# MATHEMATICAL MODELS IN MANAGEMENT SCIENCES (I): CONSUMER BEHAVIOUR AS A MARKOV PROCESS

by

T. N. Bhargava and V. N. Patankar

TECHNICAL REPORT NO. 23

TO BE PRESENTED AT THE 36th SESSION OF THE INTERNATIONAL STATISTICAL INSTITUTE, SYDNEY, AUSTRALIA, AUGUST 28-SEPTEMBER 8, 1967

PREPARED UNDER RESEARCH GRANT NO. NsG-568
(PRINCIPAL INVESTIGATOR: T. N. BHARGAVA)

FOR

NATIONAL AERONAUTICS and SPACE ADMINISTRATION





DEPARTMENT OF MATHEMATICS
KENT STATE UNIVERSITY
KENT, OHIO

May 1967

[redacted]

Mathematical Models in Management Sciences (I):
Consumer Behaviour as a Markov Process*

T. N. Bhargava and V. N. Patankar**
Kent State University, U. S. A.

Abstract

This is the first of a series of papers we propose to write on the use of mathematical models in management sciences. The aim of this paper is to present the consumer behaviour pattern, which takes place in time, as a simple Markov process but not necessarily with all the usual restrictions, such as stationarity and homogeneity. Some of the difficulties caused by the complex situations found in practice are taken into account, and methods suggested for analyzing such processes. For example, the cases where an individual consumer may be allowed to choose more than one brand, and the case where the actual amount of the particular brand bought by the individual also plays a significant role. In particular we discuss the models based on individual and collective Markov processes for the consumer purchasing behaviour.

[redacted]

---

* With the support of a NASA research grant number NsG-568 at Kent State University, U. S. A.

**On leave from Hindustan Lever Ltd., Bombay, India

In an actual market situation, a consumer is faced with the problem of purchasing a certain brand of an item in the market. We wish to study the purchasing behaviour of a consumer, and in general that of the population of consumers over a period of time, with the explicit purpose of being able to predict his future behaviour based on the past information. Various attempts in this direction have been made, some of which are listed at the end of this paper. We give a brief review of the work done so far and, in particular, study the so-called 'individual' and 'collective' Markov processes as applied to the consumer purchasing behaviour.

Let $\{X_n, n \geq 0\}$ be a family of random variables taking place in a Euclidean k-space S so that $\{X_n\}$ is a discrete time stochastic process. Let the initial probability distribution of this finite process be given by the vector $P^{(0)} = (p_i^{(0)}; i = 1, 2, \ldots, k)$, and let the process be Markovian, i.e. the transition probabilities

$$P\{X_n = j | X_{n-1} = i, X_{n-2} = h, \ldots, X_1 = g\}$$
$$= P\{X_n = j | X_{n-1} = i\} = p_{ij}^{(n)}, \text{ say.}$$

Let $P^{(n)} = (p_{ij}^{(n)}); i,j = 1, 2, \ldots, k$ be the transition probability matrix at time t = n. If $P^{(n)}$ is independent of n, the finite Markov process $\{X_n\}$ is called a finite Markov chain, and the transition probabilities $p_{ij}^{(n)}$ are simply denoted by $p_{ij}$.

At any time $t = n$, let the random variable $X_n$ represent an individual consumer X, and let it take one of the values 1, 2,..., k, i.e., the consumer chooses one and only one of the k given brands. Suppose we have a "population" of N consumers, for each of whom we define a random variable $X_{nr}$, $r = 1, 2, ..., N$, and that each of these behave in a similar way. The question which is generally asked is: To what extent the previous behaviour of the totality of consumers with respect to their choice of a particular brand affects their present (or future) choice?

There are two obvious ways of looking at the problem: (i) Study of a single consumer history, sometimes called as an "individual process"; (ii) Consideration of all the consumers in the population and prediction of the fraction of the population buying each of the brands. This is called as the "collective process."

Assuming that the process may be taken to be a Markov chain, there are still many other restrictions to be looked into, some of which are: Is $P^{(n)}$ independent of n? Can the changes in distributions of brands in time be ascribed to $P^{(n)}$ only?

Furthermore, assuming $P^{(n)}$ does not depend on time n, what do the powers of $P = P^{(n)}$ mean? That is, how to interpret $P^m$, the $m^{th}$ power of matrix P? For the individual process, the entry $p^m_{ij}$ tells us what the probability will be for a consumer to switch from brand i to brand j in time period equal to length m; in the collective process, it gives the fraction of the consumers using brand i which switch to brand j after time period m.

Finally, while the stable state solution gives the long-range predictions for the individual process, it only tells us what the so-called "equilibrium fractions" for the collective process are. These fractions, when realized, remain the same in the future.

The questions of the mean first passage times, and the reversibility of the matrix P, are rather hard to answer for the collective process; for the individual process case, the answers remain straightforward.

The individual process for stationary case has been studied and solutions presented by various people, e.g., Herniter and Magee [5]. In this, they have also considered a certain type of collective process. If $N_i(n)$ is the number of consumers choosing brand i at time n, then the vector $N(n) = (N_1(n),\ldots, N_k(n))$ defines the distribution of the total population of consumers among the k states at time t = n; we notice that $\sum_i N_i(t) = N$. Under suitable assumptions, a steady state solution may be found in [5]. There is still considerable doubt about the collective process because of so-called "aggregating;" e.g. Howard [6], who suggests a kind of "manifold convolution," and calls it a vector Markov process. We intend to probe this deeply, especially in view of what we said in the previous section. It seems natural to consider some kind of collective

Markov process; an independent study of this kind for "random digraphs" is being done by one of the authors (Bhargava) and some others e.g. see Bhargava and Fisk [2].

Another practical problem consists in investigating the violation of the homogeneity, i.e. the assumption that $P = P^{(n)}$ is the same for all consumers. Morrison [10] has developed a heterogeneous Markovian model where transition probabilities are allowed to vary in certain special ways.

Order of the process has also been widely discussed in literature. On one hand, Kuehn [8] finds the first order process inadequate to describe consumer behavior and has therefore developed higher order processes by using Bush and Mosteller Learning Models. At the other extreme, Frank [4] attributes this learning effect purely to spurious contagion due to the aggregation discussed above; he therefore develops a Bernoulli process using theory of runs.

It is only Howard [7] who has so far considered nonstationary (which he calls dynamic) process but he restricts himself to only Bernoulli model.

An interesting semi-Markov process has also been developed by Howard [6], who takes into account even those cases when a consumer fails to choose any brand at certain times. Ehrenberg [3] has shown some work in this field by giving weights to the "quantity purchased" of the brand chosen; i.e. taking the purchase rate into account. He however bases his theory on negative binomial distribution.

A model when X can be in more than one state at a time (i.e. when consumer buys more than one brand) is of practical interest and is also under investigation.

All the above and much more depends upon good testing and estimation procedures for the parameters involved in the process, as well as for stationarity. For this we refer to Anderson and Goodman [1], Morrison [11], and Massy [9].

## References

[1] Anderson, T. W. and Goodman, L. A. (1957) Statistical inference about Markov Chains, Ann. Math. Statist., vol. 28.

[2] Bhargava, T. N. and Fisk, D. L. (1967) Random digraphs and probability models, To be presented at the 36th Session of I.S.I. at Sydney.

[3] Ehrenberg, A.S.C. (1959) The pattern of consumer purchases, App. Statist., vol. 8.

[4] Frank, R. E. (1962) Brand choice as a probability process, Jr. of Bus., vol. 35.

[5] Herniter, J. D. and Magee, J. F. (1961) Customer behaviour as a Markov process, Op. Res., vol. 9.

[6] Howard, R. A. (1963) Stochastic process models for consumer behaviour, Jr. of Ad. Res., vol. 3.

[7] Howard, R. A. (1964) Dynamic inference research in control of complex systems, Technical Report No. 10, Operations Research Centre, M.I.T.

[8] Kuehn, A. (1962) Consumer brand choice-a learning process?, Jr. of Ad. Res., vol. 2.

[9] Massy, W. F. (1966) Estimation of parameters for linear stochastic learning models, Working paper 78, Graduate School of Business, Stanford University.

[10] Morrison, D. G. (1965) Stochastic models for time series with applications in marketing, Technical Report No. 8, Program in Operations Research, Stanford University.

[11] Morrison, D. G. (1966) Testing brand Switching probabilities, Jr. of Marketing Res., vol. 3.

## Abstrait

Voici le premier dans une série de papiers que nous comptons écrire sur l'emploi des modiles mathématiques dans la science des affaires. Le but de ce papier est de présenter le modèle de comportement du consommateur, qui a lieu dans le temps, comme un simple procédé Markovien mais non pas nécessairement avec les restrictions ordinaires, telles que "stationnarité" et homogénéité. On fait entrer en ligne de compte quelques--unes des difficultés causées dûes aux situations complexes actuelles trouvées en pratique et des méthodes suggérées pour analyser tels procédés. Par exemple, le cas où on peut permettre au consommateur de choiser plus d'une marque, ou le cas où la quantité actuelle d'une marque particulière achetée par l'individu joue aussi un rôle signifiant. Particulierement, nous discourons les modeles basés les procédés Markoviens individuels et collectifs pour le consommateur et ses tendances comme acheteur.