INFLUENCE OF UNCERTAINTIES IN THE ASTRONOMICAL UNIT CONVERSION AND MARS PLANETARY MASS ON EARTH—MARS TRAJECTORIES

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FOREWORD

The research presented in this report was performed for the Astrionics Laboratory of the George C. Marshall Flight Center, Huntsville, Alabama. The report is the mid-term progress report on the interplanetary navigation and guidance study task under NASA Contract NAS8-20358. The results presented here extend the scope of the navigation and guidance study completed under Contract NAS8-11198.

ABSTRACT

The influence of uncertainties in the planetary mass of Mars and the conversion of the astronomical unit to a laboratory unit on a variety of Earth-Mars trajectories is analyzed. The effect of the uncertainties in these two constants is shown in the form of deviations in the periaries radius at Mars and the resulting approach guidance velocity correction required for two guidance laws. The capability of an onboard navigation system to estimate the approach trajectory deviations is also analyzed. The navigation system consists of a 10 arc second sextant for star-planet measurements and a Kalman filter for the data smoothing.

The results of the analysis indicate deviations in the periaries radius of 10 to 20 km for an uncertainty of 150 km^3/sec^2 in Mars planetary mass. The approach velocity corrections required are on the order of 5 to 10 meters/second. The uncertainty in the astronomical unit conversion produces trajectory deviations that are quite trajectory dependent. The deviations for five heliocentric transfer angles are analyzed with the time of flight as a parameter. The curve for each transfer angle showing the distance of closest approach deviation as a function of flight time exhibits either a single or double minimum. The minimum deviation is near zero for the 180 degree transfers and increases for larger and smaller transfer angles. The minimum deviation for the 270 degree transfers is 550 km for 1000 km uncertainty in the astronomical unit conversion. This large range in the magnitude of deviations causes a corresponding large range in the approach guidance velocity corrections required. The smaller deviations require a velocity correction of 10 to 30 meters/second and the larger deviations require 50 to 70 meters/second.

These guidance velocity requirements significantly increase the total velocity requirements obtained while neglecting the two equation of motion uncertainties.

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LIST OF SYMBOLS

Matrices

$A(T,t)$	transition matrix from the state at time, t, to end constraints.
$C(T)$	point transformation from state to end point constraints.
$G(T)$	point transformation from state to $\vec{B}$ magnitude and radius of closest approach.
I	identity matrix.
P	covariance matrix of error in estimate of state.
Q	covariance matrix of random measurement noise.
$\varphi(t_2,t_1)$	vehicle state transition matrix.
$\varphi_T(t_2,t_1)$	expanded state transition matrix.
$\varphi_\mu(t_2,t_1)$	control transition matrix.

Vectors

$\vec{B}$	target miss vector.
$\vec{D}$	constraint deviation vector.
H	measurement gradient with respect to state.
K	Kalman filter gain.
$\hat{R}$, $\hat{S}$, $\hat{T}$	unit vectors of target miss coordinates.
$\vec{v}$	heliocentric vehicle velocity.
$\vec{v}_\epsilon$	heliocentric velocity of Earth.
$\vec{v}_\infty$	hyperbolic excess velocity.
$\vec{x}$	deviation state vector.
$\hat{x}$	estimate of state vector.
$\vec{\dot{x}}_g$	guidance velocity correction.

Scalar Constants

a	semi-major axis.
a_ϵ	Earth equatorial radius.
A.U.	astronomical unit.
b	$\vec{B}$ vector magnitude.
G	gravitational constant.
M_ϵ	mass of the Earth.
M_S	mass of the Sun.
R	Earth-Sun mean distance.
r	heliocentric position of vehicle.
r_ϵ	heliocentric position of Earth.
v	magnitude of $\vec{v}$.
v_ϵ	magnitude of $\vec{v}_\epsilon$.
v_∞	magntiude of $\vec{v}_\infty$.
y	measurement.
$\hat{y}$	estimate of measurement.
δ	scattering angle.
ϵ	orbit eccentricity.
μ	product of gravitational constant and mass (GM).
π	solar parallax.
π'	relative uncertainty in π.
ω	angular frequency of Earth.

Operations

$E(\ )$	expected value.
M^T	matrix transpose.
M^{-1}	matrix inverse.
$\lvert\ \rvert$	vector magnitude.

SECTION 1

INTRODUCTION

The increasing interest in both manned and unmanned exploration of the near planets, Mars and Venus, dictates a need for determining the effect of uncertainties in heliocentric and planetary constants on the navigation and guidance subsystem requirements for such missions. The two constants to be considered here are the mass of Mars and the ratio of the astronomical unit to laboratory units. The analysis and results presented are extensions and generalization of the work reported in references 1 and 2 by R. M. L. Baker Jr. and S. Herrick et.al. respectively.

The various methods used in estimating these two constants and the probable errors associated with them are described in references 3 through 8. Tables 1 and 2 summarize the results of several determinations of the planetary mass of Mars and the ratio of the astronomical unit to laboratory units. The large discrepancy between the radar measurements of the astronomical unit distance and the dynamical method using the asteroid Eros is discussed in reference 6. The discussion indicates that a plausible explanation of this discrepancy is the existence of systematic ephemeris errors that are not accounted for in the dynamic method.

The effect of the uncertainty in these two constants on the navigation and guidance subsystem requirements is analyzed using digital computer simulations of the two subsystems with a conic trajectory program. The linearized navigation and guidance theory used in the simulations is described in section 2. The results of the analyses of the uncertainties in Mars planetary mass and the ratio of the astronomical unit to laboratory units are presented in sections 3 and 4 respectively. The results show the influence of the uncertainty in each of the two constants on a variety of Earth-Mars Transfer and approach trajectories. For each of the constants the following results are obtained.

1. The approach trajectory deviations due to the uncertainty in the constant.
2. The approach guidance Δv required to correct the deviations for both fixed time of arrival and variable time of arrival guidance laws.
3. Navigation data obtained by using a 10 arc second sextant with a Kalman filter.

Section 5 summarizes the results and shows their relationship to guidance requirements that were obtained neglecting the uncertainty in these constants.

SECTION 2

NAVIGATION AND GUIDANCE THEORY

The function of the navigation system, as defined in this report, is to obtain an estimate of the vehicle state and predict the end constraint deviations. The guidance system converts the estimated end deviations into the required guidance correction based on the selected guidance law. The theoretical biasis for the computer simulations and a defintion of the data used in sections 3 and 4 to evaluate subsystem requirements are summarized in this section. A detailed description of the theory is presented in reference 9.

2.1 Navigation System

The navigation system results that are presented in sections 3 and 4 are for an onboard system using a sextant with a random error of 10 arc seconds. The measurement schedule[10]* consists of a star-planet observation every 15 minutes during approach using a repeated star sequence. Five measurements are made using a star in the trajectory plane followed by one measurement using a star normal to the trajectory plane (figure 1).

The navigation system analysis is performed using the Mark II Error Propagation Program[11]. This is an orbit determination error analysis program for a navigation system that uses a Kalman filter for processing measurement data.

The error analysis quantities are defined below along with a summary of the equations used in the Kalman trajectory estimation and end point prediction processes. A basic assumption in the theory is that linearity is satisfied in the neighborhood of a nominal trajectory. The nomeclature used in the presentation is the following.

* Superscripts refer to the list of references.

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ \mu \end{pmatrix} \qquad \varphi_T = \begin{pmatrix} \varphi & \varphi_\mu \\ 0 & I \end{pmatrix} \qquad P = E\left\{(\vec{x}-\hat{x})(\vec{x}-\hat{x})^T\right\}$$

where

$\vec{x}$ = total deviation state

μ = planetary mass

φ = vehicle state transition matrix $\left[\frac{\partial x(t)}{\partial x(0)}\right]$

φ_μ = vehicle state sensitivity to planetary mass $\left[\frac{\partial x(t)}{\partial \mu}\right]$

$\hat{x}$ = estimate of deviation state

E = expected value

P = covariance matrix of error in estimate of the state

Between the onboard observations the deviation state estimate and the error covariance matrix are propagated in time along the nominal trajectory as follows.

$$\hat{x}(t_2) = \varphi_T(t_2,t_1)\hat{x}(t_1) \qquad (1)$$

$$P(t_2) = \varphi_T(t_2,t_1)P(t_1)\varphi_T^T(t_2,t_1) \qquad (2)^*$$

At the time of an observation, the measurement information is included in the state estimate and a new covariance matrix obtained in the following manner.

* Superscript T indicates matrix Transpose.

$$\hat{x}_n = \hat{x}_o + K(y-\hat{y}) \tag{3}$$

$$P_n = P_o - KHP_o \tag{4}$$

where

$K = P_o H^T (HP_o H^T + Q)^{-1}$ (Kalman filter gain)

H = gradient of the measurement with respect to the state

y = measurement

$\hat{y}$ = estimate of measurement

Q = covariance matrix of the random measurement noise

In order to compute a guidance correction the estimated deviation state and covariance matrix of the error in estimate are propagated to the end time. They are then transformed into an estimate of the constraint deviations and error in estimate respectively. This is done to determine if the accuracy to which the constraints are known is sufficient to make a guidance correction. The prediction of the estimated end point deviations and the associated covariance matrix is shown below.

$$\hat{x}(T) = \varphi_T(T,t)\hat{x}(t) \tag{5}$$

$$P(T) = \varphi_T(T,t)\ P(t)\varphi_T^T(T,t) \tag{6}$$

In addition to the state deviation estimate and error in estimate at the end point, it is of interest for purposes of variable time of arrival guidance corrections to know the deviations in the magnitude* of $\vec{B}$ vector[12] and the related deviations in the distance of closest approach. These quantities are obtained by means of a point transformation applied to equations (5) and (6).

$$\begin{pmatrix} \delta \left|\vec{B}\right| \\ \delta(RCA) \end{pmatrix} = G(T)\ \hat{x}(T) \tag{7}$$

$$\begin{pmatrix} \sigma^2_{\left|\vec{B}\right|} & \rho\ \sigma_1\sigma_2 \\ \rho\ \sigma_1\sigma_2 & \sigma^2_{RCA} \end{pmatrix} = G(T)\ P(T)\ G^T(T) \tag{8}$$

where

$$G(T) = \begin{pmatrix} \dfrac{\partial \left|\vec{B}\right|}{\partial x(T)} \\ \dfrac{\partial (RCA)}{\partial x(T)} \end{pmatrix} \quad \text{point transformation}$$

$\left|\vec{B}\right|$ = $\left|\vec{B}\right|$ vector magnitude

RCA = radius closest approach

Equations (1) through (8) describes the processes by which the navigation system evaluation data are obtained. Examples of these data are shown in figure 9 and 22 of sections 3 and 4 respectively.

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* Only the magnitude is of interest because the guidance analysis is restricted to a two dimensional analysis.

2.2 Guidance System

The guidance system analysis is restricted to computing the velocity corrections required to correct trajectory deviations as a function of time along the approach trajectories. A detailed analysis of the selection of guidance correction times and the effects of guidance system error sources on terminal accuracy for a Mars mission is presented in reference 13. The analysis presented here includes the use of two guidance laws; (1) fixed time of arrival (FTA), and (2) variable time of arrival (VTA).

The three end constraints used with each guidance law are shown below.

$$FTA = \begin{pmatrix} X(T) \\ Y(T) \\ Z(T) \end{pmatrix}_T \qquad VTA = \begin{pmatrix} \vec{B}\cdot\hat{T} \\ \vec{B}\cdot\hat{R} \\ V_\infty \end{pmatrix}_{T + \Delta T}$$

where

T = nominal arrival time

$\vec{B}\cdot\hat{T}$, $\vec{B}\cdot\hat{R}$ = orthogonal components of the $\vec{B}$ vector

V_∞ = hyperbolic excess velocity

X, Y, Z = nominal vehicle position state at time, T

The guidance velocity correction required at time, t, is computed in the following manner. The vehicle deviation state is propagated along a nominal trajectory to the end point (equation 9) and transformed into appropriate constraint deviations (equation 10).

$$\vec{x}(T) = \varphi(T,t)\ \vec{x}(t) \tag{9}$$

$$\vec{D}(T) = C(T)\ \vec{x}(T) = C(T)\ \varphi(T,t)\ \vec{x}(t) \tag{10}$$

where

$\vec{x}$ = deviation state vector

$\vec{D}$ = constraint deviation vector

C(T) = point transformation from the state to either FTA or VTA constraints

The sensitivity of the end constraints to a velocity correction at time, t, is obtained from the partioned transition matrix.

$$\underset{3x6}{A(T,t)} = (\underset{3x3}{A_1} \mid \underset{3x3}{A_2}) = \underset{3x6}{C(T)}\ \underset{6x6}{\varphi(T,t)} \tag{11}$$

where

A_1 = sensitivity of end constraints to a position change at time, t.

A_2 = sensitivity of end constraints to a velocity change at time, t.

The velocity correction required to null the constraint deviation vector, $\vec{D}(T)$, in equation 10 is the following.

$$A_2\ \dot{\vec{x}}_g(t) + \vec{D}(T) = 0$$

or

$$\dot{\vec{x}}_g(t) = -A_2^{-1}\ \vec{D}(T) = -(A_2^{-1}\ A_1 \quad I)\ \vec{x}(t) \tag{12}$$

where

$\dot{\vec{x}}_g$ = the guidance velocity correction

The deviation state, $\vec{x}(t)$, used in the guidance analysis of section 3 and 4 is obtained by taking the difference between a nominal approach trajectory and a perturbed trajectory. The trajectory is perturbed due to an uncertainty in the planetary mass or an uncertainty in the astronomical unit conversion. Examples of the required velocity correction data are shown in figures 8 and 19 through 21.

SECTION 3

MARS' PLANETARY MASS

The target approach phase of an interplanetary trajectory is a target centered hyperbola (figure 2). The characteristics of this trajectory are determined by the vehicle velocity state relative to the target at the time the "sphere of influence"(14)* is reached and the planetary mass of the target body. The vehicle velocity state relative to the target at this time is determined by the particular heliocentric transfer trajectory that is used. The influence of an uncertainty in the planetary mass is analyzed using four approach trajectories with energies that are indicative of tranfer trajectories of interest.

The trajectory model used in the analysis is a conic section. During the approach phase of a mission, this is a good approximation to the three dimensional trajectory. The $\vec{B}$ vector and radius of closest approach (RCA) are used to describe the vehicle passage of the planet. The $\vec{B}$ vector and the associated unit vectors $\hat{R}$, $\hat{S}$, $\hat{T}$ (figure 3) are described in reference 12. The $\hat{S}$ vector is in the direction of the approach asymptote and the $\hat{R}$, $\hat{T}$ vectors are in the plane normal to the $\hat{S}$ vector and containing the $\vec{B}$ vector.

The magnitude of the vehicle velocity state at the sphere of influence, v_∞, is used as a parameter in this analysis to simulate approach trajectories of different energies. The range of values used for v_∞ is from 2 km/sec to 8 km/sec. This range covers the approach velocities resulting from practical Earth-Mars transfer trajectories. Table 3 presents examples of approach velocities for three missions. The flight path angle of the approach velocity vector is used to control the distance of closest approach. The close approach radius is varied from 5000 km to 50,000 km in the analysis.

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* A 565,000 km radius is used for the sphere of influence in the analysis.

The planetary mass used for Mars is 42915.515 km^3/sec^2. The uncertainty in the mass is assumed to be $\pm$ 200 km^3/sec^2. This is done to encompass the uncertainty of $\pm$ 10,000 inverse solar masses shown in Table 1 for the adopted value in 1961.

The results presented include the deviations in the $\vec{B}$ vector magnitude, distance of closest approach, and scattering angle, δ, due to the planetary mass uncertainty. These deviations are important when performing close approach maneuvering missions or continuing flyby missions with an Earth return.

3.1 Analysis of Effect of Mass Uncertainty

The error in planetary mass is related to an error in the semi-major axis of the approach hyperbola through the vis-viva equation.

$$a = \frac{\mu}{v_\infty^2} \tag{13}$$

where

a = semi-major axis

μ = planetary mass

v_∞ = hyperbolic excess velocity

or

$$\frac{\Delta\mu}{\mu} = \frac{\Delta a}{a} \tag{14}$$

The angle between the approach and regression asymptotes, δ, is related to the approach trajectory as follows.

$$\delta = 2 \cos^{-1}\left(\frac{1}{\epsilon}\right) = 2 \cos^{-1}\left(\frac{\mu}{\mu + r_p v_\infty^2}\right) \tag{15}$$

where

ϵ = eccentricity

r_p = periapsis radius

Figure 4 presents the scattering angle as a function of distance of closest approach and the hyperbolic excess velocity. Deviations in δ due to the uncertainty in the planetary mass may be obtained as follows. The $\vec{B}$ vector magnitude is maintained constant ie;

$$|\vec{B}| \equiv b = a\sqrt{|(1-\epsilon^2)|} \tag{16}$$

Then from equations (15) and (16)

$$\sin\left(\frac{\delta}{2}\right)\Delta\delta = -\frac{2\Delta\epsilon}{\epsilon^2} \tag{17}$$

and

$$0 = 2a(1-\epsilon^2)\,\Delta a - 2a^2\,\epsilon\Delta\epsilon$$

or

$$\Delta\epsilon = \frac{(1-\epsilon^2)}{a\epsilon}\Delta a = \frac{b^2\Delta a}{a^3\epsilon} \tag{18}$$

Substituting equation (17) into (18) yields

$$\Delta\delta = -\frac{2b^2\Delta a}{(a\epsilon)^3\sin\frac{\delta}{2}} = -\left(\frac{b}{a}\right)^2\left(\frac{2}{\epsilon^3\sin\frac{\delta}{2}}\right)\frac{\Delta\mu}{\mu} \tag{19}$$

$$\Delta\delta = -\frac{2b}{a\epsilon^2}\frac{\Delta\mu}{\mu} = -\frac{2\,ab}{a^2+b^2}\frac{\Delta\mu}{\mu}$$

in terms of v_∞ equation (19) becomes

$$\Delta\delta = -\frac{2b}{v_\infty^2}\left[\frac{\Delta\mu}{\mu^2/v_\infty^4 + b^2}\right] \tag{20}$$

Figure 5 presents the deviations in the scattering angle as a function of v_∞ and distance of closest approach. The planetary mass uncertainty, $\frac{\Delta\mu}{\mu}$, is .0047. These data were obtained by taking the difference between a nominal scattering angle and scattering angles obtained when using perturbed values of the planetary gravitational constant, μ, in a conic trajectory program. The difference results obtained in this manner for the stated planetary mass deviation agree quite well with the linear deviation expressed by equation (20).

The importance of these scattering angle deviations on a Earth return trajectory is expressed by the sensitivity of the Earth close approach distance to the scattering angle at Mars. This sensitivity for typical Mars-Earth trajectories ranges from one hundred thousand to a million kilometers for one degree variation in the scattering angle.

The deviation in close approach distance as a function of planetary mass and close approach distance (RCA) is shown in figure 6. The data in figure 6 indicate that an uncertainty in the planetary mass of the order shown in Table 2 causes close approach deviations from ± 2 km for a high energy trajectory to ± 15 km for a very low energy trajectory. The deviations on the low energy trajectory increase to ± 35 km for a close approach distance of 50,000 km. These data show that the planetary mass uncertainty is an important factor for missions requiring terminal accuracies on the order of 15 km and less.

The entry corridor at Mars with a 5 mb atmosphere is approximately 20 km[15] or ± 10 km from a nominal trajectory. This indicates that for an atmospheric entry mission, a low energy approach trajectory could have significant deviations due to the uncertainty in the planetary mass.

The statistical significance attached to the uncertainties shown in Table 1 is also a factor in determining the need for approach guidance corrections on the higher energy trajectories. If it is assumed that the uncertainties shown in the table represent one sigma values; then the deviations in figure 5 and 6 represent the maximum deviations to be expected in 68% of the cases for a selected uncertainty. It would then require the deviation numbers to be increased by a factor of 3 to include 99% of the cases. The uncertainties in Table 1 have been treated as one sigma values in the analysis.

The implications of these trajectory deviations during target approach on the navigation and guidance system requirements are analyzed in the following section.

3.2 Navigation and Guidance Analysis

The results presented in this section show the navigation and guidance requirements for controlling the approach trajectory under the influence of an uncertainty in the planetary mass. The results assume that the mid-course guidance system has controlled the vehicle to the sphere of influence perfectly. The only equation of motion uncertainty considered is the planetary mass.

The time history of the growth in the predicted deviations in close approach distance and $\vec{B}$ magnitude (equation 7) based on the state deviation is shown in figure 7. These data were obtained using a planetary mass uncertainty of 130 km^3/sec^2 and close approach distance of 5000 km. The curves all display the characteristics of having very small deviations until 4 to 8 hours before periaries. The deviations then grow rapidly to values from 2 to 15 kilometers. The approach guidance Δv required to correct these deviations is shown in figure 8 as a function of time along the trajectory. The requirements are shown for both FTA and VTA guidance laws. The ΔV required on these trajectories for each guidance law is between 1 and 10 meters/second during the last few hours. The VTA velocity requirements are smaller in all cases.

The time at which a reasonable guidance correction can be made is determined by the navigation system. The error in estimate of the end constraints must be below a predetermined level before the guidance maneuver can be executed. The selection of an entry mission at Mars with a $\pm$ 10 km entry corridor defines tolerable limits on the end constraint deviations. If it is required that the confidence in hitting the entry corridor is to be 99% (3 sigma), then the one sigma error in estimate of the close approach distance must be reduced to $\pm$ 3.3 km. The guidance correction can then be made with a 99% confidence (neglecting execution errors) of hitting the $\pm$ 10 km corridor.

The capability of an onboard navigation system using a 10 arc second sextant to estimate the end constraints is shown in figure 9. The results are shown for three nominal trajectories with different energies. The parameters being estimated include the vehicle state and the planetary mass. The analysis process used is described in section 2. The initial vehicle state uncertainty is assumed to be zero and the uncertainty in the planetary mass is 130 km^3/sec^2. The tolerable error in estimate for an entry mission is shown on figure 9 as $\pm$ 3.3 km. The times on these trajectories that this level is reached are 2 days 19 hours for the trajectory with v_∞ = 2.0 km/sec and 1 day 12 hours for the v_∞ = 4.0 km/sec trajectory. Using these correction times in figure 8, shows the guidance velocity requirements are approximately 1 meter/sec for a VTA guidance law and 3 meters/sec for a FTA guidance law. The significance of these approach corrections in terms of the total mission is discussed in section 5.

SECTION 4

ASTRONOMICAL UNIT CONVERSION

The uncertainty in the ratio of the astronomical unit (A.U.) to a laboratory unit is an important factor in the accuracy with which an interplantary mission can be performed. Reference 2 demonstrates the importance of using the basic "Gaussian" gravitational constant based on the A.U. and the solar mass in trajectory computations. This is due to the eight or nine figure accuracy[16] to which it is presently known. The same constant expressed in laboratory units is only accurate to three or four figures. The importance of the uncertainty in the ratio to an interplanetary mission is due to the fact that with an ephemeris expressed in terms of the A.U., mission analysis specifications of injection conditions at Earth are in terms of the A.U. The uncertainty in the conversion of the geocentric injection conditions from a working laboratory unit to the astronomical unit results in the Earth escape velocity being in error in units of A.U./Day. Conversely, the uncertainty in the ratio will appear in the initial heliocentric position and velocity of the Earth, the gravitational constant, and in the terminal position and velocity of Mars, if these quantities are converted from astronomical units to kilometers.

The error caused by the ratio uncertainty in the conversion of the trajectory problem totally into A.U.'s is the same as the error in converting the problem to kilometers[2]. This equivalence is shown in the next section. The computer simulation used in the analysis of the uncertainty in the ratio expresses the problem in kilometers. The geocentric hyperbolic excess velocity is assumed to be known precisely and the uncertainty in the ratio occurs in the planetary ephemeris.

The planet ephemeris model used in the analysis has the following characteristics. The planets Earth and Mars are on coplanar circular orbits about the Sun at distances of 1 A.U. and 1.53 A.U. respectively.

The uncertainty in the A.U. conversion* is included in the model in the following manner. For the Earth on a two body Keplerian orbit, the A.U., mass of the Sun, mass of the Earth, and the period of the Earth about the Sun are related by the following expression.

$$\omega^2 = \frac{G(M_s + M_\epsilon)}{(AU)^3} \tag{21}$$

where

ω = angular frequency of the Earth

M_s = mass of the Sun

M_ϵ = mass of the Earth

AU = astronomical unit

G = universal gravitational constant

It is assumed that the Earth's angular frequency is known perfectly and that the Earth's mass can be neglected with respect to the Sun's mass. Under these assumptions, the partial derivative of equation (21) becomes

$$\left(\frac{\partial GM_s}{\partial AU}\right)_{\omega = CONST} = \left(\frac{GM_s}{AU}\right) \tag{22}$$

The relationship shown in equation (22) indicates that a change in the "length" of the A.U. must be accompanied by a change in the mass of the Sun in order to maintain ω constant. In the ephemeris model used, a change in the A.U. is accompanied by changes in the radial distances of the planets and the mass of the Sun. These changes maintain the angular frequencies of the planets constant.

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* The nominal conversion factor used is 149599000. km/A.U.

The launch and target planets are positioned with an initial angular separation that will satisfy the geometry required for a specified heliocentric transfer angle for a given flight time.

The trajectory program obtains an Earth-Mars heliocentric conic trajectory with a specified flight time and transfer angle. The heliocentric conic is then patched to a Mars centered conic trajectory at the sphere of influence. The initial heliocentric velocity magnitude is then varied in a differential correction loop to obtain a specified close approach distance at Mars. This process establishes a nominal trajectory for the flight time and transfer angle. The initial heliocentric velocity vector is then separated into two parts as shown below.

$$\vec{v} = \vec{v}_{\epsilon} + \vec{v}_{\infty} \tag{23}$$

where

$\vec{v}$ = initial heliocentric vehicle velocity

$\vec{v}_{\epsilon}$ = heliocentric velocity of Earth

$\vec{v}_{\infty}$ = geocentric hyperbolic excess velocity (velocity relative to Earth at a million km)

The geocentric hyperbolic excess velocity, v_{∞}, represents the Earth departure condition measured in kilometers/sec that a mission analysis would show is required for a nominal ephemeris. This is assumed to be known precisely and is not changed. The A.U. conversion factor is then perturbed causing changes in the positions and velocities of Earth and Mars. The gravitational constant is also changed in accordance with equation (22). The result of these changes is that the initial vehicle state relative to the Sun deviates from the nominal conditions. The vehicle position is changed with the change in the Earth's position. The vehicle velocity relative to the Sun is changed through the change in the Earth's velocity in equation (23).

The perturbed heliocentric trajectory is patched to Mars and the approach trajectory differences from the nominal computed. The process is shown pictorially in figure 10.

4.1 Analysis of Effect of Uncertainty in A.U. Conversion

The ratio of the A.U. to the equatorial radius is the "solar parallax" expressed in radians (figure 11). Then the desired ratio of the kilometer to the A.U. embodied in the mean Earth distance, R, is related to the solar parallax as follows.

$$R \doteq \frac{a_\epsilon}{\pi} \tag{24}$$

where

a_ϵ = Earth equatorial radius

π = solar parallax

R = Earth-Sun mean distance

The relative uncertainty in the ratio is

$$\frac{\Delta R}{R} = - \frac{\Delta \pi}{\pi} - \frac{\Delta a_\epsilon}{a_\epsilon} \tag{25}$$

or neglecting the smaller uncertainty in a_ϵ

$$\frac{\Delta R}{R} = - \frac{\Delta \pi}{\pi} \equiv - \pi' \tag{26}$$

The effect of the relative uncertainty, π', in the ratio R will appear in the initial geocentric position and velocity of the vehicle if they are expressed in the astronomical unit. The analysis of the error resulting from the conversion of the initial state to astronomical units is presented below.

The Hohmann transfer trajectory geometry is shown in figure 12. The relative error in the major axis, 2a, can be found from the vis-viva integral which may be written:

$$v^2 = 2\mu\left(\frac{1}{r} - \frac{1}{2a}\right) = 2r_\epsilon v_\epsilon^2\left(\frac{1}{r} - \frac{1}{2a}\right) \tag{27}$$

where

r, v = heliocentric position and velocity of vehicle

r_ϵ, v_ϵ = heliocentric position and velocity of Earth

$r_\epsilon v_\epsilon^2$ = gravitational constant assuming Earth orbit is circular

$2a$ = major axis of transfer

For the Hohmann transfer, the initial heliocentric vehicle state is the following.

$$r = r_\epsilon \qquad v = v_\epsilon + v_\infty \tag{28}$$

where v_∞ is the velocity of the vehicle at about a million kilometers. In the process of conversion of the problem from kilometers to A.U.'s the position and velocity of the Earth may be assumed to be known accurately in astronomical units.

$$\Delta r = \Delta r_\epsilon = \Delta v_\epsilon = 0 \tag{29}$$

The heliocentric vehicle velocity is in error due to the fact that v_∞ although known accurately in laboratory units must be converted to A.U.'s using R as a conversion factor.

$$\Delta v = \Delta v_{\infty} = v_{\infty}\, \pi' \tag{30}$$

Then from equation (27),

$$\frac{\Delta(2a)}{(2a)^2} \quad \frac{v\, \Delta v}{r_{\epsilon} v_{\epsilon}^2} \tag{31}$$

or using equation (30)

$$\frac{\Delta(2a)}{(2a)} = \frac{2a}{r_{\epsilon}} \frac{v\, v_{\infty}}{v_{\epsilon}^2}\, \pi' \tag{32}$$

The conversion of the problem from A.U. to kilometers yields the following expression[2] for $\frac{\Delta 2a}{2a}$ that differs from equation (32).

$$\frac{\Delta 2a}{2a} = \left[\frac{2a}{r_{\epsilon}} \frac{v\, v_{\infty}}{v_{\epsilon}^2} - 1\right] \pi' \tag{33}$$

This is the uncertainty in Δa expressed in kilometers, leaving the position of Mars as an uncertainty. Since the position of Mars is well known in astronomical units, the uncertainty should be sought in $\Delta\left(\frac{2a}{R}\right)$ not $\Delta(2a)$. Then equation (32) becomes

$$\frac{\Delta\left(\frac{2a}{R}\right)}{\left(\frac{2a}{R}\right)} = \left(\frac{\Delta 2a}{2a} - \frac{\Delta R}{R}\right) = \frac{2a}{r_{\epsilon}} \frac{v\, v_{\infty}}{v_{\epsilon}^{2}}\, \pi' \tag{34}$$

which is in agreement with equation (32).

Using approximate Hohmann transfer numbers, equations (32) and (33) are evaluated to use as a point of reference for the data that follow.

$$\frac{2a}{r_\epsilon} = \frac{5}{2} \qquad \frac{v}{v_\epsilon} = \frac{13}{12} \qquad \frac{v_\infty}{v_\epsilon} = \frac{1}{12} \qquad 2a = 375.10^6 \text{ km}$$

$$\frac{2a}{r_\epsilon} \frac{v\, v_\infty}{v_\epsilon^2} = .226$$

Letting $\pi' = .67.10^{-5}$ ($\Delta R \approx -$ 1000 km)

From equation (32) the uncertainty in the semi-major axis of the transfer is the following.

$$\Delta(2a) = (.226)\ (67.10^{-5})\ (375.10^6)$$
$$\Delta(2a) = 560 \text{ km}$$
$$\Delta(a) = 280 \text{ km}$$

The uncertainty in the semi-major axis from equation (33) which leaves the position of Mars as an uncertainty is the following.

$$\Delta(2a) = (.226\text{-}1)\ (67 \cdot 10^{-5})\ (375.10^6)$$
$$\Delta(2a) = -1940 \text{ km}$$
$$\Delta(a) = -970 \text{ km}$$

The Hohmann transfer example case is illustrated in figure 13. The figure shows the significance of the "two uncertainties" in the transfer major axis.

4.2 Navigation and Guidance Analysis

The data and results presented in this section were generated using an uncertainty in the conversion of the A.U. to kilometers of $\pm$ 1000 km. This is slightly larger than the uncertainty shown in Table 2 for the 1963 adopted value.

The approach phase of a number of Earth-Mars trajectories is analyzed to determine the navigation and guidance requirements due to the uncertainty in the A.U. conversion. Five heliocentric transfer angles are used with flight times for each from 100 to 500 days. Three trajectories of interest that are listed in Table 3 are included in the analysis.

They are; 1. Hohmann transfer 180 degrees, 260 days, 2. Mariner IV trajectory 160 degrees, 228 days, and 3. High energy outbound leg of round trip trajectory[17] 270 degrees, 235 days.

The trajectory data and corresponding target approach deviations are shown in figure 14 through 18. Part A of each figure shows the flight time, launch velocity, and target approach velocity as a function of the direction of the hyperbolic approach asymptote, $\hat{S}$. Part B shows the deviation in close approach distance for a 1000 km change in the A.U. conversion to kilometers. The 160, 180, and 200 degree transfers each show two devation minimums. The 225 and 270 degree transfers each have a single minimum. The minimum deviations are near zero for the 180 degree transfer and increase with transfer angles away from 180. The minimum deviations for the 225 and 270 degree transfers are 125 km and 550 km respectively. These data show the possibility of selecting trajectories that minimize the effect of the uncertainty in the A.U. conversion on the close approach distance. The Mariner IV trajectory is one that is near a minimum. The 228 day 160 degree transfer has a deviation of 175 km for a 1000 km uncertainty in the conversion.

The position deviation state at the sphere of influence (patch point) for all the trajectories is approximately 1000 km. The close approach deviation minimums are the result of these errors at the patch point being in directions that result in cancellation or partial cancellation of the deviation in the periaries distance.

The trajectories marked with an asterisk on the 160, 180, and 270 degree transfers are analyzed to determine the approach guidance velocity required to correct the deviations. The results of this analysis are shown in figures 19, 20, and 21. The solid lines indicate the Δv required for a fixed time of arrival (FTA) and the dotted lines the requirements for a variable time of arrival (VTA). These laws are described in section 2. The curves show that the requirements for FTA are nearly the same for all the trajectories shown. A correction at the sphere of influence requires about 10 meters/second and grows to approximately 200 meters/second as periaries is approached. The Δv requirements for the VTA guidance law show a wide variation depending on the specific trajectory selected.

The trajectories that have small close approach deviations, curve 1 in figure 19 and curves 1 and 3 in figure 20, have velocity requirements that range from less than 1 meter/second at the sphere of influence to 8, 4 and 2 meters/second respectively at periaries. The remaining VTA curves in figures 19 and 20 show larger Δv requirements that range from 2 meters/second to 40 meters/second for the trajectories with larger deviations. These requirements are considerably smaller than those required with a FTA guidance law. Figure 21 shows the Δv requirements on a 270 degree transfer. Figure 18B shows the minimum deviation for this transfer is 550 km, which is much larger than other transfer minimums and the velocity requirements are correspondingly higher. The VTA velocity requirements are only slightly smaller than those required for a FTA guidance law.

The magnitude of the velocity correction required for any trajectory is dependent on the time of correction. Figure 22 shows the error in estime of the end constraints for three approach trajectories of different energies. The navigation system that was used is described in section 2. The initial error in estimate of state is assumed to be 1000 km in each of the inplane position coordinates and .2 meters/second in the velocity coordinates. These errors correspond to the actual deviations that occur at the time of patch to the target due to a 1000 km uncertainty in the A.U. conversion. Due to the onboard observations, the error in estimate of the constraints is quickly reduced to less than 100 km. It then remains relatively constant until the last few hours of the approach. The error in estimate is sufficiently small for an entry mission (3.3 km) approximately 3 or 4 hours prior to periaries on each trajectory.

Figures 19 through 21 indicate that this time corresponds to corrections of 50 to 70 meters/second for a FTA guidance law. The VTA guidance requirements at this time are less than 10 meters/second except for the 270 degree transfer where they are about 30 meters/second. A FTA guidance policy allowing for two approach corrections could reduce the total Δv required considerably from the 50-70 meters/second required for a single correction.

A factor that has been neglected in the guidance analysis is the execution errors. A correction of 70 meters/second with proportional errors of 1% would produce a .7 meter/second execution error. Three or four hours before periaries the close approach sensitivity to a velocity change is such that .7 meter/second error will cause deviations that are the same order of magntiude as the entry corridor. This factor also favors the guidance policy of two smaller approach corrections for an accurate planet passage.

SECTION 5

SUMMARY AND CONCLUSIONS

The analysis indicates that the uncertainty in Mars planetary mass produces deviations in close approach of 10 to 20 km for practical approach trajectories. The Δv required to control the deviations for a FTA guidance law is from 5 to 10 meters/second and from 1 to 5 meters/second with a VTA guidance law.

The effect of an uncertainty in the A.U. conversion to laboratory units is much more significant than the mass uncertainty. The analysis of 5 heliocentric transfer angles for various flight times shows one or two minimums in the close approach deviations for each transfer. The deviation minimums vary from near zero to 550 km for a 1000 km uncertainty in the A.U. The minimum deviations are near zero for a 180 degree transfer and increase for larger and smaller transfer angles. The guidance velocity corrections for a FTA guidance law are from 50 to 70 meters/second when using only one correction. The corrections for a VTA law vary considerably. The trajectories with small deviations (less than 100 km) require corrections from 1 to 10 meters/second. The trajectories with the larger deviations require corrections of 10 to 30 meters/second.

The guidance requirements for an Earth-Mars mission neglecting the two uncertainties that have been analyzed here are shown in Table 4.[13] The results in Table 4 for a VTA guidance law include the effects of errors in an onboard navigation system and guidance system execution errors. The approach trajectory deviations due to a planetary mass uncertainty cannot be estimated until the last few hours of the approach trajectory. It would therefore be necessary to control these deviations with the final correction. The 1 meter/second final correction shown in Table 4 would increase to a maximum of approximately 5 meter/second with a mass uncertainty of 150 km^3/sec^2. The trajectory deviations due to the uncertainty in the A.U. conversion can be estimated with an error of 30 to 40 km one day prior to periaries with a 10 arc second instrument.

This allows the possibility of making a correction at this time that will correct the deviations to an accuracy consistent with the estimate. The deviations remaining after the correction could then be removed with the final maneuver. On the trajectories with large deviations this would increase the third correction of Table 4 by approximately 10 meters/second. The final correction would be increased by 2 to 3 meters/second.

The discussion of guidance Δv requirements above is summarized in Table 5. These results were obtained by algebraically adding the velocity requirements caused by the two uncertainties in the equations of motion to those due to injection errors, navigation errors, and guidance system execution errors. This very pessimistic analysis of adding these independent effects algebraically increases the total velocity requirements from 23 meters/second to 41 meters/second.

SECTION 6
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SECTION 7

TABLES AND FIGURES

TABLE 1

MARS MASS (SUN'S MASS = 1)

Mars Mass (m^{-1})	Method	Author	Reference
3648000	Vesta	Leveau (1890)	(3)
3093500		Newcomb (1895)	(3)
3601280	Vesta	Leveau (1907)	(3)
3085000 ± 5000	Weighted Mean	de Sitter (1938)	(3)
3110000 ± 7700	Eros	Rabe (1949)	(3)
3079000 ± 5700	Deimos	Urey (1952)	(3)
3090000 ± 10000		Adopted (1961)	(3)
3088000 ± 3000	Weighted Mean	Clemence (1961)	(8)
3090000 ± 3000		Adopted (1963)	(8)

TABLE 2

ASTRONOMICAL UNIT

A. U. (KM)	Method	Author	Reference
149662400 ± 25600	Eros (Geometric)	(1941) Jones	(4)
149530300 ± 10200	Eros (Dynamical)	(1950) Rabe	(4)
149598640 ± 250	Venus (Radar)	(1962) Muhleman et.al.	(5)
149597850 ± 400	Venus (Radar)	(1962) Pettingill et.al.	(5)
149598100 ± 400	Venus (Radar)	(1962) Muhleman (Revision of Pettingill's Value)	(5)
149601000 ± 5000	Venus (Radar)	(1961) Thompson,et.al.(GB)	(5)
149599500 ± 800	Venus (Radar)	(1961) Kotelnikov (USSR)	(5)
149599244 ± 278	Mariner II Tracking	(1963) Anderson,et.al.	(7)
149599000 ± 700	Recommended Value	(1963)	(8)

TABLE 3

TYPICAL EARTH-MARS TRAJECTORIES

Transfer Trajectory	Flight Time (Days)	Heliocentric Angle (Deg)	Mars Approach Velocity (KM/SEC)
Hohmann	260	180	2.6
Mariner IV	228	160	3.1
Outbound of Round Trip	235	270	6.6

TABLE 4

GUIDANCE PERFORMANCE VTA GUIDANCE LAW

Correction	End Constraint Deviations (KM)		Δv Req'd M/Sec	Time
	$\vec{B}\cdot\hat{T}$	$\vec{B}\cdot\hat{R}$		
1	10300	2390	10.56	1^D
2	269	153	8.18	220^D
3	12.8	8.6	3.53	234^D
4	6.54	2.01	.92	$234\ 20^h$
		TOTAL	23.19	

Trajectory: 235 Days 270 Degree Transfer

TABLE 5

VELOCITY REQUIREMENTS WITH EQUATION OF MOTION UNCERTAINTIES

Correction	Δv Req'd M/Sec
1	10.56
2	8.18
3	3.53 10.00 (AU)
4	.92 5.00 (μ) 3.00 (AU)
TOTAL	41.19

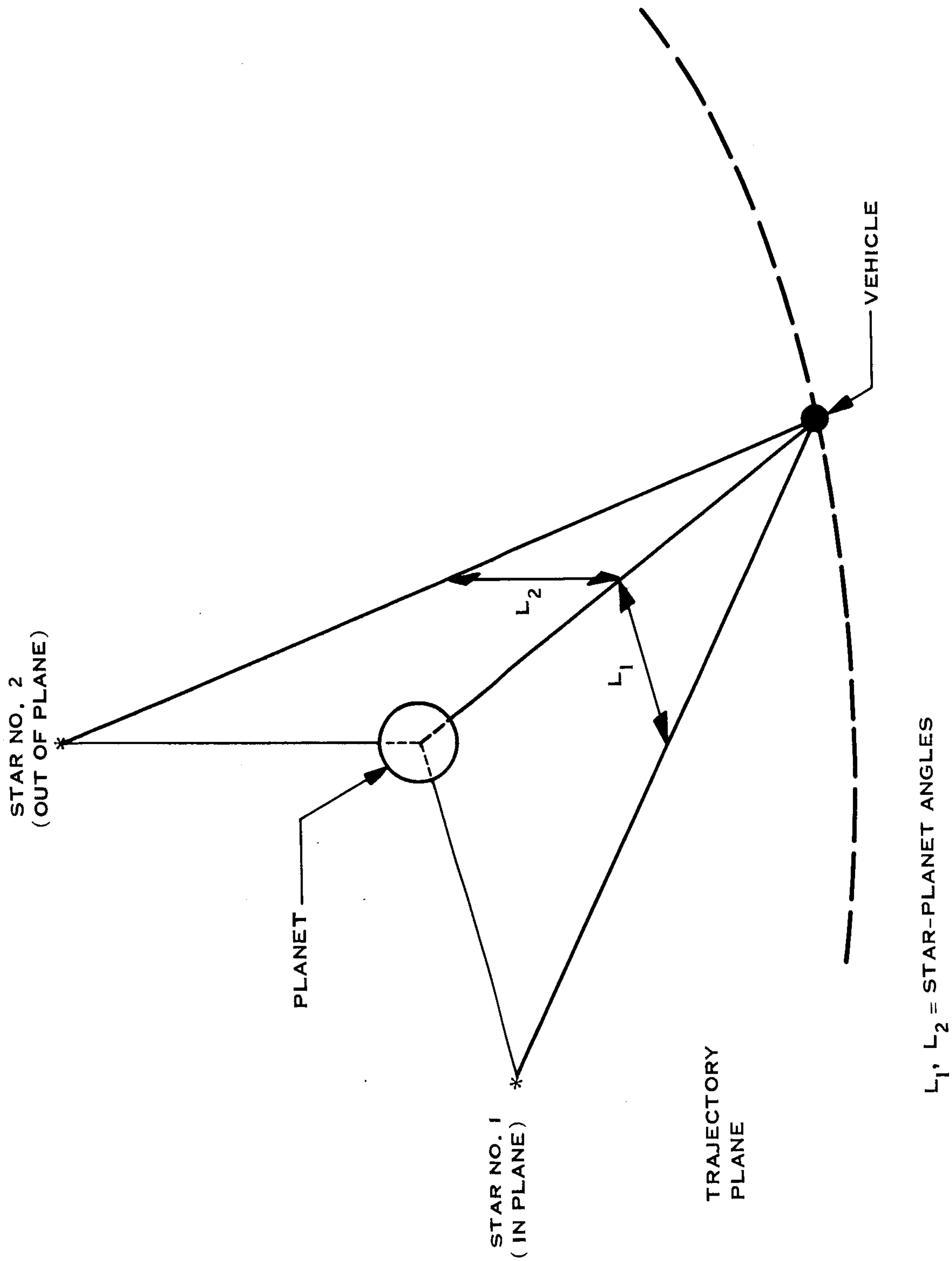
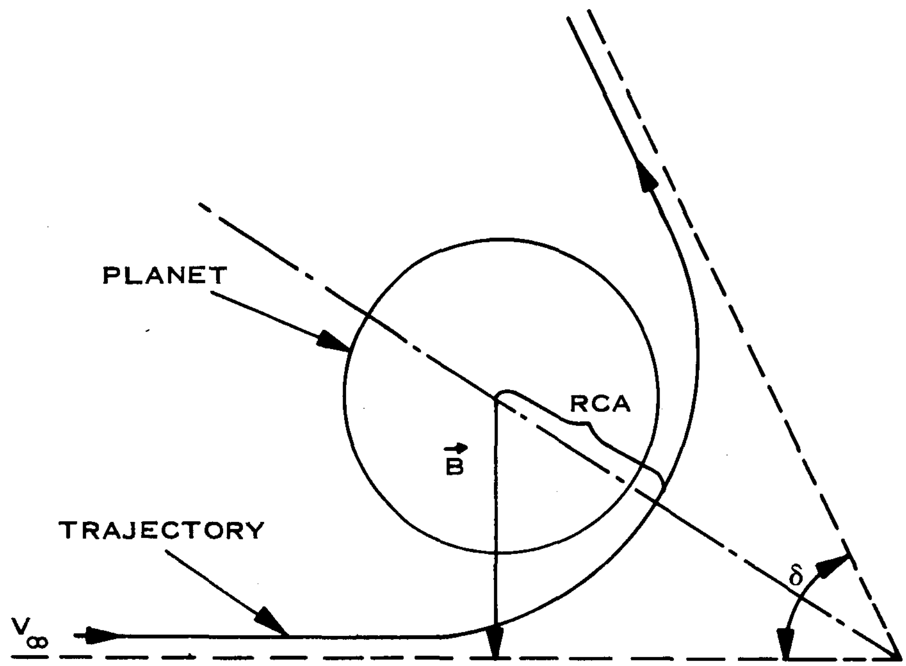


FIGURE 1 - SEXTANT IN PLANE AND OUT OF PLANE STARS



RCA = RADIUS CLOSEST APPROACH

δ = SCATTERING ANGLE

$\vec{B}$ = B VECTOR

V_∞ = VELOCITY AT INFINITY

--- = ASYMPTOTES

FIGURE 2 - APPROACH TRAJECTORY GEOMETRY

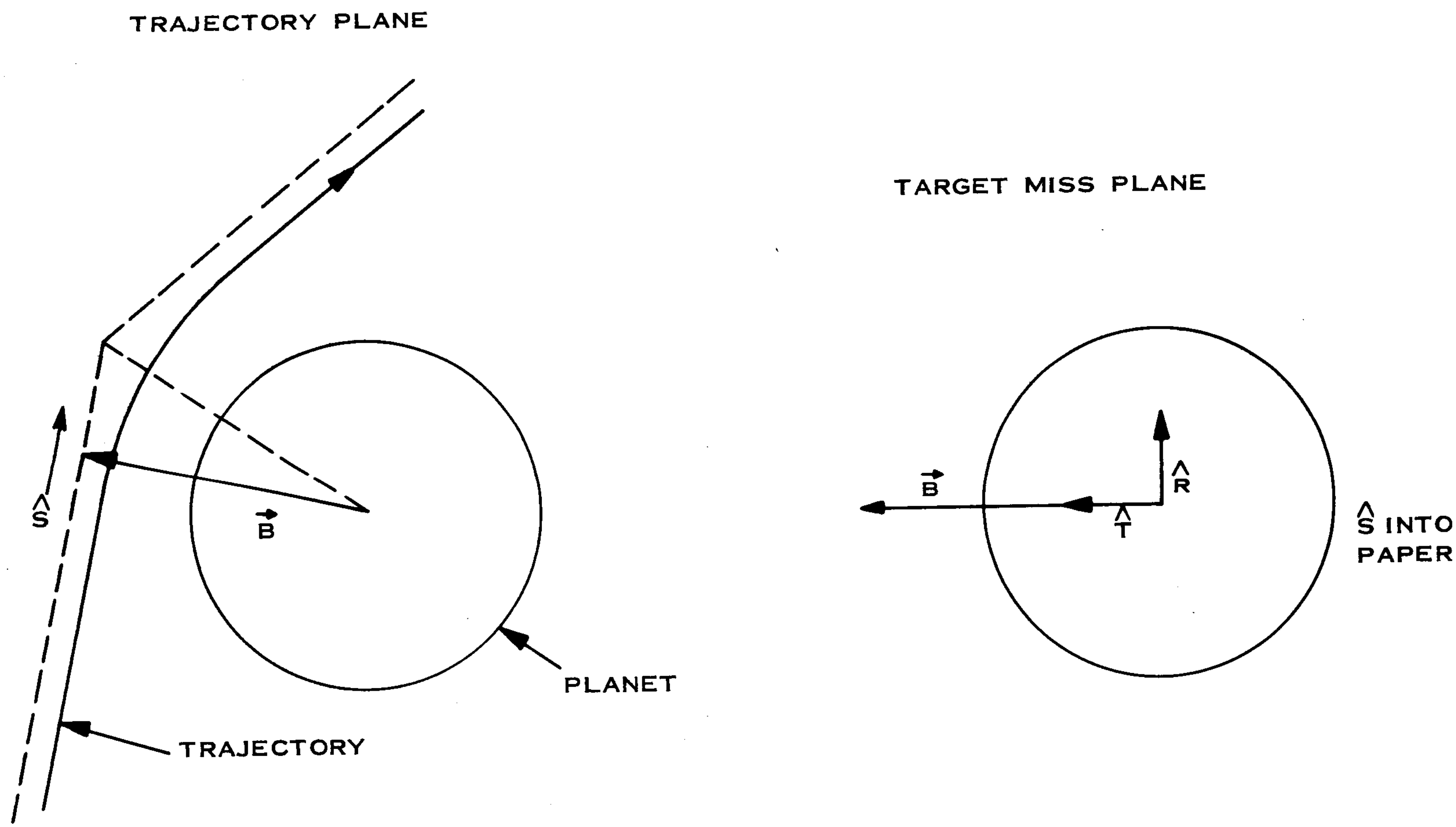


FIGURE 3 - TRAJECTORY MISS COORDINATES

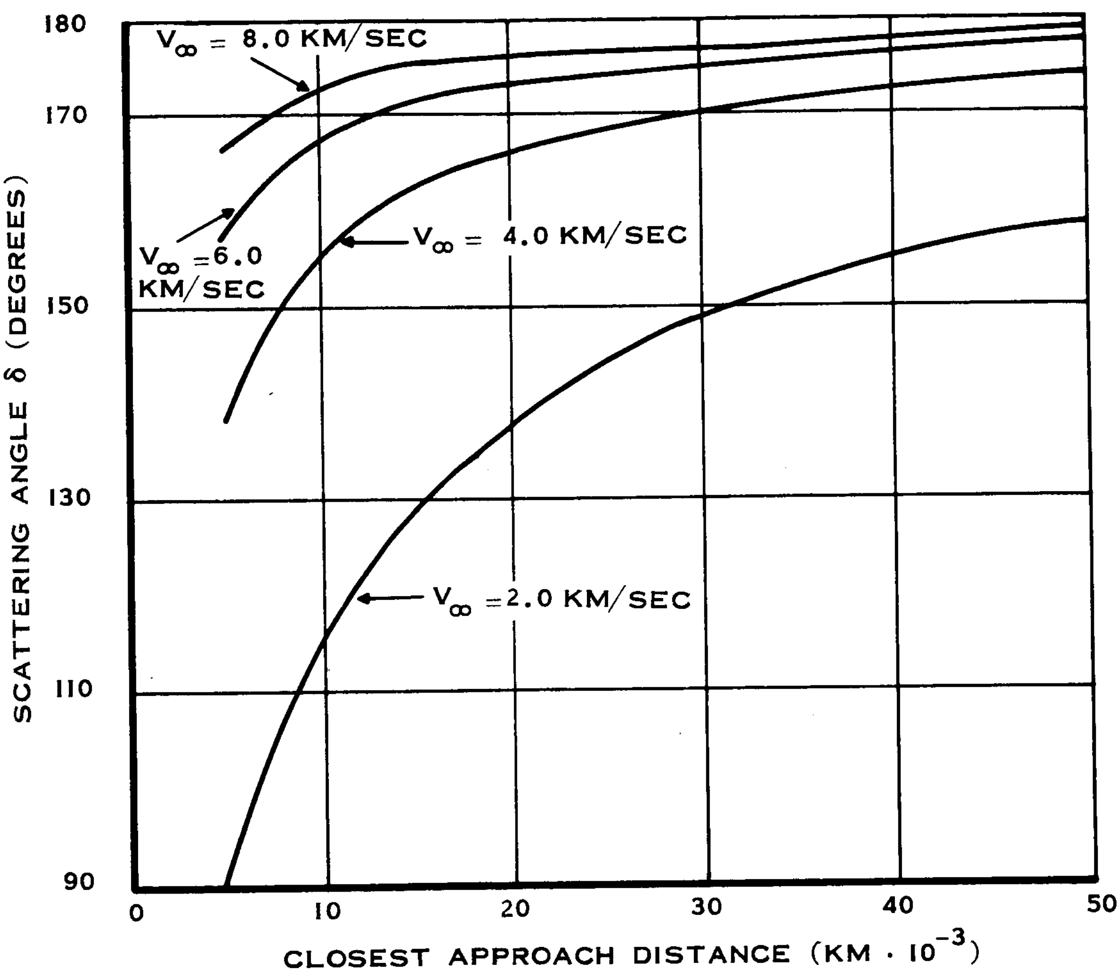


FIGURE 4 - MARS APPROACH TRAJECTORY SCATTERING ANGLE

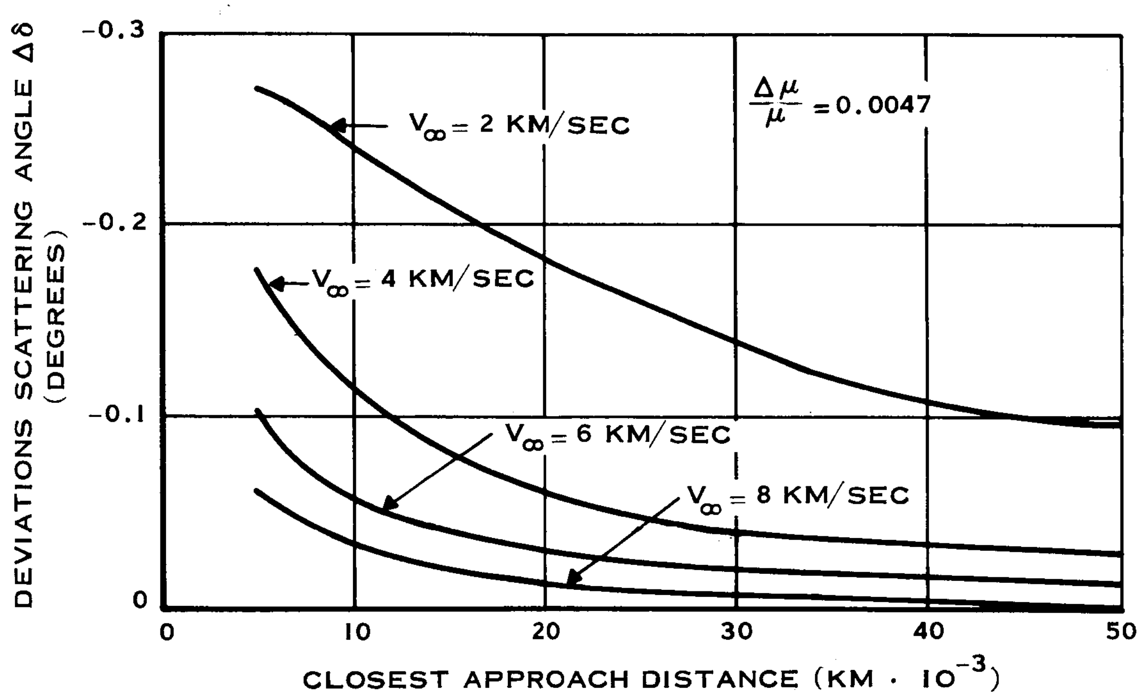


FIGURE 5 - SCATTERING ANGLE DEVIATION

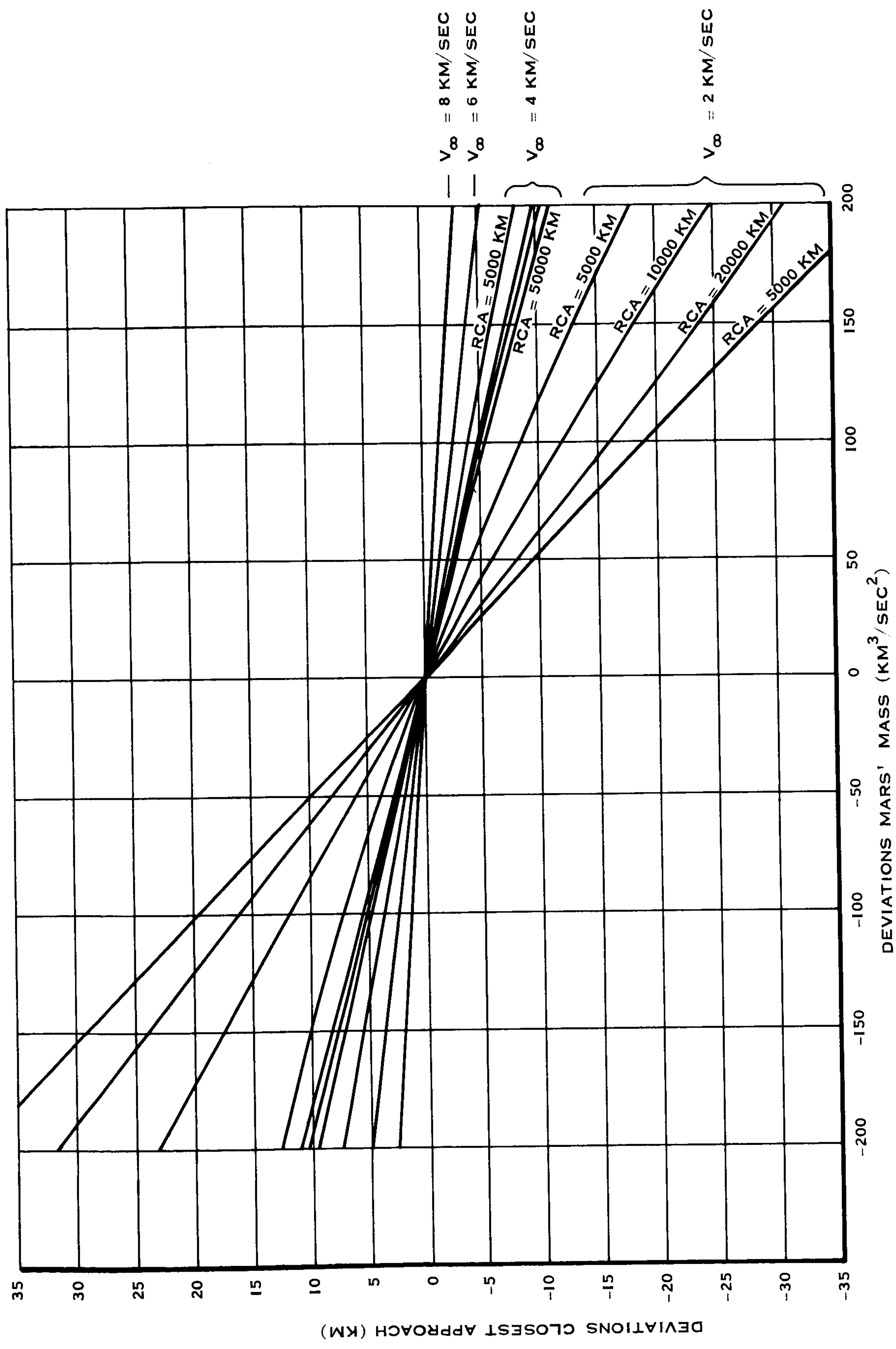


FIGURE 6 - DEVIATIONS IN CLOSEST APPROACH DISTANCE

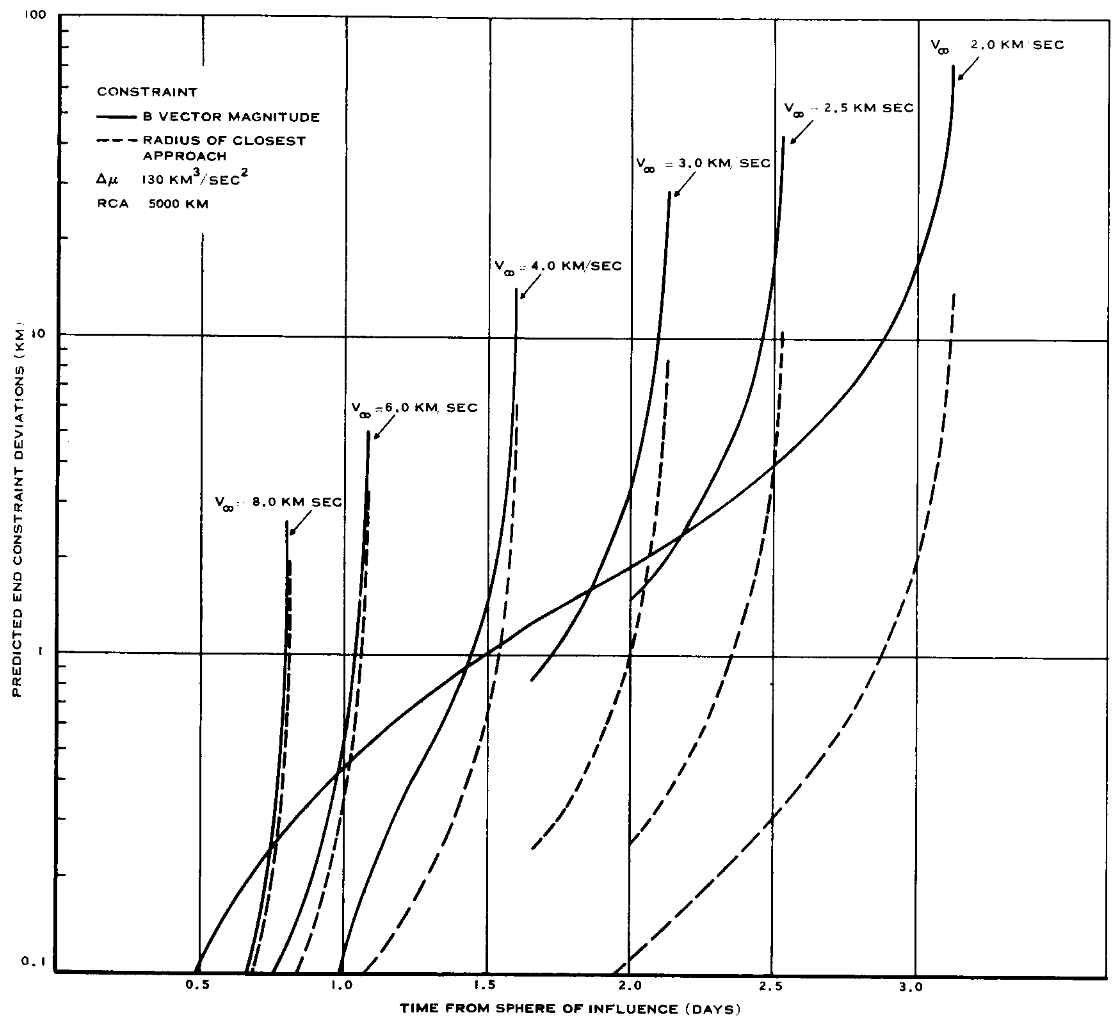


FIGURE 7 - TIME HISTORY OF PREDICTED END CONSTRAINT DEVIATIONS

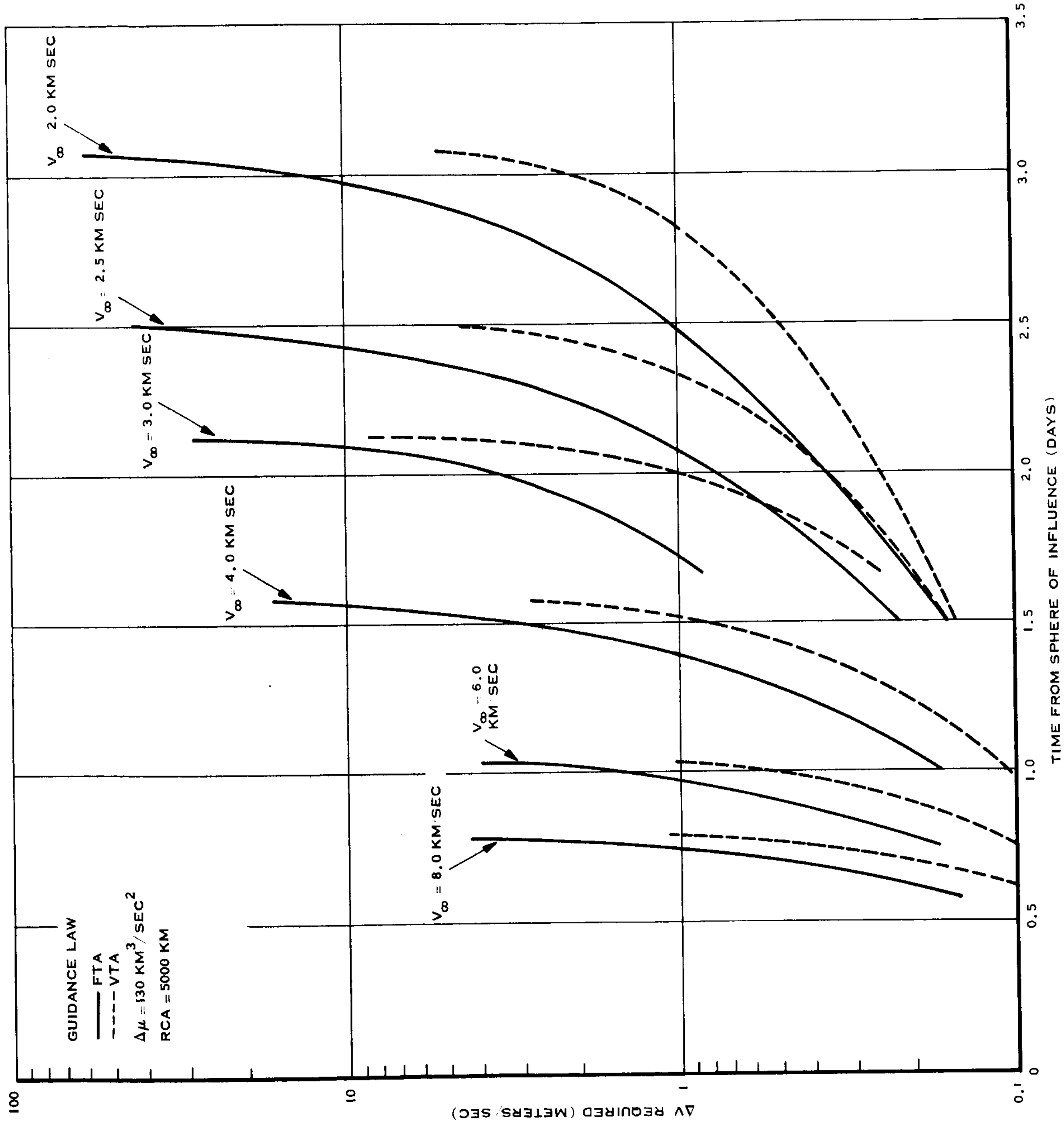


FIGURE 8 - ΔV REQUIRED FOR APPROACH GUIDANCE

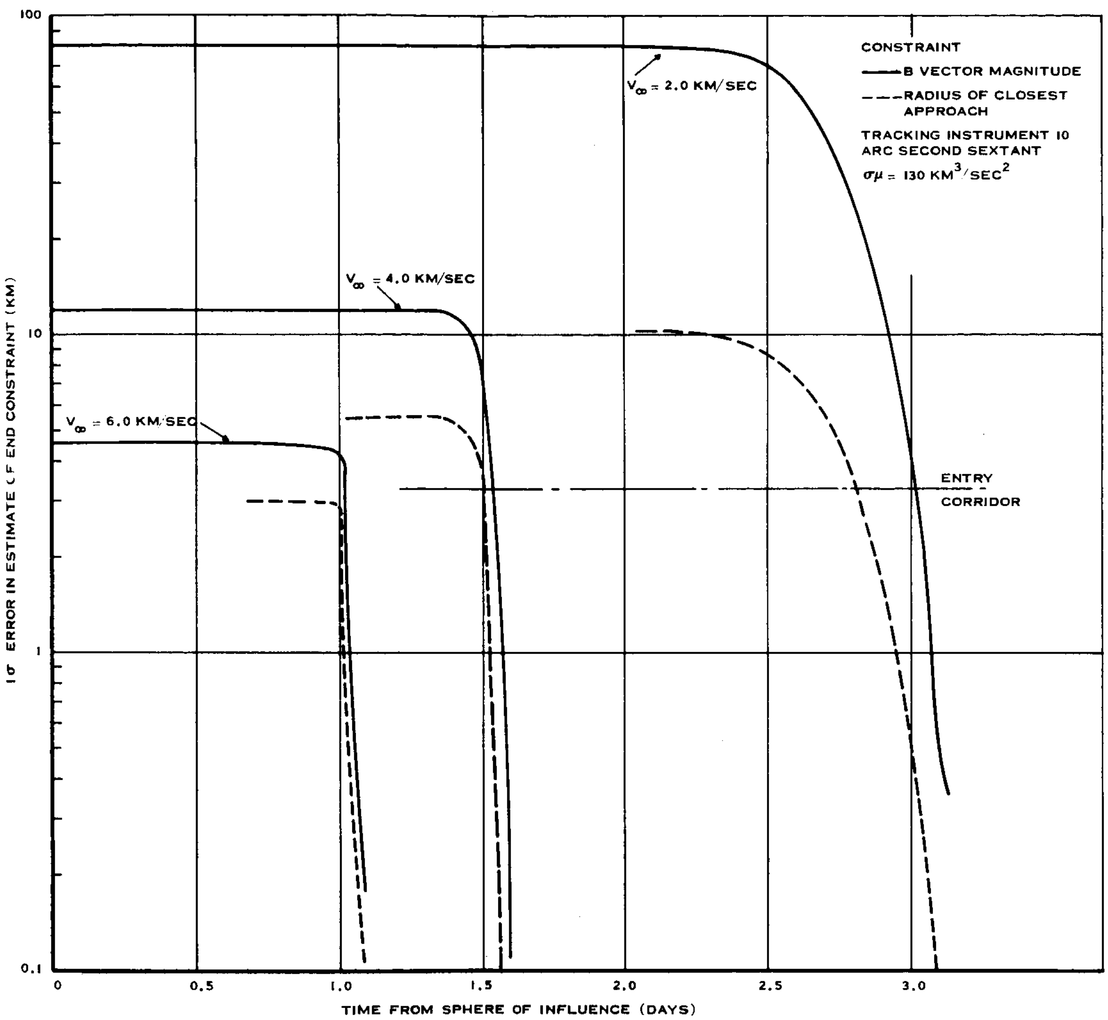


FIGURE 9 - ERROR IN ESTIMATE OF END CONSTRAINTS

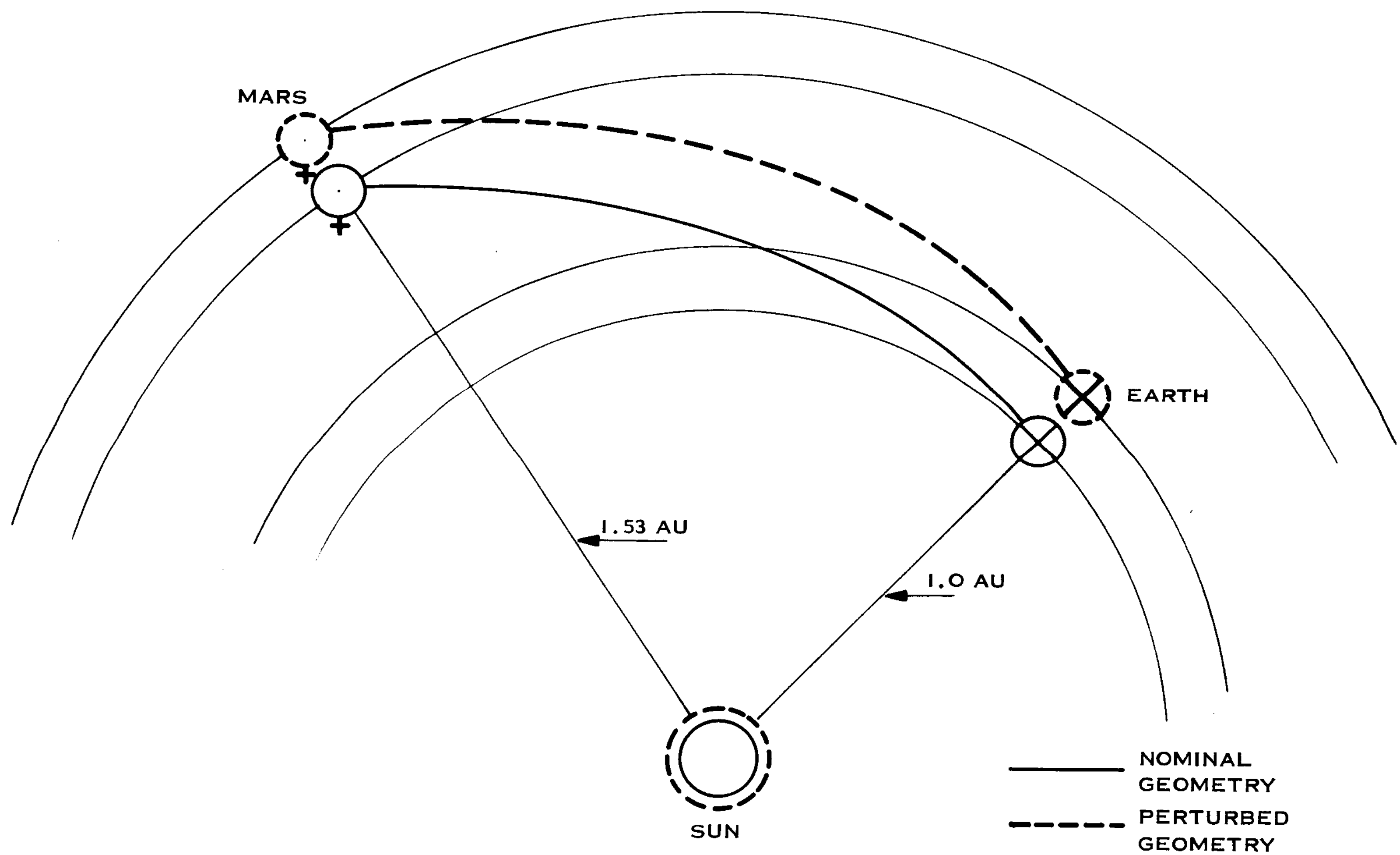
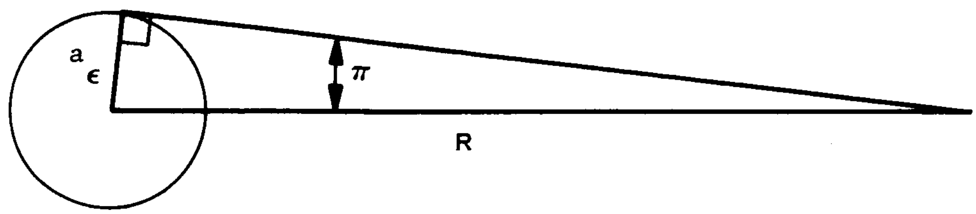


FIGURE 10 - EPHEMERIS GEOMETRY CHANGE WITH A.U. CHANGE



WHERE a_ϵ = EARTH EQUATORIAL RADIUS (KM)

R = EARTH SUN MEAN DISTANCE (KM)

π = SOLAR PARALLAX

FIGURE 11 - SOLAR PARALLAX

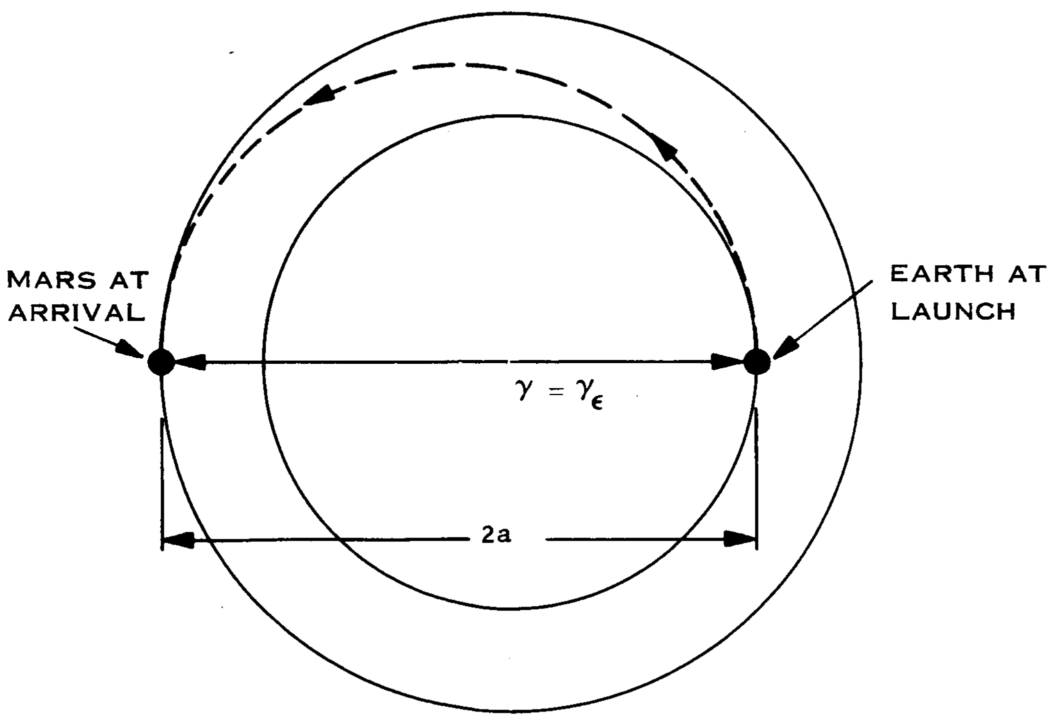
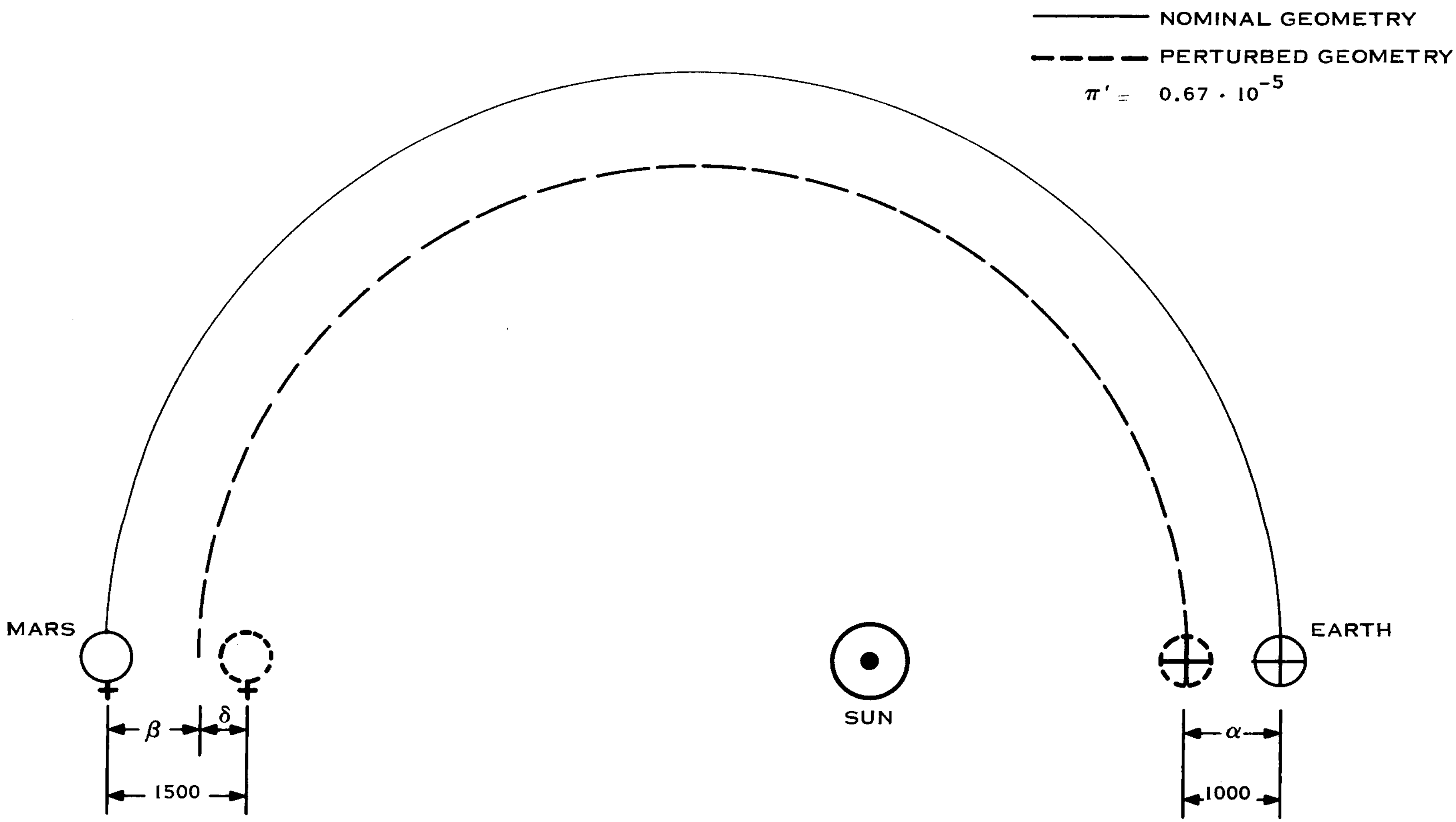


FIGURE 12 - HOHMANN TRANSFER



$\alpha = 1000$ KM $\quad \alpha + \beta = 1940$ KM $\quad \delta = 560$ KM

$\alpha + \beta = \Delta a$ TREATING MARS POSITION AS UNCERTAIN (EQ. 33)

$\delta = \Delta a$ TREATING MARS POSITION AS KNOWN (EQ. 32)

FIGURE 13 - EFFECT OF ERROR IN SOLAR PARALLAX

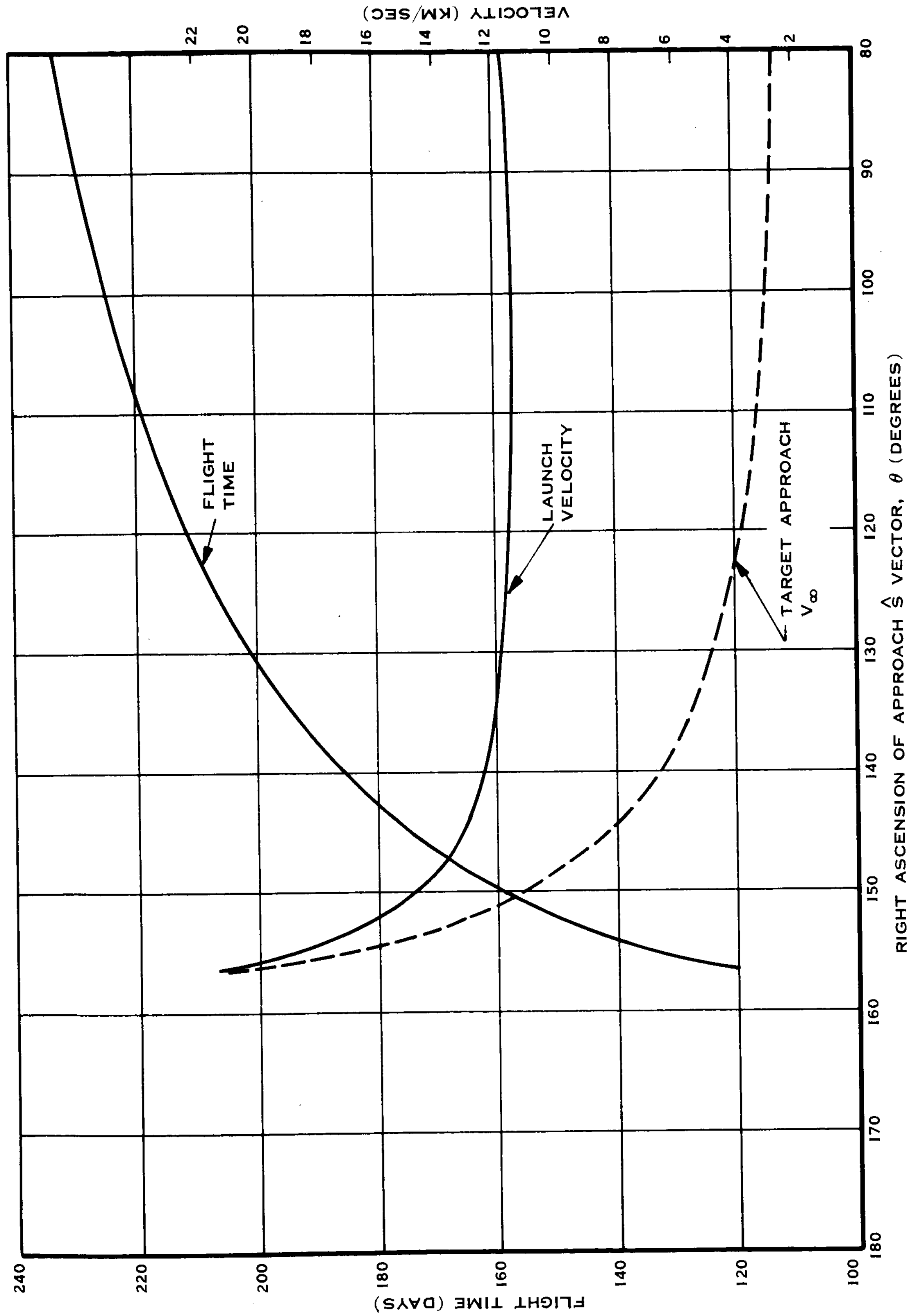


FIGURE 14A - TRAJECTORY CHARACTERISTICS - 160° TRANSFER, EARTH-MARS

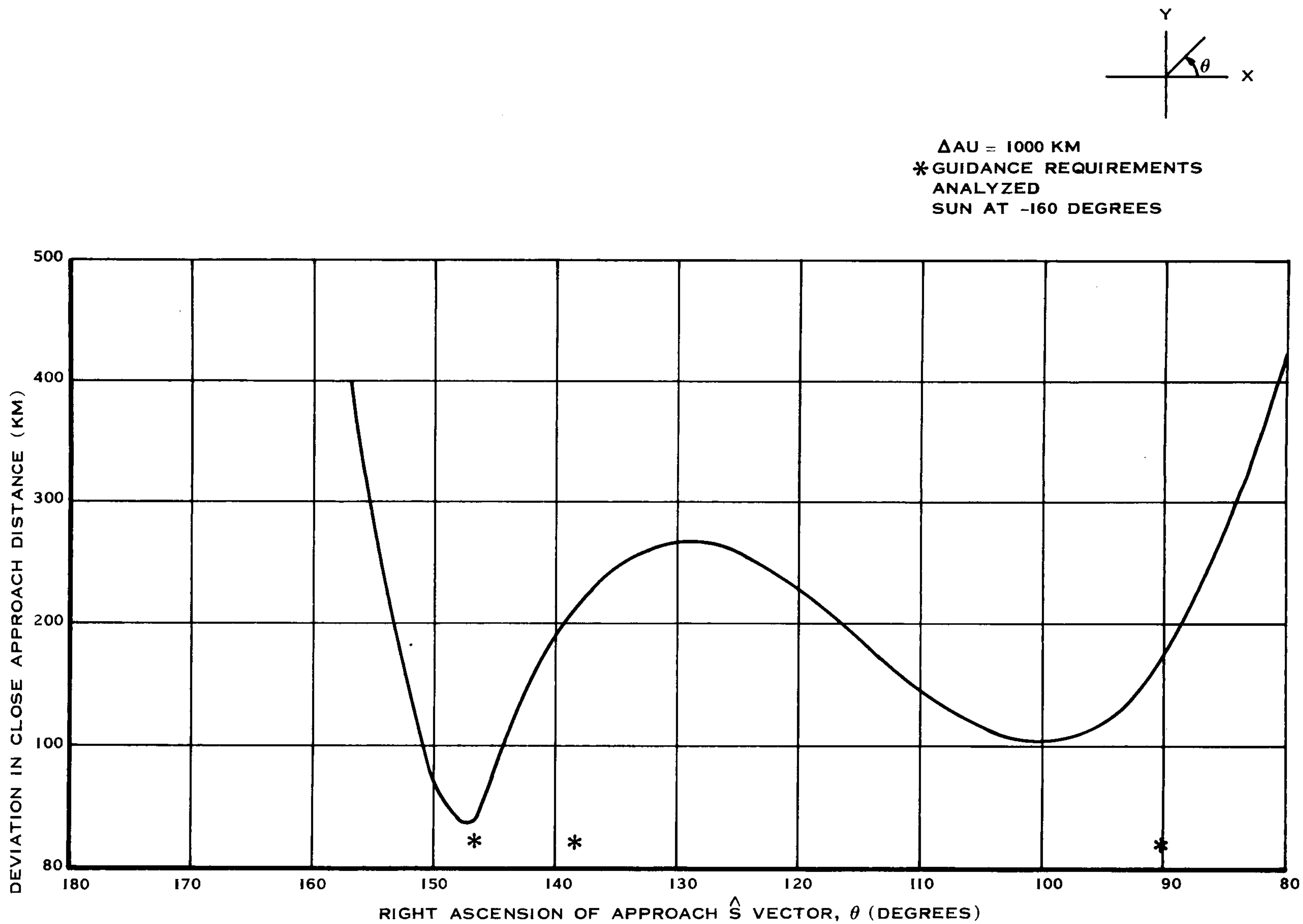


FIGURE 14B - DEVIATIONS IN CLOSE APPROACH - 160° TRANSFER, EARTH-MARS

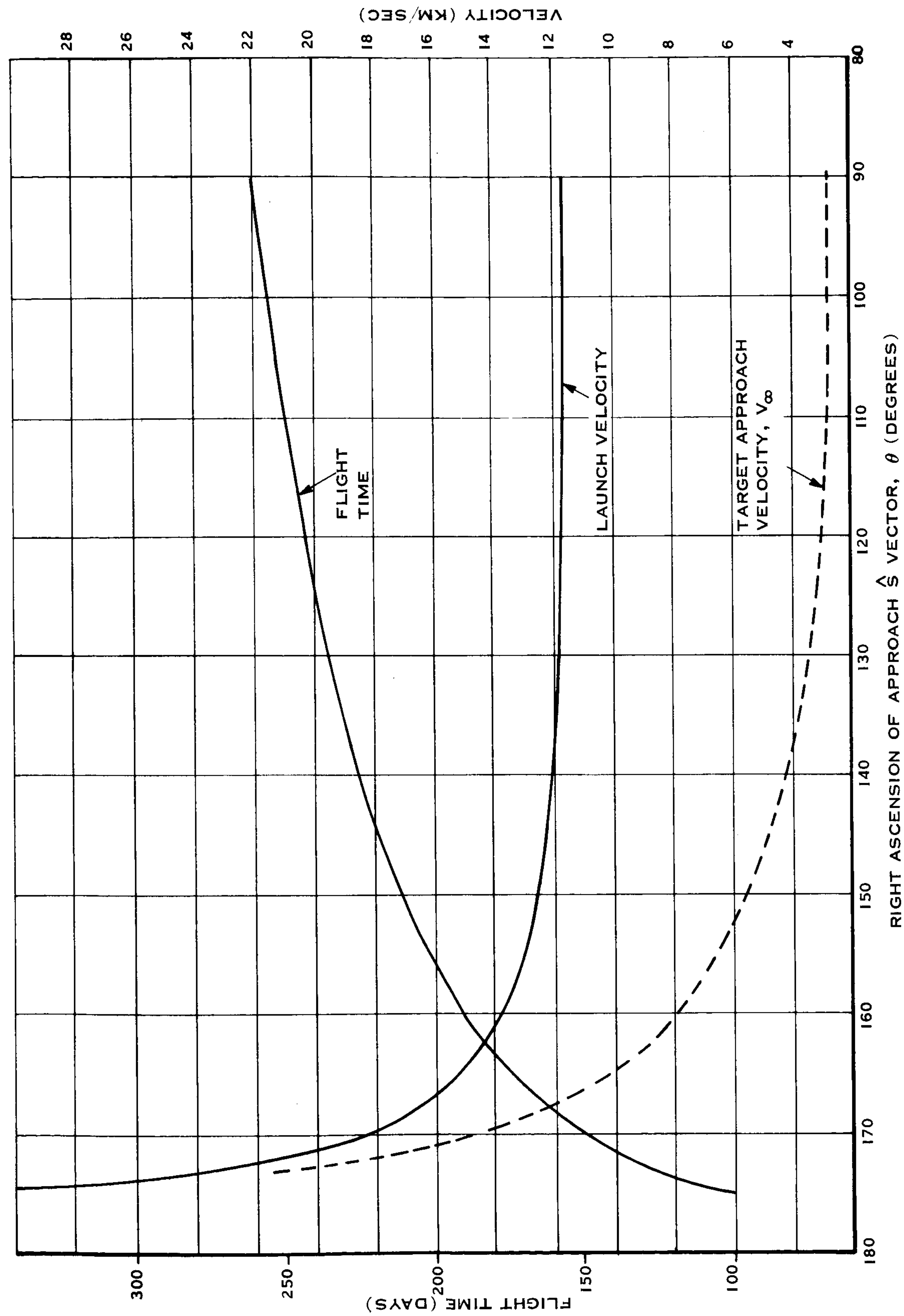


FIGURE 15A - TRAJECTORY CHARACTERISTICS - 180° TRANSFER, EARTH-MARS

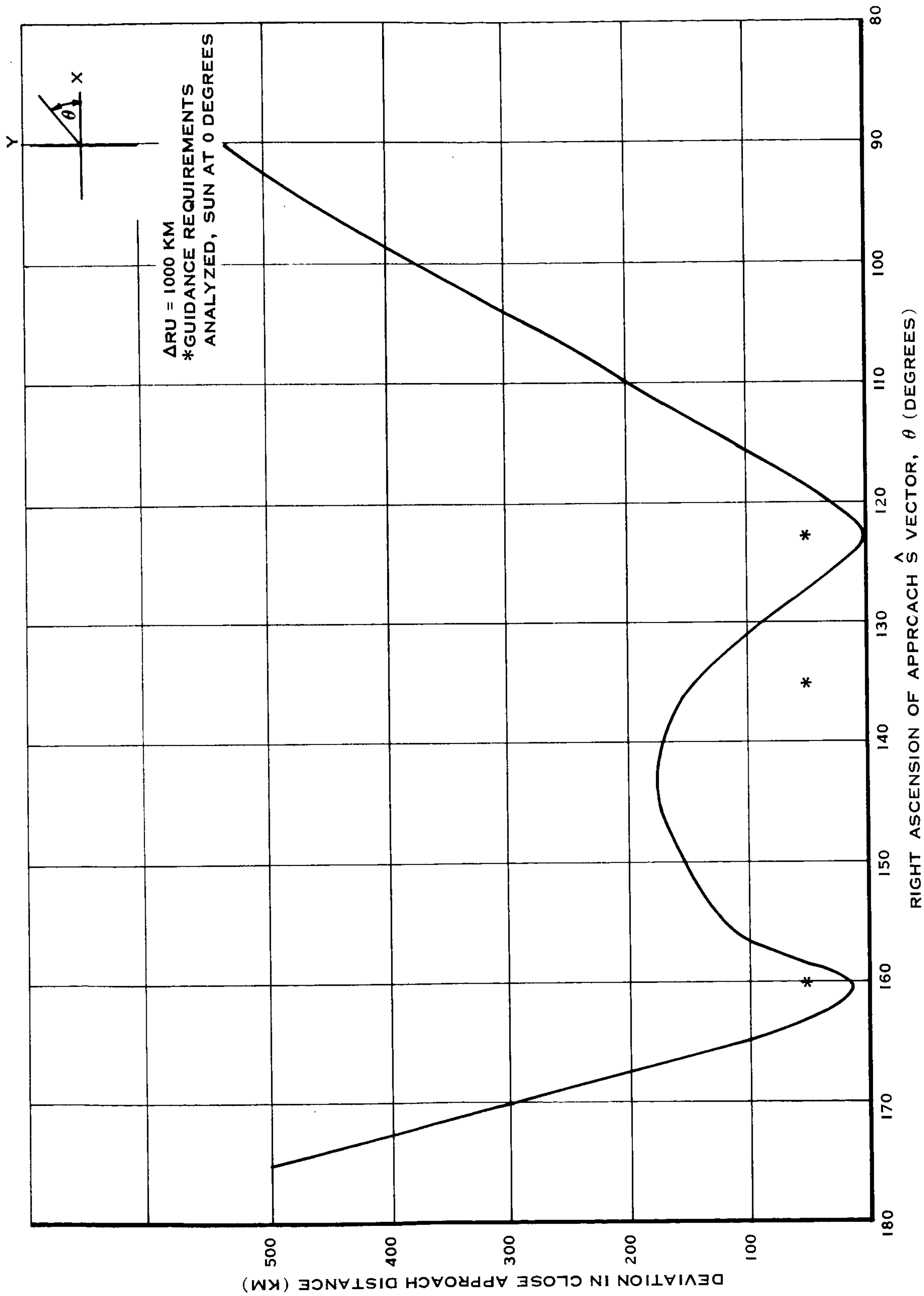


FIGURE 15B - DEVIATION IN CLOSE APPROACH - 180° TRANSFER, EARTH-MARS

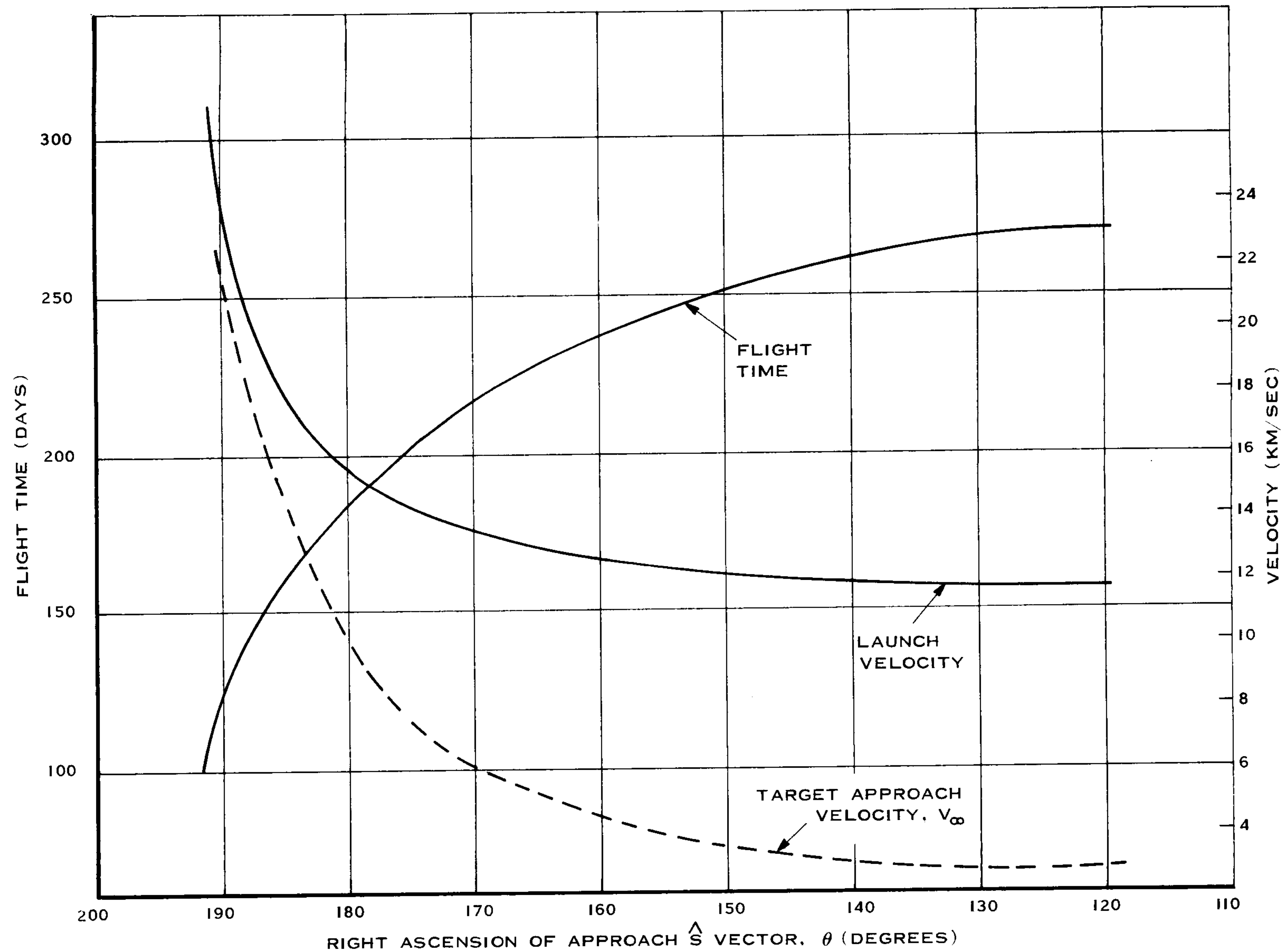


FIGURE 16A - TRAJECTORY CHARACTERISTICS - 200° TRANSFER, EARTH-MARS

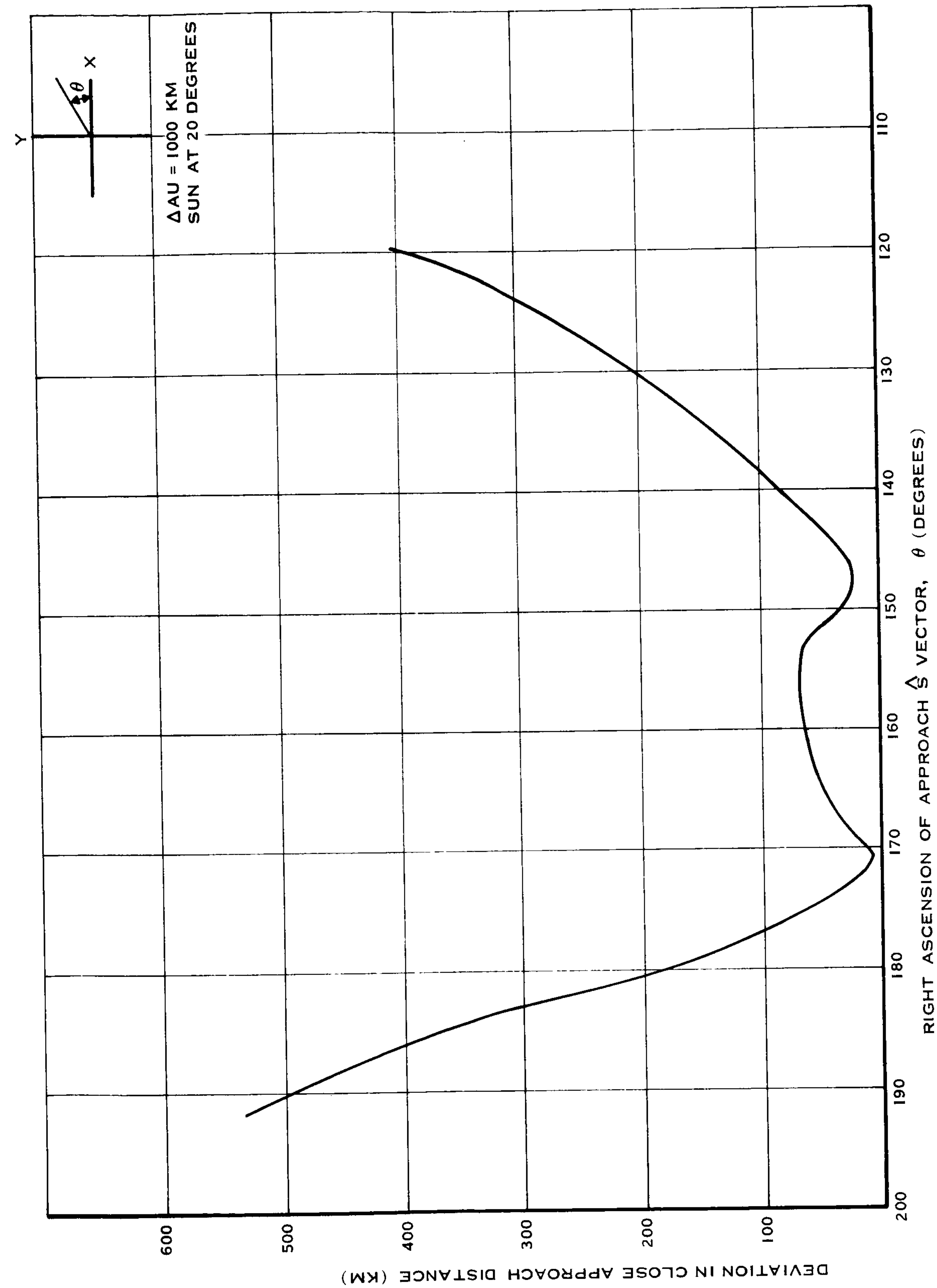


FIGURE 16B - DEVIATION IN CLOSE APPROACH - 200° TRANSFER, EARTH-MARS

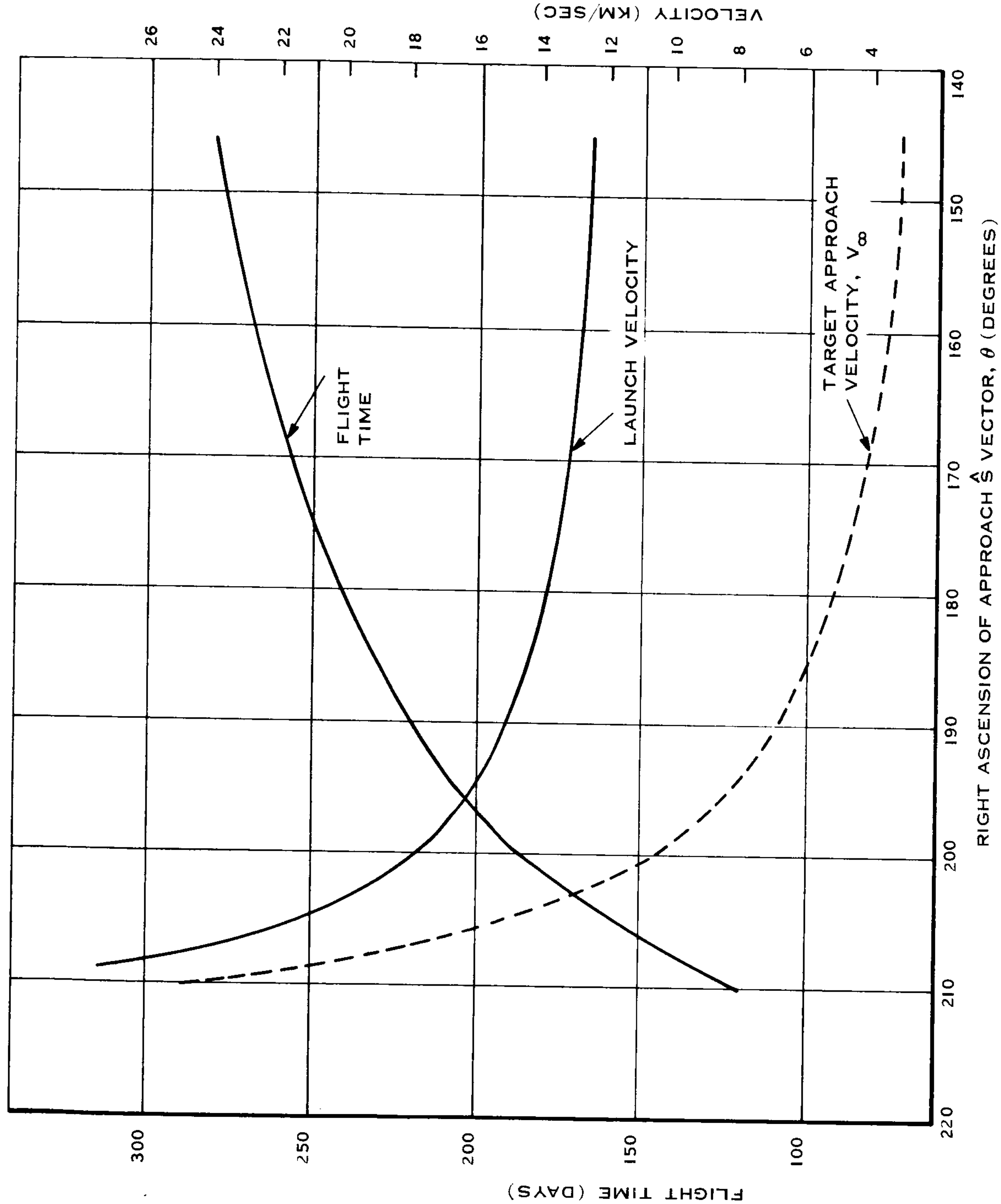


FIGURE 17A - TRAJECTORY CHARACTERISTICS - 225° TRANSFER, EARTH-MARS

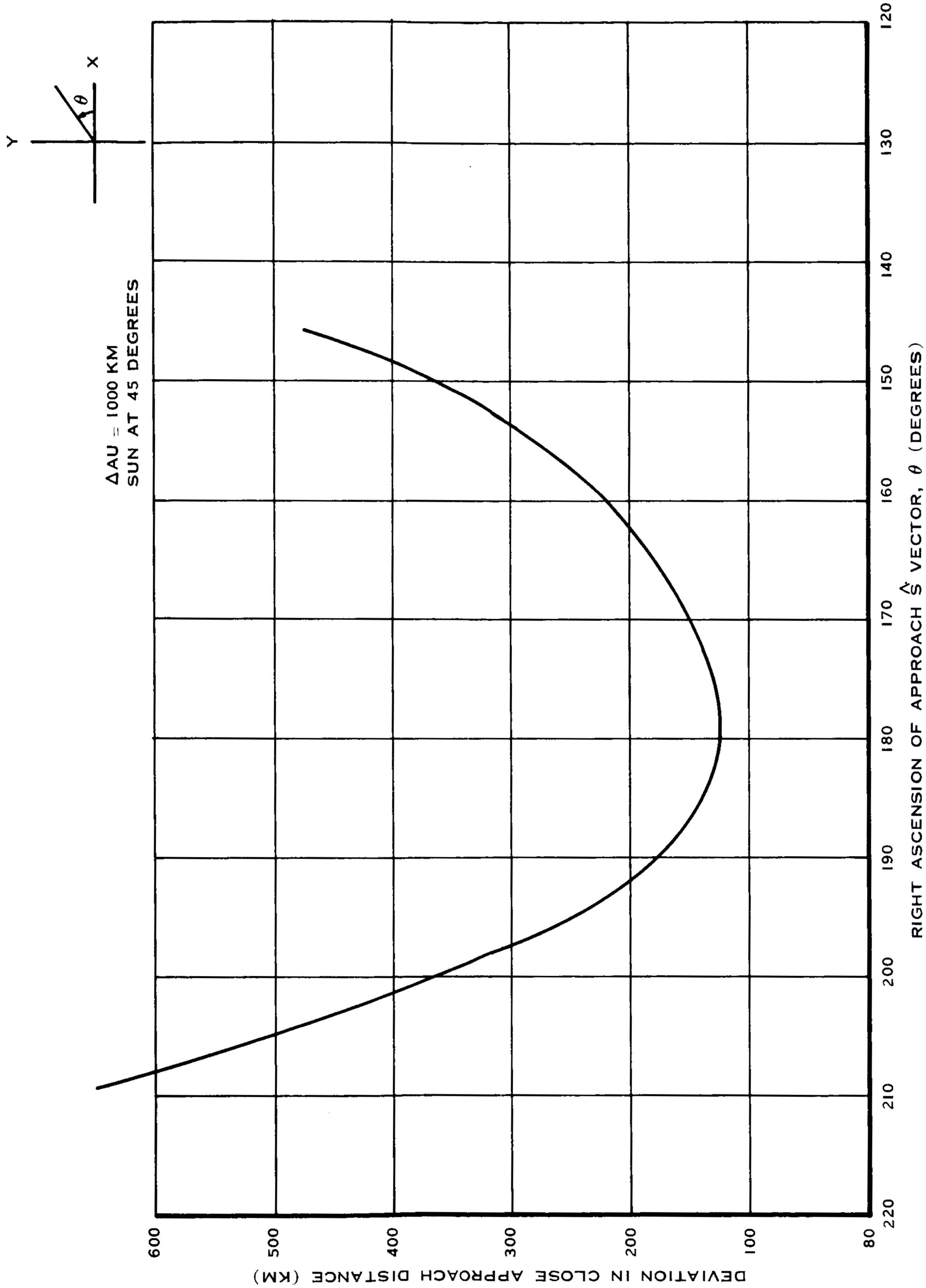


FIGURE 17B - DEVIATIONS IN CLOSE APPROACH - 225° TRANSFER, EARTH-MARS

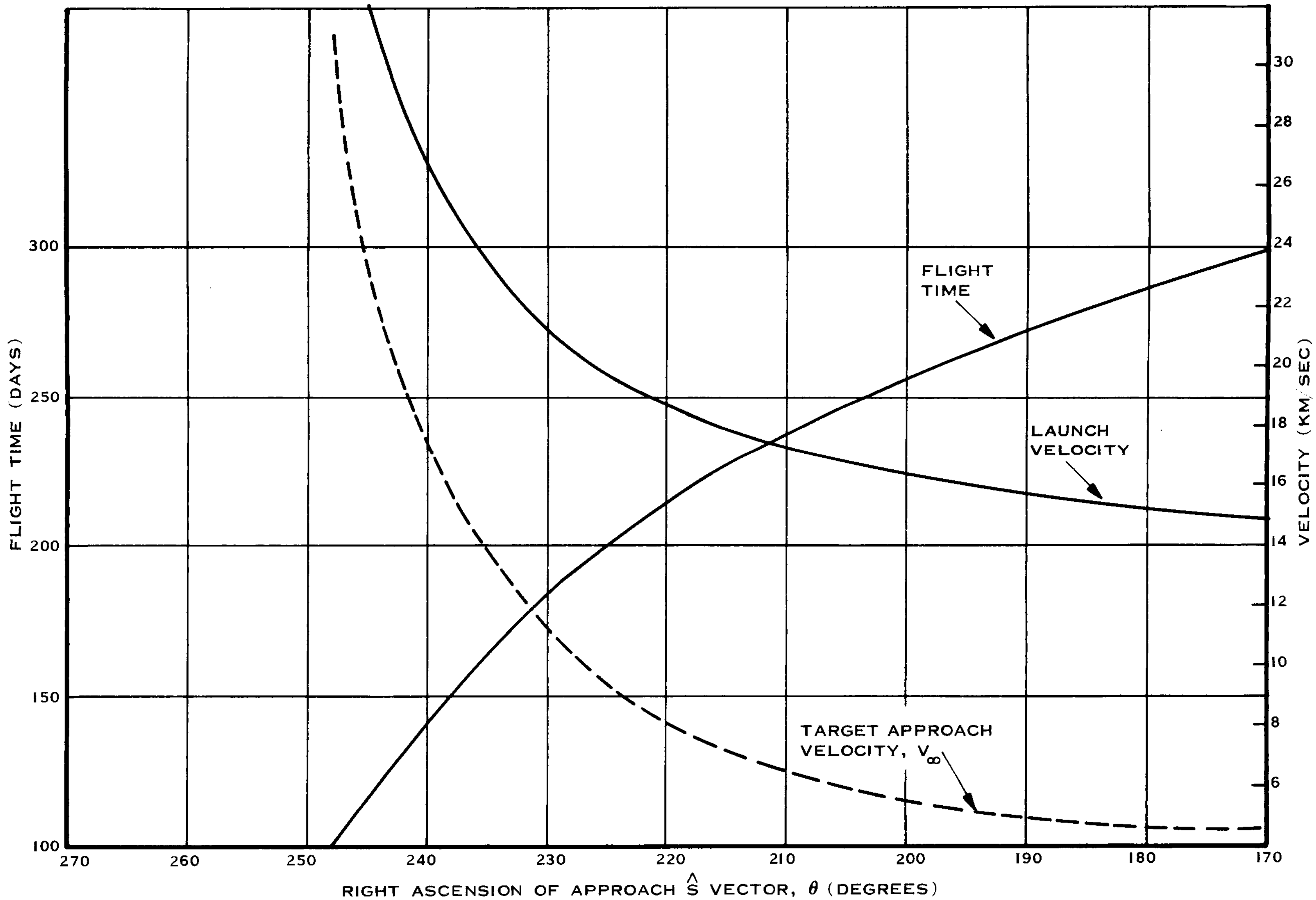


FIGURE 18A - TRAJECTORY CHARACTERISTICS - 270° TRANSFER, EARTH-MARS

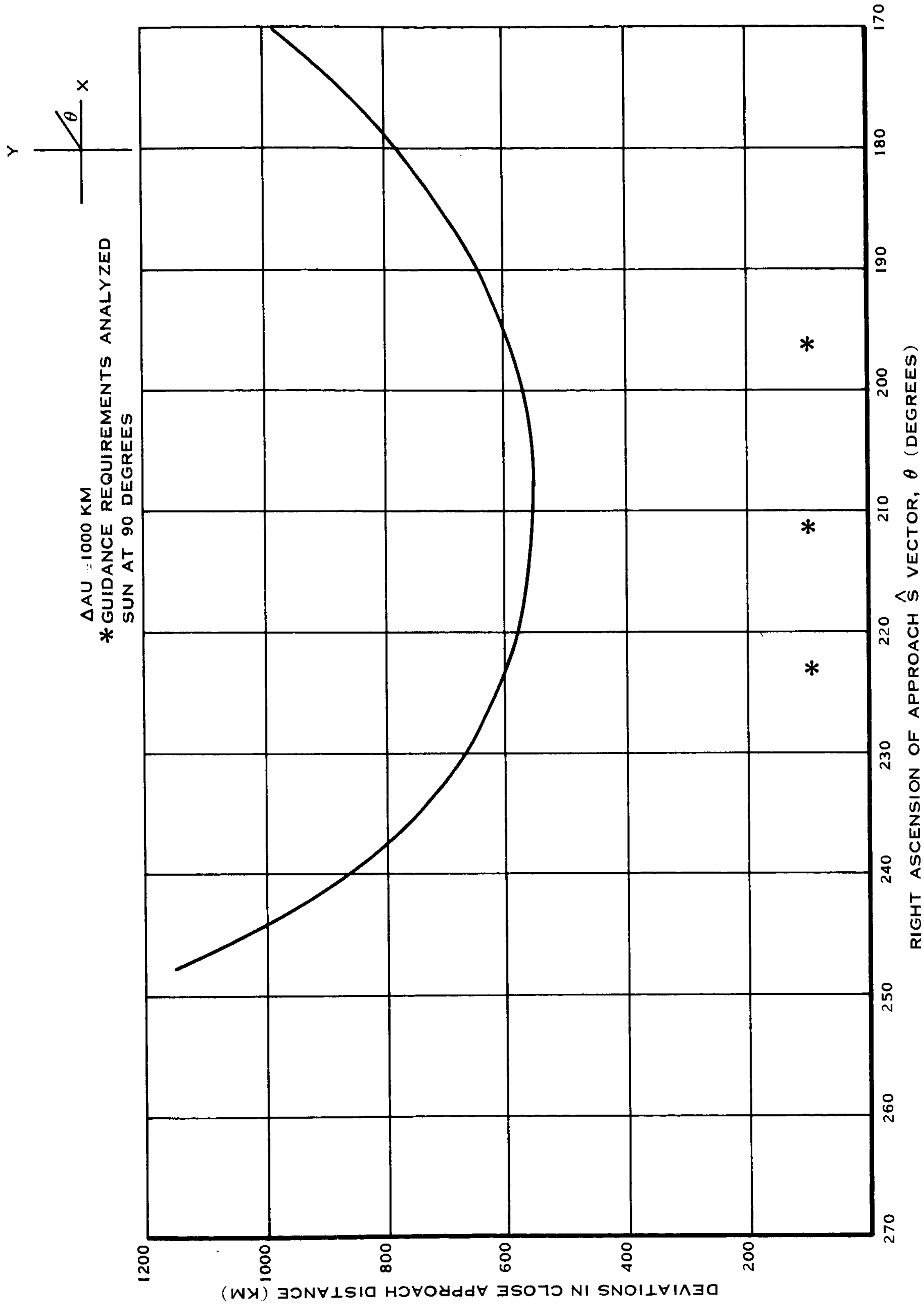


FIGURE 18B - DEVIATIONS IN CLOSE APPROACH - 270° TRANSFER, EARTH-MARS

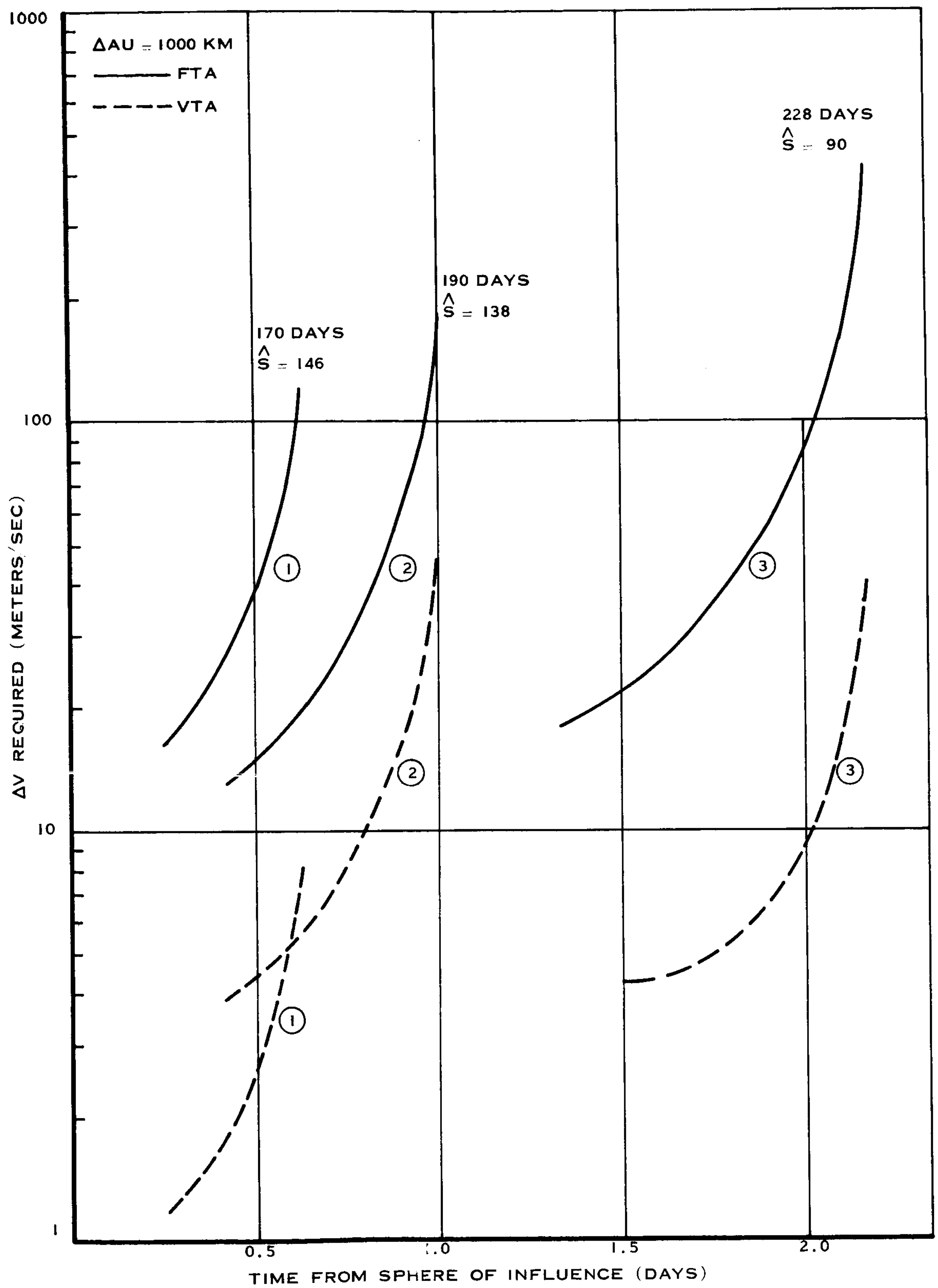


FIGURE 19 - APPROACH ΔV REQUIRED FOR 160° TRANSFER

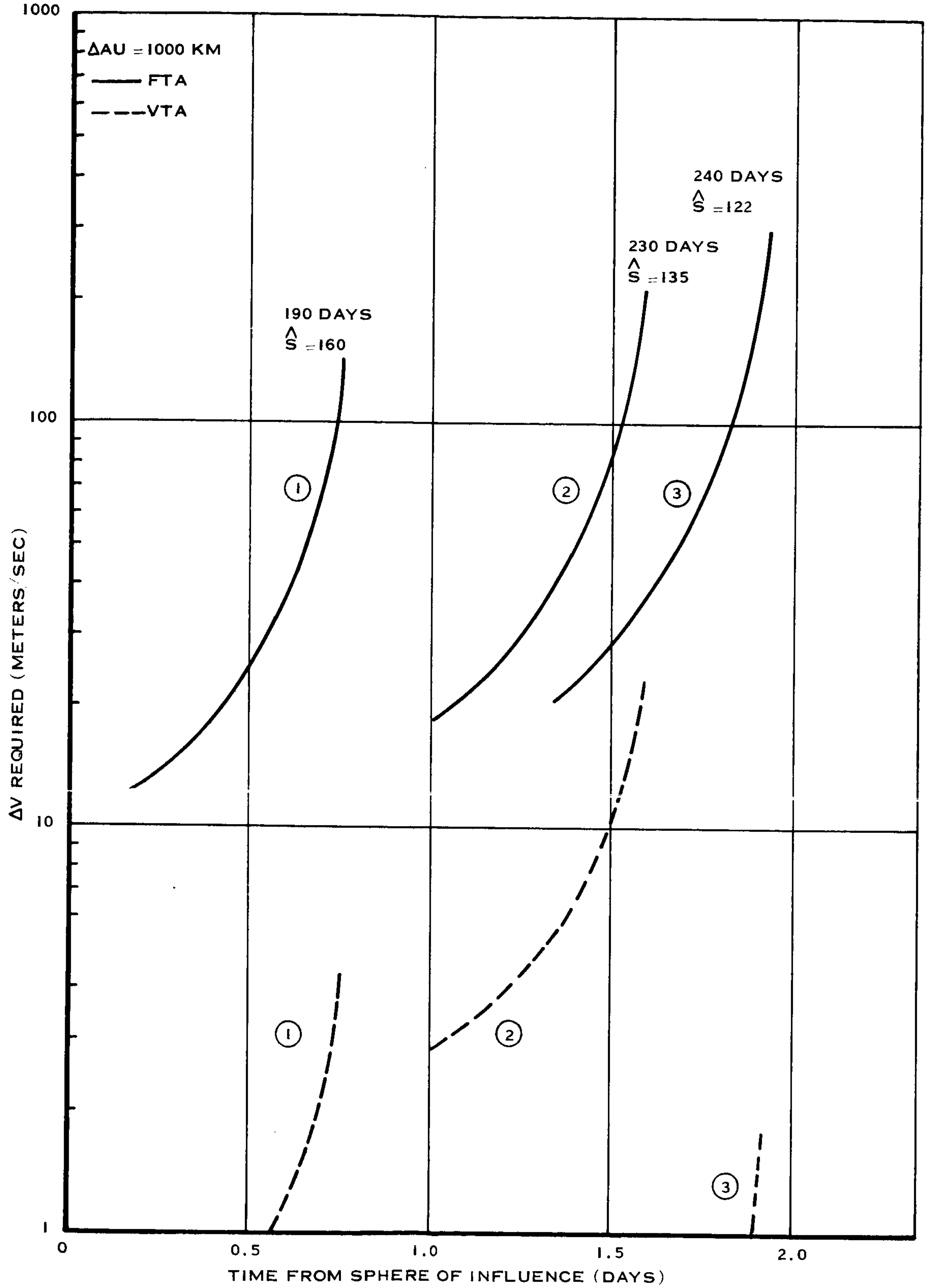


FIGURE 20 - APPROACH ΔV REQUIRED FOR 180° TRANSFER

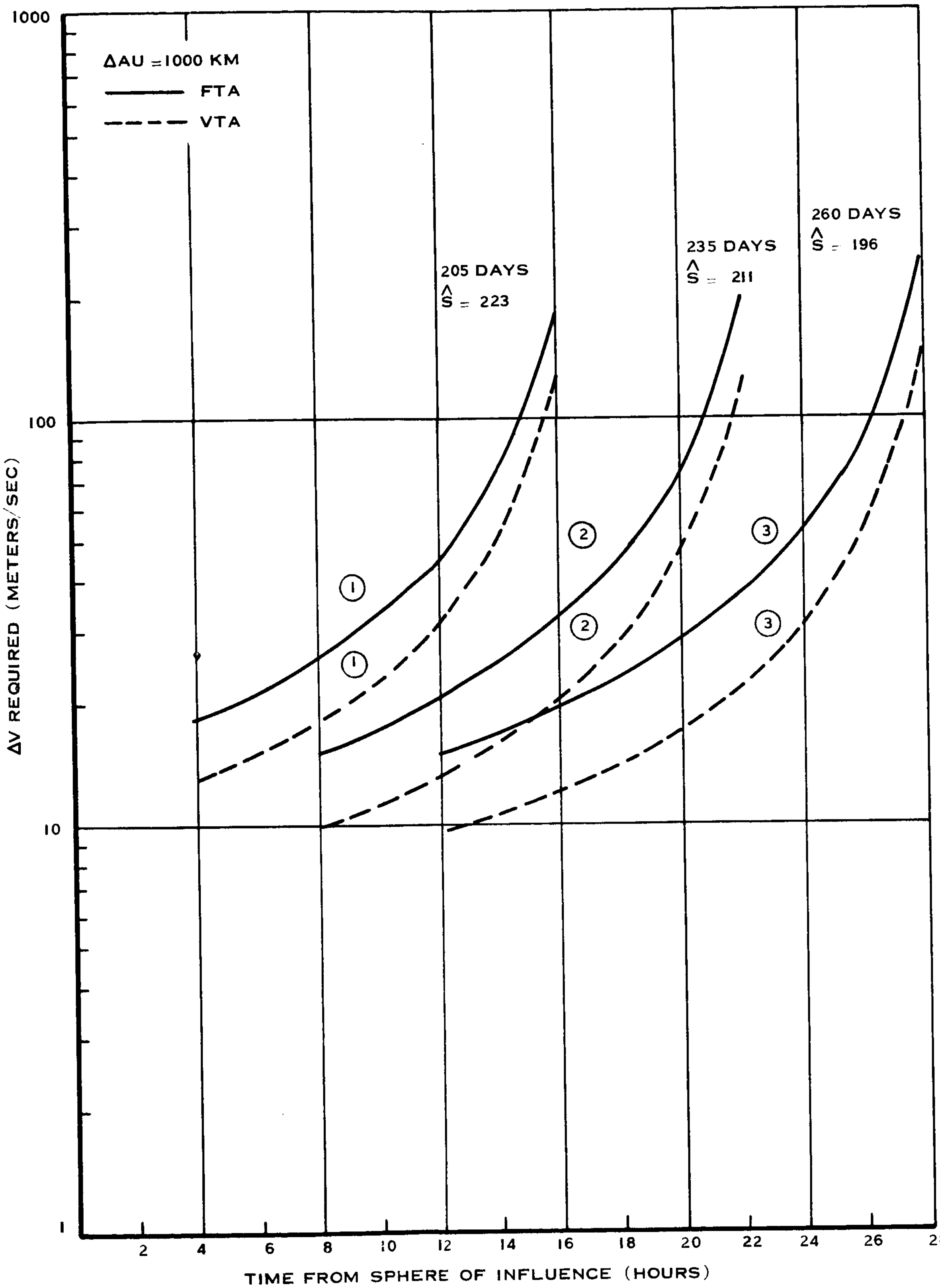


FIGURE 21 - APPROACH ΔV REQUIRED FOR 270° TRANSFER

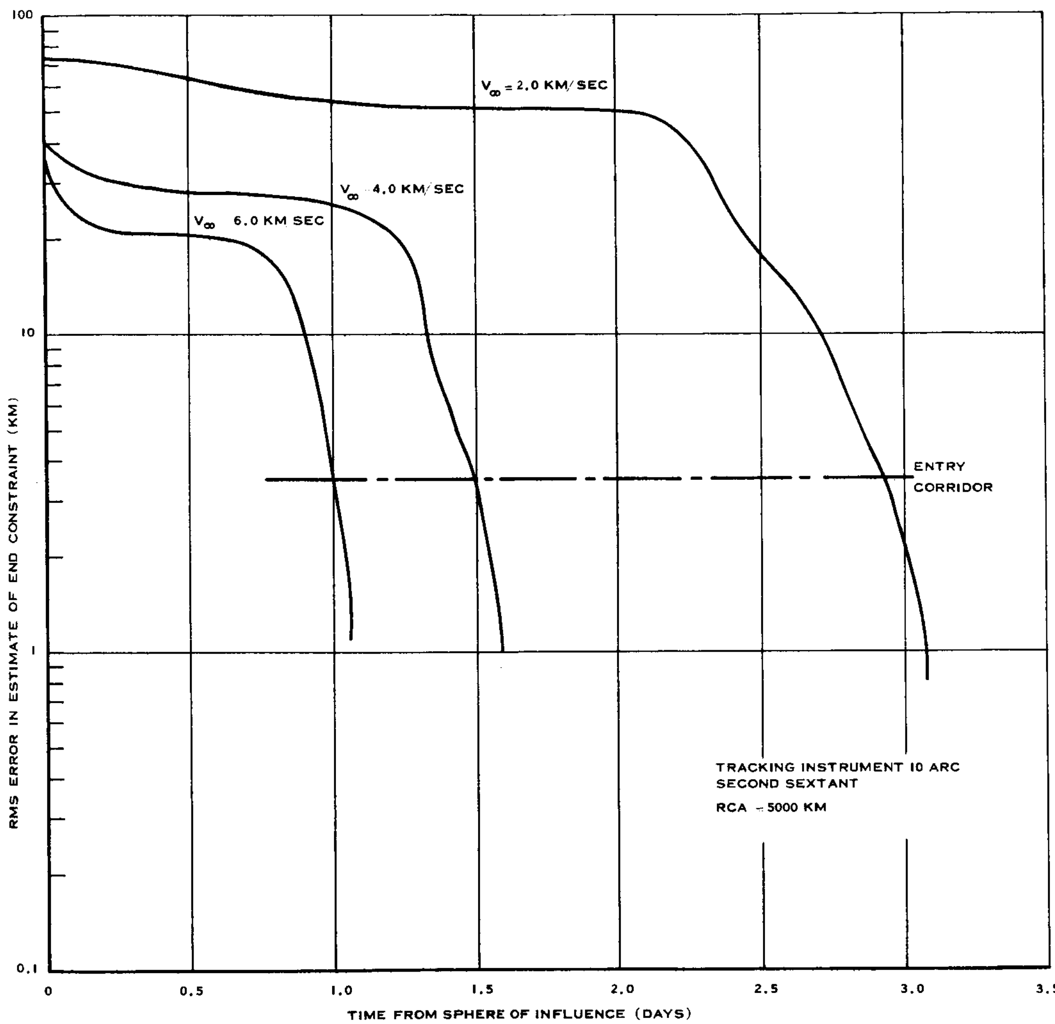


FIGURE 22 - ERROR IN ESTIMATE OF $\vec{B}$ MAGNITUDE