_{33} = .4629$

Figure 2. Simulation of Third-Order Approximation to Optimal System of Figure 1. for the Plant, $\dot{x} = -\alpha x + u$, $\alpha_o = 1$

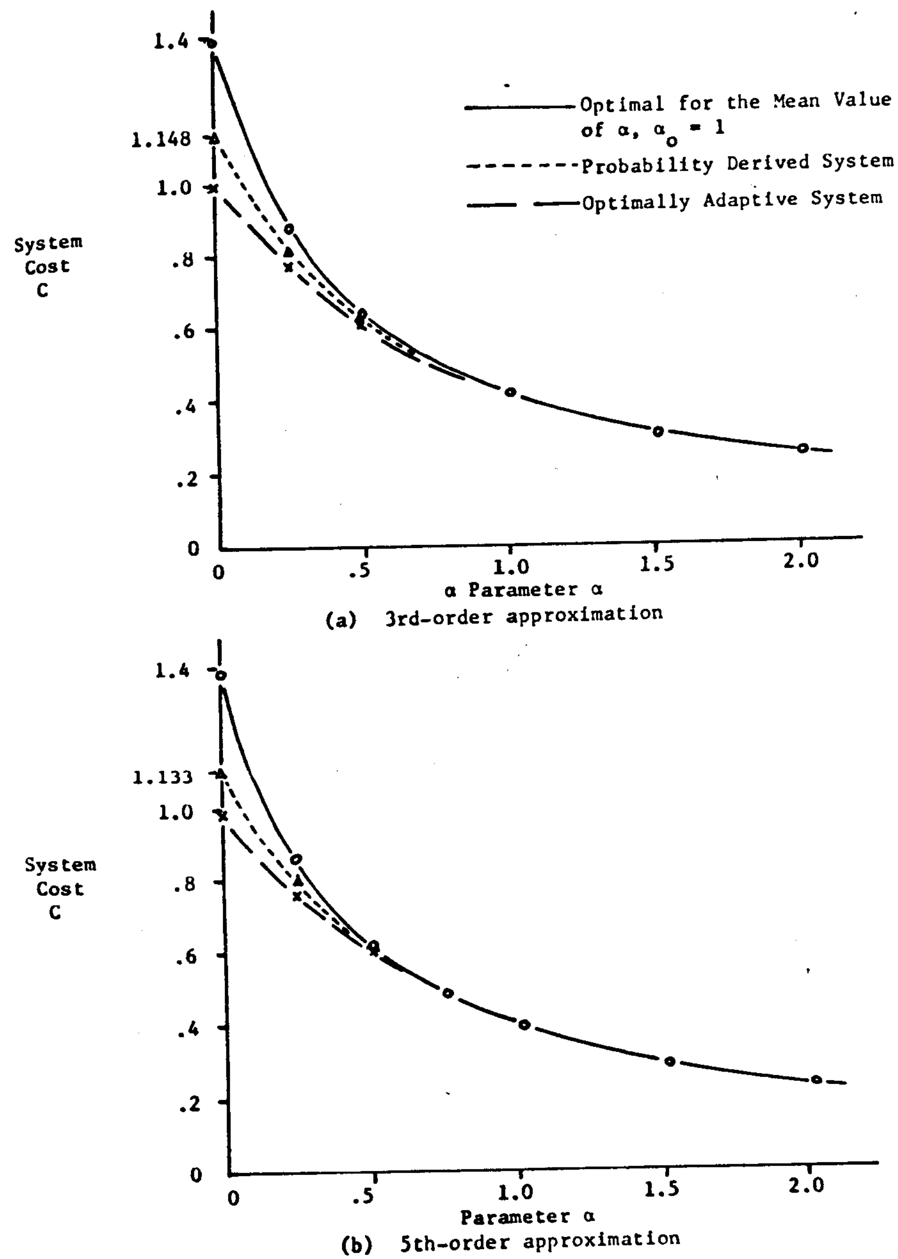


Figure 3. Comparisons of System Costs Over the Range of the Parameter α