RISK AVERSION AND BIDDING THEORY

by

D. L. Hanson and C. F. Menezes

RISK AVERSION AND BIDDING THEORY

by

D. L. Hanson[1] and C. F. Menezes[2]

1. Introduction and Summary

This paper provides a theory of individual bidding behavior in competitive sealed tender markets. The objective is to formulate a bidding model in terms of modern utility theory and to derive its basic properties. The model presented in this paper differs in important ways from the expected utility maximization bidding models independently formulated by Greismer, Levitan, and Shubik [4], and by Vernon Smith [7]. For one thing, both Greismer, et al, and Smith assume that the bidder maximizes expected utility of income. We assume that the bidder maximizes expected utility of wealth, the improvement being that utility is made to depend on both the size of the payoff and the level of initial wealth.

---

(1) Research partially sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant No. AF-AFOSR-746-65 and partially sponsored by the National Space Administration under Contract NGR-26-004-012.

(2) Research sponsored by the National Space Administration under Contract NGR-26-004-012.

A second difference relates to the form of the utility function. Greismer, et al, implicitly assume that utility is a homogeneous linear function of income. It is well known that linear utility functions, whether or not homogeneous, imply "neutrality" to risk; i.e., the individual will be indifferent between engaging in any arbitrary bet and receiving the sure option equal to the actuarial value of the bet. This type of implied behavior seems hardly consistent with intuitive evidence or observation. On the other hand, Smith assumes in places that utility is quadratic in income. This form of utility function is very prevalent in the literature on decision making under uncertainty and has been the basis for the mean-variance approach to the theory of portfolio selection. However, the quadratic utility function implies implausible behavior. As K. J. Arrow [2] has noted, it violates the principle of decreasing absolute risk aversion. It also implies that eventually wealth has negative marginal utility, so that it would be better to throw some away. Because of the implausible behavioral implications of linear and quadratic utility functions, we assume that utility is a concave function of wealth, this being the most general form of utility function which characterizes risk averse behavior.

Finally, the emphasis in this paper on deriving the formal properties of the model and giving their economic interpretation is also quite unique and turns out to be rewarding. A natural relationship emerges between the principal properties of the model and certain tools and concepts which have been developed in some branches of mathematical statistics on the one hand, and in the theory of risk aversion on the other. Specifically, an investigation of the solution properties of the model reveals the important role of the "hazard rate" or "failure rate" function, a basic concept in the mathematical theory of reliability [3]. At the same time, we find that some of the more important comparative statics properties of the model depend on the behavior of two functions, one of which has been independently established as a measure of risk aversion by K. J. Arrow [1], [2], and by J. W. Pratt [6]. As far as we know, the other function has not been interpreted as a measure of risk aversion in the literature prior to this. We establish it as such and relate it to the work of Arrow and to that of Pratt.

In Section 2 we formulate the bidding model and give conditions under which the model has a unique solution. The hazard rate function is interpreted and the expression determining the optimal bid is shown to have a straightforward behavioral meaning. Section 3 contains a summary of the work of Arrow and Pratt and some new results in the theory of risk aversion. Section 4 contains an investigation of the comparative statics properties of the model and their relation to the existence of risk aversion and the behavior of two measures of risk aversion. In contrast to the usual treatment of comparative statics in economic theory, both the direction of change in the optimal bid price and bounds on its magnitude are considered. The analysis is somewhat revealing of the nature of the substitutions between "safety" (as measured by the probability of success) and potential profits that underlie the bidder's response to a change in a specified parameter. In Section 5 we briefly outline possible directions in which the model can be extended.

2. An Expected Utility Maximization Bidding Model

This section deals with the structure and basic properties of an expected utility maximization bidding model for the sealed tender selling market. The institutional features of this market are outlined as follows: the market consists of a number of sellers competing for a single contract; each seller submits a single sealed bid; and the contract is awarded to the lowest bidder. Each seller's decision variable is his bid price. Every seller realizes that the higher his bid price the smaller the probability of getting the contract, but the larger the profits should he get it. Thus, each submitted bid reflects an attempt to balance probability and profit considerations.

To introduce the model we focus on a typical seller and denote his average cost by c . Because n , the size of the contract, is fixed in the type of market under consideration, $c = c(n)$ is a constant for any given bidding decision. Regarding the seller's beliefs about the bidding behavior of his opponents, we assume that the seller attaches a probability distribution $F(b)$ to the minimum of his competitor's bid prices b . We let p denote the bid submitted by the seller. He will get the contract if he submits a bid that is below all of the bids submitted by his competitors, that is, if $p < b$. The probability that his bid will be successful is

(2.1) $$\Pr\{p < b\} = 1 - F(p) .$$

We assume that F is continuous, so that the probability of a "tie" between bids is zero; because F is continuous, what happens in case of a tie does not affect the seller's probability of getting the contract.

The bidding situation facing the seller is equivalent to choosing p in a lottery which offers a prize of $n(p - c)$ with probability $1 - F(p)$ and a prize of zero with probability $F(p)$. Note that the prize zero corresponds to an unsuccessful bid. The utility of a prize depends on its size and on the seller's initial wealth w . In particular, the utility of the prize zero is $u(w)$. The seller, being a von Neumann-Morgenstern expected utility maximizer, chooses p so as to maximize his expected utility $E(p;c,w,n)$, where

(2.2) $$E(p;c,w,n) \equiv [1 - F(p)]u[n(p - c) + w] + F(p)u(w).$$

A rearrangement of terms reduces this to the more convenient form

(2.3) $$E(p;c,w,n) \equiv [1 - F(p)][u\{n(p - c) + w\} - u(w)] + u(w).$$

Any p which maximizes $E(p;c,w,n) \equiv E(p)$ for fixed values of the parameters c, w, and n will be called a solution of the model or an optimal bid.

Our immediate concern is whether the model has a solution, and if so, whether the solution is unique. The two theorems presented below give conditions under which there exists an optimal bid and conditions under which that optimal bid is unique. To state the theorems we need to define the number

(2.4) $$\lambda = \min\{p:F(p) = 1\} .$$

From the definition of λ it follows that if the seller submits a bid greater than or equal to λ then one of his opponents will get the contract, so $E(p) = u(w)$ for all $p \geq \lambda$. If, on the other hand, the seller's bid p is less than or equal to his unit cost c, then he has nothing to gain even if he wins the contract since $E(p) \leq u(w)$ for all $p \leq c$. Thus, for a bid p to be "reasonable," it must satisfy $c < p$ and $p < \lambda$. In order that a "reasonable" bid be available to the seller we require that $c < \lambda$.

We note from (2.3) that every bid in the interval (c, λ) gives a higher expected utility $E(p)$ than $u(w)$. We are now in a position to state our two theorems on the existence and uniqueness of an optimal bid.

<u>Theorem 1</u>[3]: (Existence Theorem) If

(A1) $c < \lambda < \infty$,

(A2) u is continuous and strictly increasing, and

(A3) F is continuous (with or without a density),

then there exists an optimal bid (not necessarily unique) in the interval (c, λ).

<u>Proof</u>: Under assumptions (A1), (A2), and (A3) we see that $E(p) = [1-F(p)][u\{n(p-c)+w\}-u(w)] + u(w)$ is a continuous function of p on the compact set $[c, \lambda]$, that $E(p) > u(w)$ for all p in (c, λ), and that $E(c) = E(p) = u(w)$. Thus there exists a number p_o (not necessarily unique) in $[c, \lambda]$ such that $E(p_o) = \max\{E(p) \mid c \leq p \leq \lambda\}$, and since $E(p) > u(w) = E(c) = E(\lambda)$ for all p in (c, λ), it follows that $p_o \neq c$ and $p_o \neq \lambda$. ∎

---

(3) The assumption $\lambda < \infty$ is not necessary in this work. In this proof we use $[1-F(p)]u\{n(p-c)+w\} \to 0$ as $p \to \lambda$. This is true if u is bounded, or if $\lambda < \infty$, or if $\int p dF(p)$ is finite and u is concave. We use $\lambda < \infty$ not only because the assumption simplifies the proof of this theorem, but also because it is difficult (if not impossible) to conceive of a situation in which $\lambda = \infty$.

Having given conditions under which there exists an optimal bid, we now show that under suitable assumptions this optimal bid is unique. Note that in the following theorem assumptions (A2a) and (A3a) imply (A2) and (A3) of Theorem 1 respectively.

Theorem 2: (Uniqueness theorem) Suppose:

(A1) $c < \lambda < \infty$;

(A2a) u is continuous, strictly increasing, and concave;

(A3a) F is absolutely continuous with density f, and the hazard rate function $f(p)/[1-F(p)]$ is a nondecreasing function of p.

Then there is a unique optimal bid p_o in the interval (c, λ), and for p in (c, λ), the expression

$$\frac{nu'[n(p - c) + w]}{u[n(p - c) + w] - u(w)} - \frac{f(p)}{1-f(p)} \tag{2.5}$$

is positive for $p < p_o$ and negative for $p > p_o$. In (2.5), the marginal utility function u' can be taken to be the right derivative of u, that is

$$u'(t) = \lim_{h \downarrow 0} \frac{u(t + h) - u(t)}{h} .$$

If $f(p)$ and $u'[n(p - c) + w]$ are continuous for p in (c, λ), then p_o is the unique zero in (c, λ) of expression (2.5).

Proof:

Because of the assumptions made about u and f,

$$E(p) = [1 - F(p)][u\{n(p - c) + w\} - u(w)] + u(w)$$

has a right derivative $D^+(E(p))$ almost everywhere and is the integral of this right derivative. (We can consider f to be the right derivative of F.) We have

$$D^+(E(p)) = -f(p)[u\{n(p-c)+w\} - u(w)] + [1-F(p)]nu'[n(p-c)+w]$$

$$\equiv (a) \cdot (b)$$

where

$$a \equiv [1-F(p)][u\{n(p-c) + w\} - u(w)]$$

and

$$b \equiv \frac{nu'[n(p-c) + w]}{u[n(p-c) + w] - u(w)} - \frac{f(p)}{1-F(p)}$$

and $u'(t) = D^+(u(t))$ is the marginal utility. From the definition of λ we see that $[1-F(p)] > 0$ for all $p < \lambda$, and since u is strictly increasing, $u\{n(p-c) + w\} - u(w) > 0$ for $p > c$. Thus the expression (a) is positive for $c < p < \lambda$.

We will show that there exists a unique p_o in the interval (c, λ) such that the expression (b) is positive for $p < p_o$ and negative for $p > p_o$. Then, for p in (c, λ), this would make $D^+(E(p))$ positive for $p < p_o$ and negative for $p > p_o$ so that $E(p)$ is strictly increasing for $p < p_o$ and strictly decreasing for $p > p_o$, and thus $E(p)$ has a unique maximum at $p = p_o$.

We note that:

(1) $u'[n(p-c) + w]$ is positive and non-increasing in p since u is concave and strictly increasing;

(2) $\lim_{p \to c} u[n(p-c) + w] - u(w) = 0$;

(3) $u[n(p-c) + w] - u(w)$ is strictly increasing in p;

so that

(4) $\dfrac{u'[n(p-c) + w]}{u[n(p-c) + w] - u(w)}$ is strictly decreasing in p for $c < p$;

and

(5) $$\lim_{p \downarrow c} \frac{u'[n(p-c) + w]}{u[n(p-c) + w] - u(w)} = \infty .$$

By assumption, $\frac{f(p)}{1-F(p)}$ is non-decreasing, so expression (b) is strictly decreasing in p. If $\lim_{p \downarrow c}(b) > 0$ and $\lim_{p \uparrow \lambda}(b) < 0$, then the desired p_o exists. But $\lim_{p \downarrow c} \frac{f(p)}{1-F(p)}$ is non-negative and finite, so it follows from (5) that $\lim_{p \downarrow c}(b) = +\infty > 0$. "Obviously" $\lim_{p \uparrow \lambda} \frac{f(p)}{1-F(p)} > 0$.

Since $\lambda < \infty$,

$$0 = \frac{1-F(\lambda)}{1-F(0)} = e^{\log (1-F(\lambda)) - \log (1-F(0))}$$

$$= \exp \int_0^\lambda \frac{d}{dt} \log (1-F(t))dt$$

$$= \exp \left[-\int_0^\lambda \frac{f(t)}{1-F(t)} \, dt \right]$$

so that $\int_0^\lambda \frac{f(t)}{1-F(t)} dt = \infty$. Since $\lambda < \infty$ and $\frac{f(t)}{1-F(t)}$ is non-decreasing we must have $\lim_{t \uparrow \lambda} \frac{f(t)}{1-F(t)} = \infty$. Thus we see that $\lim_{p \uparrow \lambda}(b) = -\infty < 0$. Hence conditions for the desired p_o to exist are satisfied.

If $u'(n(p-c) + w)$ and $f(p)$ are continuous for p in (c, λ), then expression (b) is continuous for p in (c, λ) and must assume the value zero somewhere in the interval by the mean value theorem. Since (b) is either positive or negative for each $p \neq p_o$ in (c, λ), it follows that p_o is the unique zero of expression (b) in (c, λ). ▯

In the remainder of this section we demonstrate that the expression (2.5) determining the optimal bid has a meaningful economic interpretation. To show this we define

$H(p) \equiv \frac{f(p)}{1-F(p)}$ and $G(p) \equiv \frac{nu'[n(p-c) + w]}{u[n(p-c) + w] - u(w)}$. $G(p)$ can be thought of as the rate of proportionate change in utility of profits as a function of the bid price p. Note that because of the assumptions about u, $G(p)$ is a strictly decreasing function of p. The function H plays an important role in many disciplines, particularly actuarial science and the mathematical theory of reliability, and is usually called the "hazard rate" or the "failure rate." In the context of our model, $H(p)dp$ approximately represents the probability that a bid of size $p + dp$ would be unsuccessful given that a bid of size p would have been successful. Thus $H(p)$ is the rate of proportionate increase in the probability of <u>losing</u> the contract as a function of p. It seems natural to assume that $H(p)$ is a non-decreasing function of p. This is equivalent to assuming that the <u>conditional probability that the minimum of the opponents' bids is at least $p + dp$ given that it is at least p</u> is a non-increasing function of p. (Intuitively, one might think of this, when applied to an individual, as saying that if a person is contemplating making a bid of p, then he is more likely to raise it an amount dp if p is a low bid than if p is a high bid.)

The above definitions enable us to rewrite expression (2.5) as $G(p) - H(p)$. From Theorem 2 we know there exists a unique optimal bid p_o in the interval (c, λ) such that $G(p) \gtrless H(p)$ when $p \lessgtr p_o$. Thus for bids less (greater) than p_o the rate of proportionate increase in the utility of profits exceeds (falls short of) the rate of proportionate increase in the probability of losing the contract, and expected utility can be increased by raising (lowering) the bid. Theorem 2 further states that if the marginal utility $u'[n(p-c) + w]$ and the probability density $f(p)$ are continuous functions of p in the interval (c, λ), then the optimal bid p_o is the unique solution of the equation $G(p) = H(p)$. That is, $G(p_o) = H(p_o)$ and $G(p) \gtrless H(p)$ when $p \lessgtr p_o$. Thus when marginal utility and the probability density are continuous functions of p, expected utility is maximized and the optimal bid is determined by equating the rate of proportionate increase in the utility of profits to the rate of proportionate increase in the probability of losing the contract. This is an intuitively meaningful result and is not immediately obvious from an examination of the structure of the model. Figure 1 presents a graphic illustration of the solution of the equation $G(p) = H(p)$. Figure 2 indicates what can happen when the hazard rate function is discontinuous. The same sort of thing can happen when $G(p)$ instead of $H(p)$ is discontinuous. For these cases p_o is the unique value of p for which the expression $G(p) - H(p)$ changes sign.

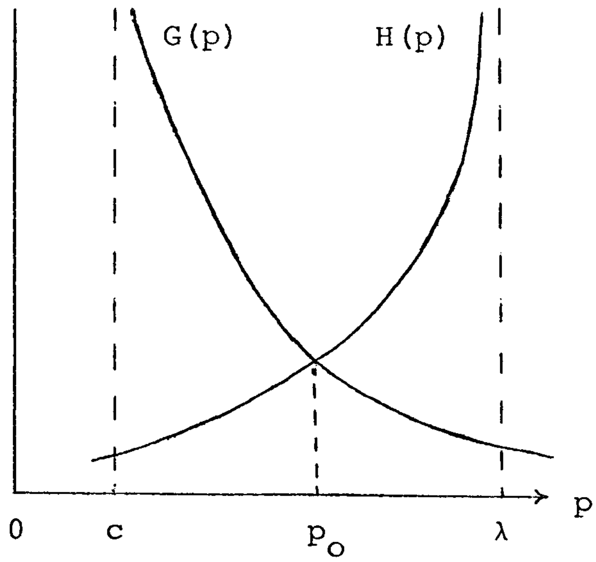

Figure 1

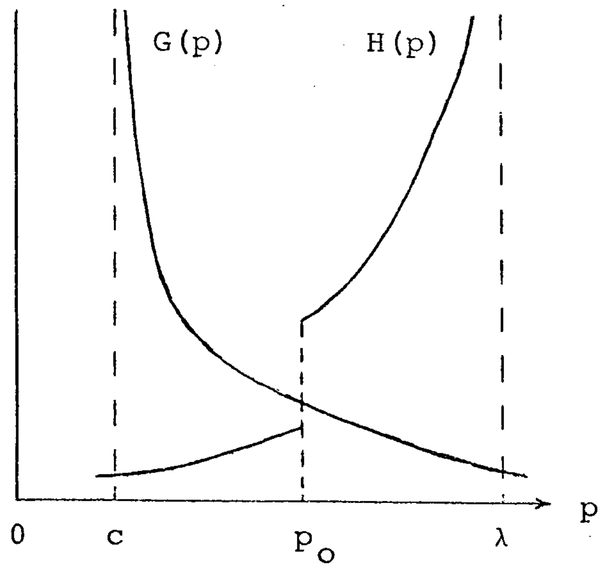

Figure 2

3. Risk Aversion and its Measurement

Let u be a utility function for wealth with marginal utility strictly positive. The purpose of this section is to demonstrate that the functions $A(t) = -u''(t)/u'(t)$ and $P(t; w) = -tu''(t+w)/u'(t+w)$ for each fixed w can be interpreted as two measures of risk aversion. We set forth the economic meanings of A and P here because, as will be shown later, some important comparative statics properties of the bidding model can be determined from the behavior of these two functions.

We begin by defining risk aversion. An individual is a risk averter if for any arbitrary risk he prefers the non-random amount equal to the actuarial value of the risk to the risk itself. Let w be his initial wealth and z, a random variable, be his income. He is risk averse if

$$(3.1) \quad u[w + E(z)] > E[u(w+z)]$$

where E is the expectation operator. A necessary and sufficient condition for (3.1) to hold for all values of w and all risks z is that the utility of wealth function u be strictly concave, or equivalently that it be the integral of a strictly decreasing marginal utility of wealth function u'. (u' may be assumed to be either the right or the left derivative of u if such is convenient.) This guarantees

(3.2) $u''(t) \leq 0$ for all $t \geq 0$

and a little more.

While (3.2) indicates the existence of (a weak form of) risk aversion, the magnitude of $u''(t)$ has in itself no meaning. The reason is that if u is a von Neumann-Morgenstern utility function, then the preference ordering represented by $E(u)$ does not change when the utility function u is replaced by the utility function $cu + b$ if c is positive. However, such transformations change the magnitude of $u''(t)$, although they do not alter its sign. Thus the sign but not the magnitude of $u''(t)$ is significant.

The foregoing suggests that a measure of risk aversion should in some sense measure the concavity of u and should remain invariant under positive linear transformations of the utility function. The functions $A(t) = -u''(t)/u'(t)$ and $P(t; w) = -tu''(t+w)/u'(t+w)$ fulfill both requirements and hence qualify as measures of risk aversion. We will show that these measures have straightforward behavioral interpretations.

A as a Measure of Risk Aversion

A is called absolute risk aversion. Its role as a measure of risk aversion was discovered independently by Kenneth J. Arrow [1], [2], and by John W. Pratt [6]. Pratt interprets A in terms of the risk premium π defined by the equation

$$u[w + E(z) - \pi] = E[u(w+z)] .$$

π can be regarded as the maximum amount, beyond the negative of the expected value of the risk itself, which an individual with wealth equal to w would pay to insure against the risk z. Pratt [5, page 125] indicates that under suitable regularity conditions

$$(3.3) \quad \pi = (\sigma^2/2)A(w + E(z)) + o(\sigma^2)$$

where σ^2 is the variance of z. (We use $o(t)$ to denote any function which is of smaller order of magnitude than t near o. In particular, $o(\sigma^2)/\sigma^2 \to 0$ as $\sigma^2 \to 0$.) Thus, when σ^2 is small, $\pi \simeq (\sigma^2/2)A(w + E(z))$. It follows that $A(w)$ is about twice the risk premium per unit of variance for "small" actuarially neutral $(E(z) = 0)$ risks. Note that in view of (3.2), the risk premium is non-negative.

Still another interpretation of A has been provided by Arrow [2, pages 33 and 34.] He considers a risk which involves winning or losing an amount h with probabilities p and $1 - p$, respectively. Given the amount of the bet h and the initial wealth w, consider the probability p' such that the individual is just indifferent between accepting and rejecting the bet. The value of p' is determined from the equation

$$u(w) = p'u(w+h) + (1-p')u(w-h)$$

using finite Taylor's series expansions of $u(w+h)$ and $u(w-h)$ about w. Under suitable regularity conditions on u

$$p' = \frac{1}{2} + \frac{h}{4}A(w) + o(h).$$

Thus for sufficiently small values of h,

(3.4) $p' \simeq \frac{1}{2} + \frac{h}{4}A(w)$.

In view of (3.2), $p' \geq \frac{1}{2}$. It follows that absolute risk aversion measures the individual's demand for more-than-fair odds.

$A(w)$ may increase, decrease, or remain constant with increasing wealth. A may be non-monotone for some utility functions and may be bounded or unbounded. Decreasing (increasing) absolute risk aversion means that the individual will pay less (more) for insurance against a given risk as his wealth increases; alternatively, that the size of favorable odds required to stake a given amount diminishes (increases) with increasing wealth.

R as a Measure of Risk Aversion

$P(t; w) = -tu''(t+w)/u'(t+w)$ has so far as we know not appeared in the literature prior to this. However, it appears to be a variant of the measure $R(t) = -tu''(t)/u'(t)$, which is called relative risk aversion by Arrow and proportional risk aversion by Pratt. The comparative statics of the bidding model do not depend on the behavior of R, but since Arrow and Pratt have provided an interpretation for R, we look at this measure in order to obtain a clue as to how to interpret P.

The interpretation of R follows quite easily from that of A. Suppose the risk premium and the risk itself are measured not in absolute terms but as proportions of initial wealth. Let $\pi^o = \pi/w$ and $z^o = z/w$ denote the proportional risk premium and the proportional risk, respectively. Then, as Pratt shows, if z^o is actuarially neutral (i.e., if $E(z^o) = 0$),

$$\pi^o = (\sigma^2/2)R(w) + o(\sigma^2),$$

where σ^2 is now the variance of z^o. A similar interpretation is provided by Arrow. Let $h = h^o w$, so that h^o is the fraction of wealth at stake. Then, Arrow shows that

$$p' = \frac{1}{2} + \frac{h^o}{4} R(w) + o(h^o).$$

Relative risk aversion may increase, decrease, or remain constant with increasing wealth. Increasing (decreasing) relative risk aversion means that the proportion of wealth spent for insurance increases (decreases) when wealth and risk are increased in the same proportion; alternatively, that the size of favorable odds demanded increases (decreases) when wealth and bet size are increased in the same proportion.

<u>P as a Measure of Risk Aversion</u>

We are now in a position to interpret the function P as a measure of risk aversion. Suppose the individual's wealth w is increased by an arbitrary amount t. Now measure the risk premium and the risk itself as proportions of t. Let $\bar{\pi} = \pi/t$ and $\bar{z} = z/t$ denote the risk premium and the risk respectively, each measured as a proportion of the increase in wealth. Under suitable regularity conditions it can be shown that

$$(3.5) \qquad \bar{\pi} = \frac{\sigma^2}{2(1+E(\bar{z}))} P[t(1+E(\bar{z}));w] + o(\sigma^2)$$

where σ^2 is the variance of $\bar{z}$. If $E(\bar{z}) = 0$ then $\bar{\pi} \simeq (\sigma^2/2)P(t;w)$.

The measure P can also be interpreted in terms of the more-than-fair odds concept. Let $h = \bar{h}t$, so that $\bar{h}$ is the fraction of <u>additional</u> wealth that is at stake Then it is easy to show that

$$(3.6) \qquad p' = \frac{1}{2} + \frac{\bar{h}}{4} P(t;w) + o(\bar{h}).$$

At a formal level the measures R and P appear to be quite similar. However, they are associated with two different types of betting situation. Relative risk aversion is relevant when the ratio of the bet size to wealth is being considered. The function P is important when the ratio of the bet size to additional wealth is under consideration. Note that if the ratio of the bet size to wealth remains constant then the ratio of the bet size to additional wealth decreases as wealth increases. Conversely, if the ratio of the bet size to additional wealth is kept constant then the ratio of the bet size to wealth must increase as wealth increases.

The following propositions and discussion are intended to provide some insight into the behavior of P. For the remainder of this section we assume that u is non-decreasing, that u is concave (but not necessarily strictly concave), that u has a continuous first derivative u', and that u' is the integral of some function u'' (possibly the regular derivative, the right derivative, or the left derivative of u').

<u>Proposition 1</u>: Fix w. If $P(t;w)$ is non-increasing in t for t in some interval $(0,t_o)$ with $t_o > 0$, then either $P(t;w) = 0$ (and consequently $u''(t+w) = 0$ for $0 < t < t_o$ or else $w = 0$.

<u>Proof</u>: $P(t;w)$ is non-negative. Assume it is non-increasing and not identically zero for $0 < t < t_o$. Then $\lim_{t \downarrow 0} P(t;w) > 0$. Now u' is non-increasing and non-negative and can't be identically zero on $(0, t_o)$ if $P(t;w)$ is to make sense. Thus we find $a > 0$, $b > 0$ such that for $0 < t \leq b$ we have $P(t,w) > a$ and $u'(t+w) > a$. Then for $0 < t \leq b$

$$u''(t+w) < -\frac{a}{t}u'(t+w) < -\frac{a^2}{t}$$

and integrating gives

$$u'(w+b) - u'(w) < \int_0^b (a^2/t)\,dt = -\infty$$

so that $u'(w) = +\infty$. Because u' is non-increasing this can happen only when $w = 0$ and then only for some utility functions. ∎

Proposition 2: Fix $w > 0$ and suppose $t_o > 0$. If $P(t;\, w)$ is monotone (strictly monotone) in t for $0 < t < t_o$, then it is non-decreasing (strictly increasing) there.

Proof: Suppose $P(t;\, w)$ is non-increasing for $0 < t < t_o$. Then by Proposition 1 we have $P(t;\, w) = 0$ for $0 < t < t_o$. Thus $P(t;\, w)$ can't be strictly decreasing for $0 < t < t_o$, and if it is non-increasing it is in fact also non-decreasing since it is a constant. ∎

These two propositions indicate that if $w > 0$ and we for some reason believe $P(w;\, t)$ to be monotone in t, then we must believe either that $P(w;\, t)$ is strictly increasing in t or that $u(t)$ is linear. If we require strict concavity of u, then we can rule out the latter. Unfortunately, fluctuations are possible. It is possible to construct a bounded or unbounded utility function with a continuous second derivative for which P is not monotone or for which R is not monotone. It would thus seem that any assumptions about the monotonicity of P must be made on the basis of either intuitive of empirical considerations.

We conclude this section with the following observation:

<u>Proposition 3</u>: If either $A(t)$ or $R(t)$ is non-decreasing then either $u''(t) \equiv 0$ (so that u is linear), or else $P(t;\ w)$ is a strictly increasing function of t for each fixed w.

4. Comparative Statics of the Bidding Model

Recall that in Section 2 an optimal bid price was defined as any value of p which maximizes expected utility $E(p;c,w,n)$ for given values of the parameters c, w, and n. It was demonstrated that under economically meaningful conditions there exists a unique optimal bid price p_o. The purpose of this section is the investigation of the change in the optimal bid price p_o caused by independent variations in the three parameters c, w, and n. Both the direction of change of p_o and bounds on its magnitude are of interest.

Throughout this section we will assume, unless specifically stated otherwise, that changes in the parameters c, w, and n will not cause the seller to revise his estimate of the bidding behavior of his competitors. Thus the probability distribution $F(p)$, and hence the hazard rate function $H(p)$, are assumed fixed under changes in c, w, and n. This is, of course, a reasonable assumption for changes in w, and also for changes in c that are internal to the firm. The assumption might be somewhat unrealistic for a change in n since such a change directly affects all firms competing for the contract. Later in this section we will comment on how our analysis must be extended in order to take into account revisions in the seller's estimate of the bidding behavior of his competitors.

The Effect on the Optimal Bid Price of a Change in Average Cost

The results of this subsection are summarized in the following theorem.

Theorem 3: Suppose conditions (A1), (A2a), and (A3a) are satisfied and that u' and f are the right derivatives of u and F respectively. If the average cost c is raised (lowered) by an amount Δc, then the new optimal bid price p_1 satisfies the inequalities

$$(4.1) \qquad p_o \leq p_1 \leq p_o + \Delta c \qquad (p_o \geq p_1 \geq p_o - \Delta c)$$

where p_o is the original optimal bid. If u' and f are continuous, then

$$(4.2) \qquad p_o < p_1 \qquad (p_o > p_1).$$

If, in addition, the hazard rate function is strictly increasing then

$$(4.3) \qquad p_1 < p_o + \Delta c \qquad (p_1 > p_o - \Delta c).$$

Proof:

Note that the basic assumptions used in this theorem are the same ones that were used in Theorem 2. We are thus guaranteed the existence and uniqueness of optimal bids for the various values of c under consideration provided only that they are all less than λ (see (2.4)).

Earlier we defined and interpreted the functions

$G(p) = \frac{nu'[n(p-c) + w]}{u[n(p-c) + w] - u(w)}$ and $H(p) = \frac{f(p)}{1-F(p)}$. From Theorems 1 and 2 we know that, for fixed c, w, and n, the expected utility $E(p; c, w, n)$ has its unique maximum at p_o, the point where the expression $G(p) - H(p)$ changes sign. Equivalently, p_o is the "crossover point" of the graphs of $G(p)$ and $H(p)$ (see Figures 1 and 2). One might therefore expect to obtain some information about the direction of change in p_o from the shifts in the graphs of $G(p)$ and $H(p)$ due to a change in the parameter c.

By assumption the graph of $H(p)$ does not change when c changes. However, for fixed p and $c < p$, we see that $u'[n(p-c) + w]$ is non-decreasing in c and $u[n(p-c) + w]$ is strictly decreasing in c, so that $G(p) = \frac{nu'[n(p-c)+w]}{u[n(p-c)+w]-u(w)}$ is a strictly increasing function of c. This argument shows that the whole graph of $G(p)$ is raised when c

increases and lowered when c decreases. (Refer back to Figures 1 and 2 to visualize this.) Thus p_o is a non-decreasing function of c. p_o is a strictly increasing function of c if both f and u' are continuous so that G and H are continuous. The various cases provide the proper inequalities between p_o and p_1.

Now let $t = p - c$ so that $G(t+c) = \frac{nu'(nt+w)}{u(nt+w) - u(w)}$ and $H(t+c) = \frac{f(t+c)}{1-F(t+c)}$. For fixed c, the "crossover point" t_o of the graphs of $G(t+c)$ and $H(t+c)$ is just $t_o = p_o - c$. The function $G(t+c)$ (and therefore its graph) does not depend on c, while $H(t+c)$ is non-decreasing in c for each fixed t. It follows (see Figure 3) that t_o is a non-increasing function of c. (t_o is a strictly decreasing function of c if H is strictly increasing and if both u' and f are continuous-- so that G and H are continuous.) In particular, if t_1 and p_1 are the "crossover points" corresponding to an average cost of $c + \Delta c$, then $p_1 - (c + \Delta c) = t_1 \leq t_o = p_o - c$ if $\Delta c > 0$, or equivalently, $p_1 \leq p_o + \Delta c$. (We get strict inequality if t_o is a strictly decreasing function of c.) The other inequalities follow by considering a decrease in c by an amount Δc. ∎

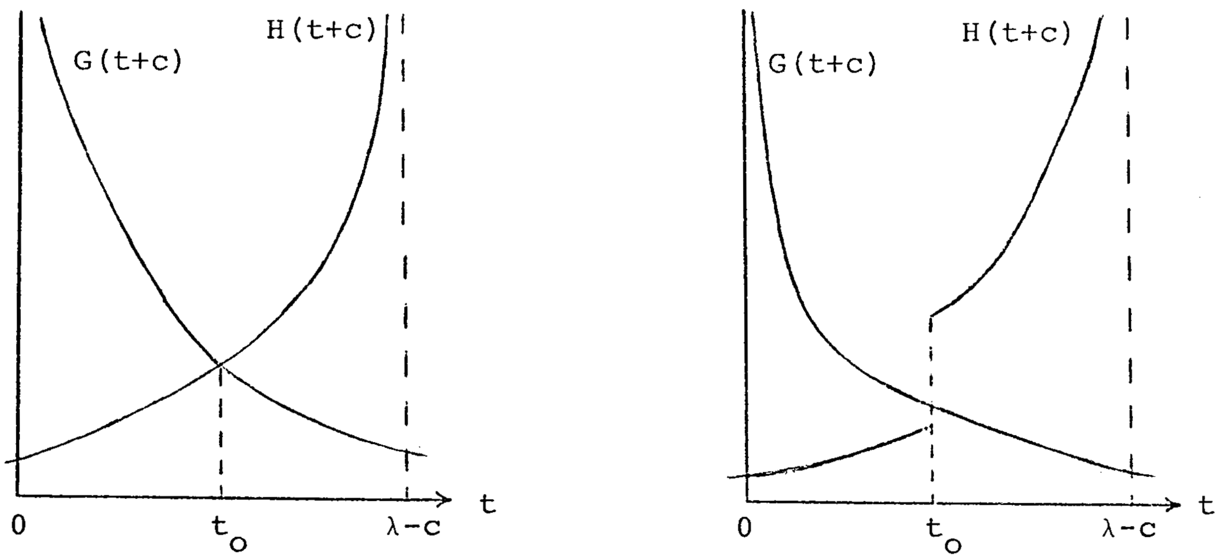

Figure 3

The conclusions of this theorem are intuitively quite appealing. Suppose average cost is increased by an amount Δc. If the bid price is left unchanged, potential average profit will decline from $(p_o - c)$ to $(p_o - c - \Delta c)$. Note that the probability of getting this smaller profit is still $1 - F(p_o)$. A smaller profit is now associated with the former probability of success, and the theorem tells us that the bidder "trades off" some of his probability of success for an increase in his potential profit. He therefore raises his bid price. However, since the bidder is risk averse, his willingness to "trade off" probability of success for potential profit declines as potential profit increases and probability of success decreases. In particular, the theorem tells us that he is totally unwilling to continue this type of "trade off" once he has obtained his old level of potential average profit.

The Effect on the Optimal Bid Price of a Change in Initial Wealth

The result of this subsection depends on an assumed monotonicity of the absolute risk aversion function A. Our result gives the direction of change in the bid price due to a change in initial wealth, but gives no bound on this change.

Theorem 4: Suppose conditions (A1), (A3a),

(A2b) u is strictly increasing, concave, and continuously differentiable,

(A4) u' is right differentiable and is the integral of its right derivative u'',

are satisfied, and that c and n are fixed. If the bidder's initial wealth is raised (lowered), and if the function A is non-increasing, then the new optimal bid price p_1 satisfies the inequality

(4.4) $p_o \leq p_1$ $\qquad\qquad$ $(p_o \geq p_1)$.

If, in addition, f is continuous and A is strictly decreasing, then the inequalities above are strict. If A is non-decreasing (strictly increasing) instead of non-increasing (strictly decreasing), then the inequalities are reversed.

Proof:

We begin by finding an equivalent expression for $G(p)$. Setting $\theta = n(p - c)$, we have

$$G(p) = \frac{1}{p - c} \cdot \frac{\theta u'(\theta+w)}{u(\theta+w) - u(w)} = \frac{1}{p - c} \cdot \frac{\theta u'(\theta+w)}{\int_0^\theta u'(t+w)\, dt}$$

$$= \{(p-c) \int_0^1 [u'(\theta\tau+w)/u'(\theta+w)]\, d\tau\}^{-1}$$

where we have made the change of variables $t = \theta\tau$. Using the fact that $a/b = \exp(\log a - \log b)$ when a and b

are positive, we obtain

$$G(p) = [(p-c)\int_0^1 \exp(\log u'(\theta\tau+w) - \log u'(\theta+w))d\tau]^{-1}$$

$$= [(p-c)\int_0^1 \exp(\int_\theta^{\theta\tau} \frac{d}{dx}\log u'(x+w)dx)d\tau]^{-1}$$

$$= [(p-c)\int_0^1 \exp(\int_\theta^{\theta\tau} -A(x+w)dx)d\tau]^{-1}$$

$$= [(p-c)\int_0^1 \exp(\theta\int_\tau^1 A(\theta y+w)dy)d\tau]^{-1}$$

where we have made the change of variables $x = \theta y$, and have noted that the reversal of the limits of integration changes the sign of the integral. Our final expression for $G(p)$ is

$$(4.5) \qquad G(p) = [(p-c)\int_0^1 \exp[n(p-c)\int_\tau^1 A(ny(p-c)+w)dy]d\tau]^{-1}.$$

By hypothesis, H is not affected by changes in w. However, from (4.5) we see that if A is non-increasing (strictly decreasing, non-decreasing, strictly increasing), then $G(p)$ is a non-decreasing (strictly increasing, non-increasing, strictly decreasing) function of w for each fixed p.

The remainder of the proof uses the arguments of Theorem 3 and will be omitted. ∎

Theorem 4 indicates the relationship between the behavior of the absolute risk aversion function A and the direction of change in the bid price due to a change in initial wealth. It might be helpful to look at this result in intuitive terms. Suppose that A decreases with wealth, i.e., an individual's willingness to engage in a bet, as measured either by the risk premium or by the favorable odds demanded, increases with wealth. This means that as his wealth increases, the bidder raises his bid price (consequently reducing $1 - F(p)$, his probability of getting the contract) in order to increase potential (and, incidentally, also expected) profit. The opposite type of argument can be used to explain the claim of Theorem 4 that the bid price decreases if A increases with wealth.

Whether A increases or decreases with wealth would seem to be an empirical rather than a theoretical issue. On the basis of intuitive evidence and casual observation we are inclined to accept Arrow's [2, page 35] hypothesis that absolute risk aversion decreases with wealth.

The Effect on the Optimal Bid Price of a Change in Contract Size

The first theorem of this subsection gives the direction of change of the optimal bid price. It depends on an assumed monotonicity of the function $P(t; w)$ introduced at the beginning of section 3. (Recall that proposition 2 of that section states that if $P(t; w)$ is monotone in t, then it must be non-decreasing.) Our other theorem provides a bound on this change.

Theorem 5: Suppose conditions (A1), (A2b), (A3a), and (A4) are satisfied and that c and w are fixed. If the contract size is increased (decreased), and if the function $P(t; w)$ is non-decreasing in t, then the new optimal bid price p_1 satisfies the inequality

(4.6) $\quad p_o \geq p_1 \qquad\qquad (p_o \leq p_1).$

If, in addition, f is continuous and $P(t; w)$ is a strictly increasing function of t, then the inequalities above are strict.

Proof:

From (4.5) and the definitions of A and P we see that

(4.7) $$G(p) = [(p-c)\int_0^1 \exp[\int_\tau^1 (1/y)P(n(p-c)y;w)dy]d\tau]^{-1}.$$

The proof of this theorem is essentially a repetition of the proof of theorem 4 using (4.7), P, and n instead of (4.5), A, and w. We omit it. □

Under the assumption that $P(t; w)$ is a non-decreasing function of t, one's intuition agrees with the theorem's description of the bidder's behavior. If the contract size is increased from n to $n + \Delta n$, then both the bidder's potential profit and his expected profit are multiplied by a factor of $(n + \Delta n)/n$ if he maintains his old bid price p_o (and if his opponents' bidding behavior is unchanged). It is intuitively reasonable that he should "trade off" some of his additional potential profit (and expected profit) for some extra probability of success (extra safety). Thus he should lower his bid price.

Note that Theorem 5 does not indicate how much of his additional potential profit the bidder will "trade off" for an increase in his probability of success. The following theorem bounds this "trade off."

Theorem 6: Suppose conditions (A1), (A2a), and (A3a) are satisfied, that c and w are fixed, and that u' and f are the right derivatives of u and F respectively. Then potential profit is a non-decreasing function of n. In particular, if the contract size n is raised (lowered) by an amount Δn, then the new optimal bid price p_1 satisfies the inequality

$$(4.8) \quad p_1 \geq p_o - \frac{\Delta n}{n+\Delta n}(p_o - c) \qquad \left(p_1 \leq p_o + \frac{\Delta n}{n+\Delta n}(p_o - c)\right).$$

If, in addition, u' and f are continuous, then the inequalities above are strict.

Proof:

The proof of this theorem is quite similar to the proof of the second part of Theorem 3. We let $s = n(p-c)$ so that

$$G(\frac{s}{n} + c) = \frac{nu'(s+w)}{u(s+w)-u(w)} \quad \text{and} \quad H(\frac{s}{n} + c) = \frac{f(\frac{s}{n} + c)}{1-F(\frac{s}{n} + c)} \quad .$$

For fixed c and n, the "crossover point" s_o of the graphs of $G(\frac{s}{n} + c)$ and $H(\frac{s}{n} + c)$ as functions of s is just $n(p_o - c)$. For each fixed s we see that $G(\frac{s}{n} + c)$ is a strictly increasing function of n, and $H(\frac{s}{n} + c)$ is a non-increasing function of n since $H(t)$ is non-increasing in t. Thus s_o is a non-decreasing function of n (and is strictly increasing if both u' and f are continuous). The inequalities are an immediate consequence of this. □

Again, the results of this theorem are quite reasonable. One would expect at least as large a total potential profit on a large order as on a small order even if the profit per unit were smaller.

Theorems 5 and 6 together indicate that if the contract size n is raised by an amount Δn, then the new optimal bid price p_1 satisfies the inequalities

(4.9) $$p_o - \frac{\Delta n}{n+\Delta n}(p_o - c) \le p_1 \le p_o.$$

The inequalities are reversed if the contract size is lowered by an amount Δn.

The preceding analysis has been based on the assumption that average cost is constant. We now consider briefly how p_o varies with n when average cost c depends on the level of output. The analysis for decreasing average cost is straightforward. Suppose an increase in contract size from n to $n + \Delta n$ decreases average cost from c to $c - \Delta c$. The effect of such a change on the optimal bid price can be decomposed into two parts. First, we have seen that an increase in contract size of Δn with cost remaining constant at c will reduce the bid price. Similarly, a net decrease in average cost of Δc with contract size constant at $n + \Delta n$ will decrease the bid price even further.

These two effects together lower the bid price more than either one does by itself. For cost as an increasing function of contract size, the two effects work in opposite directions. Whether the optimal bid price will increase or decrease depends on the size of the increases in contract size and cost, and on the specific utility function of the bidder.

Remarks about the Effect on the Optimal Bid Price of a Change in the Probability Distribution F.

Changes in F represent revisions in the bidder's beliefs about the bidding behavior of his competitors. Such changes can be dealt with if they can be expressed as appropriate changes in the hazard rate function H. Suppose, for example, that the bidder believes that the minimum of his competitors' bid prices is increased by the amount Δc (due, perhaps, to an increase of Δc in the costs of each of his competitors). One way of expressing this revision in his beliefs is by setting $F^*(p) = F(p - \Delta c)$, where F^* and F are the distribution functions expressing his new and old beliefs about the minimum of his competitors' bid prices. If H^* and H are the corresponding hazard rate functions, then $H^*(p) = H(p - \Delta c)$, and if H is non-decreasing, then using H^* instead of H amounts to lowering the whole H curve (or at least to not raising it anywhere). Thus, the intersection of the curves of H^* and G will be to the right of the intersection of the curves of H and G and will therefore result in a higher bid price. Notice that this type of argument can still be used even if the relation between F^* and F is not clearly defined so long as the bidder is willing to assume that H^* is non-decreasing and that $H^*(p) \leq H(p)$ for all p. In many cases this would be a reasonable assumption. It merely amounts to assuming that if the minimum of his competitors' bid prices is at least p, then it is at least as likely to be close to p under the old distribution F as under the new distribution F^*.

5. Possible Extensions of the Model

The model developed in this paper is formulated for competitive sealed tender selling markets. With minor modifications, the model is applicable to individual bidding behavior in sealed tender buying markets and Dutch auctions.

A promising application of the model is in the study of investment decisions associated with the submission of proposals for the acquisition of Research and Development (R & D) contracts. We briefly describe the mechanics of one kind of R & D contract market.[4] The market consists of a group of firms competing for a single contract to produce a specified

---

(4) We are indebted to Walter L. Johnson for information about the institutional features of this market.

quantity of a new product. The product is defined in terms of certain "standards of performance" and cannot be produced with existing technology. Each participating firm is required to submit a proposal on or before a given future date. The proposal consists of (1) a detailed statement of the production process the firm will use if awarded the contract and (2) a bid price. The contract is awarded to the lowest bidder from among the proposals that meet the required standards of performance.

Initially each firm must decide whether to begin the R & D work necessary for submission of a proposal. R & D costs, production costs, and the payoff are all unknown at this time. A first step toward a model explaining R & D proposal submission decisions might be the extension of the model developed in this paper to the case where production costs c are assumed to be random. Such a model would, of course, be useful in itself.

University of Missouri, Columbia

References

[1] Arrow, K. J.: "'Comment' on James S. Dusenberry, 'The Portfolio Approach to the Demand for Money and Other Assets,'" Review of Economics and Statistics, XLV, (Supplement. February 1963).

[2] ————: Aspects of the Theory of Risk Bearing, The Academic Book Store, Helsinki, Finland. 1965.

[3] Barlow, R. E., and Proschan, F.: Mathematical Theory of Reliability, John Wiley and Sons, Inc., New York, 1965.

[4] Greismer, J. H., Levitan, R. E., and Shubik, M.: "Toward a Study of Bidding Processes, Part IV: Unknown Competitive Costs," IBM Research Paper, 1966.

[5] Hernstein, I. N., and Milnor, J.: "An Axiomatic Approach to Measurable Utility," Econometrica, 21, April 1953.

[6] Pratt, J. W.: "Risk Aversion in the Small and in the Large," Econometrica, 32, 1964.

[7] Smith, V. L.: "Bidding Theory and the Treasury Bill Auction: Does Price Discrimination Increase Bill Prices?", Review of Economics and Statistics, XLVIII, No. 2, 1966.