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ON THE OPTIMAL ANGULAR VELOCITY CONTROL OF ASYMETRICAL SPACE VEHICLES*

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Atif S. Debs**

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Abstract

The problem of reducing the angular velocities of a space vehicle to zero is considered. The required control torques are found as functions of the angular momenta. Both linear and nonlinear time-invariant feedback control systems are determined. The method of solution is based on the "inverse problem of optimal control."

* This research was conducted at the M.I.T. Electronic Systems Laboratory with support extended by the Electronics Research Center of the National Aeronautics and Space Administration under research grant NGR-22-009(124).

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1. Introduction

The purpose of this paper is to derive linear and nonlinear optimal feedback systems which will drive any initial angular velocities of a space vehicle[1] to zero by the suitable application of control torques. To eliminate time-varying feedback control laws, the control problem is considered on the semi-infinite control time-interval $[0,\infty)$.

The method used for the solution is based upon the solution of an inverse optimal control problem so as to obtain a class of cost criteria and corresponding optimal controls for the space body. Since the general inverse problem of optimal control[2,3,4] is not completely resolved, this study is specialized in order to have a rigorous and consistent treatment.

Section 2 contains the differential equations satisfied by the components of the angular momentum vector. In section 3, the maximum principle and the Hamilton-Jacobi equation are used to prove that a linear time-invariant feedback system is optimal with respect to a quadratic cost functional in the angular momenta and control torques. Section 4 contains the main results which indicate the relationship between a wide class of nonlinear feedback control systems and the cost functionals that are minimized; the essential property of these controls and of the associated cost functionals is that they lead to a minimum cost which is a quadratic function of the state.

2. Definition of the Problem

Consider a body in space. Let 1,2,3 denote the body-fixed principal axes through its center of mass. Let I_1, I_2, I_3 denote the moments of inertia about the principal axes. Let $\omega_1,\omega_2,\omega_3$ be the angular velocities and let u_1,u_2,u_3 denote the control torques (generated, say, by gas jets or reaction wheels) about the principal axes. Using these variables, it is known (see, for example Athans and Falb[1] pp. 838-841) that the differential equations of the angular velocities are

$$
\begin{aligned}
I_1\dot{\omega}_1(t) &= (I_2-I_3)\omega_2(t)\omega_3(t) + u_1(t) \\
I_2\dot{\omega}_2(t) &= (I_3-I_1)\omega_3(t)\omega_1(t) + u_2(t) \\
I_3\dot{\omega}_3(t) &= (I_1-I_2)\omega_1(t)\omega_2(t) + u_3(t)
\end{aligned}
\tag{1}
$$

It is convenient to use the components of the angular momentum vector, $\underline{x}(t)$, as the state variables. So define

$$x_k(t) = I_k\omega_k(t) , \qquad k = 1,2,3 \tag{2}$$

Using these variables, (1) reduces to

$$
\begin{aligned}
\dot{x}_1(t) &= \alpha_1 x_2(t)x_3(t) + u_1(t) \\
\dot{x}_2(t) &= \alpha_2 x_3(t)x_1(t) + u_2(t) \\
\dot{x}_3(t) &= \alpha_3 x_1(t)x_2(t) + u_3(t)
\end{aligned}
\tag{3}
$$

where

$$\alpha_1 \triangleq \frac{I_2-I_3}{I_2I_3} , \qquad \alpha_2 \triangleq \frac{I_3-I_1}{I_3I_1} , \qquad \alpha_3 \triangleq \frac{I_1-I_2}{I_2I_1} \tag{4}$$

Note that

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \tag{5}$$

The fact that it is desired to generate the control torques u_1,u_2, and u_3 as a function of the state variables x_1,x_2,x_3 is denoted by $\underline{u} = \underline{u}(\underline{x})$ where $\underline{u} \triangleq (u_1,u_2,u_3)'$.

3. Linear Feedback Laws

Since the control time-interval is assumed to be infinite and since it is desired to reduce the state (angular momentum) to zero, it follows that the closed loop system must be asymptotically stable in the large. Thus, the search for optimal feedback control laws must be confined to the class of control laws that yield a stable system. The following lemma shows that this class is non-empty.

Lemma 1. The set of feedback controls driving the system (5) to the origin $\underline{0}$, in infinite time, is non-empty.

Proof : Consider the linear control law

$$u_1(t) = -x_1(t),\ u_2(t) = -x_2(t),\ u_3(t) = -x_3(t) \tag{6}$$

and the Lyapunov function $V(\underline{x})$

$$V(\underline{x}) = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2] \tag{7}$$

Use of (4), (5), and (6) yields

$$dV(\underline{x})/dt = x_1\dot{x}_1+x_2\dot{x}_2+x_3\dot{x}_3 = (\alpha_1+\alpha_2+\alpha_3)x_1x_2x_3+u_1x_1+u_2x_2+u_3x_3=-x_1^2-x_2^2-x_3^2 \tag{8}$$

so that $\dot{V}(\underline{x})$ is negative definite. This establishes that there is at least one stable control law and, hence, the lemma.

Next it will be demonstrated that there is a linear control law that minimizes a quadratic performance criterion. It should be noted that linear control laws do minimize quadratic performance criteria provided that the state differential equations are also linear. In this case, the state equations (5) are nonlinear. Few nonlinear systems admit linear optimal control laws; this is one of them.

Control Law 1 Consider the nonlinear system (5) and the quadratic cost functional (with $q > 0$)

$$J_1 = \frac{1}{2}\int_0^\infty \{q[x_1^2(t) + x_2^2(t) + x_3^2(t)] + \frac{1}{q}[u_1^2(t) + u_2^2(t) + u_3^2(t)]\}dt$$

$$= \frac{1}{2}\int_0^\infty \{q||\underline{x}(t)||^2 + \frac{1}{q}||\underline{u}(t)||^2\}dt \qquad (9)$$

Then the linear feedback control law

$$\underline{u}(t) = -q\underline{x}(t) \qquad (10)$$

is optimal. With this control, the closed loop system is asymptotically stable in the large and the optimal cost J_1^* is given by

$$J_1^* = \frac{1}{2}\,||\underline{x}(0)||^2 \qquad (11)$$

where $||\underline{x}(0)||$ is the initial magnitude of the angular momentum vector.

Elements of Proof : The proof proceeds as follows : One forms the Hamiltonian of the system with its associated canonical equation for the costate which is denoted by $\underline{p}'(t) = [p_1(t), p_2(t), p_3(t)]$. The minimization of the Hamiltonian yields the relation

$$\underline{u}(t) = -q\underline{p}(t) \qquad (12)$$

Using this relation and the resulting equations for the state and the costate one concludes the very important relation that

$$\underline{x}(t) = \underline{p}(t) \qquad (13)$$

In consequence the necessary conditions of optimality are satisfied if

$$\underline{u}(t) = -q\underline{x}(t) \qquad (14)$$

In order to prove sufficiency, one simply computes the cost of using the control (14) and shows that this cost satisfies the Hamilton-Jacobi differential equation.

4/ Nonlinear Feedback Control Laws

It was demonstrated in the previous section that a linear feedback control law is optimal for the nonlinear system (5) provided that the cost functional is quadratic in the state and control vectors. In this section, non-quadratic cost functionals are considered and optimal non-linear feedback controls are derived under the added constraint that the resulting minimum cost is a quadratic function of the initial state. Furthermore, all the suggested nonlinear control laws yield a closed-loop system which is asymptotically stable in the large.

Consider the real-valued positive definite functions, $f_k(.)$ and $g_k(.)$, $(k = 1,2,3)$ of a single variable such that

$$f_k(0) = 0, \quad g_k(0) = 0 \quad ; \quad k = 1,2,3$$

The class of cost functionals under consideration are of the form

$$J = \int_0^\infty \{q[f_1(x_1(t)) + f_2(x_2(t)) + f_3(x_3(t))] + \frac{1}{q}[g_1(u_1(t)) + g_2(u_2(t)) + g_3(u_3(t))]\}dt \tag{15}$$

where $q > 0$ is a weighting scalar.

Since nonlinear feedback controls are sought it is desired to determine the control torques as an instantaneous function of the angular momentum vector $\underline{x}$, i.e.,

$$u_1 = u_1(\underline{x}), \quad u_2 = u_2(\underline{x}), \quad u_3 = u_3(\underline{x}) \tag{16}$$

Finally demand that the minimum cost $J^*(\underline{x})$ as a function of the state is the quadratic function

$$J^*(\underline{x}) = \frac{1}{2}||\underline{x}||^2 = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2] \tag{17}$$

To establish the relations between the controls (16), the cost functional (15), and the cost (17) consider the hamiltonian function H for the optimal control problem

$$H = q[f_1(x_1) + f_2(x_2) + f_3(x_3)] + \frac{1}{q}[g_1(u_1) + g_2(u_2) + g_3(u_3)]$$

$$\alpha_1 x_2 x_3 p_1 + \alpha_2 x_3 x_1 p_2 + \alpha_3 x_1 x_2 p_3 + u_1 p_1 + u_2 p_2 + u_3 p_3 \tag{18}$$

where p_1, p_2, and p_3 are the costate variables. Since the optimal control must minimize the hamiltonian one deduces the relations

$$0 = \frac{\partial H}{\partial u_k} = \frac{1}{q}\frac{dg_k}{du_k} + p_k = \frac{1}{q}\frac{dg_k}{du_k} + \frac{\partial J^*(\underline{x})}{\partial x_k} = \frac{1}{q}\frac{dg_k}{du_k} + x_k \; , \; k = 1,2,3 \tag{19}$$

Furthermore the Hamilton-Jacobi equation

$$0 = q[f_1(x_1)+f_2(x_2)+f_3(x_3)] + \frac{1}{q}[g_1(u_1)+g_2(u_2)+g_3(u_3)]$$
$$+ \alpha_1 x_2 x_3 \frac{\partial J^*(\underline{x})}{\partial x_1} + \alpha_2 x_3 x_1 \frac{\partial J^*(\underline{x})}{\partial x_2} + \alpha_3 x_1 x_2 \frac{\partial J^*(\underline{x})}{\partial x_3}$$
$$+ u_1 \frac{\partial J^*(\underline{x})}{\partial x_1} + u_2 \frac{\partial J^*(\underline{x})}{\partial x_2} + u_3 \frac{\partial J^*(\underline{x})}{\partial x_3} \tag{20}$$

where $J^*(\underline{x})$ is the minimum cost, must hold along all optimal trajectories in R_3.

From (19) and (20) one obtains the equation (since $\alpha_1 + \alpha_2 + \alpha_3 = 0$)

$$0 = q[f_1(x_1)+f_2(x_2)+f_3(x_3)] + \frac{1}{q}[g_1(u_1)+g_2(u_2)+g_3(u_3)]+u_1x_1+u_2x_2+u_3x_3 \tag{21}$$

Since the functions $f_k(.)$ and $g_k(.)$ are positive definite, then the optimal control must have the property that

$$u_1x_1 + u_2x_2 + u_3x_3 < 0 \qquad \text{for } \underline{x} \neq \underline{0} \tag{22}$$

(Indeed this requirement guarantees the stability of the closed-loop nonlinear system).

The problem under consideration now is as follows : fix the control torques u_1, u_2, and u_3 to be some convenient and easily implementable function of the angular momenta x_1, x_2, x_3 and then determine the functions $f_i(.)$ and $g_i(.)$ associated with this control law (and, of course, the constraint (17) on the minimum cost).

One of the simplest ways of generating the control is as follows

$$u_1 = -qh_1(x_1);\; u_2 = -qh_2(x_2);\; u_3 = -qh_3(x_3) \;, \tag{23}$$

where the $h_k(.)$ are continuous and differentiable scalar-valued functions of a single variable, such that for k=1,2,3

(a) $\quad h_k(0) = 0 \qquad (24)$

(b) $\quad h_k^{-1}(.)$ exists everywhere $\qquad (25)$

(c) $\quad h_k(x_k)x_k > 0 \quad , \quad x_k \neq 0 \qquad (26)$

Clearly (26) and (23) guarantee that (22) holds. Furthermore, (24) and (23) guarantee that (21) holds for $\underline{x} = \underline{0}$.

It now remains to determine the $f_k(.)$ and $g_k(.)$. From (21) and (23) one has

$$q \sum_{k=1}^{3} f_k(x_k) + \frac{1}{q} \sum_{k=1}^{3} g_k(u_k) = q \sum_{k=1}^{3} h_k(x_k)x_k \tag{27}$$

From (19) one obtains

$$\frac{dg_k}{du_k} = \frac{dg_k}{dx_k} = -qx_k \quad ; \quad k = 1,2,3 \tag{28}$$

Since $u_k = -qh(x_k)$, (28) yields

$$\frac{dg_k}{dx_k} = q^2 x_k \frac{dh_k}{dx_k} \qquad k = 1,2,3 \tag{29}$$

which yields (since $g_k(0) = 0$)

$$g_k = q^2 \int_0^{x_k} x dh_k(x) = q^2 [x h_k(x) \Big|_{x=0}^{x=x_k} - \int_0^{x_k} h_k(x) dx] \tag{30}$$

and, so,

$$g_k = q^2 x_k h_k(x_k) - q^2 \int_0^{x_k} h_k(x) dx \tag{31}$$

Substitution of (31) into (27) yields

$$\sum_{k=1}^{3} f_k(x_k) = \sum_{k=1}^{3} \int_0^{x_k} h_k(x) dx \tag{32}$$

which implies

$$f_k(x_k) = \int_0^{x_k} h_k(x) dx \tag{33}$$

To determine the explicit dependence of $g_i(.)$ on the u_i one can simply use the inverse relationship $x_k = h_k^{-1}\left(-\frac{u_k}{q}\right)$ in Eq. (31) to obtain

$$g_k(u_k) = q^2 \left[\left(-\frac{u_k}{q}\right) h_k^{-1}\left(-\frac{u_k}{q}\right) - f_k\left(h_k^{-1}\left(-\frac{u_k}{q}\right)\right)\right] \tag{34}$$

In the remainder of this section two specific examples of the theory are presented.

<u>Example 1</u> : Suppose that each control torque u_k ($k = 1,2,3$) is generated from the corresponding angular momentum x_k ($k = 1,2,3$) by the odd <u>power-law</u>

$$u_k = -qx_k^n \quad ; \quad n \text{ odd} \quad ; \quad q > 0, \quad k = 1,2,3 \tag{35}$$

In this case, $h_k(x_k) = x_k^n$. The functions $f_k(x_k)$ are computed from (33) to obtain for $k = 1,2,3$

$$f_k(x_k) = \int_0^{x_k} x^n dx = \frac{1}{n+1} \; x_k^{n+1} \tag{36}$$

The functions $g_k(u_k)$ are found from (34)

$$g_k(u_k) = \frac{1}{n+1} \; q^{n-1/n} \; u_i^{\,n+1/n} \tag{37}$$

To recapitulate : For the system (5) and the cost functional (n odd)

$$J = \int_0^\infty \left[\frac{1}{n+1} \sum_{k=1}^{3} x_k^{n+1}(t) + \frac{nq^{n-1/n}}{q(n+1)} \sum_{k=1}^{3} u_k^{n+1/n}(t) \right] dt \tag{38}$$

the optimal control is given by

$$u_k = -qx_k^n(t) \tag{39}$$

and the minimum value of the cost functional J is given by

$$J^*(\underline{x}(t)) = \frac{1}{2} [x_1^2(t) + x_2^2(t) + x_3^2(t)] \tag{40}$$

Note that the use of this type of control functional for large n penalizes severly the system for large values of the angular momentum vector and it penalizes the control torques in an almost linear manner; this means that this criterion can be used as an approximation to the case that the control torques u_k are almost linearly related to the rate-of-flow of fuel consumed by, say, gas jets used to generate the control torques.

Example 2 Suppose that the control torques are generated by

$$u_k = -qx_k^{1/m} \quad ; \quad m \text{ odd} \quad ; \quad k = 1,2,3 \tag{41}$$

The functions $f_k(x_k)$ are then given by

$$f_k(x_k) = \int_0^{x_k} x^{1/m} dx = \frac{m}{m+1} \; x_k^{m+1/m} \tag{42}$$

and the functions $g_k(u_k)$ are given by

$$g_k(u_k) = q^2 \left[\left(- \frac{u_k}{q_k}\right)\left(- \frac{u_k}{q_k}\right)^m - \frac{m}{m+1}\left(\left(- \frac{u_k}{q}\right)^m \right)^{m+1/m} = \frac{1}{m+1} q^{1-m} u_k^{m+1} \right. \tag{43}$$

The implication is, of course, that given the system (5) and the cost functional (m odd)

$$J = \int_0^\infty \left[\frac{qm}{m+1} \sum_{k+1}^{3} x_k^{m+1/m}(t) + \frac{1}{(m+1)q^m} \sum_{k+1}^{3} u_k^{m+1}(t)\right]dt \tag{44}$$

then the optimal control is given by

$$u_k = -qx_k^{1/m}(t) \quad ; \quad k = 1,2,3 \tag{45}$$

and the minimum value of the cost functional is

$$J^*(\underline{x}(t)) = \frac{1}{2}\left[x_1^2(t) + x_2^2(t) + x_3^2(t)\right] \tag{46}$$

Note that if m is chosen large, then the cost functional (44) can be used to penalize the system very severly for using large control torques while small or large values of angular momenta are penalized almost in a proportional manner.

5. Conclusions

It has been shown that given the nonlinear differential equations that describe the behavior of the angular velocities of an arbitrary space vehicle and given a quadratic dependence of the minimum cost (as a function of the state variables), then the Hamilton-Jacobi equation together with the maximum principle can lead to classes of nonlinear feedback controllers which in turn yield the corresponding (nonquadratic) cost functionals.

Identical techniques can be used to solve the inverse optimal control problem when the minimum cost is specified to be nonquadratic, e.g. $J^*(\underline{x}) = ||\underline{x}||^{2m}$, where $m = 1,2,3,\ldots$. Distinct classes of nonlinear control systems and associated cost functionals can be obtained for each value of m.

This type of "inverse" approach to feedback system design has promise as a design aid to the engineer. It is easy to see that given the state differential equations, the engineer specifies the desired minimum cost. For each class of feedback controllers that he may wish to consider, he obtains the corresponding state and control penalty functions that define the integrand of the cost functional. He can then pick the control vs cost functional pair that reflects both feedback simplicity and a physically appealing state and control penalty. It should be noted that since the choice of the cost functional is often a subjective one, this "inverse" technique can indeed be of value, since it clearly couples well with the complexity of the feedback controllers under consideration, and since it does not violate the potential use of optimal control theory as a tool for design rather than a "straight-jacket."

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