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1. Introduction

The original formulations of the linear theory of thin shallow shells due to Marguerre[1], Vlasov[2] and Reissner[3] and subsequent treatments[4,5,6,7] have in common the two following assumptions:

(i) The tangential components of displacements may be neglected in computing the changes of curvature of the middle surface.

(ii) The transverse shear stress resultants may be neglected in the tangential equilibrium equations.

The assumption concerning the shallowness of the shell in references 1 and 3 is that:

(iii) The squares and the product of the slopes of the middle surface with regard to a reference plane are negligible with regard to unity.

In reference 2 the Gaussian curvature is neglected in Gauss' equation for the middle surface but the same final result could be obtained if assumption (ii) were used instead. In fact, in Novozhilov's treatment[6] no geometrical assumptions are made but it is assumed in addition to (i) and (ii), without being more specific however, that

(iiii) The dependent variables are rapidly varying functions.

The various derivations reduce the shallow shell problem to the solution of a system of two differential equations for the normal displacement w and the stress function ψ. In derivations not using assumption (iii) the differential operators of these equations are surface operators that reduce to the plane operators of the other derivations upon making use of the shallowness assumption.

The system of two equations for w and ψ is dual in the sense that, in the homogenous system, one equation is transformed into the other by means of the static geometric analogy[8,9].

It will be shown in this paper that by making use of the static geometric analogy Vlasov's equations of shallow shells may be established solely on assumption (i) applied in a dual form.

A study of the accuracy of the Marguerre-Vlasov equations when applied to shallow shells may be found in reference 10. Vlasov's equations, however, as mentioned earlier, apply under certain conditions to non shallow shells[6]. Their application to cylindrical shells through Donnell's equation is well known and its accuracy is studied in reference 11. The statement of Goldenweiser, however, that Vlasov's equations apply to arbitrary shells of zero Gaussian curvature is refuted by Novozhilov[6] by means of an example involving a long cylindrical shell.

On the basis of the derivation to be presented here a study of the nature of the error in Vlasov's shallow shell equations will be made and the order of magnitude of this error for non shallow shells will be established depending on the geometry of the shell and on the boundary conditions.

2. Basic Equations

In orthogonal curvilinear coordinates (ξ_1) and (ξ_2) the vector equilibrium equations take the form

$$(\alpha_2 \bar{N}_1)_{,1} + (\alpha_1 \bar{N}_2)_{,2} + \alpha_1 \alpha_2 \bar{p} = 0 \tag{1a}$$

$$(\alpha_2 \bar{M}_1)_{,1} + (\alpha_1 \bar{M}_2)_{,2} + \bar{r}_{,1} \times \alpha_2 \bar{N}_1 + \bar{r}_{,2} \times \alpha_1 \bar{N}_2 + \alpha_1 \alpha_2 \bar{m} = 0 \tag{1b}$$

where $\bar{N}_1$ and $\bar{N}_2$ are stress resultant vectors and $\bar{M}_1$ and $\bar{M}_2$ are stress couple vectors corresponding to the coordinates ξ_1 and ξ_2, respectively. $\overline{p}$ and $\overline{m}$ are force and moment load intensities per unit area of the middle surface. α_1^2 and α_2^2 are components of the surface metric tensor and are defined through the relation

$$\overline{dr}.\ \overline{dr} = \alpha_1^2\, d\xi_1^2 + \alpha_2^2\, d\xi_2^2 \tag{2}$$

A comma is used to indicate differentiation.

The homogenous system (1) is solved in terms of two vector stress functions $\overline{F}$ and $\overline{G}$ in the form

$$\alpha_2\, \overline{N}_1 = \overline{F},_2 \tag{3a}$$

$$\alpha_1\, \overline{N}_2 = -\ \overline{F},_1 \tag{3b}$$

$$\alpha_2\, \overline{M}_1 = \overline{G},_2 + \overline{r},_2 \times \overline{F} \tag{3c}$$

$$\alpha_1\, \overline{M}_2 = -\ \overline{G},_1 - \overline{r},_1 \times \overline{F} \tag{3d}$$

Strain displacement relations may be established in vector form by requiring the expression

$$\overline{N}_1 \cdot \overline{\varepsilon}_1 + \overline{N}_\varepsilon \cdot \overline{\varepsilon}_2 + \overline{M}_1 \cdot \overline{\chi}_1 + \overline{M}_2 \cdot \overline{\chi}_2 \tag{4}$$

to represent the virtual work per unit area of the middle surface of the external forces acting on an infinitesimal parallelepiped cut out of the shell[9]. This leads to the relations

$$\alpha_1\, \overline{\varepsilon}_1 = \overline{u},_1 + \overline{r},_1 \times \overline{\omega} \tag{5a}$$

$$\alpha_2\, \overline{\varepsilon}_2 = \overline{u},_2 + \overline{r},_2 \times \overline{\omega} \tag{5b}$$

$$\alpha_1\, \overline{\chi}_1 = \overline{\omega},_1 \tag{5c}$$

$$\alpha_2\, \overline{\chi}_2 = \overline{\omega},_2 \tag{5d}$$

where $\bar{u}$ is the displacement vector and $\bar{\omega}$ the rotation vector.

The analyogy between (3) and (5) is the basis of the static geometric analogy. The strain vectors satisfy compatibility equations dual of the homogenous equilibrium equations, i.e.,

$$(\alpha_2\, \bar{\chi}_2)_{,1} - (\alpha_1\, \bar{\chi}_1)_{,2} = 0 \qquad (6a)$$

$$(\alpha_2\, \bar{\varepsilon}_2)_{,1} - (\alpha_1\, \bar{\varepsilon}_1)_{,2} + \bar{r}_{,1} \times \alpha_2\, \bar{\chi}_2 - \bar{r}_{,2} \times \alpha_1\, \bar{\chi}_1 = 0 \qquad (6b)$$

The duality between the stress and strain vectors and displacement and stress function vectors is summarised in Table 1.

$\bar{N}_1$	$\bar{\chi}_2$
$\bar{N}_2$	$-\bar{\chi}_1$
$\bar{M}_1$	$\bar{\varepsilon}_2$
$\bar{M}_2$	$-\bar{\varepsilon}_1$
$\bar{F}$	$\bar{\omega}$
$\bar{G}$	$\bar{u}$

Table 1

Using as vector base the unit vectors

$$\bar{t}_1 = \frac{\bar{r}_{,1}}{\alpha_1} \qquad (7a)$$

$$\bar{t}_2 = \frac{\bar{r}_{,2}}{\alpha_2} \qquad (7b)$$

$$\bar{n} = \bar{t}_1 \times \bar{t}_2 \qquad (7c)$$

$\bar{N}_1$, $\bar{N}_2$, $\bar{F}$, $\bar{G}$, $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$, $\bar{u}$ and $\bar{\omega}$ are represented in the form

$$(\overline{\ \ }) = (\ \ )_1\, \bar{t}_1 + (\ \ )_2\, \bar{t}_2 + (\ \ )_3\, \bar{n} \tag{8a}$$

and $\bar{M}_1$, $\bar{M}_2$, $\bar{\chi}_1$, $\bar{\chi}_2$ in the form

$$(\overline{\ \ }) = -(\ \ )_2\, \bar{t}_1 + (\ \ )_1\, \bar{t}_2 + (\ \ )_3\, \bar{n} \tag{8b}$$

Scalar equations may be obtained by means of differentiation formulas for the unit vectors and may be found for example in reference 9.

For stress-strain relations a linearly elastic homogenous and isotropic material is considered. The complementary strain energy and the strain energy density functions are taken, respectively, in the forms[9]

$$\begin{aligned} W_N &= \frac{1}{2Eh}\,[N^2 - 2(1+\nu)(N_{11}\,N_{22} - N_{12}\,N_{21})] \\ &\quad + \frac{6}{Eh^3}\,[M^2 - 2(1+\nu)(M_{11}\,M_{22} - M_{12}\,M_{21})] \end{aligned} \tag{9a}$$

and

$$\begin{aligned} W_\varepsilon &= \frac{Eh}{2\,(1-\nu^2)}\,[\varepsilon^2 - 2(1-\nu)(\varepsilon_{11}\,\varepsilon_{22} - \varepsilon_{12}\,\varepsilon_{21})] \\ &\quad + \frac{Eh^3}{24(1-\nu^2)}\,[\chi^2 - 2(1-\nu)(\chi_{11}\,\chi_{22} - \chi_{12}\,\chi_{21})] \end{aligned} \tag{9b}$$

where N, M, ε and χ are defined through the notation below

$$(\ \ ) = (\ \ )_{11} + (\ \ )_{22} \tag{10}$$

The corresponding stress-strain relations are

$$\varepsilon_{ij} = \frac{\partial W_N}{\partial N_{ij}} \qquad \begin{matrix} i = 1,\ 2 \\ j = 1,\ 2,\ 3 \end{matrix} \tag{11a}$$

$$\chi_{ij} = \frac{\partial W_N}{\partial M_{ij}} \qquad \begin{matrix} i = 1,\ 2 \\ j = 1,\ 2 \end{matrix} \tag{11b}$$

or

$$N_{ij} = \frac{\partial W_\varepsilon}{\partial \varepsilon_{ij}} \qquad \begin{array}{l} i = 1, 2 \\ j = 1, 2 \end{array} \tag{11c}$$

$$M_{ij} = \frac{\partial W_\varepsilon}{\partial \chi_{ij}} \qquad \begin{array}{l} i = 1, 2 \\ j = 1, 2, 3 \end{array} \tag{11d}$$

Eqs. 11 imply a shell without transverse shear deformation and without couple stress-stress couples, i.e.,

$$\varepsilon_{13} = \varepsilon_{23} = 0 \tag{12a}$$

$$M_{13} = M_{23} = 0 \tag{12b}$$

The static geometric analogy is extended to the stress strain relations with the result that W_ε is dual of W_N through the correspondance of table 2 in which

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{13}$$

ν	$-\nu$
D	$-(Eh)^{-1}$
W_ε	$-W_N$

Table 2

3. Stress Compatibility and Strain Equilibrium Equations

Using the stress-strain relations to express appropriate parts of the tangential components of the vector compatibility equations in terms of the stress resultants and stress couples and taking account of the equilibrium equations, there results, in lines of curvature coordinates, the four equations below

$$Eh\, \alpha_2\, \chi_{13} + N_{,2} + (1 + \nu)\, \frac{\alpha_2\, N_{23}}{R_2} + (1 + \nu)\quad \alpha_2\, p_2 = 0 \qquad (14a)$$

$$-Eh\, \alpha_1\, \chi_{23} + N_{,1} + (1 + \nu)\, \frac{\alpha_1\, N_{13}}{R_1} + (1 + \nu)\quad \alpha_1\, p_1 = 0 \qquad (14b)$$

$$\frac{\alpha_2\, N_{23}}{D} - \chi_{,2} - (1 - \nu)\, \frac{\alpha_2\, \chi_{13}}{R_2} = 0 \qquad (14c)$$

$$\frac{\alpha_1\, N_{13}}{D} - \chi_{,1} - (1 - \nu)\, \frac{\alpha_1\, \chi_{23}}{R_1} = 0 \qquad (14d)$$

Eqs. (14) may also be obtained by first expressing the tangential equilibrium equations in terms of the strains then by taking account of the compatibility equation.

It is noted that if the static geometric analogy is applied to Eqs. (14a) and (14b) without the load terms there results Eqs. (14c) and (14d), respectively.

The normal components of the vector compatibility and equilibrium equations take the dual forms

$$M_{12} - M_{21} + \frac{h^2}{12}\left(\frac{N_{21}}{R_2} - \frac{N_{12}}{R_1}\right) = 0 \qquad (15a)$$

$$\varepsilon_{21} - \varepsilon_{12} - \frac{h^2}{12}\left(\frac{\chi_{12}}{R_2} - \frac{\chi_{21}}{R_1}\right) = 0 \qquad (15b)$$

$$(\alpha_2\, N_{13})_{,1} + (\alpha_1\, N_{23})_{,2} - \alpha_1\, \alpha_2\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - p_3\right) = 0 \qquad (15c)$$

$$(\alpha_2\, \chi_{23})_{,1} - (\alpha_1\, \chi_{13})_{,2} + \alpha_1\, \alpha_2\left(\frac{\chi_{22}}{R_1} + \frac{\chi_{11}}{R_2}\right) = 0 \qquad (15d)$$

Eqs. (15c) and 15d) are in their original form whereas Eqs. (15a) and

(15b) are obtained after using the stress-strain relations in the original compatibility and equilibrium equations, respectively.

Solving the 4 equations (14) for N_{13}, N_{23}, χ_{13} and χ_{23}, then letting

$$1 + \frac{h^2}{R_1^2} \approx 1 + \frac{h^2}{R_2^2} \approx 1 \tag{16}$$

and making use of the stress-strain relations

$$N = \frac{Eh}{1-\nu} \varepsilon \tag{17a}$$

and

$$\chi = \frac{M}{D(1+\nu)} \tag{17b}$$

obtain

$$\alpha_1 N_{13} = D(\chi_{,1} - \frac{\varepsilon_{,1}}{R_1}) - \frac{h^2}{12R_1} \alpha_1 p_1 \tag{18a}$$

$$\alpha_2 N_{23} = D(\chi_{,2} - \frac{\varepsilon_{,2}}{R_2}) - \frac{h^2}{12R_2} \alpha_2 p_2 \tag{18b}$$

$$\alpha_1 \chi_{23} = \frac{1}{Eh} (N_{,1} + \frac{M_{,1}}{R_1}) + \frac{1+\nu}{Eh} \alpha_1 p_1 \tag{18c}$$

$$\alpha_2 \chi_{13} = -\frac{1}{Eh} (N_{,2} + \frac{M_{,2}}{R_2}) - \frac{1+\nu}{Eh} \alpha_2 p_2 \tag{18d}$$

Substituting into Eqs. (15c-d) the result may be written in the form

$$D \Delta \chi - \frac{N_{11}}{R_1} - \frac{N_{22}}{R_2} + p_3 - D \Delta_R' \varepsilon - \frac{h^2}{12\alpha_1\alpha_2} \left[(\frac{\alpha_2 p_1}{R_1})_{,1} + (\frac{\alpha_1 p_2}{R_2})_{,2} \right] = 0 \tag{19a}$$

$$\frac{1}{Eh} \Delta N + \frac{\chi_{22}}{R_1} + \frac{\chi_{11}}{R_2} + \frac{1}{Eh} \Delta_R' M + \frac{1+\nu}{Eh\alpha_1\alpha_2} [(\alpha_2 p_1)_{,1} + (\alpha_1 p_2)_{,2}] = 0 \tag{19b}$$

where Δ is Laplace's operator in the middle surface:

$$\Delta\,(\ \ ) = \frac{1}{\alpha_1\alpha_2}\left[\left(\frac{\alpha_2(\ \ )_{,1}}{\alpha_1}\right)_{,1} + \left(\frac{\alpha_1(\ \ )_{,2}}{\alpha_2}\right)_{,2}\right] \tag{20}$$

and

$$\Delta_R'\,(\ \ ) = \frac{1}{\alpha_1\alpha_2}\left[\left(\frac{\alpha_2(\ \ )_{,1}}{\alpha_1 R_1}\right)_{,1} + \left(\frac{\alpha_1(\ \ )_{,2}}{\alpha_2 R_2}\right)_{,2}\right] \tag{21}$$

It will now be shown that the terms in ε and M in Eqs. (19) are negligible. For comparing these terms to $(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2})$ and $(\frac{X_{22}}{R_1} + \frac{X_{11}}{R_2})$, respectively, the most unfavorable case is that of a rapidly varying state stress in which it may be assumed that

$$\Delta_R'\,(\ \ ) = 0\left(\frac{\Delta(\ \ )}{R}\right) \leq 0\left(\frac{(\ \ )}{R^2 h}\right) \tag{22}$$

where R is the order of magnitude of the smaller of R_1 and R_2. With use of Eqs. (17) there comes

$$D\,\Delta_R'\,\varepsilon \leq \frac{h}{12R^2}\quad 0(N) \tag{23a}$$

$$\frac{1}{Eh}\quad \Delta_R'\,M \leq \frac{h}{12R^2}\quad 0(M) \tag{23b}$$

If it is permissible to write

$$\frac{N_{11}}{R_1}\,,\ \frac{N_{22}}{R_2} = 0\ \left(\frac{N}{R}\right) \tag{24a}$$

and

$$\frac{X_{22}}{R_1},\ \frac{X_{11}}{R_2} = 0\ \left(\frac{X}{R}\right) \tag{24b}$$

Eqs. (23) show that the terms in ε and M in Eqs. (19) are negligible with regard to $(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2})$ and $(\frac{\chi_{22}}{R_1} + \frac{\chi_{11}}{R_2})$, respectively, with a relative error of order $\frac{h}{R}$. The same conclusion is reached in the case of a rapidly varying state of stress without requiring Eqs. (24) to hold if it is admitted, as will be established later, that

$$\chi = 0\ (\frac{\sqrt{12}\ \varepsilon}{h}) \tag{25a}$$

and

$$N = 0\ (\frac{\sqrt{12}\ M}{h}) \tag{25b}$$

It is then possible to write

$$\Delta'_R\ \ \varepsilon = 0\ (\frac{\Delta\varepsilon}{R}) = 0\ (\frac{h}{\sqrt{12}R}\ \Delta\chi) \tag{26a}$$

and

$$\Delta'_R\ M = 0\ (\frac{\Delta M}{R}) = 0\ (\frac{h}{\sqrt{12}\ R}\ \Delta N) \tag{26b}$$

Eqs. 26 show that the terms in ε and M in Eqs. (19) may be neglected with a relative error of order $\frac{h}{\sqrt{12}\ R}$. The loading terms are not generally negligible in Eq. (19b) but the p_1 and p_2 terms are in Eq. (19a).

In non rapidly varying states of stress the errors due to neglecting the terms in ε and M in Eqs. (19) are even smaller. Thus Eqs. (19) may be replaced with

$$D\ \Delta\ \chi - \frac{N_{11}}{R_1} - \frac{N_{22}}{R_2} + p_3 = 0 \tag{27a}$$

$$\frac{1}{Eh}\ \Delta\ N + \frac{\chi_{22}}{R_1} + \frac{\chi_{11}}{R_2} + \frac{1+\nu}{Eh\alpha_1\alpha_2}\ [(\alpha_2\ p_1)_{,1} + (\alpha_1\ p_2)_{,2}] = 0 \tag{27b}$$

The deletion of the terms in ε, p_1 and p_2 in Eq. (19a) and of the term in M in Eq. (19b) is formally equivalent to the deletion of the same terms in Eqs. (18a-d). However, the corresponding error is of order $\leq \frac{h}{R}$ in all four equations, only if bending stresses are comparable to membrane stresses as occurs in rapidly varying states of stress. This restriction does not apply to Eqs. (27).

4. Vlasov's Shallow Shell Equations

Vlasov's shallow shell equations may be based on a single assumption applied in a dual form. This is assumption (i) of the Introduction which is restated below in a more specific way together with its dual.

The tangential components of displacement u_1 and u_2 may be neglected in computing the changes of normal curvatures and twist χ_{11}, χ_{22}, χ_{12} and χ_{21}. They may not be neglected in the changes of geodesic curvatures χ_{13} and χ_{23}.

The stress functions G_1 and G_2 may be neglected in the expressions of the in-plane stress resultants N_{11}, N_{22}, N_{12} and N_{21}. They may not be neglected in the transverse shears N_{13} and N_{23}.

The above assumption used with the conditions of no transverse shear deformation in the scalar strain-displacement relations yields

$$\omega_1 = \frac{u_{3,2}}{\alpha} - \frac{u_2}{R_2} \simeq \frac{u_{3,2}}{\alpha_2} \tag{28a}$$

$$\omega_2 = -\frac{u_{3,1}}{\alpha_1} + \frac{u_1}{R_1} \simeq -\frac{u_{3,1}}{\alpha_1} \tag{28b}$$

$$\chi_{11} = \frac{\alpha_2\,\omega_{2,1} - \alpha_{1,2}\,\omega_1}{\alpha_1\,\alpha_2} = -\frac{1}{\alpha_1\,\alpha_2}\left[\alpha_2\left(\frac{u_{3,1}}{\alpha_1}\right)_{,1} + \frac{\alpha_{1,2}}{\alpha_2}\,u_{3,2}\right] \tag{29a}$$

$$\chi_{22} = -\frac{\alpha_1\,\omega_{1,2} - \alpha_{2,1}\,\omega_2}{\alpha_1\,\alpha_2} = -\frac{1}{\alpha_1\,\alpha_2}\left[\alpha_1\left(\frac{u_{3,2}}{\alpha_2}\right)_{,2} + \frac{\alpha_{2,1}}{\alpha_1}\,u_{3,1}\right] \tag{29b}$$

The dual relations are

$$F_1 = \frac{G_{3,2}}{\alpha_2} - \frac{G_2}{R_2} \approx \frac{G_{3,2}}{\alpha_2} \tag{30a}$$

$$F_2 = -\frac{G_{3,1}}{\alpha_1} + \frac{G_1}{R_1} = -\frac{G_{3,1}}{\alpha_1} \tag{30b}$$

$$N_{11} = N_{11}^p + \frac{\alpha_1 F_{1,2} - \alpha_{2,1} F_2}{\alpha_1 \alpha_2} = N_{11}^p + \frac{1}{\alpha_1 \alpha_2} [\alpha_1 (\frac{G_{3,2}}{\alpha_2})_{,2} + \frac{\alpha_{2,1}}{\alpha_1} G_{3,1}] \tag{31a}$$

$$N_{22} = N_{22}^p - \frac{\alpha_2 F_{2,1} - \alpha_{1,2} F_1}{\alpha_1 \alpha_2} = N_{22}^p + \frac{1}{\alpha_1 \alpha_2} [\alpha_2 (\frac{G_{3,1}}{\alpha_1})_{,1} + \frac{\alpha_{1,2}}{\alpha_2} G_{3,2}] \tag{31b}$$

The superscript p in Eqs. (31) refers to a particular solution of the equilibrium equations.

From Eqs. (29) obtain

$$\chi = \chi_{11} + \chi_{22} = - \Delta u_3 \tag{32a}$$

$$N = N_{11} + N_{22} = \Delta G_3 + N^p \tag{32b}$$

and substituting into Eqs. (27) these take the form

$$D \Delta\Delta u_3 + \Delta_R G_3 = p_3 - p^* \tag{33a}$$

$$\frac{1}{Eh} \Delta\Delta G_3 - \Delta_R u_3 = - \frac{\Delta N^p}{Eh} - \frac{1+\nu}{Eh \alpha_1 \alpha_2} [(\alpha_2 p_1)_{,1} + (\alpha_1 p_2)_{,2}] \tag{33b}$$

where

$$p^* = \frac{N_{11}^p}{R_1} + \frac{N_{22}^p}{R_2} \tag{34}$$

and

$$\Delta_R() = \frac{1}{\alpha_1 \alpha_2} [(\frac{\alpha_2()_{,1}}{\alpha_1 R_2})_{,1} + (\frac{\alpha_1()_{,2}}{\alpha_2 R_1})_{,2}] \tag{35}$$

Eqs. (33) agree with those found in the literature[2,6] but contain in addition the terms in p_1 and p_2 and the terms deriving from the particular solution of the equilibrium equations. These are zero in the classical derivations.

Additional relations deriving from the basic assumption will now be obtained. χ_{12} and χ_{21} are expressed exactly as

$$\chi_{12} = - \frac{1}{\alpha_1 \alpha_2} (\alpha_2 \omega_{1,1} + \alpha_{1,2} \omega_2) - \frac{\omega_3}{R_1} \tag{36a}$$

$$\chi_{21} = \frac{1}{\alpha_1 \alpha_2} (\alpha_1 \omega_{2,2} + \alpha_{2,1} \omega_1) + \frac{\omega_3}{R_2} \tag{36b}$$

ω_1 and ω_2 are expressed in terms of the displacements through Eqs. 28. ω_3 is determined in terms of the translational displacements through Eq. (15b) which takes the form

$$2\omega_3 + \frac{1}{\alpha_1 \alpha_2} [(\alpha_1 u_1)_{,2} - (\alpha_2 u_2)_{,1}] - \frac{h^2}{12} (\frac{\chi_{12}}{R_2} - \frac{\chi_{21}}{R_1}) = 0 \tag{37}$$

It is apparent that the contribution of the terms in χ_{12} and χ_{21} in Eq. (37) to the right hand side of Eqs. (36) is of relative order $\frac{h^2}{R^2}$ and should be neglected. Upon neglecting u_1 and u_2 by the basic assumption the result is to let $\omega_3 = 0$ in Eqs. (36) which in terms of u_3 take then the form

$$\chi_{12} = \chi_{21} = - \frac{1}{\alpha_1 \alpha_2} [u_{3,12} - \frac{\alpha_{2,1}}{\alpha_2} u_{3,2} - \frac{\alpha_{1,2}}{\alpha_1} u_{3,1}] \tag{38a}$$

The dual relations are

$$N_{12} - N^p_{12} = N_{21} - N^p_{21} = - \frac{1}{\alpha_1 \alpha_2} [G_{3,12} - \frac{\alpha_{2,1}}{\alpha_2} G_{3,2} - \frac{\alpha_{1,2}}{\alpha_1} G_{3,1}] \tag{38b}$$

and yield $N_{12} = N_{21}$ if the particular solution is such that $N^p_{12} = N^p_{21}$.

5. Error in Shallow Shell Equations

The error under consideration here is that made in obtaining Eqs. (33) from Eqs. (27). The latter are obtained from Eqs. 19 with an error of order $\frac{h}{R}$ and are considered exact for the purposes of this discussion.

The source of the error is the basic assumption concerning the deletion of G_1 and G_2 in the stress-stress function relations and the deletion of u_1 and u_2 in the curvature-displacement relations.

The deletion of G_1 and G_2 from the expressions of the in-plane stress resultants would not violate the homogenous equilibrium equations if the same deletions were made in the transverse shears. This is not permissible however, although the transverse shears are not obtained through the stress functions but through Eqs. (18a-b). The result is in general a violation of the homogenous force equilibrium equations which in the tangential equations is of the same order of magnitude as the transverse shear terms. That the transverse shears are negligible in the tangential force equilibrium equations is one of the basic assumptions used in previous derivations of the shallow shell equations. Here this is implied by the basic assumption. In order however not to make the nature of the error depend on the particular solution it is necessary that N^p_{13} and N^p_{23} be also negligible in the tangential equilibrium equations. A convenient way for achieving this is to identify the particular solution of the equilibrium equations with a membrane solution.

In a way dual of the preceding the deletion of u_1 and u_2 from χ_{11}, χ_{22}, χ_{12} and χ_{21} but not from χ_{13} and χ_{23} violates in general the curvature compatibility equations by amounts which in the two tangential equations are of the same order of magnitude as the χ_{13} and χ_{23} terms.

A consequence of the preceding is that the membrane solution cannot in general be obtained exactly through stress-stress function relations of the form (31) and (38b) and the inextensional solution cannot in general be obtained exactly through strain-displacement relations of the form (29) and (38a).

A case of exception to the preceding remark is that of shells of zero Gaussian curvature. If p_1^* and p_2^* denote the fictitious load components that would be necessary to maintain tangential equilibrium without transverse shears when Eqs. (31) and (38b) are used it is found that

$$p_1^* = - \frac{1}{R_1 R_2} \frac{G_{3,1}}{\alpha_1} \tag{39a}$$

$$p_2^* = - \frac{1}{R_1 R_2} \frac{G_{3,2}}{\alpha_2} \tag{39b}$$

For shells of zero Gaussian curvature $p_1^* = p_2^* = 0$ and Eqs. (31) and (38b) satisfy identically the tangential equilibrium equations of the membrane theory. Similarly Eqs. (29) and (38a) satisfy identically the dual tangential compatibility equations of inextensional deformations. This makes it possible to obtain the membrane solution by letting $D = 0$ in Eq. (33a) and the inextensional solution by letting $\frac{1}{Eh} = 0$ in Eq. (33b).

Now the error caused by neglecting in Eq.(27a) u_1 and u_2 in χ is compared to the error of order $\frac{h}{R}$ caused by neglecting $\Delta_R' \varepsilon$ in Eq. (19a).

The strain-displacement relations for ε_{11}, ε_{22}, χ_{11} and χ_{22} take the form

$$\varepsilon_{11} = \frac{1}{\alpha_1} (u_{1,1} + \frac{\alpha_{1,2}}{\alpha_2} u_2) + \frac{u_3}{R_1} \tag{40a}$$

$$\varepsilon_{22} = \frac{1}{\alpha_2} (u_{2,2} + \frac{\alpha_{2,1}}{\alpha_1} u_1) + \frac{u_3}{R_2} \tag{40b}$$

$$\chi_{11} = \frac{1}{\alpha_1} [(\frac{u_1}{R_1} - \frac{u_{3,1}}{\alpha_1}),_1 + \frac{\alpha_{1,2}}{\alpha_2} (\frac{u_2}{R_2} - \frac{u_{3,2}}{\alpha_2})] \tag{40c}$$

$$\chi_{22} = \frac{1}{\alpha_2} [(\frac{u_2}{R_2} - \frac{u_{3,2}}{\alpha_2}),_2 + \frac{\alpha_{2,1}}{\alpha_1} (\frac{u_1}{R_1} - \frac{u_{3,1}}{\alpha_1})] \tag{40d}$$

from which it is possible to deduce

$$\varepsilon = \frac{1}{\alpha_1 \alpha_2} [(\alpha_2 u_1),_1 + (\alpha_1 u_2),_2] + (\frac{1}{R_1} + \frac{1}{R_2}) u_3 \tag{41a}$$

and

$$\chi = \frac{1}{\alpha_1 \alpha_2} [(\frac{\alpha_2 u_1}{R_1}),_1 + (\frac{\alpha_1 u_2}{R_2}),_2] - \Delta u_3 \tag{41b}$$

Assuming that R_1 and R_2 are not rapidly varying functions of the coordinates such that it is possible to write

$$(\frac{()}{R_i}),_j = 0\ (\frac{(),_j}{R_i}) \qquad i,j = 1,2 \tag{42}$$

and assuming that the terms in u_1 and u_2 in Eq. (41a) are $0(\varepsilon)$, the contribution of u_1 and u_2 to χ is seen to be $(\frac{\varepsilon}{R})$ and is negligible in Eq. (27a), as $\Delta_R' \varepsilon$ in Eq. 19a, with a relative error of order $\frac{h}{R}$.

Similarly the contribution of G_1 and G_2 to Δ_N in Eq. (27b) is in general comparable to the contribution of $\Delta_R' M$ in Eq. (19b) and of relative order $\frac{h}{R}$.

The above order of magnitude analysis fails in the case where ε is of a smaller order of magnitude than the u_1 and u_2 terms in Eq. (41a) and in the dual case concerning M. Leaving these cases for a subsequent discussion it appears from the preceding that if an error of order greater than $\frac{h}{R}$ is involved in obtaining Eqs. (33) from Eqs. (27) it is made in neglecting G_1 and G_2 in $(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2})$ and u_1 and u_2 in $(\frac{\chi_{22}}{R_1} + \frac{\chi_{11}}{R_2})$. This is in accordance with the result found earlier that the membrane and inextensional states of stress cannot, except for shells of zero Gaussian curvature, be obtained exactly through Eqs. (33a) and (33b), respectively, by letting $D = 0$ and $\frac{1}{Eh} = 0$.

In fact, the case referred to above where ε is of a smaller order of magnitude than the u_1 and u_2 terms in Eq. (41a) and the dual case concerning M occur, respectively, in inextensional deformations in which $\varepsilon = 0$ and in the homogenous membrane solution in which $M = 0$. In the first case the error caused by neglecting u_1 and u_2 in χ is not in general negligible in the term $D\Delta\chi$ in Eq. (27a) and in the second case the error caused by neglecting G_1 and G_2 in N is not in general negligible in the term $\frac{\Delta N}{Eh}$ in Eq. (27b). It appears therefore that for non shallow shells both the inextensional solution and its contribution to the equilibrium equation (27a) are in non negligible error as are both the homogenous membrane solution and its contribution to the compatibility equation (27b). While for parabolic middle surfaces the inextensional and membrane solutions are obtained exactly the error remains in their respective contributions to Eqs. (27a) and (27b).

An illustrative example of the preceding is that of a non shallow circular cylindrical shell bounded by two generatrices and two circular arcs, behaving uniformly in the longitudinal direction and as an arch in the circumferential direction. The bending of such a shell is similar to the bending of an arch and may be assumed to be inextensional. Although the general inextensional solution may be obtained exactly for u_3 and is in that case an arbitrary function of the circumferential angle, the error in neglecting the circumferential displacement in $D\Delta\chi$ in Eq. (27a) is not negligible. A correct use of Eq. (27a) would involve χ as obtained from the inextensional solution without neglecting the tangential displacement.

What is characteristic of this example is the uncoupling of the longitudinal from the circumferential action of the shell. This uncoupling will be encountered later on as a limiting case of a weak coupling.

By contrast to the membrane and the inextensional states of stress for which the terms $D\Delta\chi$ and $\frac{1}{Eh}\Delta N$ are negligible in Eqs. (27a) and (27b), respectively, there exists a state of stress, generally referred to as the edge zone state of stress or the boundary layer, in which the bending and stretching stiffness of the shell are coupled through the dual relations

$$D\Delta\chi = 0(\frac{N_{11}}{R_1}, \frac{N_{22}}{R_2}) \tag{43a}$$

and

$$\frac{1}{Eh}\Delta N = 0(\frac{\chi_{22}}{R_1}, \frac{\chi_{11}}{R_2}) \tag{43b}$$

After neglecting u_1, u_2, G_1 and G_2 Eqs. (43) take the form

$$\frac{Eh^3}{12}\Delta\Delta u_3 = 0\ (\Delta_R G_3) \tag{44a}$$

$$\frac{1}{Eh}\Delta\Delta G_3 = 0\ (\Delta_R u_3) \tag{44b}$$

The behavior of u_3 and G_3 upon differentiation as inferred from Eqs. (44) and the errors caused by neglecting u_1, u_2, G_1 and G_2 in Eqs. (27) will now be shown to depend on the nature of the operator Δ_R, on the geometrical nature of the boundary and on the boundary conditions.

Choosing for convenience R as a reference length it is possible to write

$$\Delta(\ ) = 0(\frac{\beta^2(\ )}{R^2}) \tag{45}$$

where β is a non dimensional number which may be thought of as a factor of increase upon non dimensional differentiation in at least one direction of the middle surface. If the middle surface, and consequently Δ_R, are elliptic the highest derivative terms of Δ_R contain in any system of surface coordinates second derivatives with regard to both coordinates. There follows with use of Eq. (45) that

$$\Delta_R(\ ) = 0\ (\frac{\Delta(\ )}{R}) = 0(\frac{\beta^2(\ )}{R^3}) \tag{46}$$

Eqs. (44) yield

$$\frac{Eh^3}{12}\ \frac{\beta^4}{R^4}\ u_3 = 0(\frac{\beta^2 G_3}{R^3}) \tag{47a}$$

$$\frac{1}{Eh}\ \frac{\beta^4}{R^4}\ G_3 = 0(\frac{\beta^2 u_3}{R^3}) \tag{47b}$$

For Eqs. 43 to be compatible it is necessary that

$$\beta^2 = 0(\sqrt{12}\ \frac{R}{h}) \tag{48}$$

G_3 and u_3 are then related through the relation

$$G_3 = 0(\frac{Eh^2}{\sqrt{12}}\ u_3) \tag{49}$$

and from Eqs. (49), (32) and (17) there comes

$$\chi = 0(\frac{\sqrt{12}}{h} \quad \varepsilon) \tag{50a}$$

and

$$N = 0(\frac{\sqrt{12}}{h} \quad M) \tag{50b}$$

Eq. (50b) is dual of Eq. (50a) and may also be obtained from it through the stress-strain relations.

It is consistent with Eq. 50a that the contribution of u_1 and u_2 to ε in Eq. (41a) is of the same order of magnitude as the contribution of u_3. There follows

$$u_1, u_2 = 0(\beta^{-1} \, u_3) \tag{51a}$$

Eq. (51a) makes the contribution of u_1 and u_2 to χ_{11}, χ_{22} and χ_{12} of relative order $\frac{h}{\sqrt{12}\,R}$. Similarly to Eq. (51a) it is possible to write

$$G_1, G_2 = 0(\beta^{-1} \, G_3) \tag{51b}$$

and the deletion of G_1 and G_2 in N_{11}, N_{22} and N_{12} causes a relative error of order $\frac{h}{\sqrt{12}\,R}$.

An illustrative example of the preceding is a dome like shell subjected to self equilibriating edge loads.

For hyperbolic and parabolic middle surfaces Δ_R is also hyperbolic and parabolic, respectively, and the behvaior of the state of stress depends on the geometry of the boundary and on the boundary conditions. The canonical form of the second derivative terms of a hyperbolic Δ_R is $\frac{\partial^2}{\partial_{\eta_1} \, \partial_{\eta_2}}$, where η_1 and η_2 are curvilinear coordinates corresponding to lines coinciding with the two families of asymptotic lines of the middle surface. For boundary conditions specified on an asymptotic line η_2 = constant the variation of the state of stress in the η_1 direction may be governed by the variation of the boundary conditions in the same direction. For boundary conditions such that differentiation in the η_1 direction does not change the order of magnitude, the increase in order of magnitude due to Δ_R is caused by one differentiation only as compared to two differentiations in the elliptic case. An example is a hyperbolic paraboloid bounded by generatrices.

For a parabolic Δ_R the canonical form of the second derivative terms is $\frac{\partial^2}{\partial \eta_1^2}$ where η_1 is a coordinate corresponding to the single family of asymptotic lines of the middle surface. In this case boundary conditions on asymptotic lines may be such that no change in order of magnitude is caused by Δ_R. An example is a cylindrical shell having two generatrices as part of its boundary.

Instead of no change in order of magnitude upon differentiation in the η_1 direction, there may actually be a change by a factor $O(\lambda)$ where λ is defined through the relation

$$\frac{\partial()}{\partial s} = \frac{\lambda()}{R} \tag{52}$$

s is the arclengh in the η_1 direction. For $\lambda = \beta$ the order of magnitude analysis is similar to that of the elliptic middle surface. The case $\lambda \leq 1$ however is of particular interest for its occurence in practical applications.

Letting, as before, β denote the factor of increase upon non dimensional differentiation with regard to η_2 it is possible to write

$$\Lambda(\ ) = 0(\frac{\beta^2(\ )}{R^2}) \tag{53a}$$

$$\Delta_R(\ ) = 0(\frac{\lambda^i\ \beta^{2-i}\ (\ )}{R^3}) \tag{53b}$$

where

$$i = 1 \quad \text{for the hyperbolic case} \tag{54a}$$

and

$$i = 2 \quad \text{for the parabolic case} \tag{54b}$$

Eqs. (44) take the form

$$\frac{Eh^3}{12}\ \frac{\beta^4}{R^4}\ u_3 = 0(\frac{\lambda^i\ \beta^{2-i}}{R^3}\ G_3) \tag{55a}$$

$$\frac{1}{Eh}\ \frac{\beta^4}{R^4}\ G_3 = 0(\frac{\lambda^i\ \beta^{2-i}}{R^3}\ u_3) \tag{55b}$$

and are compatible only if

$$\beta^{2+i} = 0(\lambda^i\ \mu) \tag{56}$$

where

$$\mu = \sqrt{12}\ \frac{R}{h} \tag{57}$$

G_3 and u_3 are related by the same order of magnitude relation as in the elliptic case, i.e.,

$$G_3 = 0(\frac{Eh^2}{\sqrt{12}}\ u_3) \tag{58}$$

χ and N are related to u_3 and G_3, respectively, through the relations

$$\chi = -\ \Delta u_3 = 0\ (\frac{\beta^2 u_3}{R^2}) \tag{59a}$$

$$N = \Delta G_3 = \underset{-}{O}\left(\frac{\beta^2 G_3}{R^2}\right) \tag{59b}$$

there follows

$$\chi = O\left(\frac{\sqrt{12}}{Eh^2} \quad N\right) \tag{60}$$

and using the stress-strain relation (17a)

$$\chi = O\left(\sqrt{12} \; \frac{\varepsilon}{h}\right) \tag{61}$$

Eq. (61) is the same as in the elliptic case. It should be noted however that if, as is usually the case, $\frac{\lambda}{\beta} < 1$, then

$$\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} < O\left(\frac{N}{R}\right) \tag{62a}$$

and

$$\frac{\chi_{22}}{R_1} + \frac{\chi_{11}}{R_2} < O\left(\frac{\chi}{R}\right) \tag{62b}$$

To show this, it is possible to write

$$\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} = \Delta_R \; G_3 = O\left(\frac{\lambda^i \; \beta^{2-i}}{R^3} \quad G_3\right) \tag{63}$$

whereas N is as given in Eq. (59b). There comes

$$\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} = O\left(\lambda^i \; \beta^{-i} \quad \frac{N}{R}\right) \tag{64a}$$

similarly

$$\frac{\chi_{22}}{R_1} + \frac{\chi_{11}}{R_2} = O\left(\lambda^i \; \beta^{-i} \quad \frac{\chi}{R}\right) \tag{64b}$$

For determining the order of magnitude of u_1 and u_2 in comparison to u_3

it is possible to write using Eqs. (61), (59a) and (56)

$$\varepsilon = 0(\lambda^{i}\ \beta^{-i}\ \frac{u_3}{R}) \tag{65}$$

This is another case where ε is of a smaller order of magnitude than the individual terms on the right of Eq. (41a). Eq. (65) requires then the larger of the u_1 and u_2 terms, or both, in Eq. 41a to have the same order of magnitude as $\frac{u_3}{R}$, i.e.,

$$u_1,\ u_2 = 0(\beta^{-1}\ u_3) \tag{66}$$

Eq. 66 is of the same form as in the elliptic case but β^{-1} is not as small. The relative contribution of u_1 and u_2 to the changes of curvature and twist is then, as the relative contribution of G_1 and G_2 to the in-plane stress resultants, $0(\beta^{-2})$. It is recalled that

$$\beta^{-2} = (\lambda\mu)^{-\frac{2}{3}} \quad \text{in the hyperbolic case} \tag{67a}$$

and

$$\beta^{-2} = (\lambda^{2}\ \mu)^{-\frac{1}{2}} \quad \text{in the parabolic case} \tag{67b}$$

whereas

$$\beta^{-2} = \mu^{-1} = \frac{h}{\sqrt{12}\ \ R} \quad \text{in the elliptic case} \tag{67c}$$

The case $\frac{\lambda}{\beta} << 1$ is one of weak coupling between the bending and stretching stiffnesses of the shell. An example is that of a long circular cylindrical shell bounded by two generatrices and two cross sections. If edge loads varying as sin $\frac{\pi x}{\ell}$ are specified, where ℓ is the length of the shell and x is the length coordinate along the generatrices then form Eq. (52)

$$\lambda = \frac{\pi a}{\ell} \tag{68}$$

where a is the radius of the cross section. Identifying R with a in Eq. (57) obtain

$$\mu = \sqrt{12}\ \frac{a}{h} \tag{69}$$

and from Eq. (67b) the order of the error is

$$\beta^{-2} = \frac{1}{\lambda\ \sqrt{\mu}} \tag{70}$$

The error increases with decreasing λ. For $\lambda \leq 1$ the shell is in the range called long[12]. It may be noted, however, that being long is not an intrinsic property of the shell but is related to the degree of variation of the load. The example of the cylindrical shell in a state of arch behavior discussed earlier is a limiting case in which $\lambda = 0$.

Conclusion

The derivation of Vlasov's shallow shell equations may be based on a single assumption applied in a dual form. This is that the tangential components of displacement u_1 and u_2 may be neglected in computing the changes of normal curvature and twist χ_{11}, χ_{22}, χ_{12} and χ_{21}, and that the stress functions G_1 and G_2 may be neglected in computing the in-plane stress resultants N_{11}, N_{22}, N_{12} and N_{21}.

The equations developed here agree with Vlasov's equations but contain in addition the tangential load components and a particular solution of the equilibrium equations.

The error involved in applying Vlasov's shallow shell equations to non shallow shells is generally non negligible when seeking non rapidly varying states of stress such as the membrane and inextensional states of stress. An exception to this occurs for parabolic shells.

For a state of stress satisfying Eqs. (43), which is referred to as an edge zone state of stress, the error is more significant for hyperbolic and parabolic shells than for elliptic shells if part of the boundary coincides with an asymptotic line of the middle surface and if the corresponding boundary conditions involve non rapidly varying functions. In the elliptic case the error is of the same order of magnitude as that inherent in the basic equations of thin shells whereas in the hyperbolic and parabolic cases considered above the error is of a larger order of magnitude though acceptable in practical applications if the shell is thin enough.

The orders of magnitude of the error in the three cases discussed above are obtained through Eqs. (67). When the boundary of a hyperbolic or parabolic middle surface does not coincide partly or totally with asymptotic lines, the behavior of the edge zone state of stress is similar to that of the elliptic shell. Since, in addition, the membrane and inextensional solutions of parabolic shells may be obtained exactly through Vlasov's shallow shell equations there results that these equations may be used to obtain with the same accuracy as above the total state of stress in the practically important cases where it is a superposition of the membrane, inextensional and edge zone states of stress. This may also be done with a slightly larger but often practically acceptable error if part of the boundary coincides with an asymptotic line. An illustration of the preceding is the application of Donnell's equation to circular tubes and non shallow cylindrical shell roofs.

The accuracy of Vlasov's shallow shell equations cannot be expected to experience a discontinuity when passing continuously from a shell of positive Gaussian curvature to shells of zero and negative Gaussian curvature. The orders of magnitude of the error shown in Eqs. (67) are then to be understood in the light of this remark.

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