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MULTISEGMENTED OPTIMAL TRAJECTORIES

by Thomas L. Vincent

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TABLE OF CONTENTS

	Page
LIST OF FIGURES	iv
NOMENCLATURE	v
ABSTRACT	vii
INTRODUCTION	1
AN EXTENDED PROBLEM OF BOLZA	3
The Formulation	3
Necessary Optimizing Conditions	3
Special Cases	8
APPLICATIONS	14
A Geometric Example	14
Multiple Satellite Launch Vehicle	18
Two Vehicle Rendezvous	24
DISCUSSION AND CONCLUSIONS	28
REFERENCES	30

LIST OF FIGURES

Figure		Page
1.1	Multiple Subarcs	2
2.1	Normal Corners	8
2.2	Discontinuous State Variable Corners	10
2.3	Unconnected, Non-Overlapping Trajectories	12
2.4	Branched Trajectories	12
3.1	Branched Trajectory Connecting Three Points	14
3.2	Solution For the Shortest Path Connecting the Three Points (0,0), (1,1), and (k,0), for various values of k	17
3.3	Trajectory for a Multiple Satellite Launch Vehicle	18
3.4	Coordinate System For a Multiple Satellite Launch Vehicle	19
3.5	Two Vehicle Rendezvous in the Plane	24

NOMENCLATURE

a constant

b constant

c constant

C constant

d constant

e constant

E energy per unit mass

f constant

F functional form defined on p.1

g functional form defined on p.1, p.3, acceleration due to gravity.

G functional form defined on p.4, universal gravitational constant

G' remainder of G

h length coordinate (Fig. 3.1) angular momentum per unit mass.

H functional form defined on p.4

k length coordinate (Fig. 3.1), constant defined on p. 20.

ℓ length coordinate (Fig. 3.1)

m integer, mass

M mass of earth

n integer

P mass of payload

r radius from center of earth

t independent variable, time

T thrust

u control variable, x component of velocity

v velocity, y component of velocity

V_e exhaust velocity

X steering angle (Fig. 3.4 and Fig. 3.5)

y state variable

z* functional form defined on p.4.

γ trajectory angle (Fig. 3.1 and Fig. 3.4)

λ Lagrange multiplier

μ Lagrange multiplier

ϕ functional form defined on p.1, polar angle (Fig. 3.4)

ψ functional form defined on p.1.

Subscripts

e $1 \ldots v \leq 2n + 2$

f final point

h $1 \ldots 2n$

i $1 \ldots n$

j $1 \ldots n$

k $1 \ldots m$

ℓ $1 \ldots p \leq f(n + 1)$

α 1, 3, 5 ...

β 2, 4, 6 ...

1, 2, 3, ... value of quantity at various points

Superscripts

no non-optimal controal

o optimal control

q 1 ... f/2

1,2,3 value of quantity along first, second, and third subarc.

ABSTRACT

The problem of Bolza from the calculus of variations in terms of modern control notation has been extended in scope to include situations in which a number of subarcs in the state variable trajectory may occur in a variety of ways. The subarcs are allowed to be overlapping and/or separated. This allows for several subarcs to occur in the same interval of the independent variable and for subarcs which are separated by jumps in the independent and state variables. In addition, the differential equations of constraint and the integral quantity to be extremized are allowed to be of different form from subarc to subarc.

The necessary minimizing conditions for the extended Bolza problem are obtained by extremizing a new functional which is related to it. This allows the optimizing conditions for the state and control variables to be obtained by applying the usual calculus of variations procedures and the optimizing conditions for the endpoints of the subarcs to be obtained using the ordinary theory of maxima and minima.

The results of the theory presented here may be applied to a wide range of space trajectory problems. For a number of special cases, the theory reduces to results previously obtained and recorded elsewhere. A number of example problems illustrating new applications which utilize the theory are presented in order to demonstrate the applicability of the results. The examples include the problem of inserting two payloads into separate orbits with one vehicle which has two upper stages ignited simultaneously and a two vehicle rendezvous problem.

SECTION I

INTRODUCTION

Perhaps one of the most useful formulations of a problem in the calculus of variations is that of the problem of Bolza. This problem as formulated by Bliss[1], may be written in terms of the modern concepts of state and control variables.[2,3] Using modern notation, the problem of Bolza may be stated as follows: Among all continuous state variable and piecewise continuous control variable functions

$$y_i(t) \text{ and } u_k(t) , \qquad i = 1,\ldots,n; \quad k = 1,\ldots,m \tag{1.1}$$

which satisfy differential equations,

$$\dot{y}_i = \phi_i(y_j \; u_k, \; t), \qquad j = 1,\ldots,n \tag{1.2}$$

and endpoint conditions of the form

$$\psi_e(t_1, \; t_f, \; y_{i1}, \; y_{if}) = 0, \qquad e = 1,\ldots,v \leqq 2n + 2 \tag{1.3}$$

find the set which minimizes a sum of the form

$$g(t_1, \; t_f, \; y_{i1}, \; y_{if}) + \int_{t_1}^{t_f} F(y_i \; u_k, \; t)dt. \tag{1.4}$$

It is seen with the above formulation, that the problem of Bolza is limited to continuous state variables defined over a single interval from t_1 to t_f and with boundary conditions (1.3) specified only at the endpoints of this interval.

In attempting to formulate a given problem in flight mechanics or space mechanics as a problem of Bolza, it becomes apparent that the assumption of continuous state variables over a single time interval may become somewhat restrictive. For example, the flight of a multi-staged rocket vehicle represents a situation in which the state variable, mass, will be discontinuous. Numerous other examples can be cited in which a solution will require more than one subarc. It is desirable therefore to investigate an extended Bolza problem with the possibility of unconnected and/or over lapping subarcs such as shown in Figure (1.1) for the case of three subarcs.

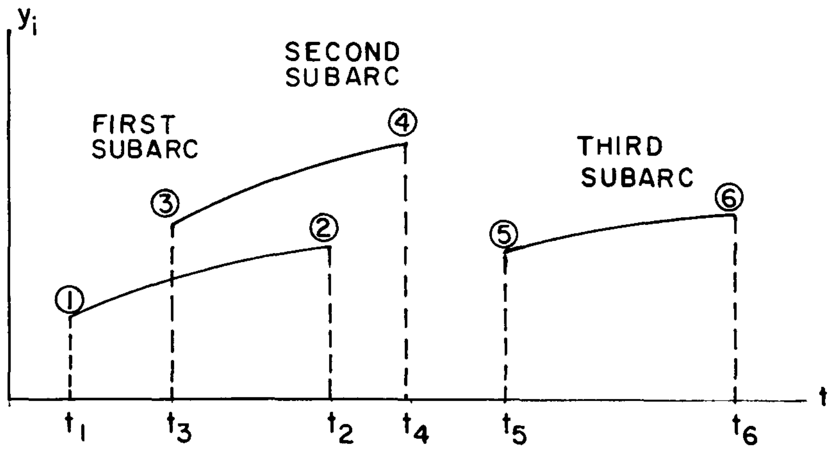

FIGURE 1.1 MULTIPLE SUBARCS

In Section II, the development of necessary conditions for situations such as depicted in Figure 1.1 will be obtained which will also allow for the possibility of different constraint equations (1.2) along each subarc. Two different sets of conditions will be developed; necessary conditions to be satisfied along each subarc, and necessary conditions to be satisfied at the beginning and end of each subarc.

The situation depicted in Figure 1.1 is quite general and for certain special cases the optimizing conditions contained in Section II reduced to results previously obtained. For example, if the state variables and time are continuous at each corner [$y_{i2} = y_{i3}$ and $t_2 = t_3$ etc.] and if the equations of constraint are the same for each subarc, with no further conditions imposed, then the results reduce to the well known Erdmann-Weierstrass corner conditions in control notation. If the state variables are discontinuous, but the time is continuous, then the results of Mason[4] are obtained and if the state variables and time are discontinuous but not overlapping the results of Vincent[5] are equivalent to those presented here.

Since numerous examples have already been developed for the special cases mentioned above, the examples presented in Section III will illustrate only the situation of overlapping subarcs. In addition to a geometric example there is presented the examples of a multiple satellite vehicle launch and a two vehicle rendezvous.

In the material which follows "endpoint" will refer to the initial point and final point of the entire trajectory only. The term "cornerpoint" will be used to designate all other endpoints of the subarcs. In Figure 1.1 the points 1 and 6 are "endpoints" and the points 2, 3, 4, and 5 are "cornerpoints". The total number of points (endpoints plus cornerpoints) and hence the final point will be designated by f , and even integer.

SECTION II

AN EXTENDED PROBLEM OF BOLZA

The Formulation

Introductory Remarks - The extended problem of Bolza will be closely related to the original problem of Bolza and for the case of a single subarc, they are identical. The extended problem of Bolza is formulated as follows: Among all state and control variable functions,

$$y_i(t) \quad \text{and} \quad u_k(t) \qquad i = 1,\ldots,n; \quad k = 1,\ldots,m \tag{2.1}$$

between the intervals (t_α, t_β) where $\alpha = (2q - 1)$, $\beta = 2q$ and q takes on values 1,..., f/2, which between the intervals satisfy differential equations of the form

$$\dot{y}_i = \phi_i^{\,q}(y_j\ u_k\ t)\ , \qquad j = 1\ldots n \tag{2.2}$$

and prescribed endpoint and corner point conditions of the form

$$g_\ell(t_\alpha,\ t_\beta,\ y_{i\alpha},\ y_{i\beta}) = 0, \qquad \ell = 1,\ldots,p \leq f(n + 1) \tag{2.3}$$

find the set which will minimize a sum of the form

$$g(t_1,\ t_f,\ y_{i1},\ y_{if}) + \sum_{q=1}^{f/2} \int_{t_\alpha}^{t_\beta} F^q(y_i,\ u_k,\ t)dt. \tag{2.4}$$

The range on the subscripts i, j, k, ℓ, α and β will be as given above for the remainder of the material presented in this report. Hence the range on these subscripts will not be repeated in what follows.

Necessary Optimizing Conditions

Introduction of Lagrange Multipliers - Consider now a new functional z* obtained from equations (2.2) and (2.3) and the functional form (2.4) by:

(a) Multiplying equations (2.2) by the variables λ_i, integrating over the appropriate interval from t_α to t_β and adding to the integral in expression (2.4).

(b) Multiplying equations (2.3) by the parameters μ_ℓ and adding to the function g in expression (2.4).

This results in the definition of z* given by

$$z^* = g + \mu_\ell g_\ell + \sum_{q=1}^{f/2} \int_{t_\alpha}^{t_\beta} (F^q - \lambda_i \phi_i^q + \lambda_i \dot{y}_i)dt. \tag{2.5}$$

The hypothesis is now made that the trajectory $y_i(t)$ and the control $u_k(t)$ between the intervals t_α and t_β and endpoints/cornerpoints $y_{i1},\ldots,y_{if}$; $t_1,\ldots,t_f$ which make the functional (2.4) take on a minimal value subject to the constraints (2.2) and (2.3) will also minimize (2.5). Under this hypothesis, if all of the functions and endpoints/cornerpoints are found which minimize z* also satisfy the constraints (2.2) and (2.3), then among these functions and endpoints/cornerpoints must also be the solution to the extended problem of Bolza as formulated by equations (2.2) and (2.3) and the functional form (2.4). It will be shown later that any functions and endpoints which minimize z* must satisfy the conditions (2.2) and (2.3), hence it is of interest to examine minimizing solutions to z*.

For convenience, the following functions are defined:

$$G(t_\alpha, t_\beta, y_{i\alpha}, y_{i\beta}, \mu_\ell) = g + \mu_\ell g_\ell \quad , \tag{2.6}$$

$$H^q[y_i(t), u_k(t), \lambda_i(t)] = \lambda_i \phi_i^q - F^q. \tag{2.7}$$

Thus equation (2.5) may be written as

$$z^* = G + \sum_{q=1}^{f/2} \int_{t_\alpha}^{t_\beta} [-H^q + \lambda_i \dot{y}_i]dt. \tag{2.8}$$

It is noted that the functional z* is not only a function of the paths $[y_i(t), u_k(t), \lambda_i(t)]$ connecting the point α to β but also the quantities $[t_1,\ldots,t_f, y_{i1},\ldots,t_{if}, \mu_\ell]$ associated with the various points. The necessary optimizing conditions for extremizing z*, and hence the original problem, are obtained by applying the general principle that the optimizing conditions which determine the path with all of the endpoints/cornerpoints fixed will remain unchanged if the endpoints/cornerpoints are considered as free. Hence, two sets of optimizing criteria will, in general, have to be satisfied: conditions relating to the path and conditions relating to the endpoints/cornerpoints.

Optimal path conditions - The optimizing conditions related to the path are obtained by fixing all of the endpoints/cornerpoints so that z* becomes

$$z^* = C + \sum_{q=1}^{f/2} \int_{t_\alpha}^{t_\beta} [-H^q + \lambda_i \dot{y}_i]\, dt, \tag{2.9}$$

where C is a constant. Since all of the points are fixed, the sum of the integrals will be a minimum if each individual integral is minimized. Between any two points α and β, equation (2.9) is a functional of a well known form in the calculus of variations[6], and the following Euler equations represent necessary conditions for extremizing z* in the interval (t_α, t_β)

$$\frac{\partial H^q}{\partial y_i} + \dot{\lambda}_i = 0 \tag{2.10}$$

$$-\frac{\partial H^q}{\partial \lambda_i} + \dot{y}_i = 0 \tag{2.11}$$

$$\frac{\partial H^q}{\partial u_k} = 0. \tag{2.12}$$

Solving the above set of equations for $q = 1 \ldots f/2$ yields a path $[y_i(t), u_k(t), \lambda_i(t)]$ between the points α and β. It is noted that equations (2.11) are just the equations of constraint (2.2), hence the optimal path for z* satisfies the same constraints as required for the functional (2.4).

The total solution between t_1 and t_f is obtained by joining together the several continuous trajectories $y_i(t)$ or subarcs between points α and β each of which satisfies the above Euler equations.

Equations (2.10) and (2.12) have as a first integral

$$\frac{dH^q}{dt} = \frac{\partial H^q}{\partial t} \tag{2.13}$$

A further necessary condition for minimizing the functional (2.8) is given by the Weierstrass E function condition. In modern state variable, control variable notation, this condition becomes[7]

$$H^{qo} > H^{qno} \tag{2.14}$$

where H^{qo} represents the function H^q evaluated with respect to optimal control and H^{qno} represents the function H evaluated with respect to an admissible non-optimal control variable. It is assumed that $t_\beta > t_\alpha$.

<u>Optimal endpoint/cornerpoint conditions</u> - The optimizing conditions related to the endpoints/cornerpoints may now be obtained by noting that for each subarc a family of trajectories passing through the points α and β, must contain 2n arbitrary constants $y_i = y_i(t, C_h^q)$, $h = 1 \ldots 2n$. It will be assumed that for each subarc, such a family is given and that fixing n of the constants in the family $y_i = y_i(t, C_h^q)$, will result in a central field with the center of the pencil located at one of the points α or β. With the center of the pencil at one point, it is assumed that the other point is associated with unique values of the remaining n constants.

Before substituting the above trajectory into the integral (2.8), it is important to examine how y_i changes with respect to t and C_h^q. Changes in y_i along a particular trajectory (C_h^q fixed) are given by

$$dy_i = \frac{\partial y_i}{\partial t} \, dt \quad , \tag{2.15}$$

whereas arbitrary changes in y_i are given by

$$dy_i = \frac{\partial y_i}{\partial t} dt + \frac{\partial y_i}{\partial C_h^{\,q}} dC_h^{\,q} \quad . \tag{2.16}$$

Note that repeated indices here on q does not imply summation. By substituting the trajectory $y_i = y_i(t\ C_h^{\,q})$ into the integral contained in equation (2.8) the expression for z* becomes

$$z^* = G(t_\alpha, t_\beta, y_{i\alpha}, y_{i\beta}, \mu_\ell) + \sum_{q=1}^{f/2} \int_{t_\alpha}^{t_\beta} \{-H^q [y(t, C_h^{\,q}), t] + \lambda_i \frac{\partial y_i(t, C_h^{\,q})}{\partial t} \} dt. \tag{2.17}$$

With this substitution, z* becomes a function of the endpoints/cornerpoints $(t_1, \ldots, t_f;\ y_{i1}, \ldots, y_{if})$ and the parameters μ_ℓ and $C_h^{\,q}$. From the theory of ordinary maxima and minima, a necessary condition that z* be an extremum with respect to these quantities is given by dz* = 0. This differential is given by

$$dz^* = \frac{\partial z^*}{\partial \mu_e} d\mu_e + \frac{\partial z^*}{\partial y_{i\alpha}} dy_{i\alpha} + \frac{\partial z^*}{\partial y_{i\beta}} dy_{i\beta} + \frac{\partial z^*}{\partial t_\alpha} dt_\alpha + \frac{\partial z^*}{\partial t_\beta} dt_\beta + \frac{\partial z^*}{\partial C_h^{\,q}} dC_h^{\,q} \tag{2.18}$$

The first three partial derivatives are easily evaluated

$$\frac{\partial z^*}{\partial \mu_\ell} = \frac{\partial G}{\partial \mu_\ell} = g_\ell \tag{2.19}$$

$$\frac{\partial z^*}{\partial y_{i\alpha}} = \frac{\partial G}{\partial y_{i\alpha}} \tag{2.20}$$

$$\frac{\partial z^*}{\partial y_{i\beta}} = \frac{\partial G}{\partial y_{i\beta}} \tag{2.21}$$

The other partial derivatives are evaluated using Leibniz' formula[8]

$$\frac{\partial z^*}{\partial t_\alpha} = \frac{\partial G}{\partial t_\alpha} + H_\alpha^{\,q} - \lambda_{i\alpha} \left. \frac{\partial y_i}{\partial t} \right|_\alpha \tag{2.22}$$

$$\frac{\partial z^*}{\partial t_\beta} = \frac{\partial G}{\partial t_\beta} - H^q_\beta + \lambda_{i\beta} \left. \frac{\partial y_i}{\partial t} \right|_\beta \tag{2.23}$$

$$\frac{\partial z^*}{\partial c^q_h} = \sum_{q=1}^{f/2} \int_{t_\alpha}^{t_\beta} \left[- \frac{\partial H^q}{\partial y_i} \frac{\partial y_i}{\partial c^q_h} + \lambda_i \frac{\partial^2 y_i}{\partial t \partial c^q_h} \right] dt \tag{2.24}$$

By noting that along the trajectory

$$\frac{d}{dt} \left(\lambda_i \frac{\partial y_i}{\partial c^q_h} \right) = \lambda_i \frac{\partial^2 y_i}{\partial c^q_h \partial t} + \dot{\lambda}_i \frac{\partial y_i}{\partial c^q_h} \tag{2.25}$$

equation (2.24) may be written as

$$\frac{\partial z^*}{\partial c^q_h} = \sum_{q=1}^{f/2} \int_{t_\alpha}^{t_\beta} - \left[\frac{\partial H^q}{\partial y_i} + \dot{\lambda}_i \right] \frac{\partial y_i}{\partial c^q_h} \, dt + \lambda_{i\beta} \left. \frac{\partial y_i}{\partial c^q_h} \right|_\beta - \lambda_{i\alpha} \left. \frac{\partial y_i}{\partial c^q_h} \right|_\alpha \tag{2.26}$$

By substititing equations (2.19) - (2.23) and (2.26) into equation (2.18) and utilizing equation (2.16), the following result is obtained

$$dz^* = g_\ell \, d\mu_\ell + \left(\frac{\partial G}{\partial y_{i\alpha}} - \lambda_{i\alpha} \right) dy_{i\alpha} + \left(\frac{\partial G}{\partial y_{i\beta}} + \lambda_{i\beta} \right) dy_{i\beta} + \left(\frac{\partial G}{\partial t_\alpha} + H^q_\alpha \right) dt_\alpha$$

$$\left(\frac{\partial G}{\partial t_\beta} - H^q_\beta \right) dt_\beta + \left\{ \sum_{q=1}^{f/2} \int_{t_\alpha}^{t_\beta} - \left[\frac{\partial H^q}{\partial y_i} + \dot{\lambda}_i \right] \frac{\partial y_i}{\partial c^q_h} \right\} dc^q_h \tag{2.27}$$

If the family of trajectories $y_i = y\,(t, c^q)$ between the points α and β are stationary curves, then by equation (2.10), the integral term in equation (2.27) is identically zero.

By setting the coefficients of the various differential terms equal to zero (by hypothesis, all of the variables contained in z* are independent) the following necessary endpoint/cornerpoint conditions are obtained

$$g_\ell = 0 \quad , \tag{2.28}$$

$$\lambda_{i\alpha} = \frac{\partial G}{\partial y_{i\alpha}} \quad , \tag{2.29}$$

$$\lambda_{i\beta} = - \frac{\partial G}{\partial y_{i\beta}} \quad , \tag{2.30}$$

$$H^q_\alpha = - \frac{\partial G}{\partial t_\alpha} \quad , \tag{2.31}$$

$$H^q_\beta = \frac{\partial G}{\partial t_\beta} \quad . \tag{2.32}$$

Equations (2.29) - (2.32) are the same as those obtained by Mason[9] for the case in which the state variables y_i and multipliers λ_i maintain the same identity for each of the subarcs. Mason considers a more general case where this identity need not be maintained and obtains his results by mapping the several subarcs into a single interval.

Note that equation (2.28) is identical to the endpoint/corner conditions (2.3), thus the optimal endpoints/cornerpoints for z* satisfy the same conditions as required for the functional (2.4).

Special Cases

Because of the general way in which the various subarcs were assumed to lie as shown in Figure 1.1, the results presented here are applicable to a number of different situations, some of which have been previously investigated. A number of special cases may be specified. For brevity in the following discussion only 3 subarcs will be assumed for each case. Extension to more subarcs is obvious.

Case I - Normal Corners - Assume that the subarcs are of such a nature that the state variables and time are continuous at points of discontinuity in the control as shown in Figure (2.1).

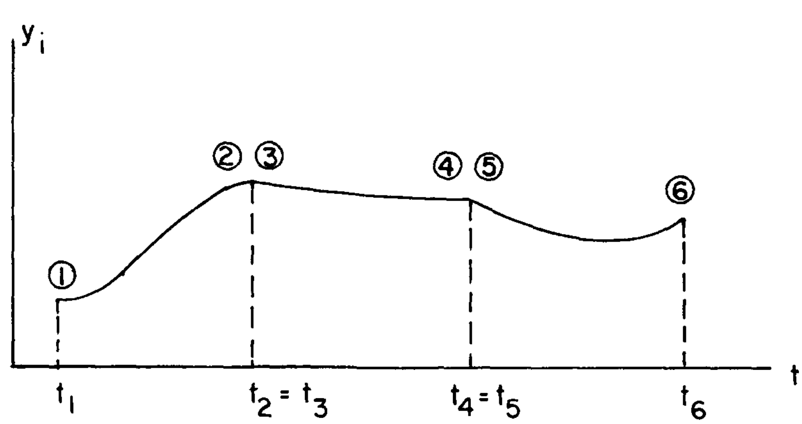

FIGURE 2.1 NORMAL CORNERS

By incorporating as corner conditions

$$t_2 - t_3 = 0 \quad , \tag{2.33}$$

$$t_4 - t_5 = 0 \quad , \tag{2.34}$$

$$y_{i2} - y_{i3} = 0 \quad , \tag{2.35}$$

$$y_{i4} - y_{i5} = 0 \quad , \tag{2.36}$$

the G function may be written as

$$G = \mu_1(t_2 - t_3) + \mu_2(t_4 - t_5) + \mu_3(y_{i2} - y_{i3}) + \mu_4(y_{i4} - y_{i5}) + G' , \tag{2.37}$$

where G' is composed of the remaining endpoint/cornerpoint conditions. It is assumed that G' does not contain t_3, t_5, y_{i3} or y_{i5} since any condition on these points can be replaced by a condition on t_2, t_4, y_{i2} and y_{i4}. From equations (2.29) - (2.32), the following conditions are obtained for the cornerpoints.

First Corner Point

$$\lambda_{i2} = - \mu_3 - \frac{\partial G'}{\partial y_{i2}} \quad , \tag{2.38}$$

$$\lambda_{i3} = - \mu_3 \quad , \tag{2.39}$$

$$H_2^1 = \mu_1 + \frac{\partial G'}{\partial t_2} \quad , \tag{2.40}$$

$$H_3^2 = \mu_1 \quad . \tag{2.41}$$

Second Corner Point

$$\lambda_{i4} = - \mu_4 - \frac{\partial G'}{\partial y_{i4}} \quad , \tag{2.42}$$

$$\lambda_{i5} = -\mu_4 \quad , \tag{2.43}$$

$$H_4^2 = \mu_2 + \frac{\partial G'}{\partial t_4} \quad , \tag{2.44}$$

$$H_5^2 = \mu_2 \quad . \tag{2.45}$$

Eliminating the constant multipliers yields the following conditions for the first corner point

$$\lambda_{i2} = \lambda_{i3} - \frac{\partial G'}{\partial y_{i2}} \quad , \tag{2.46}$$

$$H_2^1 = H_3^2 + \frac{\partial G'}{\partial t_2} \quad . \tag{2.47}$$

For the second corner point equations (2.42) - (2.45) yield

$$\lambda_{i4} = \lambda_{i5} - \frac{\partial G'}{\partial y_{i4}} \quad , \tag{2.48}$$

$$H_4^2 = H_4^5 + \frac{\partial G'}{\partial t_4} \quad . \tag{2.49}$$

If G' is composed of endpoint conditions only, the H function and Lagrange multipliers are continuous at each corner point. If in addition $H^1 = H^2 = H^3 = H$, then these results are equivalent to the well known Erdmann-Weierstrass corner conditions. A derivation of these conditions using control notation is given by Lutz.[10]

<u>Case II - Discontinuous State Variable corners</u> - For the situation in which $t_2 = t_3$, $t_4 = t_5$ as shown in Figure (2.2) the results of Mason, Dickerson and Smith are easily obtained.[4]

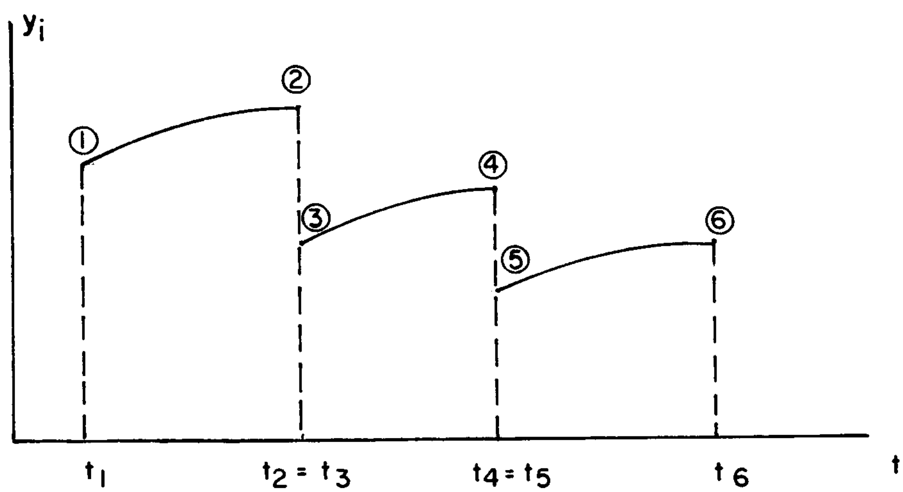

FIGURE 2.2 DISCONTINUOUS STATE VARIABLE CORNERS

In this case, two of the restrictions are given by

$$g_1 = t_2 - t_3 = 0 \quad , \tag{2.50}$$

$$g_2 = t_4 - t_5 = 0 \quad . \tag{2.51}$$

The G function in this case is given by

$$G = \mu_1(t_2 - t_3) + \mu_2(t_4 - t_5) + G' \quad , \tag{2.52}$$

where G' is composed of the remaining endpoint and cornerpoint conditions. It is assumed that G' does not contain t_3 or t_5 (any conditions on t_3 or t_5 can be expressed in terms of t_2 and t_4). Among the endpoint/cornerpoint conditions given by equations (2.29) - (2.32) will be the following

$$H_2^1 = +\mu_1 + \frac{\partial G'}{\partial t_2} \quad , \quad H_4^2 = +\mu_2 + \frac{\partial G'}{\partial t_4} \quad , \tag{2.53}$$

$$H_3^2 = +\mu_1 \quad , \quad H_5^3 = +\mu_2 \quad . \tag{2.54}$$

Eliminating μ_1 and μ_2 between these equations yields

$$H_2^1 = + H_3^2 + \frac{\partial G'}{\partial t_2} \quad , \tag{2.55}$$

$$H_4^2 = + H_5^3 + \frac{\partial G'}{\partial t_4} \quad . \tag{2.56}$$

The corner conditions related to the Lagrange multipliers remain unchanged from equations (2.29) and (2.30).

Case III - Unconnected, Non Overlapping Corners - If both the state variables and time are discontinuous at the cornerpoints as depicted in Figure (2.3), then the endpoint/cornerpoint conditions remain unchanged and are given by equations (2.29) - (2.32).

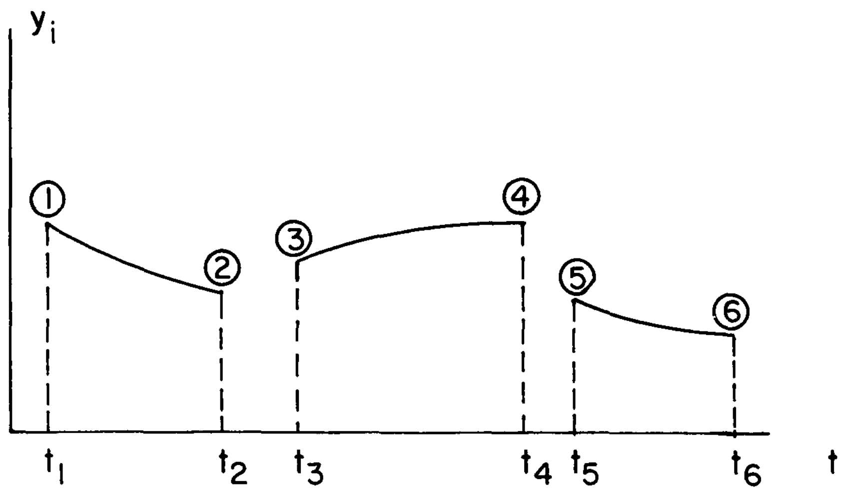

FIGURE 2.3 UNCONNECTED, NON OVERLAPPING, TRAJECTORIES

Vincent and Mason[5] have previously investigated problems of this nature which can result from the imposition of restrictions on the control variable. They have shown that if the dynamical equations of constraint containing a single control variable can be analytically integrated along any segment of the trajectory along which the control law is restricted, then the restricted segment can be effectively eliminated and a trajectory such as shown in Figure (2.3) is obtained. The results presented here can be applied directly to such a situation.

Case IV - Branched Trajectories - The situation in which an endpoint of one trajectory lies on another trajectory is termed branched trajectories because of the appearance of the trajectories as shown in Figure 2.4.

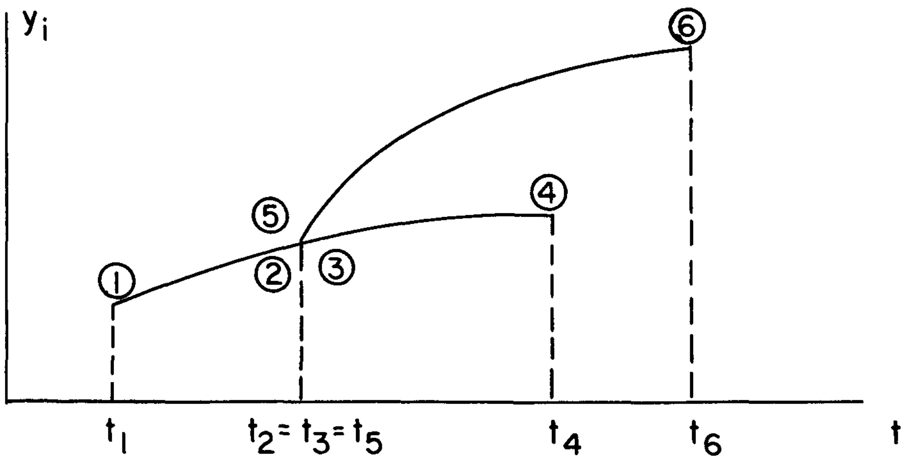

FIGURE 2.4 BRANCHED TRAJECTORIES

Among the corner conditions that are specified for this problem will be the following

$$t_2 = t_3 \quad , \tag{2.57}$$

$$t_2 = t_5 \quad , \tag{2.58}$$

$$y_{i2} = y_{i3} \quad , \tag{2.59}$$

$$y_{i2} = y_{i5} \quad . \tag{2.60}$$

The G function may be written as

$$G = \mu_1(t_2 - t_3) + \mu_2(t_2 - t_5) + \mu_3 (y_{i2} - y_{i3}) + \mu_4(y_{i2} - y_{i5}) + G' \tag{2.61}$$

where G' is composed of the remaining endpoint/cornerpoint conditions but does not contain y_{i3}, y_{i5}, t_3 or t_5. Among the endpoint/cornerpoint conditions given by equations (2.29) - (2.32) will be the following

$$\lambda_{i2} = -\mu_3 - \mu_4 - \frac{\partial G'}{\partial y_{i2}} \quad , \tag{2.62}$$

$$\lambda_{i3} = - \mu_3 \quad , \tag{2.63}$$

$$\lambda_{i5} = - \mu_4 \quad , \tag{2.64}$$

$$H_2^1 = \mu_1 + \mu_2 + \frac{\partial G'}{\partial t_2} \quad , \tag{2.65}$$

$$H_3^2 = \mu_1 \quad , \tag{2.66}$$

$$H_5^3 = \mu_2 \quad . \tag{2.67}$$

Eliminating the multipliers between these equations yields the following conditions to be satisfied at the branch point.

$$\lambda_{i2} = \lambda_{i3} + \lambda_{i5} - \frac{\partial G'}{\partial y_{i2}} \tag{2.68}$$

$$H_2^1 = H_3^2 + H_5^3 + \frac{\partial G'}{\partial t_2} \tag{2.69}$$

Problems involving branched trajectories or overlapping trajectories represent new situations to which the theory presented here may be readily applied. Two examples involving branched trajectories and one involving an overlapping trajectory are presented in Section III in order to illustrate applications of the theory.

SECTION III

APPLICATIONS*

A Geometric Example

A Minimum Distance Problem with Branches - The extended problem of Bolza presented in Section II provides a modus operandi for determining optimal trajectories with variables which may be multivalued because the various subarcs overlap.

In order to exhibit the salient features of branched trajectory optimization, a simple minimum distance problem will first be examined. The problem is to determine the shortest path, possibly branched which connects three noncolinear points. Figure (3.1) shows a candidate path with a branch occurring at point 2.

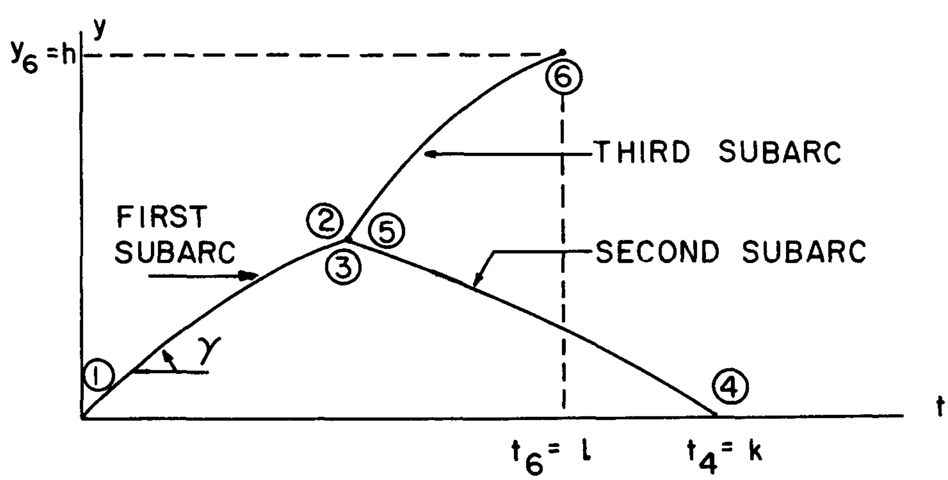

FIGURE 3.1 BRANCHED TRAJECTORY CONNECTING THREE POINTS

Point 1 is assumed to lie at the origin, point 4 on the t axis and point 6 somewhere in the first quadrant. The three branches correspond to three

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* Numerous applications to the theory presented in Section II, including those presented here are to be found in Mason's Dissertation.[9]

subarcs of the extended problem of Bolza.

The kinematical relation to be satisfied at every point along the various subarcs is given by

$$\dot{y} = \tan \gamma \quad , \tag{3.1}$$

where γ is the trajectory angle shown in Figure 3.1 and the independent variable t is displacement along the horizontal axis. The quantity to be extremized in this case is total arc length given by

$$I = \int_{t_1}^{t_2} \sec \gamma \, dt + \int_{t_3}^{t_4} \sec \gamma dt + \int_{t_5}^{t_6} \sec \gamma dt. \tag{3.2}$$

With the subarcs numbered as shown in Figure 3.1, the endpoint/cornerpoint contitions take the form

$$g_1 = t_1 = 0 \quad , \tag{3.3}$$

$$g_2 = y_1 = 0 \quad , \tag{3.4}$$

$$g_3 = t_4 - k = 0 \quad , \tag{3.5}$$

$$g_4 = y_4 = 0 \quad , \tag{3.6}$$

$$g_5 = t_6 - \ell = 0 \quad , \tag{3.7}$$

$$g_6 = y_6 - h = 0 \quad , \tag{3.8}$$

$$g_7 = t_3 - t_2 = 0 \quad , \tag{3.9}$$

$$g_8 = y_3 - y_2 = 0 \quad , \tag{3.10}$$

$$g_9 = t_5 - t_2 = 0 \quad , \tag{3.11}$$

$$g_{10} = y_5 - y_2 = 0 \quad . \tag{3.12}$$

<u>Necessary Conditions</u> - The H function for this problem will be the same for each subarc (ie. $H^1 = H^2 = H^3 = H$) and is given by

$$H = \lambda \tan \gamma - \sec \gamma. \tag{3.13}$$

Hence along each subarc, in addition to equation (3.1), the following Euler equations must be satisfied:

$$\lambda = \text{constant}, \tag{3.14}$$

and

$$\lambda \sec^2\gamma - \sec\gamma \tan\gamma = 0 \quad . \tag{3.15}$$

Thus

$$\sin\gamma = \lambda = \text{constant} \quad . \tag{3.16}$$

Equation (3.16) establishes that the three subarcs are straight line segments. The slope of the line segments may now be determined from the endpoint/cornerpoint conditions. The G' function for this case is given by

$$G' = \mu_1 t_1 + \mu_2 y_1 + \mu_3(t_4 - k) + \mu_4 y_4 + \mu_5(t_6 - \ell) + \mu_6(y_6 - h) \tag{3.17}$$

Application of the endpoint/cornerpoint conditions, equations (2.29) - (2.32) at the endpoints 1, 4, and 6 yield no useful information. However, the following information is obtained from equations (2.59) and (2.60) for the branch point

$$\lambda_2 = \lambda_3 + \lambda_5 \quad , \tag{3.18}$$

and

$$H_2 = H_3 + H_5 \quad . \tag{3.19}$$

<u>Solution</u> - Substituting equation (3.16) into equation (3.13) yields

$$H = -\cos\gamma. \tag{3.20}$$

This information along with equation (3.16) may now be substituted into equations (3.18) and (3.19). Squaring and adding the resultant expressions yields the information

$$\cos(\gamma_5 - \gamma_3) = -\frac{1}{2} \quad . \tag{3.21}$$

Thus the second and third subarc intersect at an angle of 120 degrees. Assuming $\gamma_5 > \gamma_3$ gives the result

$$\gamma_5 = \gamma_3 + 120^\circ \quad . \tag{3.22}$$

Substituting equation (3.22) into equation (3.19) yields

$$\cos\gamma_2 = \cos\gamma_3 + \cos(\gamma_3 + 120^\circ) \quad . \tag{3.23}$$

By use of obvious trigimetric identities the above equation can be shown to reduce to

$$\cos \gamma_2 = \cos(\gamma_3 + 60^\circ) \quad . \tag{3.24}$$

Thus

$$\gamma_3 = \gamma_2 - 60^\circ \quad , \tag{3.25}$$

and from equation (3.22)

$$\gamma_5 = \gamma_2 + 60^\circ \quad . \tag{3.26}$$

The solution for the coordinates of the branch point may now be obtained by integrating the constraint equation on each subarc and using the appropriate boundary conditions in evaluating the constants of integration. For the problem shown in Figure 3.1, integration of equation (3.1) for the first, second, and third subarc become

$$y_2 = \tan \gamma_2 t_2 \quad , \tag{3.27}$$

$$h - y_2 = \tan(\gamma_2 + 60^\circ)(\ell - t_2) \quad , \tag{3.28}$$

$$-y_2 = \tan(\gamma_2 - 60^\circ)(k - t_2) \quad . \tag{3.29}$$

By fixing the coordinates of point $\mathcal{G}(\ell, h)$ the above three equations may be solved for t_2, y_2, and γ_2, thus fixing the coordinates of the branch point.

It is interesting to examine the geometric solution to this problem. Figure 3.2 illustrates the solution for $\ell = h = 1$ for various values of k.

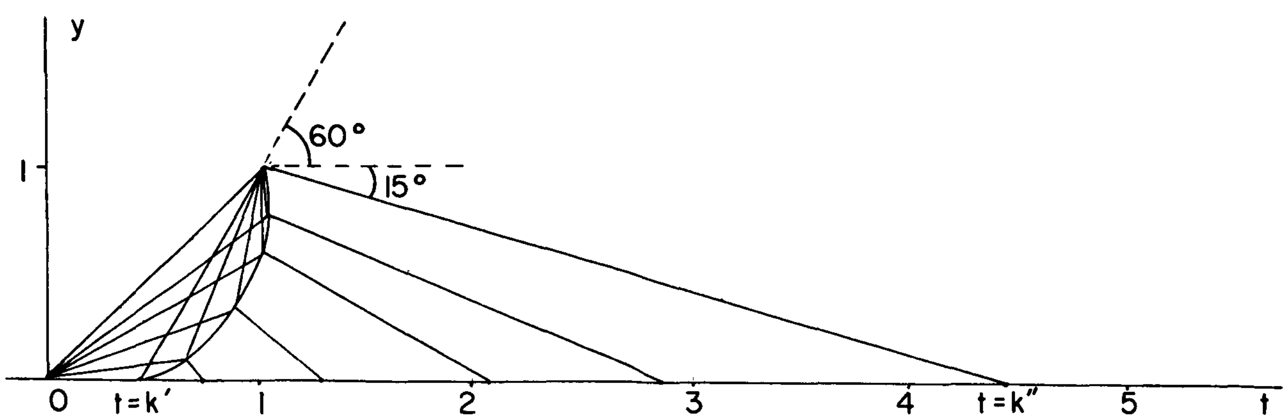

FIGURE 3.2 SOLUTION FOR THE SHORTEST PATH CONNECTING THE THREE POINTS (0,0), (1,1), AND (k,0), FOR VARIOUS VALUES OF k

It is apparent from the geometry of the above figure that three branches are obtained if $k' < k < k''$ where

$$k' = \frac{\tan 60^{\circ} - 1}{\tan 60^{\circ}} \quad \text{and} \quad k'' = \frac{1 + \tan 15^{\circ}}{\tan 15^{\circ}} \quad . \tag{3.30}$$

The solution to the problem degenerates to two subarcs at $k = k'$ and $k = k''$. For k less than k' or greater than k" the three-subarc solution is replaced by a two subarc solution.

Multiple Satellite Launch Vehicle

Description of Problem - Consider the problem of designing a multistage rocket capable of inserting two payloads into different orbits in a single launching. Figure 3.3 shows the representative trajectory for a rocket vehicle which splits into two stages at the branch point. From the branch point on, each stage carries its own payload. It will be assumed that when the vehicle splits into two stages, the structural mass of the first stages will be discarded. Hence the trajectory will be branched with respect to the variables r, ϕ, v, and γ, but will be discontinuous with respect to the mass.

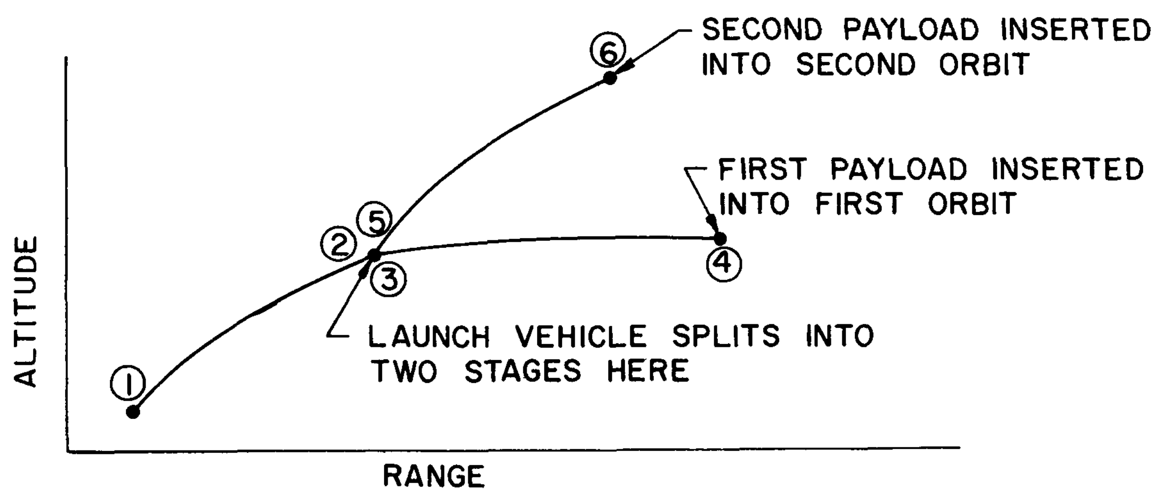

FIGURE 3.3 TRAJECTORY FOR A MULTIPLE SATELLITE LAUNCH VEHICLE

Using the nomenclature defined in Figure 3.4 assuming only the gravitational and thrust forces shown, the dynamical and kinematical relations for each subarc become

$$\dot{r} = v \sin \gamma \quad , \tag{3.31}$$

$$\dot{\phi} = \frac{v}{r} \cos \gamma \quad , \tag{3.32}$$

$$\dot{v} = \frac{Tq}{m} \cos X - \frac{GM}{r^2} \sin \gamma \quad , \tag{3.33}$$

$$\dot{\gamma} = \frac{T^q}{mv} \sin X + \left(\frac{v}{r} - \frac{GM}{r^2 v}\right) \cos \gamma \quad , \tag{3.34}$$

$$\dot{m} = - \frac{T^q}{V_e^{\,q}} \quad , \tag{3.35}$$

where q = 1, 2, 3.

It is assumed that for each stage the thrust T^q and exhaust velocity $V_e^{\,q}$ are constants. Hence there are 5 state variables (r, ϕ, v, γ, m) and one control variable X.

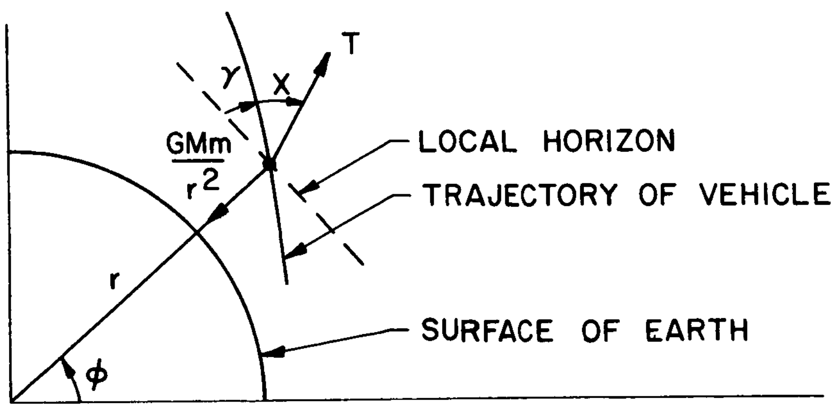

FIGURE 3.4 COORDINATE SYSTEM FOR A MULTIPLE SATELLITE LAUNCH VEHICLE

The performance criteria in this case will be the minimization of initial weight. Thus

$$g = m_1 \quad . \tag{3.36}$$

It will be assumed that the initial state, except for the mass is fixed.

$$g_1 = t_1 = 0 \quad , \tag{3.37}$$

$$g_2 = r_1 - c_1 = 0 \quad , \tag{3.38}$$

$$g_3 = \phi_1 - c_2 = 0 \quad , \tag{3.39}$$

$$g_4 = v_1 - c_3 = 0 \quad , \tag{3.40}$$

$$g_5 = \gamma_1 - c_4 = 0 \quad . \tag{3.41}$$

The terminal states for the two payloads will be assumed to be orbits defined by specification of the final energy per unit mass and angular momentum per unit mass for each vehicle.

$$g_6 = \frac{1}{2} v_4^2 - \frac{GM}{r_4} - E^2 = 0 \quad , \tag{3.42}$$

$$g_7 = r_4 v_4 \cos \gamma_4 - h^2 = 0 \quad , \tag{3.43}$$

$$g_8 = \frac{1}{2} v_6^2 - \frac{GM}{r_6} - E^3 = 0 \quad , \tag{3.44}$$

$$g_9 = r_6 v_6 \cos \gamma_6 - h^3 = 0 \quad , \tag{3.45}$$

Statements must be made as to how the mass is related at the branch point. Following the methods of Mason, Dickerson, and Smith,[4] if the structural weight of each stage is assumed proportional to the weight of fuel and, hence, the burning time (mass flow rate is constant), then the mass dropped at the end of the first stage is given by

$$g_{10} = m_2 - m_3 - m_5 - k^1(t_2 - t_1) = 0, \tag{3.46}$$

where k^1 is the constant of proportionality for the first stage. Similarly if P^2 and P^3 are the desired values for the two payloads, the final masses are related to the burning times of their stages by

$$g_{11} = m_4 - k^2(t_4 - t_2) - P^2 = 0 \quad , \tag{3.47}$$

$$g_{12} = m_6 - k^3(t_6 - t_2) - P^3 = 0 \quad . \tag{3.48}$$

Additional requirements must now be given to assure that the state variable subarcs with respect to r, ϕ, v, and γ fit together at the branch point; the following boundary conditions are statements to that effect.

$$g_{13} = t_3 - t_2 = 0 \quad , \tag{3.49}$$

$$g_{14} = t_5 - t_2 = 0 \quad , \tag{3.50}$$

$$g_{15} = r_3 - r_2 = 0 \quad , \tag{3.51}$$

$$g_{16} = r_5 - r_2 = 0 \quad , \tag{3.52}$$

$$g_{17} = \phi_3 - \phi_2 = 0 \quad , \tag{3.53}$$

$$g_{18} = \phi_5 - \phi_2 = 0 \quad , \tag{3.54}$$

$$g_{19} = v_3 - v_2 = 0 \quad , \tag{3.55}$$

$$g_{20} = v_5 - v_2 = 0 \quad , \tag{3.56}$$

$$g_{21} = \gamma_3 - \gamma_2 = 0 \quad , \tag{3.57}$$

$$g_{22} = \gamma_5 - \gamma_2 = 0 \quad . \tag{3.58}$$

Optimum Switching Time - By substituting into the H function defined by equation (2.7) (in this case $F^q = 0$) the necessary conditions related to the optimal path of the rocket vehicle for each stage may be obtained from equations (2.10) - (2.12). These equations may then be solved for the optimal stearing angle X. For brevity these equations will not be discussed here, except to note that since H is not explicitly a function of time, it is constant along each subarc. Rather, the following analysis will illustrate how the corner conditions may be used to determine the optimum switching time.

The G' function for this case is given by

$$G' = m_1 + \mu_1(t_1) + \mu_2(r_1 - C_1) + \ldots + \mu_{12}[m_6 - k^3(t_6 - t_5) - P^3] \; . \tag{3.59}$$

Application of the endpoint/cornerpoint conditions, equations (2.29) - (2.32) at the endpoints 1, 4, and 6, yields

Point 1:

$$\lambda_{r1} = \mu_2 \quad , \tag{3.60}$$

$$\lambda_{\phi 1} = \mu_3 \quad , \tag{3.61}$$

$$\lambda_{v1} = \mu_4 \quad , \tag{3.62}$$

$$\lambda_{\gamma 1} = \mu_5 \quad , \tag{3.63}$$

$$\lambda_{m1} = 1 \quad , \tag{3.64}$$

$$H_1' = - \mu_1 - \mu_{20} k_1 \quad , \tag{3.65}$$

Point 4:

$$\lambda_{r4} = - \mu_6 \frac{GM}{r_4^2} - \mu_7 v_4 \cos \gamma_4 \quad , \tag{3.66}$$

$$\lambda_{\phi 4} = 0 \quad , \tag{3.67}$$

$$\lambda_{v4} = -\mu_6 v_4 - \mu_7 r_4 \cos \gamma_4 \quad , \tag{3.68}$$

$$\lambda_{\gamma 4} = \mu_7 r_4 v_4 \sin \gamma_4 \quad , \tag{3.69}$$

$$\lambda_{m4} = -\mu_{11} \quad , \tag{3.70}$$

$$H_4^{\,2} = -\mu_{11} k^2 \quad , \tag{3.71}$$

Point 6:

$$\lambda_{r6} = -\mu_8 \frac{GM}{r_6^2} - \mu_9 v_6 \cos \gamma_6 \quad , \tag{3.72}$$

$$\lambda_{\phi 6} = 0 \quad , \tag{3.73}$$

$$\lambda_{v6} = -\mu_8 v_6 - \mu_9 r_6 \cos \gamma_6 \quad , \tag{3.74}$$

$$\lambda_{\gamma 6} = \mu_9 r_6 v_6 \sin \gamma_6 \quad , \tag{3.75}$$

$$\lambda_{m6} = \mu_{12} \quad , \tag{3.76}$$

$$H_6^{\,3} = -\mu_{12} k^3 \quad . \tag{3.77}$$

Application of equations (2.62) and (2.63) at the branch point yield

$$\lambda_{r2} = \lambda_{r3} + \lambda_{r5} \quad , \tag{3.78}$$

$$\lambda_{\phi 2} = \lambda_{\phi 3} + \lambda_{\phi 5} \quad , \tag{3.79}$$

$$\lambda_{v2} = \phi_{v3} + \lambda_{v5} \quad , \tag{3.80}$$

$$\lambda_{\gamma 2} = \lambda_{\gamma 3} + \lambda_{\gamma 5} \quad , \tag{3.81}$$

$$H_2^1 = H_3^2 + H_5^3 - \mu_{10} k^1 + \mu_{11} k^2 + \mu_{12} k^3 \quad . \tag{3.82}$$

Finally application of the endpoint/cornerpoint conditions (2.29) - (2.32) at the points 2, 3, and 5 for the mass yields

$$\lambda_{m2} = -\mu_{10} \quad , \tag{3.83}$$

$$\lambda_{m3} = -\mu_{10} \quad , \tag{3.84}$$

$$\lambda_{m5} = -\mu_{10} \quad , \quad . \tag{3.85}$$

Equations (3.60) - (3.63) and (3.65) yield no usable information. The constant multipliers in equations (3.66) (3.68) and (3.67) may be eliminated to yield

$$\lambda_{r4} v_4 \sin \gamma_4 - \lambda_{v4} \frac{GM}{r_4^2} \sin \gamma_4 + \lambda_{\gamma 4} \left[\frac{v_4}{r_4} - \frac{GM}{V_4 r_4^2}\right] \cos \gamma_4 = 0 \quad . \tag{3.86}$$

Similarly the constant multipliers in equations (3.72), (3.74) and (3.75) maybe eliminated to yield

$$\lambda_{r6} v_6 \sin \gamma_6 - \lambda_{v6} \frac{GM}{r_6^2} \sin \gamma_6 + \lambda_{\gamma 6} \left[\frac{v_6}{r_6} - \frac{GM}{V_6 r_6^2}\right] \cos \gamma_6 = 0 \quad . \tag{3.87}$$

Equations (3.70) and (3.71) combine to yield

$$H_4^2 - \lambda_{m4} k^2 = 0 \quad , \tag{3.88}$$

and equations (3.76) and (3.77) combine to yield

$$H_6^3 - \lambda_{m6} k^3 = 0 \quad . \tag{3.89}$$

Substituting equations (3.83), (3.70), and (3.76) into equation (3.82) yields

$$H_2^1 = \lambda_{m2} k^1 + (H_3^2 - \lambda_{m4} k^2) + (H_5^3 - \lambda_{m6} k^3) \quad . \tag{3.90}$$

Since H is constant along each subarc

$$H_3^2 = H_4^2 \text{ and } H_5^3 = H_6^3 \quad . \tag{3.91}$$

Hence equations (3.88) and (3.89) may be used to reduce equation (3.90) to

$$H_2^1 - \lambda_{m2} k^1 = 0 \quad . \tag{3.92}$$

Finally equations (3.83), (3.84) and (3.85) yield the result.

$$\lambda_{m2} = \lambda_{m3} \quad , \tag{3.93}$$

$$\lambda_{m2} = \lambda_{m5} \quad . \tag{3.94}$$

The results given by equations (3.88), (3.89) and (3.92) are of the same form as results given by Mason, Dickerson, and Smith[4] for the multistage booster optimization problem. The actual solution to the problem presented here is numerically quite difficult. A procedure using the above conditions would be as follows:

1. Guesses are made for the unknown initial values of the Lagrange multipliers.
2. The equations of motion plus the Euler equations for the first subarc are integrated until the condition (3.92) is met.
3. The changes in the Lagrange multiplier's for the next two subarcs are made in accordance with equations (3.78) - (3.81) and (3.93), (3.94).
4. The equations of motion and the Euler equations are integrated on the second and third subarc until conditions (3.88) and (3.89) are met.
5. At these points, checks must be made to see if conditions (3.86) and (3.87) have been met. If so, a solution has been obtained. If not, steps 1 - 4 must be repeated until they are.

Two Vehicle Rendezvous

Description of Problem - In order to demonstrate the technique of solving a rendezvous problem using the methods developed here for overlapping trajectories, a simplified rendezvous situation will be assumed. Motion will be confined to a plane and in addition, the two vehicles are assumed to be of constant mass and operate in a uniform gravitational field under a constant thrust force. The situation is depicted in Figure 3.5

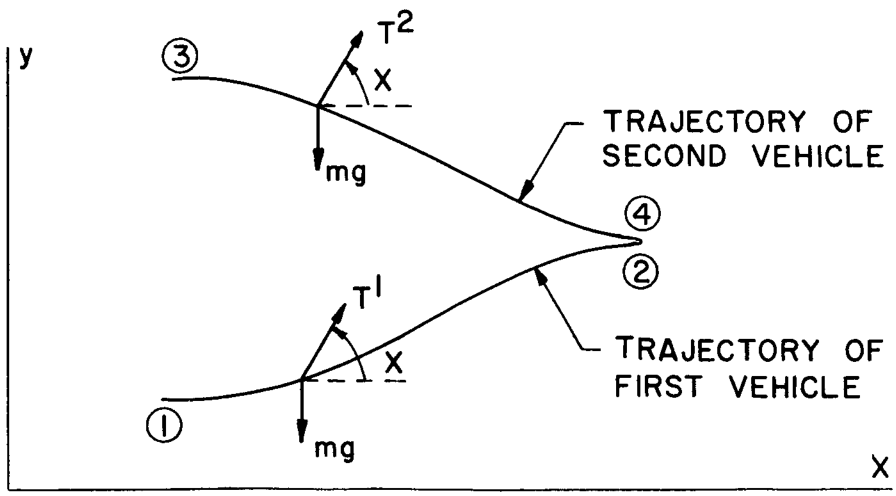

FIGURE 3.5 TWO VEHICLE RENDEZVOUS IN THE PLANE

The kinematical and dynamical equations of constraint for each vehicle are given by

$$\dot{x} = u \quad , \tag{3.95}$$

$$\dot{y} = v \quad , \tag{3.96}$$

$$\dot{u} = (\frac{T}{m})^q \cos X \quad , \tag{3.97}$$

$$\dot{v} = (\frac{T}{m})^q \sin X - g \quad , \tag{3.98}$$

where $q = 1, 2$.

It will be assumed that a minimum time rendezvous is required, hence

$$g = t_2 \quad . \tag{3.99}$$

The initial state of the two vehicles will be assumed fixed at time zero. Hence, no useful information from the endpoint/corner conditions will be obtained for points 1 and 3. To bring the two vehicles together, the time, coordinates and velocity must be matched at the branch point. These requirements are given by the following conditions

$$g_1 = t_2 - t_4 = 0 \quad , \tag{3.100}$$

$$g_2 = x_2 - x_4 = 0 \quad , \tag{3.101}$$

$$g_3 = y_2 - y_4 = 0 \quad , \tag{3.102}$$

$$g_4 = u_2 - u_4 = 0 \quad , \tag{3.103}$$

$$g_5 = v_2 - v_4 = 0 \tag{3.104}$$

<u>Optimum Stearing Angles</u> - The H function for each subarc is given by

$$H^1 = \lambda_x u + \lambda_y v + \lambda_u (\frac{T}{m})^1 \cos X + \lambda_v [(\frac{T}{m})^1 \sin X - g] \quad , \tag{3.105}$$

$$H^2 = \lambda_x u + \lambda_y v + \lambda_u (\frac{T}{m})^2 \cos X + \lambda_v [(\frac{T}{m})^2 \sin X - g] \quad . \tag{3.106}$$

Applying Euler equations (2.10) and (2.12) to each subarc yields identical necessary conditions

$$\dot{\lambda}_x = 0 \quad , \tag{3.107}$$

$$\dot{\lambda}_y = 0 \quad , \tag{3.108}$$

$$\dot{\lambda}_y = -\lambda_x \quad , \tag{3.109}$$

$$\dot{\lambda}_v = -\lambda_y \quad , \tag{3.110}$$

$$\tan X = \frac{\lambda_v}{\lambda_u} \quad . \tag{3.111}$$

That part of the G function applicable to the branch point is given by

$$G = t_2 + \mu_1(t_2 - t_4) + \mu_2(x_2 - x_4) + \mu_3(y_2 - y_4) + \mu_4(u_2 - u_4) + \mu_5(v_2 - v_4) \tag{3.112}$$

Application of the endpoint/cornerpoint conditions (2.29) - (2.32) at the points 2 and 4 yields

Point 2:

$$\lambda_{x2} = -\mu_2 \quad , \tag{3.113}$$

$$\lambda_{y2} = -\mu_3 \quad , \tag{3.114}$$

$$\lambda_{u2} = -\mu_4 \quad , \tag{3.115}$$

$$\lambda_{v2} = -\mu_5 \quad , \tag{3.116}$$

$$H_2 = 1 + \mu_1 \quad , \tag{3.117}$$

Point 4:

$$\lambda_{x4} = \mu_2 \quad , \tag{3.118}$$

$$\lambda_{y4} = \mu_3 \quad , \tag{3.119}$$

$$\lambda_{u4} = \mu_4 \quad , \tag{3.120}$$

$$\lambda_{v4} = \mu_5 \quad , \tag{3.121}$$

$$H_4 = -\mu_1 \quad . \tag{3.122}$$

Eliminating the constant Lagrange multipliers between these equations yields

$$\lambda_{x2} = -\lambda_{x4} \quad , \tag{3.123}$$

$$\lambda_{y2} = -\lambda_{y4} \quad , \tag{3.124}$$

$$\lambda_{u2} = -\lambda_{u4} \quad , \tag{3.125}$$

$$\lambda_{v2} = -\lambda_{v4} \quad , \tag{3.126}$$

$$H_2 = 1 - H_4 \quad . \tag{3.127}$$

The multipliers λ_x and λ_y are constants on each subarc and if

$$\lambda_x = a \quad , \tag{3.128}$$

$$\lambda_y = b \quad , \tag{3.129}$$

on the first subarc then by equations (3.123) and (3.124)

$$\lambda_x = -a \quad , \tag{3.130}$$

$$\lambda_y = -b \quad , \tag{3.131}$$

on the second subarc. Integrating equations (3.109) and (3.110) on the first subarc yields

$$\lambda_u = c - at \quad , \tag{3.132}$$

$$\lambda_v = d - bt \quad , \tag{3.133}$$

and for the second subarc

$$\lambda_u = e + at \quad , \tag{3.134}$$

$$\lambda_v = f + bt \quad . \tag{3.135}$$

Writing equations (3.132) - (3.135) for the rendezvous point and applying equations (3.125) and (3.126) yields the result

$$e = -c \tag{3.136}$$

$$f = -d \tag{3.137}$$

Thus the control law for the first subarc is given by

$$\tan X = \frac{d - bt}{c - at} \tag{3.138}$$

and for the second subarc

$$\tan X = \frac{-(d - bt)}{-(c - at)} \tag{3.139}$$

Thus it is concluded that the two vehicles thrust in parallel but opposite directions.

SECTION IV

DISCUSSION AND CONCLUSIONS

The theory presented in Section II is developed for problems which are to be optimized over a number of subarcs which may or may not be overlapping. The method used in this section is based on the concept of extremizing a functional which is related to the problem of Bolza, but is in such a form that standard calculus of variations techniques can be used to obtain necessary optimizing conditions related to the path and ordinary maxima and minima techniques can be used to obtain necessary optimizing conditions related to the endpoints/cornerpoints.

The results presented in Section II are equivalent to the results obtained by Mason[9] although by a completely different procedure. Mason's results were obtained by extending Denbows[12] method for handling problems with boundary conditions which specify restrictions at corner points lying between the ends of a trajectory. In order to develop necessary conditions, Denbow transformed his problem into the standard Bolza problem for which a fairly complete theory is available. He then inverted the transformation in order to obtain necessary conditions for the original problem. Mason, using state, control variable notation carried Denbow's work another step to include, as was done here, problems whose variables are defined over disjoint intervals. The procedure presented here represents a somewhat simpler method for obtaining the necessary endpoint/cornerpoint conditions than the procedure used by Mason.

The endpoint/cornerpoint conditions obtained in Section II unify the results of previous investigations which may be obtained as special cases. The same results may be used to handle problems with normal corners, discontinuous state variable corners, and unconnected corners, all of which may have the specification of additional restrictions at the cornerpoints.

The applications in Section III by no means exhausts the uses to which the theory of Section II can be applied but are presented to illustrate the procedure of using the corner conditions for a solution. The results of Section II can be applied to any optimization problem composed of more than one subarc.

It is interesting to note that the results obtained for the two vehicle rendezvous could also have been obtained from Issacs[13] theory of differential games. In that context this problem could be called the "two player rendezvous" and falls under the heading of cooperative games. The term "cooperative arises from the fact that both vehicles are attempting to minimize the same quantity, namely the time to rendezvous. It is apparent that cooperative games of the nature as presented in this example may be treated as overlapping trajectories with a common final point.

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