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AN APPROACH TO THE PROBLEM OF RECONSTRUCTING POLYHEDRA FROM TWO OR MORE OF THEIR PERSPECTIVE PROJECTIONS

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ABSTRACT

This report is an investigation into the problem of reconstructing a three-dimensional geometrical description of polyhedral objects from two or more of their perspective projections. A classification scheme for subdividing the problem according to projection and object characteristics is presented. Some basic techniques for the reconstruction are described.

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LIST OF BASIC SYMBOLS

$\mathcal{S}$	object set
C_P	center of projection
Π_P	picture plane
$\mathcal{R}(\mathcal{S}, C_P)$	set of rays
C_PC_V	axis of projection
C_V	center of vision
$\mathcal{W}(\mathcal{S}, C_P, \Pi_P)$	wire-frame perspective projection
i, j, ...	used as superscript to identify a term with the i^{th}, j^{th}, ... module
$C_P^iC_P^j$	line through C_P^i and C_P^j
$\lvert C_P^iC_P^j \rvert$	distance from C_P^i to C_P^j
x, y, z	overall three-dimensional coordinate system
O	origin of x, y, z
x^i, y^i	picture plane coordinate system for module i
V_N	n^{th} object vertex
$\overline{V_NV_M}$	vector from V_N to V_M.
Π_A^{ij}	plane determined by C_P^j and $C_P^iC_V^i$
δ^i	angle between $C_P^iC_V^i$ and $C_P^iC_P^j$
β^{ij}	dihedral angle between Π_A^{ij} and Π_A^{ji}
$\mathcal{P}^i$	projection perimeter
τ^i	projection pyramid
$\mathcal{U}^i$	union of all τ^i for single i.

I. INTRODUCTION

This report is concerned with the problem of reconstructing an object from two or more of its perspective projections. By "reconstructing" we mean obtaining a complete three-dimensional geometric description of the object. This problem is encountered in a variety of situations. Thus, an object might no longer be physically available, as in the case of views from high-speed photography, pictures of deceased people, or a photograph of a burning building. Also, the object might be in a location in which the only way of obtaining information about the object is through photographs. Examples of this might be objects in outer space or in the deep ocean. As a third example, a computer might receive views of an object, with the views obtained by means of television cameras. These cameras might be the "eyes" of a robot that allow it to observe and analyze its surroundings, a task that is of considerable current interest.[1]

A perspective projection of an object is uniquely defined once the center of projection and the picture plane are specified relative to the object. The converse is, of course, not true; that is, given the center of projection and the picture plane, a perspective projection does not uniquely define the object. To obtain a unique description of an object, two or more projections are necessary. If the available projections are inadequate for yielding a unique description of the object, then a <u>true</u> reconstruction of the object is not possible. However, it may be possible through the use of other information about the nature of the object (for example, knowledge of symmetry, similarity, and balance) to reconstruct the object such that it satisfies all the constraints imposed by this information and the available perspective projections.

A discussion of these areas of investigation as applied to photographs, may be found in a previous report by the author.[2] The present report is concerned with the reconstruction of polyhedral* objects from two or more perspective projections. Different types of perspective projections are described and compared. A classification system is introducted for various projection and object characterisitics, and some basic reconstruction techniques are presented for use in a reconstruction procedure.

* A polyhedron is a finite set of polygons arranged in space in such a way that every side of each polygron belongs to just one other polygon, with the restriction that no subset has the same property. A simply-connected polyhedron is one that may be continuously deformed into a sphere.

II. PERSPECTIVE PROJECTIONS

There are four types of perspective projections. In order to describe them clearly, several terms must be defined. Let an object set, $\mathcal{S}$, consist of one or more polyhedra. Any point not in the convex hull* of $\mathcal{S}$ but a finite distance away** may then be chosen as a center of projection, C_P. The picture plane, Π_P, may be chosen as any plane not containing C_P and satisfying the condition that there exist a plane through C_P and parallel to Π_P that does not intersect the convex hull of $\mathcal{S}$. C_P and Π_P determine a perspective projection module. A set of these modules is called a projection system. Given $\mathcal{S}$, the rays, $\mathcal{R}(\mathcal{S}, C_P)$, of the perspective projection module are all the straight lines through C_P that intersect lines of $\mathcal{S}$. The axis, C_PC_V, of the module is the perpendicular to Π_P through C_P. The center of vision, C_V, is the intersection of C_PC_V with Π_P. A wire-frame perspective projection, $\mathcal{P}(\mathcal{S}, C_P, \Pi_P)$, is the intersection of the rays, $\mathcal{R}(\mathcal{S}, C_P)$, with Π_P (see Fig. 1).

An important special kind of perspective projection is the visible perspective projection, $\mathcal{R}(\mathcal{S}, C_P, \Pi_P)$. Consider each ray as directed away from C_P. Define the visible rays $\mathcal{R}_v(\mathcal{S}, C_P)$, as the subset of $\mathcal{R}(\mathcal{S}, C_P)$ that contains only those rays that intersect lines of $\mathcal{S}$ before

*The convex hull is the intersection of all convex objects that contain the given object set.

**The case of the center of projection at infinity (parallel projection) is not considered. This case has its own special features and is best treated separately as done by Smith.[7]

intersecting any other points of $\mathcal{S}$. Then a visible perspective projection is the intersection of $\mathcal{R}_V(\mathcal{S}, C_P)$ with Π_P (see Fig. 2).

The most common example of a visible perspective projection is a photograph. The center of the lens corresponds to the center of projection, and the film is coincident with the picture plane. Generally, (but not necessarily!) the axis of the projection is also the axis of the lens and the center of vision is the center of the photograph.

A third type of perspective projection is the <u>conventional perspective projection</u>, $\mathcal{P}_C(\mathcal{S}, C_P, \Pi_P)$. This type of projection contains all the projected edges of the wire-frame perspective projection. However, the lines that are not common to both the wire-frame perspective projection and visible perspective projection are shown dashed to indicate that they are hidden (see Fig. 3). This may be specified as follows:

$$\begin{aligned} \mathcal{P}_C(\mathcal{S}, C_P\Pi_P) = \mathcal{P}(\mathcal{S}, C_P\Pi_P) \cap \mathcal{P}_V(\mathcal{S}, C_P, \Pi_P) \quad & \text{(solid lines)} \\ \mathcal{P}(\mathcal{S}, C_P\Pi_P) \cap \mathcal{P}'_V(\mathcal{S}, C_P, \Pi_P) \quad & \text{(dashed lines)} \end{aligned}$$

Finally, a total perspective projection, $\mathcal{P}_T(\mathcal{S}, C_P, \Pi_P)$, is an extension of the conventional perspective projection. In this type of projection, the number of surfaces hiding each hidden line is indicated. This may be done by using dashes of different lengths or by attaching tags to the lines (see Fig. 4).

The types of perspective projections listed in increasing order of the amount of information that they provide about the object set is as follows:

1) Visible perspective projection

2) Wire-frame perspective projection

3) Conventional perspective projection

4) Total perspective projection.

III. BASIC FEATURES OF THE RECONSTRUCTION PROCEDURE

There are five main features to the reconstruction procedure. First, the projections are considered in pairs. Second, the projections are classified according to the characteristic of each pair of projections and their corresponding modules. The first two classifications, based on projection configurations and object-set-to-projection orientations, will remain in the final procedure since they do not presuppose knowledge of the object-set type. The third, classification by object-set type, is employed only as a guide to be used in the step-by-step development of the algorithm. The third feature involves the use of vertex classification and grouping. Much of the reconstruction is concerned with the matching of vertices in two projections and then determining the location of the object vertices. The next two features are involved only with the visible perspective projections. Fourth, is the reconstruction of the parts of the object set that are visible in only one of the projections. Such parts are built onto the previously reconstructed commonly visible parts using given or assumed object-set properties where necessary. Finally, there is the restoration of any completely hidden parts of the object set. If sufficient non-projective information is not available, then this is accomplished using assumed object-set properties.

IV. CLASSIFICATION OF OBJECT SET AND MODULE CONFIGURATIONS

In order to render the problem more tractable, the pairs of projections are classified according to their module arrangements and the relationship between the object set and the modules. The characteristics that determine the classifications are described below.

A. Perspective Projection Module Configurations

Given two modules, the i^{th} and the j^{th}, there are many ways in which they can be arranged. The general configuration is shown in Fig. 5. The following notation is used:

C_P^i = center of projection of the i^{th} module

$C_P^i C_P^j$ = line through C_P^j and C_P^i

$C_P^i C_V^i$ = axis of the i^{th} module

δ^i = angle between $C_P^i C_V^i$ and $C_P^i C_P^j$

π_A^{ij} = plane determined by $C_P^i C_V^i$ and C_P^j

β^i = dihedral angle between π_A^{ij} and π_A^{ji}

The pair of modules may be classified according to the arrangement of their axes. This arrangement may be either coplanar ($\beta=0°, \pi_A^{ij}=\pi_A^{ji}$), or non-coplanar ($\beta>0°$). In addition, when the axes are coplanar, they may be further classified as:

1) Collinear axes

(a) $\beta = 0°$

(b) $\delta^i = \delta^j = 0$

(c) $C_P^i \neq C_P^j$

2) Parallel axes

(a) $\beta = 0^\circ$

(b) $\delta^i = \delta^j \neq 0^\circ$

(c) $C_P^i \neq C_P^j$

3) Non-parallel axes

(a) $\beta = 0^\circ$

(b) $\delta^i \neq \delta^j$

The possible parallel axis and non-parallel axis arrangements are shown in Figs. 6 and 7.

B. Object-Set-to-Module Orientation

The characteristics to be described here are based on the orientation of the object set with respect to the modules. The first is concerned with object sets containing two or more polyhedra. The projection perimeter, $\mathcal{B}^i$, of a polyhedron consists of those edges, or portions of edges, that bound the projection of the polyhedron in the i^{th} module. A projection pyramid, τ^i, is defined as the pyramid with base $\mathcal{B}^i$ and triangular sides which are defined by C_P^i and each line in $\mathcal{B}^i$. Each pair of polyhedra in the object set can then be classified according to the relationships of their projection pyramids in each module:

1) $\tau_1^i \cap \tau_2^i = \emptyset$ ($\emptyset$ = empty set + C_P^i)

2) $\tau_1^i \cap \tau_2^i \neq \emptyset$

This is a classification based on whether or not the projections of the polyhedra overlap.

The second characteristic involves a pair of modules. Let $\mathcal{U}^i$ be defined as the union of all τ^i. For the parallel, non-parallel and non-planar axis configuration there are two mutually exclusive relations for $\mathcal{U}^i$ and $\mathcal{U}^j$:

1) $\mathcal{U}^i \cap \mathcal{U}^j \supset \{c_p^i, c_p^i\}$

2) $\mathcal{U}^i \cap \mathcal{U}^j \not\supset \{c_p^i, c_p^j\}$

C. <u>Object-Set Types</u>

This group of characteristics is based on the composition of the object set. First, there may be one or more polyhedra in the object set. Second, the polyhedra may be convex or non-convex, simply connected or multiply-connected. Since simply-connected polyhedra are uniquely defined by their vertices and edges,* their reconstruction is simpler than the reconstruction of multiply-connected polyhedra.

*A proof is given in the Appendix.

V. DATA FORMAT AND COORDINATE SYSTEMS

The input data for the algorithm consists of three parts. First, there is the data describing the edge-vertex configuration presented in the projection. This data is input as a list of vertex coordinates, $V_N^i(x^i,y^i)$, plus an incidence matrix*, $\mathcal{I}^i$, for each projection (See Fig.8). The vertices may be numbered in any manner convenient for the quantizing system used. This type of data has been successfully obtained by others.[3,4,5] The second part consists of the data describing the projection system. In general this would include the $C_P C_V$ orientations, the C_P to Π_P distances, and the locations of the C_P's. The third part of the input data consists of given or assumed non-projective information about the object set, necessary to compensate for any information not obtainable from the projections. For example, it might be known that the object is symmetric about some plane. This data will be handled in the form of appropriate codes which are developed as the reconstruction procedure is expanded to include additional situations.

The vertices and edges in the i^{th} picture plane are referenced to a two-dimensional coordinate system, (x^i,y^i) or (ρ^i,θ^i) with origin at

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* The edge-vertex configuration may be considered as an undirected linear graph. Given such a graph with v vertices and e edges, there are three commonly used matrix representations. The <u>incidence matrix</u> (v by e) contains a "1" at each position representing an edge incident on a vertex, and "0's" elsewhere. The <u>connectivity matrix</u> (v by v) contains a "1" at each position representing the connection of two vertices, and "0's" elsewhere. Finally, the <u>matrix of a line graph</u> (v by v) is similar to the connectivity matrix except that the names of the edges replace the "1's".

C_V^i. A three-dimensional coordinate system, (x,y,z), is used to link the centers of projection, the picture planes, and the object set. This is illustrated in Fig.9.

VI. PRE-RECONSTRUCTION DATA ANALYSIS

The input data must be preprocessed before reconstruction may begin. First, the given projections must be classified in pairs. This is done by comparing the values of δ_1, δ_2, β, and the locations of C_P^i, and τ^i with the values given in the classification rules stated previously. To determine the number of objects appearing separately, the incidence matrix is arranged in quasi-diagonal form so that each of the submatrices on the diagonal represent a separate object.

Second, for the visible, conventional, and total perspective projections, the visible projected vertices are grouped into ordered sets representing the closed loops that can be formed by the projected edges. This is accomplished by a search for cycles* in the incidence matrix[6]. The cycles, corresponding to visible faces in the case of convex objects, and potential visible faces in the case of non-convex objects, are then found in the following manner:

1) Arrange the cycles in order of increasing number of vertices. Let m be the minimum number of vertices in any set.
2) All cycles with m vertices are faces or potential faces. Add these cycles to the set of faces sets and to the set of test sets.

* A cycle is any closed, non-intersecting sequence of edges.

3) Remove from further consideration any remaining cycles which contain a set of the test group as a subset.

4) If there remain cycles to be examined, form all combinations $C=(A\cap B')\cup(A'\cap B)$ where A and B are sets in the set of test sets that have at least one edge in common. Substitute these sets for those in the set of test sets.

5) Remove from further consideration any remaining cycles which contain a test set as a subset.

6) Repeat steps 4 and 5 until no cycles remain to be examined or all possible C sets are formed.

7) If there are still cycles to be classified, let m=m+1 and return to step 2.

The perimeter(s) of the projection are the set(s) remaining in the test group upon completion of the above steps. As an example, consider the data of Fig. 8. All the cycles are found by an algorithm such as the one given by Welch[6]. These cycles are:

(1) 1341

(2) 14521

(3) 34763

(4) 45874

(5) 134521

(6) 136741

(7) 1258741

(8) 3458763

(9) 13678521

(10) 13478521

(11) 13674521

(12) 13678541

(13) 125876341

Now let F be the set of faces and T the set of test sets. The first pass through the rules given above results in:

1) m=3

2) F = {(1)}

3) T = {(1)}

4) Cycles (5),(6),(10),(12), and (13) are removed from further consideration.

In the second pass, the results up to step 4 are:

1) m=4

2) F = {(1),(2),(3),(4)}

3) T = {(2),(3),(4)}

4) Cycles (7),(8),(13) are removed from consideration.

In step 5, two new test sets (14) and (15) are formed:

(14) 3458763

(15) 1478521

Since no additional cycles can be eliminated, m=7 (next cycle still in cycle list); on the next pass the process ends after step 2 since there are no more cycles to be checked. The face set, F = {(1),(2),(3),(4),(9)} now contains all the visible faces, {(1),(2),(3),(4)}, as well as the projection perimeter, {(9)}, which is still in T.

The third part of the data preparation is the determination of the correspondence between the vertices shown in each pair of projections. Some of the methods for establishing this correspondence are general, but others are applicable in only one or two cases. Some examples are:

1) Vertex Classification

This method is applicable to all configurations. The vertices in each projection may be classified according to: 1) the number of incident edges, 2) incidence with $\mathcal{B}^i$ and, 3) the type of object set vertex that the projected vertex could represent. There are many choices for the set of basic vertex types. The one being considered here is shown in Fig. 10.

2) Polar Coordinate Ordering

Vertex matching in the collinear configuration is facilitated by transforming the picture plane coordinates into polar form since the θ-coordinates of the projection of the same object vertex are constant from projection to projection. To overcome quantization limitations, the following method is used for each projection. First, the vertices are arranged in ρ-sets, each set containing all the vertices whose ρ coordinates differ by less than $\Delta\rho$ (the minimum allowable distance between ρ-sets as determined by the encoding grid size and the resolution of the projection). Then each ρ-set is subdivided into θ-sets, where the $\Delta\theta$ between sets is inversely proportional to ρ. When this process is completed, an initital matching of the vertices in the two projections can be obtained by matching corresponding sets.

3) Maximum Height-Change Grouping

In the parallel and non-parallel configurations, the plane π_A^{ij}, is determined by $C_P^i C_V^i$ and $C_P^j C_P^i$, and divides the vertices into two disjoint sets. If a vertex is above this plane in one projection then it must also be above it in the other projection. In conjunction with this division, a grouping of the vertices can be made according to a maximum height-change criterion. If a vertex is at a height y^i in the i^{th} projection, this criterion specifies the range of heights, $y^j = y^i + \Delta y$, that must be searched in the j^{th} projection for the corresponding vertex.

Consider a visible orthographic projection* on π_A^{ij} of the projection configuration as shown in Fig. 11a (note that only the outlines of the projection pyramids are shown). Let A,B,C,D,E, and F be defined as shown. Now let:

$$MIN^j = \begin{cases} |C_P^j E| & \text{if } \tau^j \supset C_P^j F \\ MIN(|C_P^j C|, \ |C_P^j D|) & \text{if } \tau^j \not\supset C_P^j F \end{cases}$$

and

$$MAX^i = MAX(|C_P^i B|, \ |C_P^i D|)$$

* A visible orthographic projection is the orthographic counterpart of a visible perspective projection. An orthographic projection is a parallel projection with the picture plane perpendicular to the rays.

Then:

MAX^i - MIN^j = maximum distance, further from C_P^j than from C_P^i, that the π_A^{ij} plane projection of the object vertex can be located.

Now let y_M^j be the maximum height of a vertex in the j^{th} projection. Let y_V be the actual height of the corresponding object vertex above π_A^{ij}. Assume that the projection of this vertex on π_A^{ij} is at a distance MIN^j from C_P^j, the worst case (see Fig. 11b). Furthermore, adjust the scale so that the center of projection to picture plane distances are the same for both projections and call this distance $|C_P C_V|$. Then:

$$y_V = (MIN^j)(y_M^j)/|C_P C_V|$$

and:

$$\Delta y = y_M^j\ (1-MIN^j/MAX^i)$$

where Δy is the maximum difference in height between the projections of the same object vertex in the i^{th} and j^{th} projections. This property can now be used as a vertex grouping criterion.

VII. RECONSTRUCTION TECHNIQUES

Once the steps outlined in the previous section have been completed, the actual reconstruction of the object set may begin. Different techniques must be used depending on the visibility conditions.

For a vertex that appears in two or more projections, a direct determination of its three-dimensional coordinates is possible. The following definitions are necessary (see Fig. 12):

V_1 = objection vertex 1.

V_1^i = projection of V_1 in the i^{th} module.

$\overline{OV_1^i}$ = vector from origin O to V_1^i.

$\overline{OC_P^j}$ = vector from origin O to C_P^j.

$C_P^jV_1$ = projecting ray of vertex V_1 for C_P^j.

R_1^j = a point on $C_P^jV_1$.

In the ideal case the projecting rays actually intersect at the object vertex. This intersection can be found as follows:

Let:

$\overline{OR_1^i}$ = the vector from origin O to any point on $C_P^iV_1$.

$\overline{OR_1^j}$ = the vector from origin O to any point on $C_P^jV_1$.

Then:

$$\overline{OR_1^i} = A\ \overline{OC_P^i} + (1-A)\ \overline{OV_1^i}$$

$$\overline{OR_1^j} = B\ \overline{OC_P^j} + (1-B)\ \overline{OV_1^j}$$

Where A and B are scalers.

Equating $\overline{OR_1^i}$ and $\overline{OR_1^j}$ yields:

$$A\ \overline{OC_P^i} + (1-A)\ \overline{OV_1^i} = \overline{OR_1} = B\ \overline{OC_P^j} + (1-B)\ \overline{OV_1^j}$$

which can be solved for the desired object vertex.

In most practical cases, the projecting rays will not intersect due to unavoidable quantization errors in the data giving the locations of the center of projection and the projection of the object vertex. In this case the desired object vertex will be assumed to be the midpoint of the shortest mutual perpendicular to the two projecting rays.

Let:

$\overline{C_P^iV_1^i}$ = vector from C_P^i to V_1^i.

$\overline{C_P^jV_1^j}$ = vector from C_P^j to V_1^j.

$\overline{V_1^iV_1^j}$ = vector from V_1^i to V_1^j.

The unit vector perpendicular to both $\overline{C_P^iV_1^i}$ and $\overline{C_P^jV_1^j}$ is

$$n = (\overline{C_P^iV_1^i} \times \overline{C_P^jV_1^j}) \ / \ |\overline{C_P^iV_1^i} \times \overline{C_P^jV_1^j}|$$

Then the minimum distance between $C_P^iV_1^i$ and $C_P^jV_1^j$ is $|n \cdot V_1^iV_1^j|$

The equation:

$$\overline{OR_1^i} - \overline{OR_1^j} = |n \cdot \overline{V_1^iV_1^j}|n$$

can now be solved to determine the desired shortest mutual perpendicular.

For visible perspective projections there are two more situations to be considered. For vertices and edges that appear in only one projection, other approaches than that given above must be taken. One of the possibilities that has been partially investigated is the reversal of the methods used in the mechanical drawing of perspective projections. There are many of these techniques and not all of them have been examined for application to this problem. The part of the object set reconstructed from two or more projections would be useful here as a start.

The last part of the reconstruction involves the restoration of those parts of the object set that are not visible in any of the projections. If no suitable nonprojective information is given, this will be accomplished by one of the following:

1) Use of symmetry about the vertex perimeter.
2) Use of the minimum number of edges and vertices necessary to complete the reconstruction and to be consistent with the reconstructed visible part.
3) Extrapolate the description on the basis of symmetry exhibited by the reconstructed visible part.
4) Assume similarity to some known object set.

VIII. CONCLUSIONS

An approach has been presented for developing computer procedures that can reconstruct polyhedral objects from sets of their perspective projections. A classification scheme has been introduced for subdividing the problem, thus facilitating the development of the reconstruction procedures. A computer program implementation of the reconstruction procedures will be necessary before the effectiveness of the procedures can be evaluated. Tests with many different types of object sets and module configurations will be required.

In attempting to overcome the ambiguity problem inherent in the object-set data from a solitary projection, a new problem has been introduced. This is the problem of determining the correspondence between projected vertices and edges in different projections of the same object set. The success of any reconstruction procedures will be dependent on the solution of this problem.

As shown in the appendix, a multiply-connected polyhedron is not always uniquely defined by its vertices and edges alone. Sometimes the faces must also be specified. This makes the reconstruction of multiply-connected polyhedra considerably more difficult than the reconstruction of simply-connected polyhedra.

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APPENDIX: A THEOREM ON THE UNIQUENESS OF POLYHEDRA

Theorem: Given the vertices and edges of a simply-connected polyhedron, there is only one possible set of faces that can be chosen that will yield a polyhedron.

Proof:

Consider Euler's Formula for Polyhedra:

$$V - E + F = 3 - h$$

where:

V = the number of vertices.

E = the number of edges.

F = the number of faces.

h = the connectivity number.

For simply-connected polyhedra $h = 1$, and for multiply-connected polyhedra $h \geq 2$. Since h is determined solely by the edges, it is not possible to change the connectivity of a polyhedron by changing only the faces. Therefore, faces cannot be chosen for the given edges and vertices that would yield a multiply-connected polyhedron.

It only remains to be shown that the edges and vertices define a unique simply-connected polyhedron. This is equivalent to proving that, given a simply-connected polyhedron, no other set of faces is possible for the vertices and edges of the given polyhedron. Since the polyhedron is simply-connected, it has an inside and an outside, and the faces separate the inside from the outside. Now consider any set of edges that form a non-intersecting closed path and are coplanar but do not bound a face of the given polyhedron. Only sets which meet these conditions may be chosen as new faces. From the definition of simply-connected polyhedra, such a set of edges must separate the remaining edges of the polyhedron into two disjoint sets. But since a face separates the inside from the outside, one of these sets must be completely outside or completely inside the new polyhedron, which is a contradiction.

The theorem cannot be extended to multiply-connected polyhedra. For example, the edges and vertices shown in Fig. 13 can be fitted with faces that yield both polyhedra shown in Fig. 14.

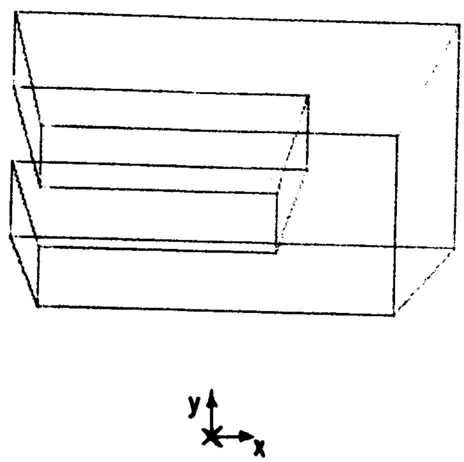


FIG. 1

WIRE-FRAME PERSPECTIVE PROJECTION OF A SLOTTED BLOCK

(FIGURE DRAWN BY COMPUTER)

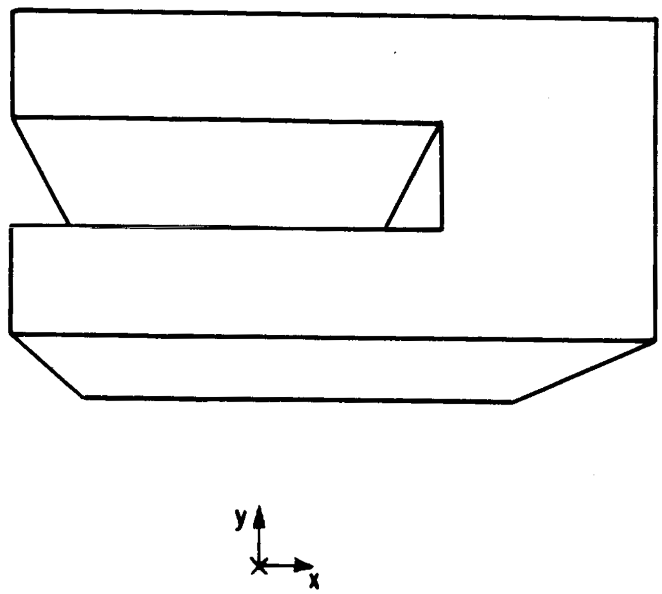


FIG. 2

VISIBLE PERSPECTIVE PROJECTION
OF A SLOTTED BLOCK

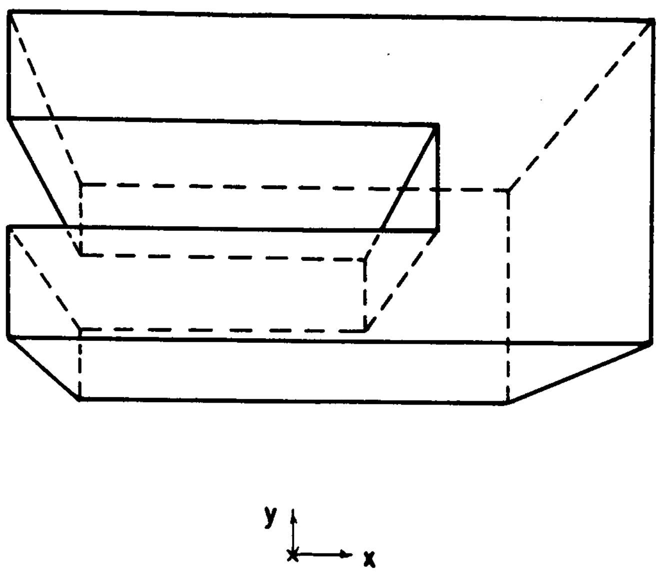


FIG. 3

CONVENTIONAL PERSPECTIVE PROJECTION OF A SLOTTED BLOCK

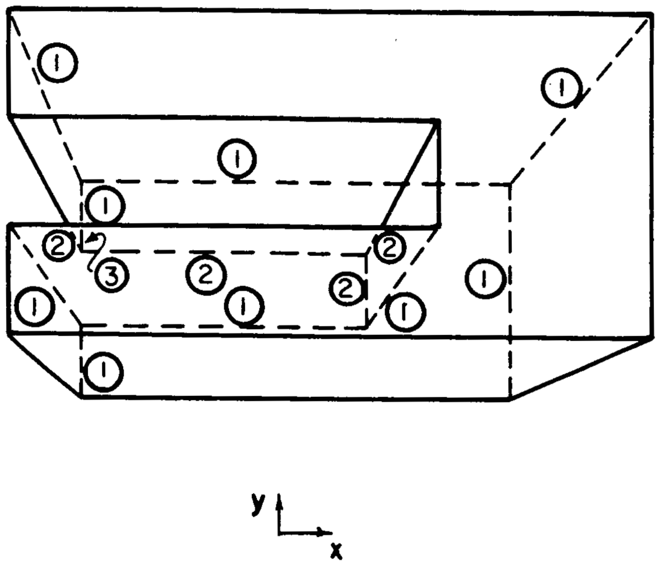


FIG. 4

TOTAL PERSPECTIVE PROJECTION
OF A SLOTTED BLOCK

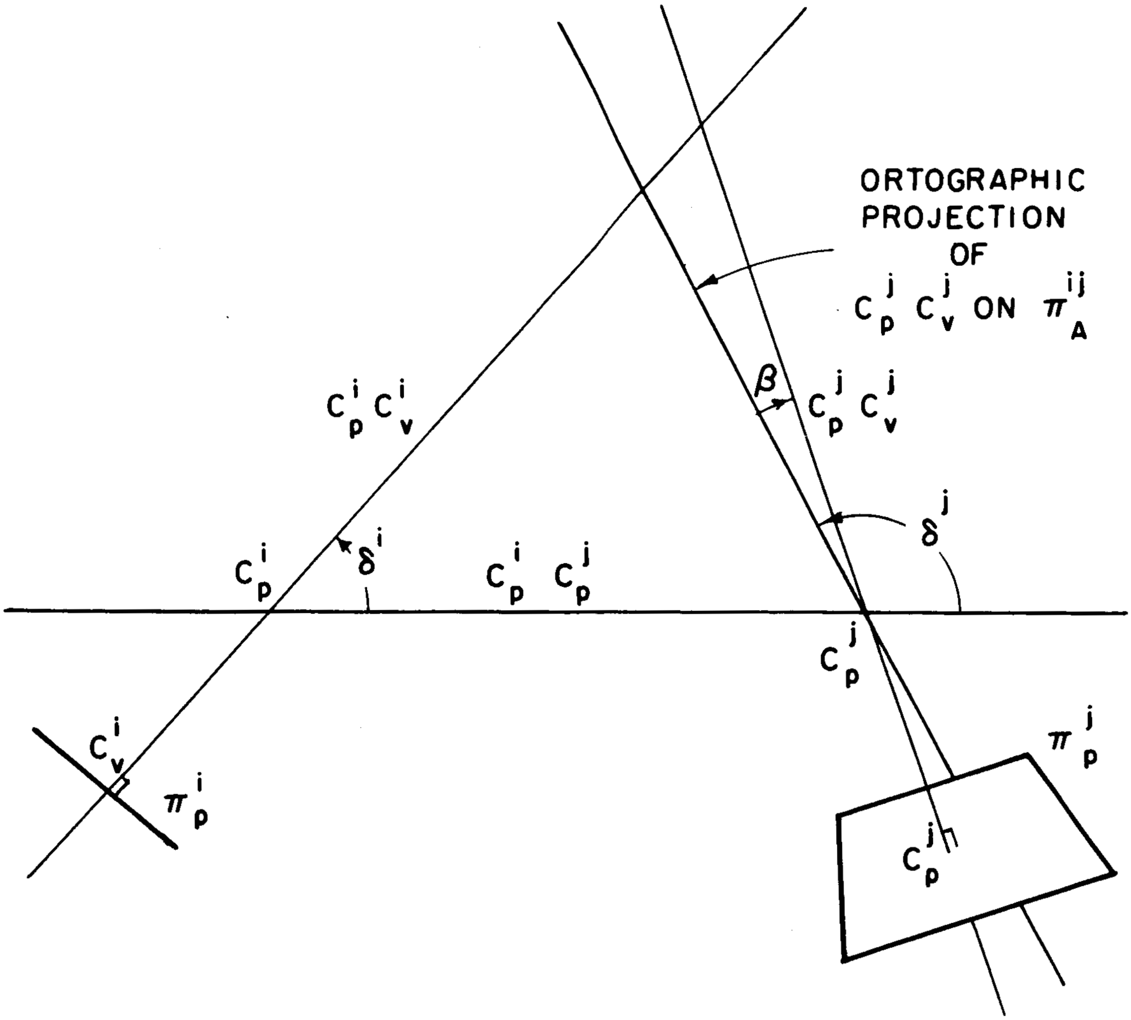


FIG. 5

MODULE CONFIGURATION FOR A PAIR OF PROJECTIONS

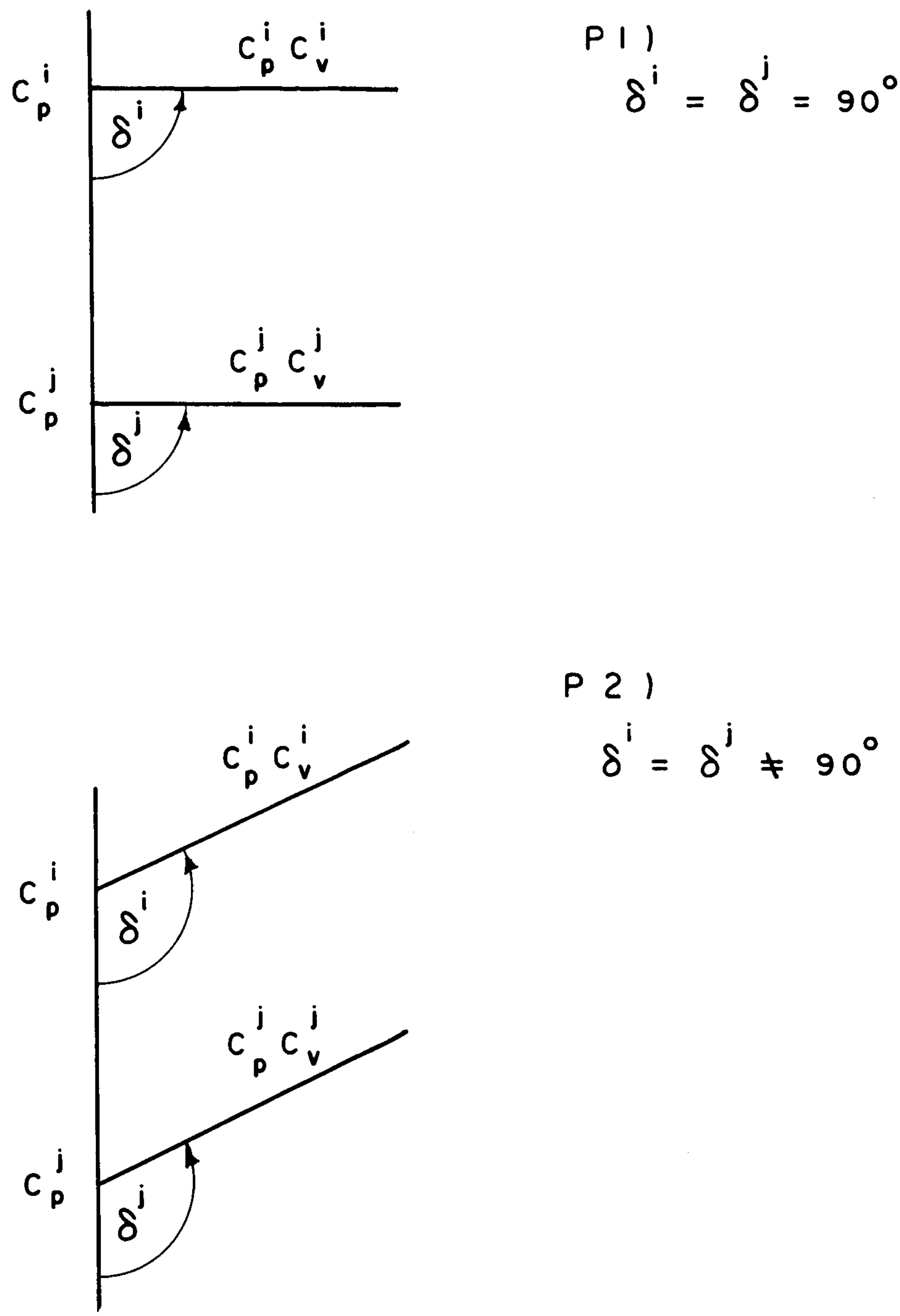
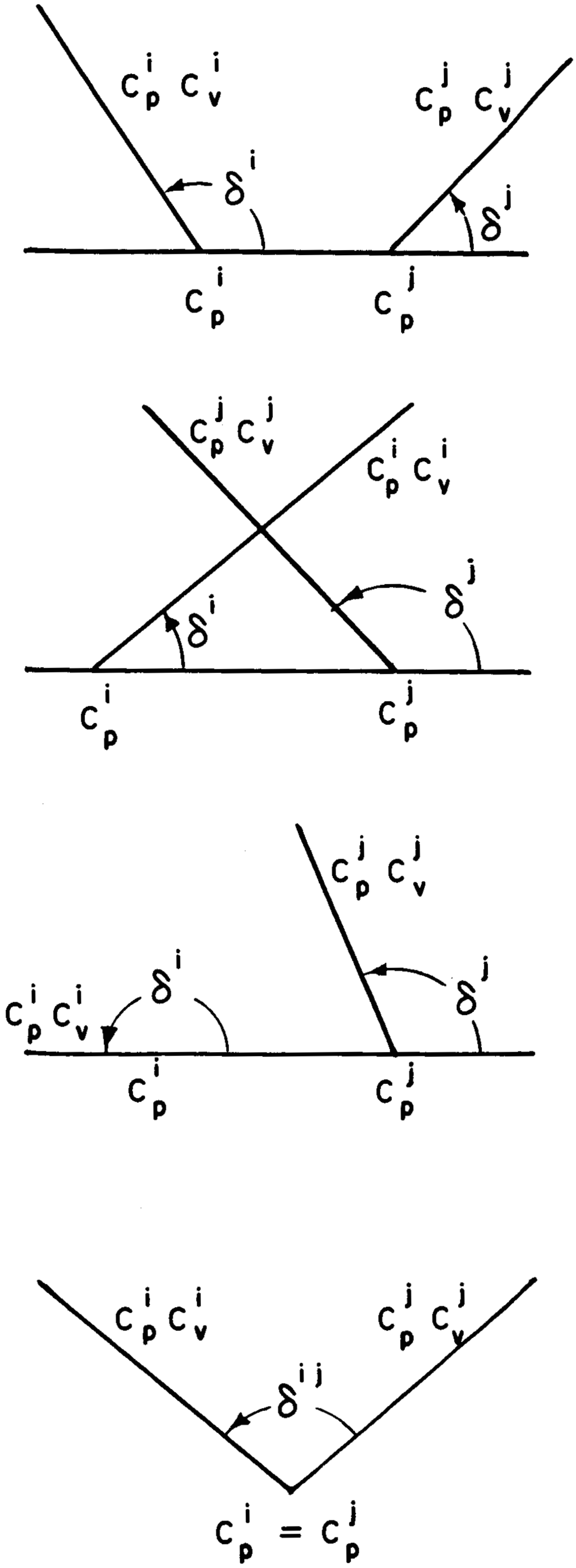


FIG. 6

PARALLEL-AXIS CONFIGURATIONS



S 1) $\delta^i > \delta^j$

S 2) $\delta^i < \delta^j$

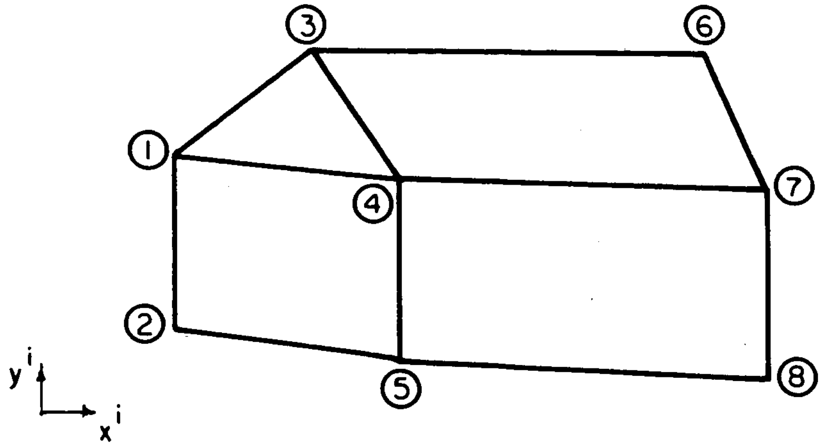
S 3) $\delta^i = 180°$

$180° > \delta^j > 0°$

S 4) $180° > \delta^{ij} > 0°$

FIG. 7

NON-PARALLEL AXIS CONFIGURATIONS



Vertex	x^i	y^i
1	-40	5
2	-40	-14
3	-20	15
4	0	0
5	0	-20
6	55	14
7	60	-1
8	60	-19

$$\mathcal{J} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

FIG. 8

ILLUSTRATION OF VISIBLE PERSPECTIVE PROJECTION DATA: PROJECTION, VERTEX LIST, AND INCIDENCE MATRIX

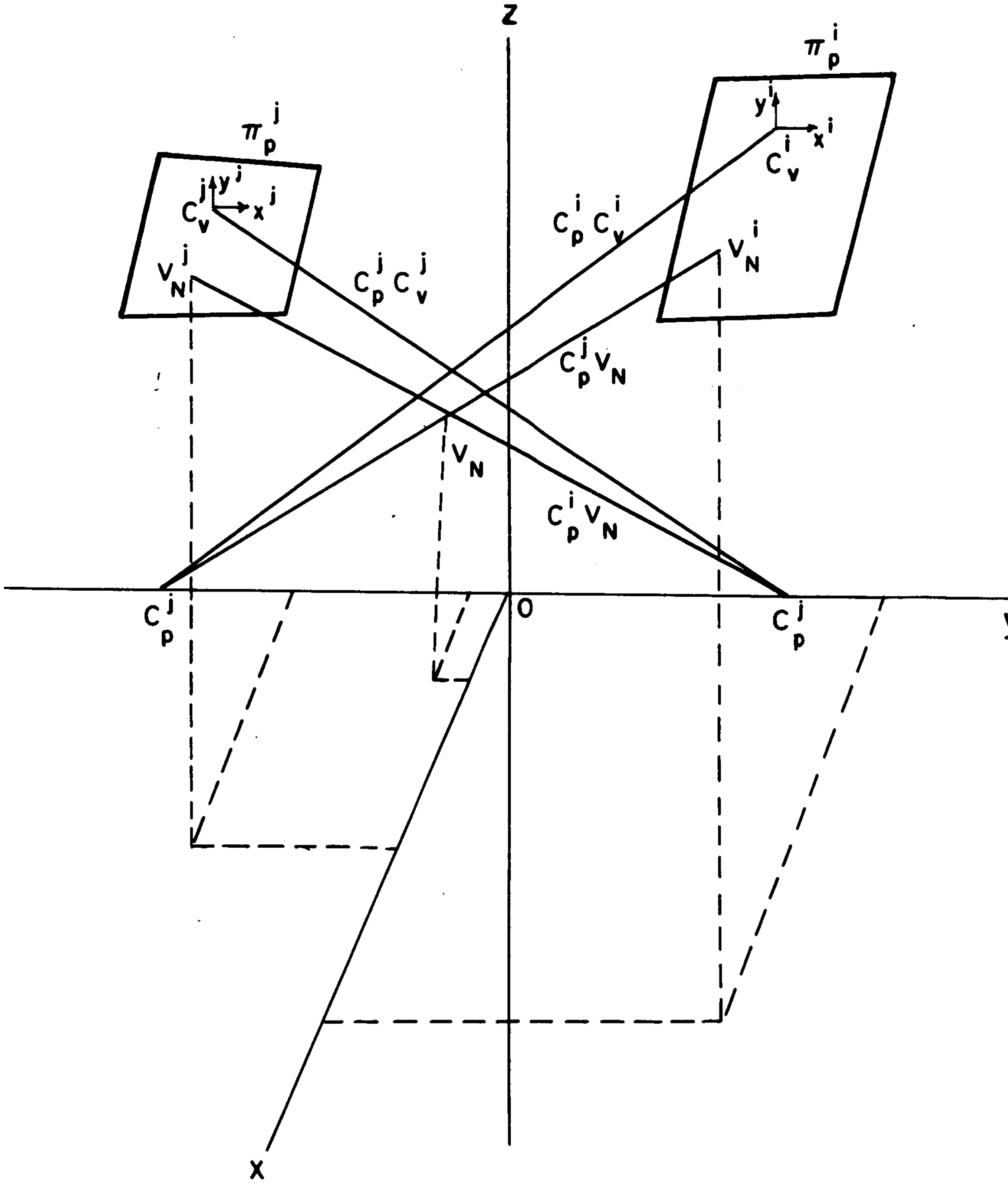
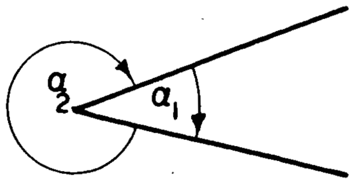


FIG. 9

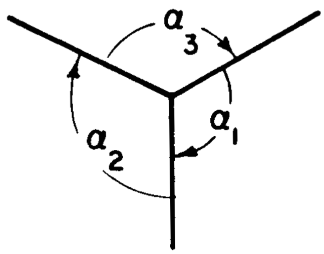
ILLUSTRATION OF COORDINATE SYSTEMS AND NOTATION



A) 2 EDGES

$\alpha_1 < 180^\circ$

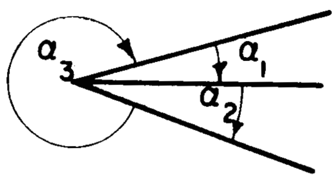
$\alpha_2 > 180^\circ$



B) 3 EDGES

$\alpha_i < 180^\circ$

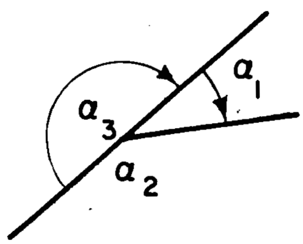
$\sum_{i=K}^{L} \alpha_i \neq 180^\circ \quad K, L = 1, 2, 3$



C) 3 EDGES

$\alpha_3 > 180^\circ$

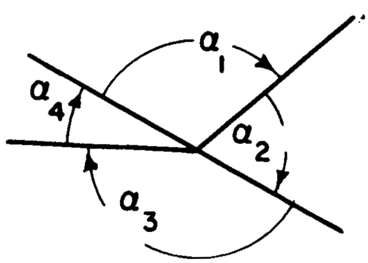
$\alpha_1, \alpha_2 < 180^\circ$



D) 3 EDGES

$\alpha_3 = 180^\circ$

$\alpha_1, \alpha_2 < 180^\circ$



E) 4 EDGES

$\alpha_i \neq 180^\circ$

$\alpha_1 + \alpha_2 = 180^\circ$

$\alpha_3 + \alpha_4 = 180^\circ$

FIG. 10

BASIC VERTEX TYPES

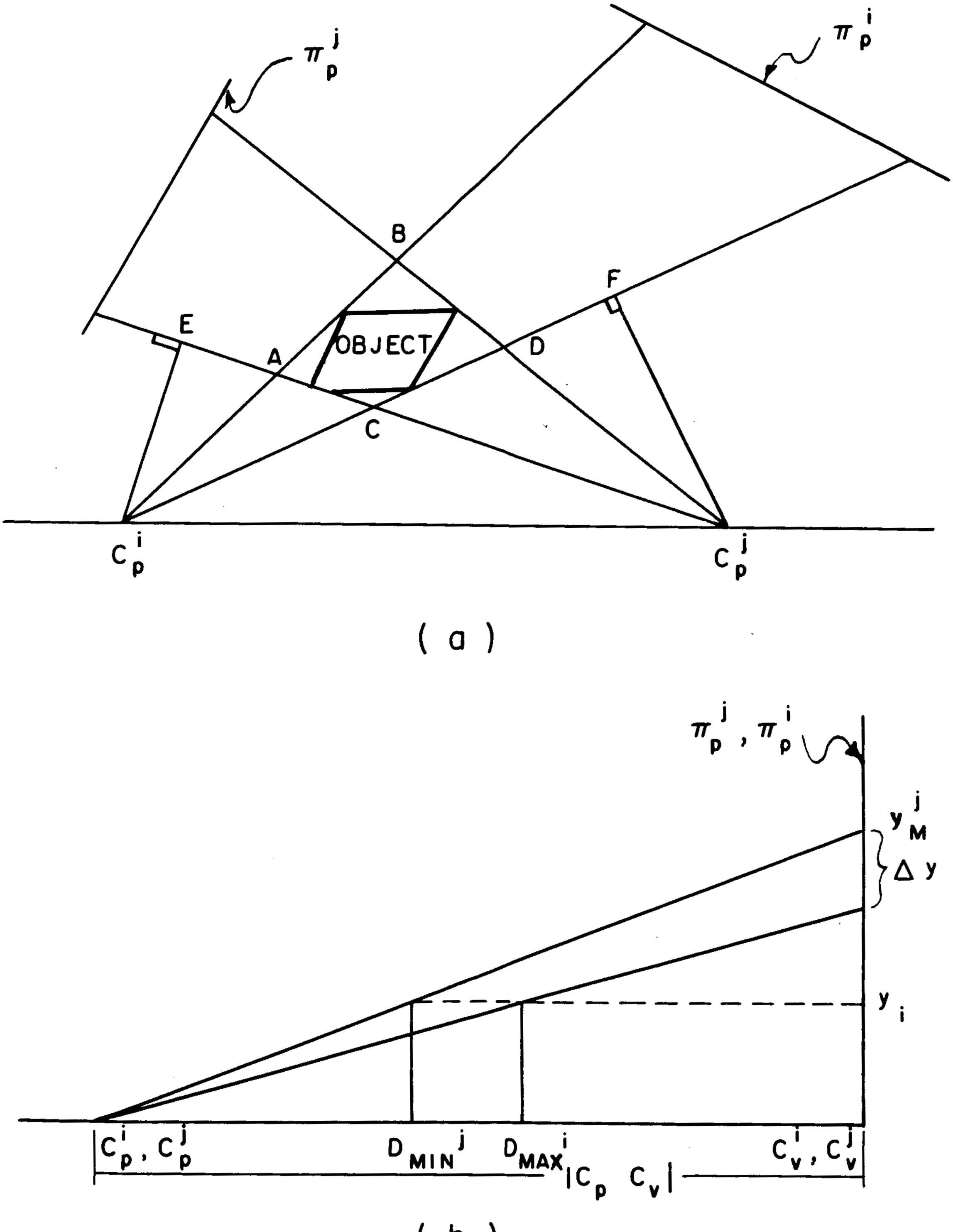


FIG. 11

DETERMINATION OF MAXIMUM HEIGHT CHANGE

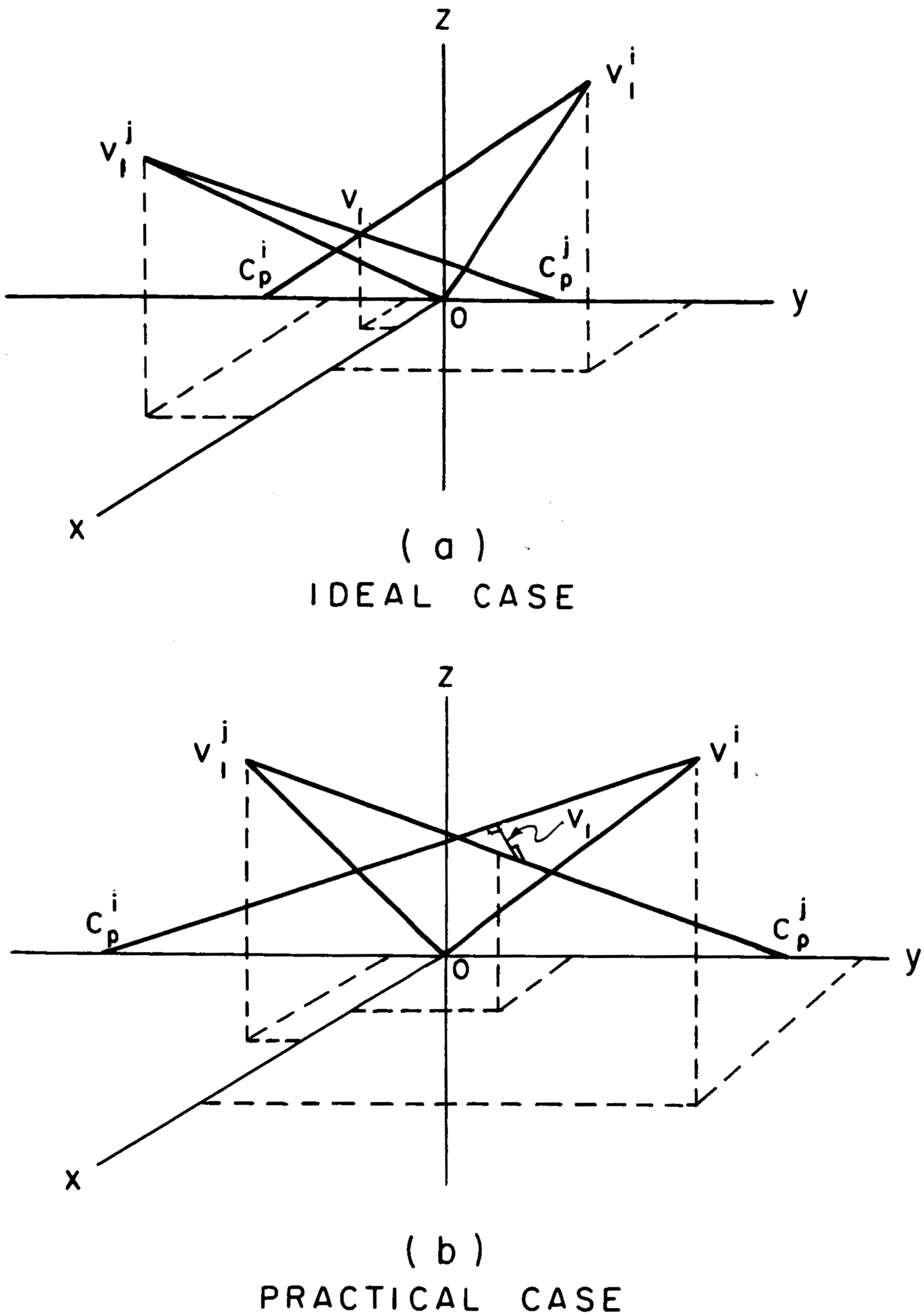


FIG. 12

DETERMINATION OF OBJECT VERTICES

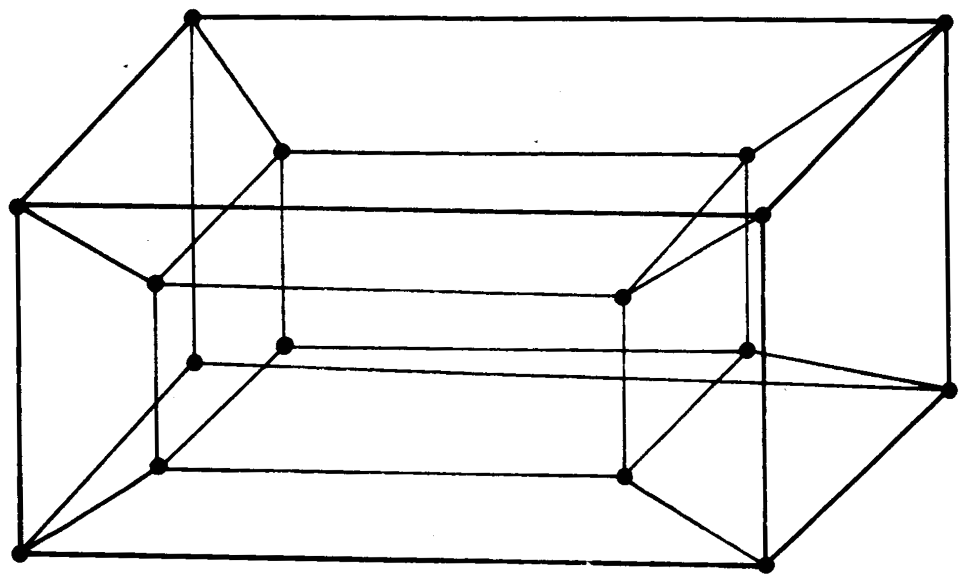


FIG. 13

VERTICES AND EDGES OF A MULTIPLY-CONNECTED POLYHEDRON

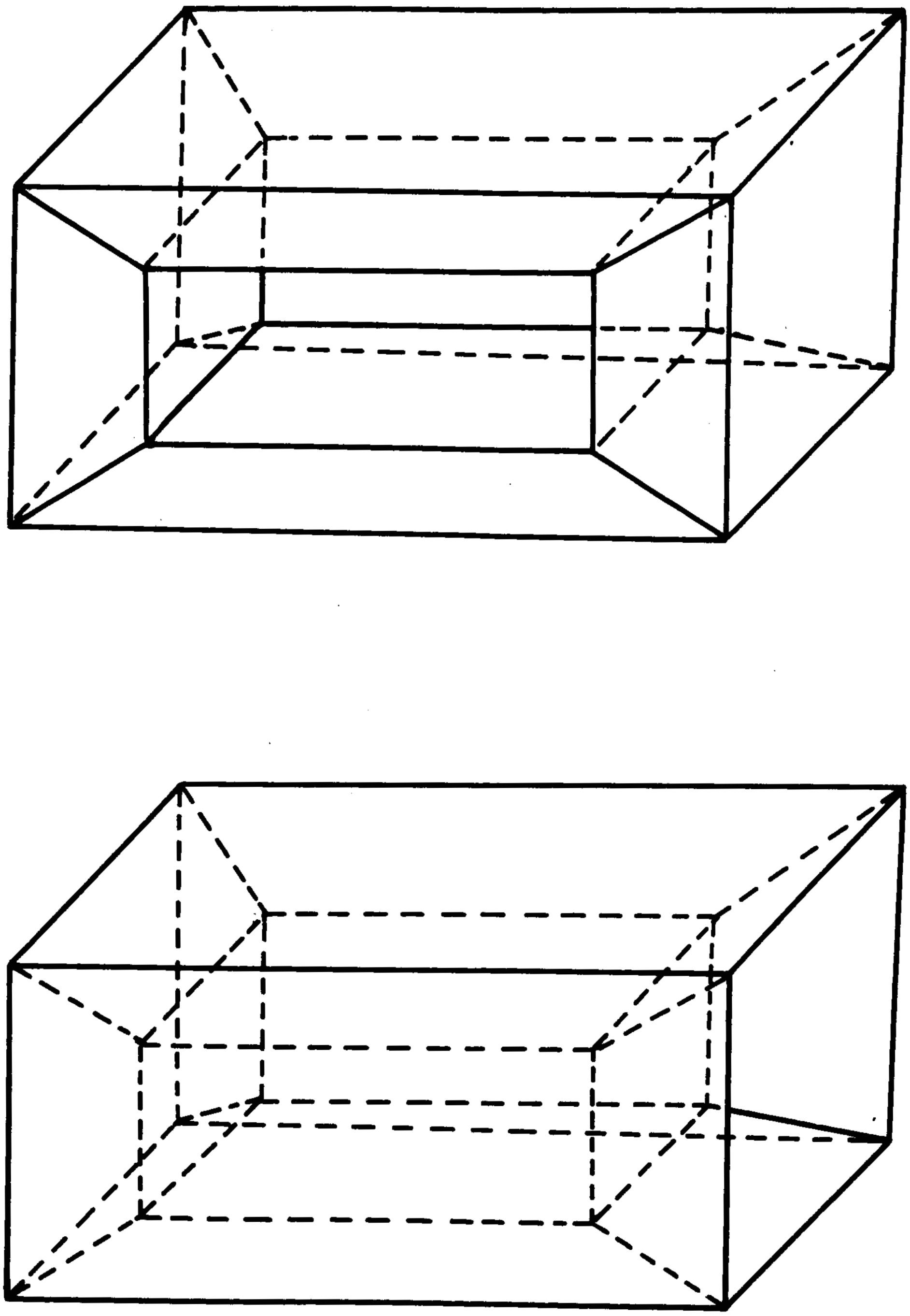


FIG. 14

TWO DIFFERENT MULTIPLY-CONNECTED POLYHEDRA WITH THE SAME SET OF VERTICES AND EDGES