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PURDUE UNIVERSITY
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OPTIMIZATION OF STOCHASTIC CONTROL PROCESSES WITH RESPECT TO PROBABILITY OF ENTERING A TARGET MANIFOLD

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For

Jet Propulsion Laboratory

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OPTIMIZATION OF STOCHASTIC CONTROL PROCESSES WITH RESPECT TO PROBABILITY OF ENTERING A TARGET MANIFOLD

ABSTRACT

In this report the problem of generating an open-loop control to drive a stochastically disturbed system from a given initial state to a target manifold is considered. The control is to be optimum with respect to a performance index which contains a probability term and an energy term. The probability term is the probability of entering the target manifold during some instant of a specified time interval.

Two approaches to this problem are investigated. The first approach is to obtain a closed form approximation to the probability term by generating an approximate solution to a diffusion equation, and then applying Pontryagin's Maximum Principle. The second approach is to formulate an appealing one-parameter class of controls, and search this class over its parameter. In this approach, the probability term is found by observing a digital simulation of the system.

A sample problem is formulated and both approaches are applied to it. Numerical results are obtained and compared.

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CHAPTER 1

INTRODUCTION

1.1 Outline of the Chapter

This chapter begins with a brief review of some of the recent work in the field of stochastic control. The purpose of this review is to indicate the position of this thesis in the context of the field.

Following the review of recent work in the field is a brief description of the problem to be investigated and an outline of the remainder of the thesis.

The chapter ends with some comments on notation.

1.2 Review of Stochastic Control

Kushner [14] considers a system defined by

$$dx = f(x, u)\, dt + dz \tag{1-1}$$

where x is an m dimensional vector

u is an $\ell \leq m$ dimensional control

z is some suitable random process, e.g., a step function with random step sizes, or a Brownian motion process.

with a cost criterion

$$E \int_0^T g(x, u)\, dt \tag{1-2}$$

where g is quadratic in x and u, and

T is fixed.

He gives a method of correction to the optimal deterministic control when the effects of the random process are small. He extends this work [15], [16], [17], [18], to develop a stochastic maximum principle, complete with adjoint equations and a stochastic version of the Hamiltonian, for the minimization of

$$E[C' x(T)] \tag{1-3}$$

subject to

$$dx = f(x, u)dt + \sigma(x, u)dZ \tag{1-4}$$

where c and x are m dimensional vectors

u is an $\ell \leq m$ dimensional control

$\sigma(x, u)$ is a weighting matrix

Z is a sample vector of a suitable random process.

Later, Kushner [19] solves essentially the same problem by a technique involving a stochastic version of Lyapunov functions. In this paper, the Z process appearing in equation (1-4) is assumed to be a sample function of a Weiner process. In this case, equation (1-4) is interpreted as an Ito equation [1].

The theory of stochastic stability referred to in [19] is developed by Wonham [20] and Kushner [21]. In these papers various types of stability are defined and discussed.

A feature common to the above papers on optimization is that the performance indices are expectations of a random variable corresponding to performance indices commonly occurring in deterministic optimal control. A second category of stochastic optimal control problems is typified by one investigated by Mishchenko [4], [11]. Mishchenko considers a controlled point in state space in pursuit of a second point.

The state variables of the pursued point are sample functions of a Markov process. One term of the performance index is the probability of "capture" of the pursued point during a specified time interval. The pursued point is considered "captured" if the controlled point is brought within some specified spherical neighborhood of it. Mischenko shows that the probability of "capture" is the solution to the Kolmogorov backward diffusion equation, and develops an estimate of the solution to this equation as a functional on the control.

1.3 The Problem to be Investigated

The problem to be investigated may be described as follows. It is desired to generate an open loop control which drives a linear system which is disturbed by Gaussian white noise from a known initial state to some manifold containing the origin. The control is to be chosen to extremize a performance index containing two terms: an energy term and a probability term. The probability term is the probability of entering a specified neighborhood of the target manifold within a specified time interval. This problem is related to the work reviewed in section 1.2 as follows. In its formulation this problem resembles those treated by Wonham and Kushner in that it begins with a stochastically disturbed system and seeks to solve a problem which has counterparts in deterministic optimal control theory. It is related to Mischenko's problem in that similar methods may be utilized to determine the probability term of the performance index.

1.4 Outline of the Thesis

The distinctive feature of the problem investigated in this thesis is the probability term in the performance index. Broadly speaking, the work reported is intended to describe the mathematical theory on which the problem is based, and to investigate two approachs to its solution.

The first approach begins by formulating the probability term of the performance index as the solution to a Kolmogorov backward diffusion equation. An estimate of the solution is constructed and used as the probability term in the performance index. The performance index is then maximized by an application of Pontryagin's maximum principle, and solution of the resulting two-point boundary value problem.

The second approach is to select an appealing one-parameter subclass of controls, and find the optimum control in the subclass. The probability term in the performance index is determined by observing the performance of a digital simulation of the system, thus avoiding the solution of the Kolmogorov equation. The one-parameter subclass is then searched for the optimum control. Thus the solution of the two-point boundary value problem is avoided.

Chapter 2 assumes the concept of the Ito equation, and proceeds to develop the properties of its solution which are required for the remainder of the work. It gives a review of some pertinent theorems and their proofs. Its purpose is to collect known results necessary to the remainder of the work in a sufficiently precise form for correct application to the present problem. The proofs are presented to indicate the degree to which the results rely on the linearity and time invariance of the Ito equation.

Chapter 3 contains a precise statement of the problem to be investigated. It shows how the methods of Mishchenko [4], [11] may be applied here. The limitations and difficulties of this approach are described.

In Chapter 4 Mishchenko's techniques are applied to the present problem. An estimate of the accuracy of the resulting estimate of the probability term of the performance index is obtained.

Chapter 5 begins by applying the results of Chapter 4 and Pontryagin's maximum principle to obtain a formulation of the present problem as a two-point boundary value problem. An algorithm for the solution of this two-point boundary value problem is presented, and a brief discussion of its convergence properties is presented.

The material in Chapters 4 and 5 reveals the difficulties and complexities involved in the approach to the problem described in them. In Chapter 6 an alternative approach is investigated. This approach is as follows:

1. An appealing one-parameter subclass of controls is selected.
2. The probability term in the performance index is estimated by digital simulation, and the performance index is plotted as a function of the control subclass parameter.
3. The optimum control in the class is determined by an examination of this plot.

This approach yields only a suboptimal solution. A sample problem is formulated and numerical results are obtained for a certain subclass of controls. An examination of the results suggests a second subclass. Numerical results are obtained for this subclass also, and the two sets of results are compared.

The algorithm given in Chapter 5 is applied to the sample problem stated in Chapter 6. The numerical results are presented in Chapter 7.

The accuracy of the estimate of the probability term developed in Chapter 4 is checked.

Chapter 8 summarizes the results of the previous chapters and draws some conclusions. Some avenues for future investigation are suggested.

1.5 Notation

The following comments on notation apply throughout the remainder of the thesis, except as noted.

1. Vectors are taken to be column vectors unless noted otherwise.
2. The transpose of a vector or matrix is denoted by a "prime." That is, A' is the transpose of the matrix A.
3. The norm of a vector with respect to a matrix is denoted and defined as follows:

$$\|x\|_A = \sqrt{x'Ax} \tag{1-5}$$

where x is an m-dimensional vector

A is an m x m positive semidefinite real symmetric matrix.

4. It is frequently necessary to partition vectors into two parts. For this purpose the following notation is usually employed:

$$x = \begin{bmatrix} \tilde{x} \\ \hat{x} \end{bmatrix} \tag{1-6}$$

where x is an m-dimensional vector

$\tilde{x}$ is a k-dimensional vector

$\hat{x}$ is an m-k-dimensional vector

5. The gradient operator is defined as follows

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_m} \end{bmatrix} \qquad (1\text{-}7)$$

where x is an m-dimensional vector

f(x) is a scalar function.

6. The operator $\nabla_y \nabla_x$ is defined as follows:

$$\nabla_y \nabla_x f(x, y) = \begin{bmatrix} \frac{\partial^2 f}{\partial y_1 \partial x_1} & \frac{\partial^2 f}{\partial y_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial y_1 \partial x_m} \\ \vdots & & & \vdots \\ \frac{\partial^2 f}{\partial y_m \partial x_1} & \cdots & \cdots & \frac{\partial^2 f}{\partial y_m \partial x_m} \end{bmatrix} \qquad (1\text{-}8)$$

CHAPTER 2

MATHEMATICAL FOUNDATIONS

2.1 Introduction

From a mathematical point of view, the problem to be dealt with is the control of a stochastic process which is defined as the solution of a linear, time invariant Ito equation (Wonham[2]). The primary purpose of this chapter is to establish for use in later chapters some of the properties of such a process. A secondary purpose of the chapter is to discuss the possibility of establishing similar results for solutions of time varying and nonlinear Ito equations.

The properties of the Ito equation solution are stated in theorems. The proofs of these theorems are sketched briefly, or merely referenced. Most of the theorems are known. The only reason for spending any time here on proofs is to indicate the degree to which they rely on the linearity or time invariance of the Ito equation.

Theorems 2-1 through 2-6 establish the statistical properties of the Ito equation solution. These properties include its conditional mean and covariance, its Markov property, its transition density, and a diffusion equation satisfied by its transition density. Theorem 2-7 establishes that a certain sequence of discrete approximations to the Ito equation converges in a certain sense to the solution of the Ito equation. This theorem will serve as a basis for digital simulation.

2.2 Definition of the Ito Equation

The concept and some of the properties of Ito equations are given by Doob [1], Wonham [2] and Ito [3]. The theorems in this chapter refer to the solution x(t) of the linear, time invariant Ito equation

$$dx_t = (Ax + Bu)\,dt + c\,dn_t \qquad (2\text{-}1)$$

where

x is an m-dimensional vector

A is an m x m matrix

B is an m x ℓ_u matrix

u is an ℓ_u-dimensional control vector, $\ell_u \leq m$

t is time

c is an m x k matrix

n is a k-dimensional ($k \leq m$) Brownian motion process, i.e., one whose increments are Gaussianly distributed and satisfy

$$E[n(t_2) - n(t_1)] = 0 \qquad (2\text{-}2)$$

$$E\left\{[n(t_4) - n(t_3)][n(t_2) - n(t_1)]'\right\} = 0 \qquad (2\text{-}3)$$

for $t_4 > t_3 > t_2 > t_1$

and

$$E\left\{[n(t_2) - n(t_1)][n(t_2) - n(t_1)]'\right\} = W(t_2 - t_1) \qquad (2\text{-}4)$$

where E denotes expectation

()' is the transpose of ()

W is an k x k matrix with rank k.

2.3 Properties of the Solution

The first property of $x(t)$ to be stated is the form of its solution. This is as follows.

Theorem 2-1. Let $y(t)$ be defined as

$$y(t) = \Phi_A(t,t_o)\, x\,(t_o) + \int_{t_o}^{t} d\alpha\, \Phi_A(t,\alpha)\, Bu(\alpha) + \int_{t_o}^{t} \Phi_A(t,\alpha)\, Cdn_\alpha \qquad (2\text{-}5)$$

where $\Phi_A(t,t_o)$ is defined as the solution to

$$\frac{\partial\, \Phi_A}{\partial t} = A\, \Phi_A,\quad \Phi_A(t,t_o) = I_m \qquad (2\text{-}6)$$

I_m is the identity matrix of rank m.

The last integral in (2-5) is a stochastic integral as defined by Doob [1], page 278.

Then $y(t)$ is the solution of (2-1). The proof of theorem 2-1 follows that of the corresponding theorem for ordinary differential equations, except that the following lemma must first be established.

Lemma

Let $\dot{\Phi}_A(\beta,\alpha) = \left[\frac{\partial}{\partial t}\, \Phi_A(t,\alpha)\right]_{t=\beta}$. Then

$$\int_{\alpha=t_o}^{t} \Phi_A(t,\alpha)\, Cdn_\alpha = \int_{t_o}^{t} d\beta \int_{\alpha=t_o}^{\beta} \dot{\Phi}_A(\beta,\alpha)\, Cdn_\alpha + \int_{\alpha=t_o}^{t} Cdn_\alpha \qquad (2\text{-}7)$$

Proof of the Lemma: On pages 430 and 431 of Doob [1] it is shown that the order of integration of iterated integrals of the type on the right hand side of (2-7) is interchangeble. Thus

$$\int_{t_o}^{t} d\beta \int_{\alpha=t_o}^{t} \dot{\Phi}_A(\beta,\alpha)\ Cdn_\alpha = \int_{\alpha=t_o}^{t} \left\{ \int_{\alpha}^{t} d\beta\ \dot{\Phi}_A(\beta,\alpha) \right\} Cdn_\alpha \tag{2-8}$$

But by definition,

$$\int_{\alpha}^{t} d\beta\ \dot{\Phi}_A(\beta,\alpha) = \Phi_A(t,\alpha) - \Phi_A(\alpha,\alpha) \tag{2-9}$$

Then, since $\Phi_A(\alpha,\alpha)$ is the identity matrix, (2-8) and (2-9) combine to prove the lemma.

Proof of Theorem 2-1

The proof of Theorem 2-1 may now be carried out. It follows directly from (2-5) that

$$y(t_o) = x(t_o) \tag{2-10}$$

the next step is to compute dy_t from (2-5). From (2-7) and (2-6),

$$d \int_{\alpha=t_o}^{t} \Phi_A(t,\alpha)\ Cdn_\alpha = \left\{ A \int_{\alpha=t_o}^{t} \Phi_A(t,\alpha)\ Cdn_\alpha \right\} dt + Cdn_t \tag{2-11}$$

Then from (2-5), (2-6) and (2-11),

$$dy_t = \Big\{ A\ \Phi_A(t,t_o)\ x(t_o) + A \int_{t_o}^{t} d\alpha\ \Phi_A(t,\alpha)\ Bu(\alpha)$$

$$+ Bu(t) + A \int_{\alpha=t_o}^{t} \Phi_A(t,\alpha)\ Cdn_\alpha \Big\} dt + Cdn_t \tag{2-12}$$

Substitution of (2-5) into (2-12) yields

$$dy_t = (Ay + Bu)\ dt + Cdn_t \tag{2-13}$$

A comparison of (2-13) with (2-1), and (2-10) completes the proof.

Theorem 2-2. Let $x(t)$ and $\Phi_A(t,t_o)$ be as defined in Theorem 2-1. Then

$$E\left\{x(t_2)\,|x(t_1)\right\} = \Phi_A(t_2,t_1)\;x(t_1) + \int_{t_1}^{t_2} d\alpha\;\Phi_A(t_2,\alpha)\;Bu(\alpha) \tag{2-14}$$

where $E\left\{x(t_2)\,|x(t_1)\right\}$ is the expectation of $x(t_2)$ given $x(t_1)$.

Proof The proof follows directly from the definition of a stochastic integral (see pages 425-428 of Doob [1]) and (2-5). (See also Wonham [2] page 104).

Theorem 2-3. Let $x(t)$ and $\Phi_A(t,t_o)$ be as defined in theorem 2-1. Then

$$\text{cov}\left\{x(t_2)\,|x(t_1)\right\} = \int_{t_1}^{t_2} d\alpha\;\Phi_A(t_2,\alpha)\;CWC'\;\Phi_A'(t_2,\alpha) \tag{2-15}$$

where $\text{cov}\left\{x(t_2)\,|x(t_1)\right\} =$

$$E\left\{\left[x(t_2) - E\,x(t_2)\right]\left[x(t_2) - E\,x(t_2)\right]'|x(t_1)\right\} \tag{2-16}$$

Proof From (2-5) and (2-14),

$$\text{cov}\left\{x(t_2)\,|x(t_1)\right\} = E\left\{\int_{\alpha=t_1}^{t_2}\Phi_A(t_2,\alpha)\;Cdn_\alpha\int_{\beta=t_1}^{t_2} dn_\beta'C'\;\Phi_A'(t_2,\beta)\right. \tag{2-17}$$

Wonham [2] rigorously justifies the formal manipulations

$$E\;dn_\alpha\;dn_\beta' = Wd\alpha,\quad \alpha = \beta \tag{2-18}$$

$$E\;dn_\alpha\;dn_\beta' = 0\quad,\ \alpha \neq \beta \tag{2-19}$$

(A heuristic explanation of this manipulation is apparent from (2-3) and (2-4).) Then (2-15) follows from (2-17), (2-18), and (2-19).

Corollary Let $x(t)$ and $\Phi_A(t,t_o)$ be as defined in theorem 2-1. Let

$$Q(t,t_o) = \text{cov}\left\{x(t)\,|x(t_o)\right\} \tag{2-20}$$

then

$$\frac{\partial Q}{\partial t} = AQ + QA' + CWC', \; Q(t_o, t_o) = 0 \tag{2-21}$$

Proof The proof follows immediately from differentiation of (2-15) and substitution from (2-6).

Theorem 2-4. Let

$$P = \left[\sqrt{\lambda_1}\, v_1 \middle| \sqrt{\lambda_2}\, v_2 \middle| \; --- \; \middle| \sqrt{\lambda_k}\, v_k \right] \tag{2-22}$$

where the λ_i are the eigenvalues of W and the v_i are the corresponding orthonormal eigenvectors. Then $\mathrm{cov}[x(t_2)|x(t_1)]$ has rank m if and only if the matrix

$$[CP|\; ACP \;|\; A^2CP \;|\; --- \;|\; A^{m-1}CP] \tag{2-23}$$

has rank m.

The proof given here makes use of the concept of controllability. This concept is defined as follows.

Definition the pair $[\bar{A}(t), \bar{B}(t)]$ are said to be controllable at time t_o if and only if for every $\bar{x}$ there exists a finite t_1 and a control $\bar{u}$ such that $x(t_1) = 0$ subject to

$$\dot{x} = \bar{A}(t)\, x + \bar{B}(t)\, u, \; x(t_o) = \bar{x} \tag{2-24}$$

Proof of Theorem 2-4

Before applying the concept of controllability, a certain relationship between P and W must be established. Note first that

$$(P')^{-1} = \left[\frac{v_1}{\sqrt{\lambda_1}} \middle| \frac{v_2}{\sqrt{\lambda_2}} \middle| \; --- \middle| \frac{v_k}{\sqrt{\lambda_k}}\right] \tag{2-25}$$

Then

$$P = W(P')^{-1} \tag{2-26}$$

from which

$$W = PP' \tag{2-27}$$

From (2-15) and (2-27)

$$\text{cov}\left[x(t_2)|x(t_1)\right] = \int_{t_1}^{t_2} d\alpha\ \Phi_A(t_2,\alpha)\ CPP'C'\Phi_A'(t_2,\alpha) \quad (2\text{-}28)$$

From (2-28) and the theorem of section 11.7 of Zadeh and Desoer [5 p. 513],

> the pair [A,CP] is controllable if and only if the rank of $\text{cov}[x(t_2)|x(t_1)]$ is m. (2-29)

Zadeh and Desoer also show (see section 11.3 of [5]) that for linear time invariant systems,

> the pair [A,CP] is completely controllable if and only if the matrix of (2-23) has rank m. (2-30)

Then (2-29) and (2-30) combine to complete the proof.

Theorem 2-5. Let x(t) be the solution to (2-1). If Q has rank m, then x(t) is a Markov process and

$$p_{x(t_2)|x(t_1)}(\xi_2|\xi_1) = \frac{1}{(2\pi)^{m/2}(\det Q)^{1/2}} \exp\left\{-\frac{1}{2}||\xi_2-\mu||^2_{Q^{-1}}\right\} \quad (2\text{-}31)$$

where $p_{x(t_2)|x(t_1)}(\xi_2|\xi_1)$ = probability density associated with the event $x(t_2) = \xi_2$ given that $x(t_1) = \xi_1$. (2-32)

$$Q = \text{cov}[x(t_2)|x(t_1)] \quad (2\text{-}33)$$

$$\mu = E[x(t_2)|x(t_1)] \quad (2\text{-}34)$$

Proof It is clear from (2-5) and the definition of n that x(t) is Gaussianly distributed. Then (2-31) follows directly.

Since x(t) is Gaussianly distributed, x(t) is a Markov process if

$$E[x(t_2)|x(t)] = E[x(t_2)|x(t_1)] \quad (2\text{-}35)$$

and

$$\text{cov}[x(t_2)|x(t)] = \text{cov}[x(t_2)|x(t_1)] \quad (2\text{-}36)$$

for $t \leq t_1 < t_2$.

But (2-35) and (2-36) follow directly from (2-5).

Theorem 2-6. Let x(t) be a solution to (2-1) and $p_{x(t_2)|x(t_1)}(\xi_2|\xi_1)$ be as defined in (2-32). Then

$$\left[\frac{\partial}{\partial t} + \sum_{i=1}^{m} [A\ \xi + Bu(t)]_i \frac{\partial}{\partial \xi_i} + \frac{1}{2}\sum_{i,j=1}^{m} (CWC')_{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j}\right] p_{x(t_2)|x(t)}(\xi_2|\xi)=0 \tag{2-37}$$

Proof The proof follows directly from (2-31) and Doob [1] (for the scalar case) and Mishckenko [4] for the vector case.

Theorem 2-7. Let x(t) be the solution to (2-1). Let $x^{\ell}(t)$ be defined for all t as the solution to

$$\dot{x}^{\ell} = A\ x^{\ell} + Bu + \frac{1}{2}\sqrt{\frac{\ell}{T}} \sum_{i=0}^{\ell-1} \left\{ sgn(t - \frac{iT}{\ell}) - sgn\left[t - \frac{(i+1)T}{\ell}\right]\right\} CN^{i} \tag{2-38}$$

where
$$sgn(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ +1, & t > 0 \end{cases} \tag{2-39}$$

the N^i, $i=1,2,\ldots,\ell$, are k dimensional Gaussian random vectors with

$$E[N^i] = 0 \tag{2-40}$$

$$E[(N^i)(N^j)'] = 0,\ i \neq j \tag{2-41}$$

$$E[(N^i)(N^i)'] = W \tag{2-42}$$

Then x^{ℓ} is a Gaussianly distributed process with

$$E[x^{\ell}(T)|x^{\ell}(0)] = E[x^{\ell}(T)|x^{\ell}(t),\ t < 0] \tag{2-43}$$

$$cov[x^{\ell}(T)|x^{\ell}(0)]\Big|_{x^{\ell}(0)=x(0)} = cov[x^{\ell}(T)|x^{\ell}(t),\ t<0] \tag{2-44}$$

$$E[x^{\ell}(T)|x^{\ell}(0)]\Big|_{x^{\ell}(0)=x(0)} = E[x(T)|x(0)] \tag{2-45}$$

$$\lim_{\ell \to \infty} = \operatorname{cov}[x^{\ell}(T)|x^{\ell}(0)]\Big|_{x^{\ell}(0)=x(0)} = \operatorname{cov}[x(T)|x(0)] \tag{2-46}$$

<u>Proof</u> Since (2-38) is an ordinary linear differential equation, $x^{\ell}(T)$ is

$$x^{\ell}(T) = \Phi_A(T,0)\, x^{\ell}(0) + \int_0^T d\alpha\, \Phi_A(T,\alpha)\, Bu(\alpha)$$

$$+ \frac{1}{2}\int_0^T d\alpha \sum_{i=0}^{\ell-1} \left\{\operatorname{sgn}(\alpha - \frac{iT}{\ell}) - \operatorname{sgn}[\alpha - \frac{(i+1)T}{\ell}]\right\} \frac{\Phi_A(T,\alpha) CN^i \sqrt{\ell}}{\sqrt{T}} \tag{2-47}$$

From (2-47) it follows that since the N^i are Gaussian, x^{ℓ} is Gaussianly distributed. Equations (2-43) and (2-44) follow directly from (2-47), (2-40), (2-41) and (2-42), (i.e. the independence of the N^i). From (2-40) and (2-47),

$$E[x^{\ell}(T)|x^{\ell}(0)=x(0)] = \Phi_A(T,0)\, x(0) + \int_0^T d\alpha\, \Phi(T,\alpha)\, Bu(\alpha) \tag{2-48}$$

then (2-45) is established by comparison of (2-48) and (2-14). Let

$$F^i = \frac{1}{2}\int_0^T d\alpha \left\{\operatorname{sgn}(\alpha - \frac{iT}{\ell}) - \operatorname{sgn}\left[\alpha - \frac{(i+1)T}{\ell}\right]\right\} \Phi_A(T,\alpha) \tag{2-49}$$

Then from (2-47) and (2-48),

$$x^{\ell}(T) - E[x^{\ell}(T)|x^{\ell}(0) = x(0)] = \frac{\sqrt{\ell}}{\sqrt{T}} \sum_{i=0}^{\ell-1} F^i\, CN^i \tag{2-50}$$

and hence from (2-41) and (2-42)

$$\operatorname{cov}[x^{\ell}(T)|x^{\ell}(0)=x(0)] = \frac{\ell}{T} \sum_{i=0}^{\ell-1} F^i\, CWC'(F^i)' \tag{2-51}$$

Applying the law of the mean to (2-49),

$$F^i = \tilde{\Phi}_A^i \frac{T}{\ell} \tag{2-52}$$

where $\left[\tilde{\Phi}_A^i\right]_{jq} = \left[\Phi_A(T,\alpha_{jq})\right]_{jq}$ for some $\alpha_{jq} \in \left[\frac{iT}{\ell}, \frac{(i+1)T}{\ell}\right]$ (2-53)

Then (2-51) becomes

$$\text{cov}\left[x^\ell(T) \middle| x^\ell(0)=x(0)\right] = \sum_{i=0}^{\ell-1} \tilde{\Phi}_A^i \; CWC'(\tilde{\Phi}_A^i)' \; \frac{T}{\ell} \tag{2-54}$$

From (2-53), and letting $i = \frac{\ell\tau}{T}$, (2-55)

$$\lim_{\ell \to \infty} \tilde{\Phi}_A^{\frac{\ell\tau}{T}} = \Phi_A(T,\tau) \tag{2-56}$$

Then by (2-15), (2-54) yields (2-46).

2.4 Extensions to More General Ito Equations.

Equation (2-1) is a linear Ito equation with constant coefficients. As stated by theorem 2-1, the solution of (2-1) is of the same form as an ordinary linear differential equation with constant coefficients. Theorems 2-2 through 2-5 and 2-7 follow directly from the solution form established in theorem 2-1. It is natural to ask whether analagous results can be established for the solutions of more general Ito equations. The answer to this question is discussed in the following paragraphs.

First relax the constant coefficient requirement. That is, suppose that A, B, and C are suitably well behaved functions of time. Then the form of (2-5) would remain essentially unaltered, and theorem 2-2 and 2-3 would follow as before. (Note that W may also be time varying.) Theorem 2-4 does not follow. The rank of $\text{cov}[x(t_2)|x(t_1)]$ may be determined only by an examination of equation (2-15). Theorem 2-5, since it follows from Theorems 2-1, 2-2, and 2-3, follows as before. Theorems 2-6 and 2-7 remain unaltered.

Consider next the case

$$dx_t = f(x,t)dt + c(x,t)dn_t \tag{2-57}$$

where f and c are nonlinear function of x and t. Theorem 2-1 is no longer available, and hence neither are theorems 2-2, 2-3, 2-4, or equation (2-31). The solution to (2-57) does retain the Markov property under suitable (and usefully broad) restrictions on f and c (See Doob [1]). Equation (2-37) also remains valid, with $[A\xi + Bu]$ replaced by $f(\xi,t)$ (See Doob [1] and Mishchenko [4]).

Wong and Zakai [6] have discussed the question of a theorem analogous to theorem 2-7 for equations of the form (2-57). They concluded that in the scalar case the following theorem holds:

<u>Theorem 2-8</u>. Let $x(t)$ be a solution to equation (2-57), with all scalar variables. Let

$$n^{\ell}(t) = \frac{1}{2}\sqrt{\frac{\ell}{T}} \sum_{i=1}^{\ell} \left\{\operatorname{sgn}\left(t - \frac{iT}{\ell}\right) - \operatorname{sgn}\left[t - \frac{(i+1)T}{\ell}\right]\right\} \left\{N^i + \frac{(N^{i+1} - N^i)\ell}{T}\left(t - \frac{iT}{\ell}\right)\right\} \tag{2-58}$$

where N^i is as defined in theorem 2-7. Let $x^{\ell}(t)$ be defined by

$$dx_t^{\ell} = f(x^{\ell},t)\,dt + c(x^{\ell},t)\,dn_t^{\ell} \tag{2-59}$$

If

i) $\frac{\partial c}{\partial x}$ is continuous in x and t

ii) $f(x,t)$ is continuous in t

iii) $|f(x,t) - f(x_o,t)| \leq K\,|x - x_o|$

for some K independent of x, x_o, and t,

iv) $E[x^{\ell}(0)^4] < \infty$

Then

$$\underset{\ell \to \infty}{\mathbf{l.i.m.}}\, x^{\ell}(t) = x^{\infty}(t) \tag{2-60}$$

where $x^{\infty}(t)$ satisfies the Ito equation

$$dx_t^{\infty} = [f(x^{\infty},t) + \frac{1}{2}\, c(x^{\infty},t)\, \frac{\partial c}{\partial x^{\infty}}]\, dt + c(x^{\infty},t)\, dn_t . \tag{2-61}$$

Stated in words, theorem 2-8 says that if the Brownian motion process, n, in (2-57) is approximated by a piecewise linear function which converges to n as the grid gets smaller, then the resulting solution converges in the mean, but not to the solution of (2-57). It converges instead to the solution of (2-61). Note that (2-61) differs from (2-57) only by the term $C(x^{\infty},t)\, \frac{\partial c}{\partial x^{\infty}}$. Thus if c is not a function of x, the sequence of approximate solutions to (2-57) converges to the solution of (2-57) as the grid becomes finer.

Theorem 2-8 raises a question of particular interest to engineers. Suppose a physical system is modeled according to (2-57). How should it be simulated? The answer depends on whether the disturbance is more accureately described as a Brownian motion process, or as some approximation to it.

The methods of Wong and Zakai do not easily extend to the vector case of (2-57). They have some discussion of this case, but do not reach any definite conclusions.

CHAPTER 3

DEFINITION OF THE BASIC PROBLEM AND ITS RELATION TO MISHCHENKO'S PROBLEM

3.1 Introduction

This chapter deals with four related subjects:

1. A precise definition of the basic problem is stated and discussed.
2. Mishchenko's work (Chapter VII of [4]) on a related problem is summarized.
3. The equivalence of the two problems is discussed.
4. Some of the limitations and difficulties in the application of Mishchenko's results to the basic problem are discussed.

3.2 Basic Problem Definition

The basic problem to be considered here is the choice of a control $u(t)$ for the system defined by the stochastic differential equation.

$$dx_t = [Ax + Bu]\, dt + C dn_t, \quad x(0) = \bar{x} \tag{3-1}$$

The control is to maximize

$$I = \text{Prob}\,\{x(t) \in S \text{ for some } t \in [0,T]\} - \int_0^T dt\, \|\, u(t)\, \|_U^2 \tag{3-2}$$

where

$$S = \{x \mid \|\, \tilde{x}\, \|_{\tilde{\alpha}^{-1}} < r(\hat{x})\} \tag{3-3}$$

The following restrictions and definitions apply:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \tag{3-4}$$

$$\tilde{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \qquad \hat{x} = \begin{bmatrix} x_{k+1} \\ \vdots \\ x_m \end{bmatrix} \tag{3-5}$$

A is m x m and has rank m (3-6)

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_\ell \end{bmatrix} \qquad \ell \leq m \tag{3-7}$$

B is m x ℓ (3-8)

$$n = \begin{bmatrix} n_1 \\ \vdots \\ n_k \end{bmatrix} \tag{3-9}$$

$$E \{[n(t_2) - n(t_1)] [n(t_2) - n(t_1)]'\} = W \mid t_2 - t_1 \mid \tag{3-10}$$

W has rank k (3-11)

C is m x k (3-12)

$$CWC' = \begin{bmatrix} \tilde{a} & 0 \\ & \\ 0 & 0 \end{bmatrix} \tag{3-13}$$

$\tilde{a}$ is k x k and has rank k (3-14)

Prob denotes probability.

$$\bar{r} \geq r(\hat{\xi}) \geq \underline{r} > 0 \tag{3-15}$$

$$||\nabla_{\hat{\xi}}\, r(\hat{\xi})|| \leq K_r\, r^{\beta}(\hat{\xi}),\ \beta > 1 \qquad (3\text{-}16)$$

where K_r, $\bar{r}$ and $\underline{r}$ are constants.

The rank of $[B, AB, A^2B, \text{---}, A^{m-1}B]$ is m (3-17)

The rank of $[C, AC, A^2C, \text{---}, A^{m-1}C]$ is m (3-18)

U is real, symmetric, and positive definite (3-19)

Several comments on the above restrictions will be made:

1. The restriction that $\tilde{a}$ have rank k imposes no loss of generality. Neither does the restriction on the form of CWC' imposed by (3-13). Any system may be transformed so as to meet these requirements.
2. The restrictions (3-15) and (3-16) are required for estimating the error of the approximation to the probability term of I (see equation (5-1)) developed in Chapter 4.
3. Restriction (3-17) is made to insure controllability for the approach described in Chapter 6.
4. Restriction (3-18) guarantees the nonsingularity of $\text{Cov}\,[x(t_2)\,|\,x(t_1)]$. This restriction can always be met by reformulation of the problem.

Stated in words, the problem is to bring a stochastically disturbed system from an initial state to a terminal manifold within a stated time. The performance criterion expresses a compromise between energy used and probability of hitting the target manifold.

3.3 Mishchenko's Problem

Consider now the problem solved by Mishchenko [4]. In this problem

Mishchenko considers two points in Euclidean m-space. One, whose coordinates are given by the vector w, is controlled by the function u:

$$\dot{w} = g(w,t,u) \tag{3-20}$$

The other, whose coordinates are given by the vector v, moves randomly. Specifically, v(t) is a Markov process whose transition density satisfies

$$\left\{\frac{\partial}{\partial t_1} + \sum_i b^i \frac{\partial}{\partial \xi_1^i} + \sum_{i,j} a_{ij} \frac{\partial^2}{\partial \xi_1^i \partial \xi_1^j}\right\} p_{v(t_2)|v(t_1)}(\xi_2|\xi_1) = 0 \tag{3-21}$$

where $\xi_j^i = i^{\underline{th}}$ component of the vector ξ_j (3-22)

$p_{v(t_2)|v(t_1)}(\xi_2|\xi_1)$ = **probability density associated with the event $v(t_2) = \xi_2$ given that $v(t_1) = \xi_1$**

(3-23)

Mishchenko [4] then derives an estimate for the function

$$\psi(t_1,\xi_1,t_2) = \text{Prob}\left\{[v(t) - w(t)] \in S_\epsilon \text{ for some } t \in [t_1, t_2] \text{ given } v(t_1) = \xi_1\right\} \tag{3-24}$$

where $S_\epsilon = \left\{x \;\middle|\; ||x|| < \epsilon\right\}$ (3-25)

and ϵ is a positive constant.

3.4 Equivalence Between the Basic Problem and Mishchenko's Problem

The problem posed by equations (3-1) through (3-19) will now be put in the form of Mishchenko's problem. Let

$$d\,z_t = A\,z\,dt + C\,d\,n_t, \qquad z(0) = \bar{x} \tag{3-26}$$

$$dy_t = A\,y\,dt - B\,u\,dt, \qquad y(0) = 0 \tag{3-27}$$

Then

$$x = z - y \tag{3-28}$$

According to theorems 2-5 and 2-6, Z(t) is a Markov process with

According to theorems 2-5 and 2-6, z(t) is a Markov process with

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{m} [A\xi + Bu]_i \frac{\partial}{\partial \xi^i} + \frac{1}{2} \sum_{i,\ j=1}^{m} (CWC')_{ij} \frac{\partial^2}{\partial \xi^i \partial \xi^j}\right\}$$

$$p_{x(t_2) \mid x(t)}(\xi_2 \mid \xi) = 0 \tag{3-29}$$

then

$$\text{Prob}\ \{x(t) \in S_\epsilon \text{ for some } t \in [0,T]\} = \psi(0,\bar{x},T) \tag{3-30}$$

The left hand side of (3-30) is identical to the probability term of (3-2), except for the definition of the target manifold. It turns out that Mishchenko's estimation technique works better for the manifold S' than for S_ϵ. As a matter of fact the form of S was motivated by computational difficulties imposed by the form of S_ϵ.

3.5 Some Limitations and Difficulties in Mishchenko's Work

The discussion up to this point has shown that Mishchenko's estimation technique may be used to furnish an estimate of the probability term of (3-2). Having obtained this estimate it is possible to apply the maximum principle and arrive at the desired control. Several points deserve comment here:

1. Mishchenko shows that his estimate is accurate to within $o(\epsilon^{n-2})$. This means that the estimate is useful for systems of order three and higher only. For the k-dimensional target S, the error is $o(\epsilon^{k-2})$. Although Mishchenko does not give actual bounds for the error for the S_ϵ case, bounds for the error in the S case are worked out in Chapter 4.

2. The application of the maximum principle leads to a two point

boundary value problem. Moreover, the estimate of the probability term, and hence the Hamiltonian, contains m and k dimensional integrals.

3. Mishchenko assumes that the a matrix has rank m. This is not the usual case in problems of the type considered here. The necessary modification to the estimation technique is carried out in Chapter 4.

CHAPTER 4

APPLICATION AND MODIFICATION OF MISHCHENKO'S ESTIMATE

4.1 Introduction

The purpose of this Chapter is to estimate

$$\psi(t_1,\xi,t_2) = \text{Prob}\{x(t)\in S \text{ for some } t\in[t_1,t_2]|z(t_1) = \xi\} \qquad (4\text{-}1)$$

where x, S, and z are as defined in Chapter 3, and ξ is an m dimensional vector. This estimate is made by extending and modifying Mishchenko's technique. The modifications and extensions are as follows:

1. The definition of the target manifold is changed.
2. The requirement that the matrix a have rank m is relaxed.
3. A bound for the error of the estimate is developed.

The technique will be applied specifically to the problem whose form is defined by (3-29) and (3-30). The work will be carried out in three steps:

1. An estimate of ψ will be constructed in transformed coordinates.
2. A bound on the error of this estimate will be derived.
3. The expression for the estimate in the original coordinate system will be stated.

4.2 Construction of the Estimate

Mishchenko shows that if

$$\left[\frac{\partial}{\partial t_1} + \sum_i A\,\xi_1^i \frac{\partial}{\partial \xi_1^i} + \sum_{i,j} a_{ij} \frac{\partial^2}{\partial \xi_1^i \partial \xi_1^j}\right]\left[p_{z(t_2)|z(t_1)}(\xi_2|\xi_1)\right] = 0 \qquad (4\text{-}2)$$

then

$$\left[\frac{\partial}{\partial t_1} + \sum_i A\,\xi^i \frac{\partial}{\partial \xi^i} + \sum_{i,j} a_{ij} \frac{\partial^2}{\partial \xi^i \partial \xi^j}\right] \psi(t_1,\xi,t_2) = 0 \qquad (4\text{-}3)$$

The initial and boundary conditions are

$$\lim_{t_1 \to t_2} \psi(t_1,\xi,t_2) = 0, \quad ||\tilde{\xi}-\tilde{y}(t_1)||_{\tilde{a}^{-1}} > r[\hat{\xi}-\hat{y}(t_1)] \qquad (4\text{-}4)$$

$$\lim_{||\tilde{\xi}-\tilde{y}(t_1)||_{\tilde{a}^{-1}} \to r[\hat{\xi}-\hat{y}(t_1)]} \psi(t_1,\xi,t_2) = 1, \quad t_1 < t_2 \qquad (4\text{-}5)$$

where y is as defined in Chapter 3, and

$\tilde{\xi}$ and $\tilde{y}$ are k dimensional vectors

$\hat{\xi}$ and $\hat{y}$ are m-k dimensional vectors

$$\xi = \begin{bmatrix} \tilde{\xi} \\ \hat{\xi} \end{bmatrix} \qquad y = \begin{bmatrix} \tilde{y} \\ \hat{y} \end{bmatrix} \qquad (4\text{-}6)$$

The first step toward the solution of (4-3) through (4-5) is the change of coordinates:

$$\xi = \bar{\xi} + y(t_1) \qquad (4\text{-}7)$$

$$\phi(t_1,\bar{\xi},t_2) = \psi(t_1,\xi,t_2) \qquad (4\text{-}8)$$

Thus

$$\frac{\partial \psi}{\partial \xi_i} = \sum_{j=1}^{m} \frac{\partial \phi}{\partial \bar{\xi}_j} \frac{\partial \bar{\xi}_j}{\partial \xi_i} = \frac{\partial \phi}{\partial \bar{\xi}_i} \qquad (4\text{-}9)$$

$$\frac{\partial\psi}{\partial t_1} = \frac{\partial\Phi}{\partial t_1} + \sum_{j=1}^{m} \frac{\partial\Phi}{\partial\bar{\xi}_j}\frac{\partial\bar{\xi}_j}{\partial t_1} = \frac{\partial\Phi}{\partial t_1} - \sum_{j=1}^{m} \frac{\partial\Phi}{\partial\bar{\xi}_j}\frac{d}{dt_1} y_j(t_1) \tag{4-10}$$

$$\frac{\partial\psi}{\partial t_1} = \frac{\partial\Phi}{\partial t_1} - \sum_{j=1}^{m} \frac{\partial\Phi}{\partial\bar{\xi}_j} [A\ y(t_1) - B\ u(t_1)]_j \tag{4-11}$$

$$\frac{\partial^2\psi}{\partial\xi_i\partial\xi_j} = \frac{\partial^2\Phi}{\partial\bar{\xi}_i\partial\bar{\xi}_j} \tag{4-12}$$

Substitution of (4-7) through (4-12) into (4-3) through (4-5) yields

$$\frac{\partial\Phi}{\partial t_1} + \sum_{i=1}^{m} [A\ \bar{\xi} + B\ u]_i \frac{\partial\Phi}{\partial\bar{\xi}_i} + \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} \frac{\partial^2\Phi}{\partial\bar{\xi}_i\partial\bar{\xi}_j} = 0 \tag{4-13}$$

$$\lim_{t_1 \to t_2} \Phi(t_1,\bar{\xi},t_2) = 0, \quad \text{for} \quad ||\tilde{\xi}||_{\tilde{a}^{-1}} > r(\hat{\xi}) \tag{4-14}$$

$$\lim_{||\tilde{\xi}||_{a^{-1}} \to r(\hat{\xi})} \Phi(t_1,\bar{\xi},t_2) = 1, \quad \text{for} \quad t_1 < t_2 \tag{4-15}$$

The next step in Mishchenko's procedure is to find a particular solution, $\Phi_o=\Phi_o(t_1,\bar{\xi},t_2)$, to

$$\frac{\partial\Phi_o}{\partial t_1} + \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} \frac{\partial\Phi_o}{\partial\bar{\xi}_i\partial\bar{\xi}_j} = 0 \tag{4-16}$$

$$\lim_{t_1 \to t_2} \Phi_o(t_1,\bar{\xi},t_2) = 0, \quad \text{for} \quad ||\tilde{\xi}||_{\tilde{a}^{-1}} > r(\hat{\xi}) \tag{4-17}$$

where $\tilde{\xi}$ is a k dimensional vector

$\hat{\xi}$ is an m-k dimensional vector

$$[\ldots] \qquad \text{(4-17a)}$$

Mishchenko implicitly assumes that $[a_{ij}]$ has rank m. Instead suppose that $[a_{ij}]$ is of rank $k \leq m$, and that $[a_{ij}]$ is in the form

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1k} & \\ a_{21} & a_{22} & \cdots & a_{2k} & \\ \cdots & \cdots & \cdots & \cdots & 0 \\ a_{k1} & a_{k2} & \cdots & a_{kk} & \\ \hdashline & & 0 & & 0 \end{array}\right] = \begin{bmatrix} \tilde{a} & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(4-18)}$$

([illegible] is [illegible] obtainable through a non-singular linear transformation. [illegible] Chapter 4 of [10].)

Let [illegible] orthonormal eigenvectors of $[\tilde{a}_{ij}]$ be

$$\begin{bmatrix} \alpha_1^1 \\ \alpha_1^2 \\ \vdots \\ \alpha_1^k \end{bmatrix} = \alpha_1, \quad \begin{bmatrix} \alpha_2^1 \\ \alpha_2^2 \\ \vdots \\ \alpha_2^k \end{bmatrix} = \alpha_2, \quad \ldots \quad \begin{bmatrix} \alpha_k^1 \\ \alpha_k^2 \\ \vdots \\ \alpha_k^k \end{bmatrix} \qquad \text{(4-[illegible])}$$

[illegible] corresponding eigenvalues be $\lambda_1, \lambda_2, \ldots, \lambda_k$ [illegible]

$$P = \begin{bmatrix} \frac{\alpha_1^1}{\sqrt{\lambda_1}} & \frac{\alpha_1^2}{\sqrt{\lambda_1}} & \cdots & \frac{\alpha_1^k}{\sqrt{\lambda_1}} & 0 & \cdots & 0 \\ \frac{\alpha_2^1}{\sqrt{\lambda_2}} & \frac{\alpha_2^2}{\sqrt{\lambda_2}} & \cdots & \frac{\alpha_2^k}{\sqrt{\lambda_k}} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\alpha_k^1}{\sqrt{\lambda_k}} & \frac{\alpha_k^2}{\sqrt{\lambda_k}} & \cdots & \frac{\alpha_k^k}{\sqrt{\lambda_k}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \cdots & 0 \\ \vdots & & & & & \ddots & \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \cdots & 1 \end{bmatrix} \tag{4-20}$$

For convenience P may be written as

$$P = \begin{bmatrix} \tilde{\alpha}' & 0 \\ 0 & I_{m-k} \end{bmatrix} \tag{4-21}$$

where I_{m-k} is the identity matrix of rank m-k.

Let

$$\bar{\bar{\xi}} = P\,\bar{\xi} \tag{4-22}$$

and

$$\bar{\phi}_o(t_1,\bar{\bar{\xi}},t_2) = \phi_o(t_1,\bar{\xi},t_2) \tag{4-23}$$

Then

$$\frac{\partial \phi_o}{\partial \bar{\xi}_i} = \sum_{\ell=1}^{m} \frac{\partial \bar{\phi}_o}{\partial \bar{\bar{\xi}}_\ell} \frac{\partial \bar{\bar{\xi}}_\ell}{\partial \bar{\xi}_i} = \sum_{\ell=1}^{m} \frac{\partial \bar{\phi}_o}{\partial \bar{\bar{\xi}}_\ell} p_{\ell i} \tag{4-24}$$

and

$$\frac{\partial^2 \phi_o}{\partial \bar{\xi}_i \partial \bar{\xi}_j} = \sum_{q=1}^{m} \sum_{\ell=1}^{m} p_{\ell i} \frac{\partial^2 \bar{\phi}_o}{\partial \bar{\bar{\xi}}_\ell \partial \bar{\bar{\xi}}_q} \frac{\partial \bar{\bar{\xi}}_q}{\partial \bar{\xi}_j} \tag{4-25}$$

$$\frac{\partial^2 \bar{\phi}_o}{\partial \bar{\xi}_i \partial \bar{\xi}_j} = \sum_{q=1}^{m} \sum_{\ell=1}^{m} P_{\ell i} \frac{\partial^2 \bar{\bar{\phi}}_o}{\partial \bar{\bar{\xi}}_\ell \partial \bar{\bar{\xi}}_q} P_{qj} \tag{4-26}$$

$$\frac{\partial \bar{\phi}_o}{\partial t_1} = \frac{\partial \bar{\bar{\phi}}_o}{\partial t_1} \tag{4-27}$$

Then

$$\sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 \bar{\phi}_o}{\partial \bar{\xi}_i \partial \bar{\xi}_j} = \sum_{i,j,q,\ell=1}^{m} P_{\ell i}\, a_{ij}\, P_{qj} \frac{\partial^2 \bar{\bar{\phi}}_o}{\partial \bar{\bar{\xi}}_\ell \partial \bar{\bar{\xi}}_q} \tag{4-28}$$

Since

$$\sum_{i,j=1}^{m} P_{\ell i}\, a_{ij}\, P_{qj} = \left\{ P[a_{ij}]P' \right\}_{\ell q} \tag{4-29}$$

where P' = transpose of P, equation (4-28) may be written as

$$\sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 \bar{\phi}_o}{\partial \bar{\xi}_i \partial \bar{\xi}_j} = \sum_{q,\ell=1}^{m} \left\{ P[a_{ij}]P' \right\}_{\ell q} \frac{\partial^2 \bar{\bar{\phi}}_o}{\partial \bar{\bar{\xi}}_\ell \partial \bar{\bar{\xi}}_q} \tag{4-30}$$

Using the notation of (4-19) and (4-21), since

$$\tilde{a}\, \alpha_j = \lambda_j\, \alpha_j, \quad j=1,\ldots,k \tag{4-31}$$

it follows that

$$[a_{ij}]P' = \begin{bmatrix} \tilde{a} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\alpha} & 0 \\ 0 & I_{m-k} \end{bmatrix} \tag{4-32}$$

$$[a_{ij}]P' = \begin{bmatrix} \sqrt{\lambda_1}\,\alpha_1 & \sqrt{\lambda_2}\,\alpha_2 & \cdots & \sqrt{\lambda_k}\,\alpha_k & 0 \ldots 0 \end{bmatrix} \tag{4-33}$$

$$P[a_{ij}]P' = \begin{bmatrix} \tilde{\alpha} & 0 \\ 0 & I_{m-k} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1}\,\alpha_1 & \cdots & \sqrt{\lambda_k}\,\alpha_k & 0 \ldots 0 \end{bmatrix} \tag{4-34}$$

$$P[a_{ij}]P' = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \tag{4-35}$$

where I_k = identity matrix of rank k. Then (4-30) becomes

$$\sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 \Phi_o}{\partial \bar{\xi}_i \partial \bar{\xi}_j} = \sum_{\ell=1}^{k} \frac{\partial^2 \bar{\bar{\Phi}}_o}{\partial \bar{\bar{\xi}}_\ell^{\,2}} \tag{4-36}$$

From (4-27) and (4-36), (4-16) and (4-17) become

$$\frac{\partial \bar{\bar{\Phi}}_o}{\partial t_1} + \sum_{\ell=1}^{k} \frac{\partial^2 \bar{\bar{\Phi}}_o}{\partial \bar{\bar{\xi}}_\ell^{\,2}} = 0 \tag{4-37}$$

$$\lim_{t_1 \to t_2} \bar{\bar{\Phi}}_o(t_1, \bar{\bar{\xi}}, t_2) = 0, \quad \text{for } ||\tilde{\bar{\bar{\xi}}}|| > r(\hat{\bar{\bar{\xi}}}) \tag{4-38}$$

The solution of (4-37) and (4-38) begins by finding a solution $\bar{\bar{\Phi}}_o = \bar{\bar{\Phi}}_o(\bar{\bar{\xi}})$ to

$$\sum_{\ell=1}^{k} \frac{\partial^2 \bar{\bar{\Phi}}_o(\bar{\bar{\xi}})}{\partial \bar{\bar{\xi}}_\ell^{\,2}} = 0 \tag{4-39}$$

$$\lim_{||\tilde{\bar{\bar{\xi}}}|| \to r(\hat{\bar{\bar{\xi}}})} \bar{\bar{\Phi}}_o(\bar{\bar{\xi}}) = 1 \tag{4-40}$$

At this point the motivation for the choice of manifolds becomes apparent. Mishchenko's original choice of a manifold which is spherical in $\bar{\xi}$ coordinates results in one which is ellipsoidal in $\bar{\bar{\xi}}$ coordinates. As a result, this solution to (4-39), (4-40) is expressed in terms of a Fredholm integral equation of the second kind. The domain of the integral is the surface of the ellipsoid

$$||\bar{\xi}||_{\tilde{a}}^{2} = \epsilon^2 \tag{4-41}$$

The solution of this equation presents monumental numerical difficulties. With the choice of the manifolds specified by (4-4) and (4-5), the solution to (4-39) and (4-40) is

$$\bar{\bar{\Phi}}_o(\tilde{\bar{\bar{\xi}}},\hat{\bar{\bar{\xi}}}) = \frac{r^{k-2}(\frac{\Delta}{\xi})}{||\tilde{\bar{\bar{\xi}}}||^{k-2}} \tag{4-42}$$

This agrees with Mishchenko's results [4] for the case where $[a_{ij}]$ is the identity matrix.

The fundamental solution to (4-37) is

$$\bar{\bar{h}}(t_1,\tilde{\bar{\bar{\xi}}},t_2,\tilde{\bar{\bar{\eta}}}) = \frac{\exp\left\{-\frac{||\tilde{\bar{\bar{\eta}}}-\tilde{\bar{\bar{\xi}}}||^2}{4(t_2-t_1)}\right\}}{(2\pi)^{k/2}\ [2(t_2-t_1)]^{k/2}} \tag{4-43}$$

where $\tilde{\bar{\bar{\eta}}}$ is a k dimensional vector

A particular solution to (4-37) and (4-38), then, is

$$\bar{\Phi}_o(t_1,\bar{\bar{\xi}},t_2) = \frac{r^{k-2}(\hat{\bar{\bar{\xi}}})}{||\tilde{\bar{\bar{\xi}}}||^{k-2}} - \int_{||\tilde{\bar{\bar{\eta}}}|| \geq 0} d\tilde{\bar{\bar{\eta}}}_1\ldots d\tilde{\bar{\bar{\eta}}}_k\ \bar{\bar{h}}(t_1,\tilde{\bar{\bar{\xi}}},t_2,\tilde{\bar{\bar{\eta}}})\bar{\bar{\Phi}}_o(\tilde{\bar{\bar{\eta}}},\hat{\bar{\bar{\xi}}})$$

$$\text{for } t_2 > t_1 \tag{4-44}$$

The next step is to transform coordinates back to $\bar{\xi}$, and thus get an expression for Φ_o. From (4-17a), (4-20), and (4-21),

$$\hat{\bar{\bar{\xi}}} = \hat{\bar{\xi}} \tag{4-45}$$

$$\tilde{\bar{\bar{\xi}}} = \tilde{\alpha}'\ \tilde{\bar{\xi}} \tag{4-46}$$

From (4-46),

$$||\tilde{\bar{\bar{\xi}}}||^2 = \tilde{\bar{\xi}}'\ \tilde{\alpha}\,\tilde{\alpha}'\ \tilde{\bar{\xi}} \tag{4-47}$$

It follows from the definition of $\tilde{\alpha}$ that

$$\tilde{\alpha}' \, \tilde{a} \, \tilde{\alpha} = I_k \tag{4-48}$$

Then

$$\tilde{\alpha} \, \tilde{\alpha}' = \tilde{a}^{-1} \tag{4-49}$$

and (4-47) becomes

$$\|\tilde{\tilde{\xi}}\|^2 = \|\tilde{\xi}\|^2_{\tilde{a}^{-1}} \tag{4-50}$$

Then (4-44) becomes

$$\phi_o(t_1,\overline{\xi},t_2) = \frac{r^{k-2}(\hat{\overline{\xi}})}{\|\tilde{\overline{\xi}}\|^{k-2}_{\tilde{a}^{-1}}} - \int_{\|\tilde{\eta}\|_{\tilde{a}^{-1}} \geq 0} d\tilde{\tilde{\eta}}_1 \ldots d\tilde{\tilde{\eta}}_k \; \overline{h}(t_1,\tilde{\tilde{\xi}},t_2,\tilde{\tilde{\eta}}) \; \tilde{\tilde{\phi}}_o(\tilde{\tilde{\eta}},\hat{\tilde{\xi}}) \tag{4-51}$$

where $\tilde{\eta}$ is a k dimensional vector

$$\overline{h}(t_1,\tilde{\xi},t_2,\tilde{\tilde{\eta}}) = \frac{\exp\left\{-\dfrac{\|\tilde{\eta} - \tilde{\xi}\|^2_{\tilde{a}^{-1}}}{4(t_2-t_1)}\right\}}{(2\pi)^{k/2}[\det \tilde{a}]^{1/2}[2(t_2-t_1)]^{k/2}} \tag{4-52}$$

and

$$\tilde{\phi}_o(\tilde{\eta},\hat{\xi}) = \frac{r^{k-2}(\hat{\tilde{\xi}})}{\|\tilde{\eta}\|^{k-2}_{\tilde{a}^{-1}}} \tag{4-53}$$

Next, let $\phi^*(t_1,\overline{\xi},t_2)$ be a solution to (4-13), and let

$$\phi^*(t_1,\overline{\xi},t_2) = \phi_1(t_1,\overline{\xi},t_2) + \phi_o(t_1,\overline{\xi},t_2) \tag{4-54}$$

Substitution into (4-13) shows that ϕ_1 must satisfy

$$\frac{\partial \phi_1}{\partial t_1} + \sum_{i=1}^{m} [A\,\bar{\xi} + B\,u]_i \frac{\partial \phi_1}{\partial \bar{\xi}_i} + \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 \phi_1}{\partial \bar{\xi}_i \partial \bar{\xi}_j}$$

$$= - \sum_{i=1}^{m} [A\,\bar{\xi} + B\,u]_i \frac{\partial \phi_o}{\partial \bar{\xi}_i} \qquad (4\text{-}55)$$

The fundamental solution of (4-13) is $\bar{q}(t_1,\bar{\xi},t_2,\bar{\eta})$ = probability density associated with the event

$$\{x(t_2) = \bar{\eta} | x(t_1) = \bar{\xi} \text{ and } x(t) \notin S \text{ for any } t \in [t_1,t_2]\} \qquad (4\text{-}56)$$

where $\bar{\eta}$ is an m dimensional vector. Then a solution to (4-55) is

$$\phi_1(t_1,\bar{\xi},t_2) = \int_{t_1}^{t_2} d\sigma \int_{||\tilde{\eta}||_{\tilde{a}^{-1}} \geq 0} d\bar{\eta}_1 \ldots d\bar{\eta}_m\, \bar{q}(t_1,\bar{\xi},\sigma,\bar{\eta}) \sum_{i=1}^{m} [A\bar{\eta} + Bu]_i \frac{\partial \phi_o}{\partial \bar{\eta}_i} \qquad (4\text{-}57)$$

where the arguments of ϕ_o are as follows

$$\phi_o = \phi_o(t_1,\bar{\eta},\sigma) \qquad (4\text{-}58)$$

and

$$\tilde{\eta} = \begin{bmatrix} \bar{\eta}_1 \\ \vdots \\ \bar{\eta}_k \end{bmatrix}, \quad \hat{\eta} = \begin{bmatrix} \bar{\eta}_{k+1} \\ \vdots \\ \bar{\eta}_m \end{bmatrix} \qquad (4\text{-}59)$$

Consider now the initial and boundary conditions of (4-54). Equations (4-17), (4-51) through (4-53), and (4-57) clearly show that

$$\lim_{t_1 \to t_2} \phi^*(t_1,\bar{\xi},t_2) = 0, \quad ||\tilde{\xi}||_{\tilde{a}^{-1}} > r(\hat{\xi}) \qquad (4\text{-}60)$$

Mishchenko argues that since $\bar{q}(t_1,\bar{\xi},t_2,\bar{\eta})$ is the probability density defined in (4-56),

$$\lim_{||\tilde{\xi}||_{\tilde{a}^{-1}} \to r(\hat{\xi})} \bar{q}(t_1,\bar{\xi},t_2,\bar{\eta}) = 0, \quad \text{for } t_1 < t_2 \tag{4-61}$$

Then

$$\lim_{||\tilde{\xi}||_{\tilde{a}^{-1}} \to r(\hat{\xi})} \Phi_1(t_1,\bar{\xi},t_2) = 0, \text{ for } t_1 < t_2 \tag{4-62}$$

and hence

$$\lim_{||\tilde{\xi}||_{\tilde{a}^{-1}} \to r(\hat{\xi})} \Phi^*(t_1,\bar{\xi},t_2) = \lim_{||\tilde{\xi}||_{\tilde{a}^{-1}} \to r(\hat{\xi})} \Phi_o(t_1,\bar{\xi},t_2), \text{ for } t_1 < t_2 \tag{4-63}$$

where Φ_o is given in (4-51).

4.3 A Bound on the Error of the Estimate

The error between $\Phi(t_1,\bar{\xi},t_2)$, the solution to (4-13) through (4-15), and $\Phi^*(t_1,\bar{\xi},t_2)$ will now be estimated. Let the error term be

$$\Phi_\epsilon(t_1,\bar{\xi},t_2) = \Phi(t_1,\bar{\xi},t_2) - \Phi^*(t_1,\bar{\xi},t_2) \tag{4-64}$$

It is clear that $\Phi_\epsilon(t_1,\bar{\xi},t_2)$ is also a solution to (4-13). First, from (4-14), (4-60), and (4-64),

$$\lim_{t_1 \to t_2} \Phi_\epsilon(t_1,\bar{\xi},t_2) = 0, \text{ for } ||\tilde{\xi}||_{\tilde{a}^{-1}} > r(\hat{\xi}) \tag{4-65}$$

From (4-63), (4-64), (4-51), and (4-15),

$$\lim_{||\tilde{\xi}||_{\tilde{a}^{-1}} \to r(\hat{\xi})} \Phi_\epsilon(t_1,\bar{\xi},t_2) = \lim_{||\tilde{\xi}||_{\tilde{a}^{-1}} \to r(\hat{\xi})} \int_{||\tilde{\eta}||_{\tilde{a}^{-1}} \geq 0} d\bar{\eta}_1 \ldots d\bar{\eta}_k \, \bar{h}(t_1,\tilde{\xi},t_2,\tilde{\eta})\tilde{\Phi}(\tilde{\eta},\hat{\xi})$$

$$\text{for } t_1 < t_2 \tag{4-66}$$

A bound will now be placed on (4-66). From (4-45), (4-46), (4-50), (4-53), (4-52) and (4-43), equation (4-66) becomes

$$\lim_{||\tilde{\bar{\xi}}||_{\tilde{a}^{-1}} \to r(\hat{\bar{\xi}})} \Phi_\epsilon(t_1,\bar{\xi},t_2) = \lim_{||\tilde{\bar{\bar{\xi}}}|| \to r(\hat{\bar{\bar{\xi}}})} \int_{||\tilde{\bar{\bar{\eta}}}|| > 0} d\tilde{\bar{\bar{\eta}}}\, \bar{\bar{h}}(t_1,\tilde{\bar{\bar{\xi}}},t_2,\tilde{\bar{\bar{\eta}}}) \frac{r^{k-2}(\hat{\bar{\bar{\xi}}})}{||\tilde{\bar{\bar{\eta}}}||^{k-2}} \tag{4-67}$$

Then from (A-1), (A-24), (A-25) and (4-43),

$$0 < \lim_{||\tilde{\bar{\xi}}||_{\tilde{a}^{-1}} \to r(\hat{\bar{\xi}})} \Phi_\epsilon(t_1,\bar{\xi},t_2) < \begin{cases} \dfrac{K_{k-2}}{2^{k-2} K_t^{\frac{k-2}{2}}} r_{k-2}^{\frac{k-2}{2}}(\hat{\bar{\bar{\xi}}}) & ,t_2-t_1 > K_t r(\hat{\bar{\xi}}) \\ \bar{V}, & t_2-t_1 \leq K_t r(\hat{\bar{\xi}}) \end{cases} \tag{4-68}$$

Then $\Phi_\epsilon(t_1,\bar{\xi},t_2)$ satisfies the hypotheses of the Theorem of Appendix A. Then $\Phi_\epsilon(t_1,\bar{\xi},t_2)$ is bounded according to (A-54):

$$0 < \Phi_\epsilon(t_1,\bar{\xi},t_2) < \Delta(\bar{\xi},\underline{r}) + \frac{K_{k-2}}{2^{k-2} K_t^{\frac{k-2}{2}}} r^{\frac{k-2}{2}}(\hat{\bar{\bar{\xi}}})\, \Phi(t_1,\bar{\xi},t_2)$$

$$\text{for } \bar{\xi} \in S_{Bu}(t_1,\underline{R},t_2) \tag{4-69}$$

where Δ is defined by (A-162)

S_{Bu} is defined by (A-56)

The final approximation involves the substitution of $\bar{p}$ for $\bar{q}$ in (4-57). The error may be estimated as follows. Let

$$\phi_1(t_1,\bar{\xi},t_2) = \tilde{\phi}_1(t_1,\bar{\xi},t_2) - \int_{t_1}^{t_2} d\sigma \int_{\|\bar{\eta}\|_{\tilde{a}^{-1}} \geq 0} d\bar{\eta}[\bar{p}(t_1,\bar{\xi},\sigma,\bar{\eta}) - \bar{q}(t_1,\bar{\xi},\sigma,\bar{\eta})] \sum_{i=1}^{m} [A\bar{\eta}+Bu(\sigma)]_i \frac{\partial\phi_o}{\partial\bar{\eta}_i} \tag{4-70}$$

where

$$\tilde{\phi}_1(t_1,\bar{\xi},t_2) = \int_{t_1}^{t_2} d\sigma \int_{\|\tilde{\bar{\eta}}\|_{\tilde{a}^{-1}} \geq 0} d\bar{\eta}\, \bar{p}(t_1,\bar{\xi},\sigma,\bar{\eta}) \sum_{i=1}^{m} [A\bar{\eta}+Bu]_i \frac{\partial\phi_o}{\partial\bar{\eta}_i} \tag{4-71}$$

and arguments of ϕ_o are as in (4-58).

$\bar{p}(t_1,\bar{\xi},\sigma,\bar{\eta})$ = probability density associated with the event

$$\{x(\sigma) = \bar{\eta} | x(t_1) = \bar{\xi}\}, \text{ for } t_1 < \sigma \tag{4-72}$$

Let

$$\tilde{\phi}_\epsilon(t_1,\bar{\xi},t_2) = \phi_1(t_1,\bar{\xi},t_2) - \tilde{\phi}_1(t_1,\bar{\xi},t_2) \tag{4-73}$$

Then the error term $\tilde{\phi}_\epsilon$ may be estimated as follows:

$$|\tilde{\phi}_\epsilon(t_1,\bar{\xi},t_2)| \leq \int_{t_1}^{t_2} d\sigma \int_{\|\bar{\eta}\|_{\tilde{a}^{-1}} \geq 0} d\bar{\eta}[\bar{p}(t_1,\bar{\xi},\sigma,\bar{\eta}) - \bar{q}(t_1,\bar{\xi},\sigma,\bar{\eta})][\|A\bar{\eta}+Bu\| \, \|\nabla_{\bar{\eta}}\phi_o(t_1,\bar{\eta},\sigma)\|] \tag{4-74}$$

Comparison of (4-72) with (4-56) yields

$$\bar{p}(t_1,\bar{\xi},\sigma,\bar{\eta}) \geq \bar{q}(t_1,\bar{\xi},\sigma,\bar{\eta}) \geq 0 \tag{4-75}$$

Application of the law of the mean, and (4-75) to (4-74) yields

$$|\tilde{\phi}_\epsilon(t_1,\bar{\xi},t_2)| \leq (t_2-t_1) \int_{\|\tilde{\bar{\eta}}\|_{\tilde{a}^{-1}} \geq 0} d\bar{\eta}\, \bar{p}(t_1,\bar{\xi},\bar{\sigma},\bar{\eta}) \cdot [\|A\bar{\eta}+Bu\| \, \|\nabla_{\bar{\eta}}\phi_o(t_1,\bar{\eta},\bar{\sigma})\|$$

$$\text{for some } \bar{\sigma} \epsilon [t_1,t_2] \tag{4-76}$$

From (4-76)

$$|\tilde{\Phi}_{\epsilon}(t_1,\bar{\xi},t_2)| \leq (t_2-t_1)\,||A\bar{\eta}+Bu||\ ||\nabla_{\bar{\eta}}\Phi_o(t_1,\bar{\eta},t_2)||$$

$$\text{for } \bar{\sigma} = t_1 \qquad (4\text{-}77)$$

The estimation of $|\Phi_{\epsilon}(t_1,\bar{\xi},t_2)|$ for $\bar{\sigma}\in(t_1,t_2)$ proceeds as follows.

Equations (4-72), (3-26) through (3-28), and Theorems 2-2, 2-3, and 2-5 show that

$$\bar{p}(t_1,\bar{\xi},\bar{\sigma},\bar{\eta}) = \frac{\exp\left\{-\frac{1}{2}||\bar{\eta}-\bar{\mu}(t_1)||_{\bar{Q}^{-1}(\bar{\sigma},t_1)}\right\}}{(2\pi)^{m/2}[\det\ \bar{Q}(\bar{\sigma},t_1)]^{1/2}} \qquad (4\text{-}78)$$

where

$$\bar{\mu}(\bar{\sigma}) = \left\{\exp[A(\bar{\sigma}-t_1)]\right\}\bar{\xi} + \int_{t_1}^{\bar{\sigma}} d\alpha \left\{\exp[A(\bar{\sigma}-\alpha)]\right\} Bu(\alpha) \qquad (4\text{-}79)$$

$$Q(\bar{\sigma},t_1) = 2\int_{t_1}^{\bar{\sigma}} d\alpha \left\{\exp[A(\bar{\sigma}-\alpha)]\right\} [a_{ij}] \left\{\exp[A(\bar{\sigma}-\alpha)]\right\}' \qquad (4\text{-}80)$$

Estimating as in (A-114),

$$\bar{p}(t_1,\bar{\xi},\sigma,\bar{\eta}) \leq \left[\frac{\bar{\lambda}_{\bar{\sigma}t_1}}{\underline{\lambda}_{\bar{\sigma}t_1}}\right] g(\bar{\lambda}_{\bar{\sigma}t_1}t_1,\bar{\mu},\bar{\lambda}_{\bar{\sigma}t_1}\bar{\sigma},\bar{\eta}) \qquad (4\text{-}81)$$

where $\bar{\lambda}_{\bar{\sigma}t_1}$ and $\underline{\lambda}_{\bar{\sigma}t_1}$ are respectively the maximum and minimum eigenvalues of $\bar{Q}(\sigma,t_1)$:

$$\bar{Q}(\bar{\sigma},t_1) = \frac{Q(\bar{\sigma},t_1)}{2(\bar{\sigma}-t_1)} \qquad (4\text{-}82)$$

From (4-76) and (4-81)

$$|\tilde{\Phi}_{\epsilon}(t_1,\bar{\xi},t_2)| \leq (t_2-t_1)\int_{||\bar{\eta}||_{\tilde{a}^{-1}}\geq 0} \left\{ d\bar{\eta} \left[\frac{\bar{\lambda}_{\bar{\sigma}t_1}}{\underline{\lambda}_{\bar{\sigma}t_1}}\right]^{m/2} g(\underline{\lambda}_{\bar{\sigma}t_1}t_1,\bar{\mu},\underline{\lambda}_{\bar{\sigma}t_1}t_1,\bar{\eta})\cdot\right.$$

$$\cdot \|A\bar{\eta}+Bu(\bar{\sigma})\| \; \|\nabla_{\bar{\bar{\eta}}} \bar{\phi}_o\| \} \tag{4-83}$$

Next let

$$\bar{\bar{\eta}} = P\,\bar{\eta} \tag{4-84}$$

$$\bar{\bar{\mu}} = P\,\mu \tag{4-85}$$

Then (4-83) becomes

$$|\tilde{\phi}_\epsilon(t_1,\bar{\xi},t_2)| \le \frac{t_2-t_1}{\det P} \int_{\|\tilde{\bar{\bar{\eta}}}\| \ge 0} d\bar{\bar{\eta}} \left[\frac{\bar{\lambda}_{\bar{\sigma} t_1}}{\underline{\lambda}_{\bar{\sigma} t_1}} \right] g(\underline{\lambda}_{\bar{\sigma} t_1} t_1, P^{-1}\bar{\bar{\mu}}, \underline{\lambda}_{\bar{\sigma} t_1} \bar{\sigma}, P^{-1}\bar{\bar{\eta}})$$

$$\cdot \|AP^{-1}\bar{\bar{\eta}}+Bu(\bar{\sigma})\| \; \|P'\nabla_{\bar{\bar{\eta}}} \bar{\phi}_o(t_1,\bar{\bar{\eta}},\bar{\sigma})\| \tag{4-86}$$

From (B-35)

$$\|P^{-1}\,\bar{\bar{\eta}} - P^{-1}\,\bar{\bar{\mu}}\| \ge \frac{1}{K_P}\,\|\bar{\bar{\eta}} - \bar{\bar{\mu}}\| \tag{4-87}$$

where

$$K_P = m \max_{1 \le i \le m} \|P\,e_i\| \tag{4-88}$$

where e_i is the m dimensional vector whose i^{th} element is one and whose other elements are zero

Then from (4-87) and (A-2),

$$g(\underline{\lambda}_{\bar{\sigma} t_1} t_1, P^{-1}\,\bar{\bar{\mu}}, \underline{\lambda}_{\bar{\sigma} t_1} \bar{\sigma}, P^{-1}\,\bar{\bar{\eta}}) \le$$

$$K_P^{m/2}\; g(K_P\,\underline{\lambda}_{\bar{\sigma} t_1} t_1, \bar{\bar{\mu}}, K_P\,\underline{\lambda}_{\bar{\sigma} t_1} \bar{\sigma}, \bar{\bar{\eta}}) \tag{4-89}$$

From (B-30),

$$\|P'\nabla_{\bar{\bar{\eta}}} \bar{\phi}_o(t_1,\bar{\bar{\eta}},\bar{\sigma})\| \le K_{P'} \|\nabla_{\bar{\bar{\eta}}} \bar{\phi}_o(t_1,\bar{\bar{\eta}},\bar{\sigma})\| \tag{4-90}$$

where

$$K_{P'} = m \max_{1 \leq i \leq m} \|P' e_i\| \tag{4-91}$$

and

$$\|AP^{-1} \bar{\bar{\eta}}\| \leq K_{AP^{-1}} \|\bar{\bar{\eta}}\| \tag{4-92}$$

where

$$K_{AP^{-1}} = m \max_{1 \leq i \leq m} \|AP^{-1} e_i\| \tag{4-93}$$

and

$$\|Bu(\sigma)\| \leq K_B \|u(\sigma)\| \tag{4-94}$$

where

$$K_B = m \max_{1 \leq i \leq m} \|B e_i\| \tag{4-95}$$

Substitution into (4-86) from (4-89), (4-90), (4-91), (4-93), and (D-16) yields

$$|\tilde{\Phi}_\epsilon(t_1,\xi,t_2)| \leq \frac{K_{P'}\left[K_P \frac{\bar{\lambda}_{\bar{\sigma}t_1}}{\underline{\lambda}_{\bar{\sigma}t_1}}\right]^{m/2}}{\det P}(t_2-t_1)\int_{\|\tilde{\bar{\bar{\eta}}}\| \geq 0} d\bar{\bar{\eta}}\, g(K_P\underline{\lambda}_{\bar{\sigma}t_1} t_1, \bar{\bar{\mu}}, K_P\underline{\lambda}_{\bar{\sigma}t_1}\bar{\sigma}, \bar{\bar{\eta}})$$

$$\left[K_{AP^{-1}}(\|\tilde{\bar{\bar{\eta}}}\| + \|\hat{\bar{\bar{\eta}}}\|) + K_B\|u(\sigma)\|\right]\left[\frac{(k+2)(\bar{V}+1)F^{k-2}}{\|\bar{\bar{\eta}}\|^{k-1}} + \frac{(k-2)(\bar{V}+1)F^{k-3+\beta}}{\|\bar{\bar{\eta}}\|^{k-2}}\right]$$

$$\text{for } \bar{\sigma} \in (t_1, t_2) \tag{4-96}$$

From (4-96), (A-1), (A-22), (A-26), and (A-38)

$$|\tilde{\Phi}_\epsilon(t_1, \xi, t_2)| \leq$$

$$K_{P'} \frac{(t_2-t_1)}{\det P}\left[K_P \frac{\bar{\lambda}_{\bar{\sigma}t_1}}{\underline{\lambda}_{\bar{\sigma}t_1}}\right]^{m/2} \left\{\frac{[F_1\sqrt{\bar{\sigma}-t_1} + F_2\|\hat{\bar{\bar{\mu}}}\| + F_3\|u(\bar{\sigma})\|]F^{k-2}}{\|\bar{\bar{\mu}}\|^{k-1}} + \right.$$

$$+ \frac{F_4 \bar{r}^{-k-2} + \left[F_5\sqrt{\bar{\sigma}-t_1} + F_6||\tilde{\bar{\mu}}|| + F_7||u(\bar{\sigma})||\right]\bar{r}^{-k-3+\beta}}{||\tilde{\bar{\mu}}||^{k-2}}$$

$$+ \frac{F_8\ \bar{r}^{-k-3+\beta}}{||\tilde{\bar{\mu}}||^{k-3}} \Bigg\}, \quad \text{for } \bar{\sigma} \in (t_1, t_2] \tag{4-97}$$

where

$\beta > +1$

$$F_1 = K_{AP^{-1}}(k-2)\bar{V}(\bar{V}+1)\ k_{\rho,m-k}\sqrt{\tfrac{\lambda}{-}\bar{\sigma}\, t_1}\, K_P \tag{4-98}$$

$$F_2 = K_{AP^{-1}}(k-2)\bar{V}(\bar{V}+1)\ \bar{k}_{\rho,m-k} \tag{4-99}$$

$$F_3 = K_B(k-2)\bar{V}(\bar{V}+1) \tag{4-100}$$

$$F_4 = K_{AP^{-1}}(k-2)\bar{V}(\bar{V}+1) \tag{4-101}$$

$$F_5 = K_{AP^{-1}}(k-2)(\bar{V}+1)\bar{V}k_{\rho,m-k}\sqrt{\tfrac{\lambda}{-}\bar{\sigma}\, t_1}\, K_P \tag{4-102}$$

$$F_6 = K_{AP^{-1}}(k-2)(\bar{V}+1)\bar{V}\bar{k}_{\rho,m-k} \tag{4-103}$$

$$F_7 = K_B(k-2)(\bar{V}+1)\bar{V} \tag{4-104}$$

$$F_8 = K_{AP^{-1}}(k-2)(\bar{V}+1)\bar{V} \tag{4-105}$$

From (4-77), (4-90), (D-9), and (D-10),

$$\tilde{\Phi}_\epsilon(t_1, \bar{\xi}, t_2) = 0 \text{ for } \bar{\sigma} = t_1 \tag{4-106}$$

The solution whose error has been estimated from (4-54) is $\Phi_0 + \tilde{\Phi}_1$. An examination of the error estimates shows that the error has been shown to be $O(\bar{r}^{-k-2})$, while Mishchenko [11] claims that it is $o(\bar{r}^{-k-2})$. As a matter of fact, the error is $o(\underline{r}^{k-2})$. This may be shown as follows. First, the error terms of (4-69) are $o(\underline{r}^{k-2})$. It has been concluded also (see (4-74) through (4-96)) that

$$\int_{t_1}^{t_2} d\sigma \int_{||\tilde{\overline{\eta}}||_{\tilde{a}^{-1}} \geq 0} d\overline{\eta}\, \overline{p}(t_1,\overline{\xi},\sigma,\overline{\eta})[||A\overline{\eta}+Bu||\ ||\nabla_{\overline{\eta}}\phi_o(t_1,\overline{\eta},\sigma)]$$
$$= O(\underline{r}^{k-2}) \qquad (4\text{-}107)$$

From (4-75) and (4-107), there exist positive constants $K_{\overline{p}}$ and $K_{\overline{q}}$ such that

$$\int_{t_1}^{t_2} d\sigma \int_{||\tilde{\overline{\eta}}||_{\tilde{a}^{-1}} \geq 0} d\overline{\eta}[\overline{p}(t_1,\overline{\xi},\sigma,\overline{\eta})-\overline{q}(t_1,\overline{\xi},\sigma,\overline{\eta})][||A\overline{\eta}+Bu||\cdot||\nabla_{\overline{\eta}}\phi_o||]$$
$$\leq (K_{\overline{p}}-K_{\overline{q}})\,\underline{r}^{k-2} \qquad (4\text{-}108)$$

From the definition of $\overline{p}$ and $\overline{q}$,

$$\lim_{\underline{r}\to 0} \overline{p}(t_1,\overline{\xi},\sigma,\overline{\eta})=\overline{q}(t_1,\overline{\xi},\sigma,\overline{\eta}) \qquad (4\text{-}109)$$

Then

$$\lim_{\underline{r}\to 0} (K_{\overline{p}} - K_{\overline{q}}) = 0 \qquad (4\text{-}110)$$

then from (4-74), (4-108), and (4-110),

$$|\tilde{\phi}_\epsilon(t_1,\overline{\xi},t_2)| = o(\underline{r}^{k-2}) \qquad (4\text{-}111)$$

4.4 The Estimate in ξ Coordinates

The final step is to transform from $\overline{\xi}$ coordinates back to ξ coordinates. Let

$$\psi_o(t_1,\xi,t_2) = \phi_o(t_1,\overline{\xi},t_2) \qquad (4\text{-}112)$$

$$\psi_1(t_1,\xi,t_2) = \tilde{\phi}_1(t_1,\overline{\xi},t_2) \qquad (4\text{-}113)$$

$$\overline{\psi}(t_1,\xi,t_2) = \psi_o(t_1,\xi,t_2) + \psi_1(t_1,\xi,t_2) \qquad (4\text{-}114)$$

The from (4-7), (4-51) through (4-53),

$$\psi_o(t_1,\xi,t_2) = \frac{r^{k-2}(\hat{\xi}-\hat{y}(t_1))}{\|\tilde{\xi}-\tilde{y}(t_1)\|^{k-2}_{\tilde{a}^{-1}}}$$

$$- \int_{\|\tilde{\eta}\|_{\tilde{a}^{-1}}\geq 0} d\tilde{\eta}_1 \ldots d\tilde{\eta}_k \; h(t_1,\xi,t_2,\tilde{\eta})\bar{\psi}_o(\tilde{\eta},\hat{\xi}) \qquad (4\text{-}115)$$

where

$$h(t_1,\xi,t_2,\tilde{\eta}) = \frac{\exp\{-\dfrac{\|\tilde{\eta}-\tilde{\xi}+\tilde{y}(t_1)\|^2_{\tilde{a}^{-1}}}{4(t_2-t_1)}\}}{(2\pi)^{k/2}[\det \tilde{a}]^{1/2}[2(t_2-t_1)]^{k/2}} \qquad (4\text{-}116)$$

$$\bar{\psi}_o(\tilde{\eta},\hat{\xi}) = \frac{r^{k-2}[\hat{\xi}-\hat{y}(t_1)]}{\|\tilde{\eta}\|^{k-2}_{\tilde{a}^{-1}}} \qquad (4\text{-}117)$$

where $\tilde{\eta}$ is a k dimensional vector. From (4-71),

$$\psi_1(t_1,\xi,t_2)=\int_{t_1}^{t_2} d\sigma \int_{\|\tilde{\eta}\|_{\tilde{a}^{-1}}\geq 0} d\eta \; p_{z(\sigma)|z(t_1)}(\eta|\xi) \sum_{i=1}^{m}\{A[\eta-y(\sigma)]+Bu(\sigma)\} \frac{\partial\psi_o}{\partial\eta_i} \qquad (4\text{-}118)$$

where η is the m dimensional vector with

$$\eta = \begin{bmatrix} \tilde{\eta} \\ \hat{\eta} \end{bmatrix} \qquad (4\text{-}119)$$

and the arguments of ψ_o are as follows: $\psi_o = \psi_o(\sigma,\eta,t_2)$ (4-120)

The expression for $\frac{\partial\psi_o}{\partial\eta_i}$ is

$$\frac{\partial\psi_o}{\partial\eta_i} = \sum_{j=1}^{m} \frac{\partial\bar{\psi}_o}{\partial\bar{\eta}_j}\frac{\partial\bar{\bar{\eta}}_j}{\partial\eta_i} \qquad (4\text{-}121)$$

where

$$\bar{\bar{\eta}} = P[\eta - y(\sigma)] \qquad (4\text{-}122)$$

From (4-122),

$$\frac{\partial \bar{\bar{\eta}}_j}{\partial \eta_i} = P_{ji} = P'_{ij} \tag{4-123}$$

From (4-21), (4-121), and (4-123),

$$\frac{\partial \psi_o}{\partial \eta_i} = \sum_{j=1}^{k} \tilde{\alpha}_{ij} [\nabla_{\tilde{\bar{\bar{\eta}}}} \bar{\Phi}_o(\sigma,\bar{\bar{\eta}},t_2)]_j, \; i \le k \tag{4-124}$$

From (D-9), (4-21), (4-45), (4-46), (4-50), and (4-116), and (4-122)

$$\left[\nabla_{\tilde{\bar{\bar{\eta}}}} \bar{\Phi}_o(\sigma,\bar{\eta},t_2)\right]_j = -(k-2)r^{k-2} (\hat{\eta}-\hat{y}(\sigma)) \left\{ \frac{\sum_{\ell=1}^{k} \tilde{\alpha}'_{j\ell}[\tilde{\eta}-\tilde{y}(\sigma)]_\ell}{||\tilde{\eta}-\tilde{y}(\sigma)||^k_{\tilde{a}^{-1}}} - \int_{||\tilde{\zeta}||_{\tilde{a}^{-1}} \ge 0} d\tilde{\zeta} \; \frac{\sum_{\ell=1}^{k} \tilde{\alpha}'_{j\ell} \tilde{\zeta}_\ell}{||\tilde{\zeta}||^k_{\tilde{a}^{-1}}} h(\sigma,\tilde{\eta},t_2,\tilde{\zeta}) \right\}_j \tag{4-125}$$

where $\tilde{\zeta}$ is a k dimensional vector.

Note that

$$\sum_{j=1}^{k} \tilde{\alpha}_{ij} \sum_{\ell=1}^{k} \tilde{\alpha}'_{j\ell} = \sum_{\ell=1}^{k} \sum_{j=1}^{k} \tilde{\alpha}_{ij} \tilde{\alpha}'_{j\ell} \tag{4-126}$$

$$\sum_{j=1}^{k} \tilde{\alpha}_{ij} \sum_{\ell=1}^{k} \tilde{\alpha}'_{j\ell} = \sum_{\ell=1}^{k} (\tilde{\alpha}\,\tilde{\alpha})_{i\ell} \tag{4-127}$$

then from (4-21), (4-49), (4-123), (4-124), (4-125), and (4-127),

$$\nabla_{\tilde{\eta}} \psi_o = -(k-2)\, r^{k-2} [\hat{\eta}-\hat{y}(\sigma)]\, \tilde{a}^{-1} \left\{ \frac{\tilde{\eta}-\tilde{y}(\sigma)}{||\tilde{\eta}-\tilde{y}(\sigma)||^k_{\tilde{a}^{-1}}} - \int_{||\tilde{\zeta}||_{\tilde{a}^{-1}} \ge 0} d\tilde{\zeta} \frac{\tilde{\zeta}}{||\tilde{\zeta}||^k_{\tilde{a}^{-1}}} h(\sigma,\tilde{\eta},t_2,\tilde{\zeta}) \right\} \tag{4-128}$$

Similarly, from (D-10),

$$\nabla_{\hat{\eta}}\psi_o = (k-2)r^{k-3}(\hat{\eta}-\hat{y}(\sigma)) \left\{ \frac{1}{||\tilde{\eta}-\tilde{y}(\sigma)||^{k-2}_{\tilde{a}^{-1}}} \right.$$

$$\left. - \int_{||\tilde{\zeta}||_{\tilde{a}^{-1}} \geq 0} d\tilde{\zeta} \frac{h(\sigma,\tilde{\eta},t_2,\tilde{\zeta})}{||\tilde{\zeta}||^{k-2}_{\tilde{a}^{-1}}} \right\} \nabla_{\hat{\eta}} r(\hat{\eta}-\hat{y}(\sigma))\Big\} \qquad (4\text{-}129)$$

CHAPTER 5

THE TWO-POINT BOUNDARY VALUE PROBLEM

5.1 Introduction

The purpose of this chapter is to use the results obtained so far to reduce the problem stated by equations (3-1) through (3-19) to a two-point boundary value problem, and to present an algorithm for the solution of this problem.

5.2 Two-Point Boundary Value Problem Formulation

It was shown in Chapter 3 that the problem stated by (3-1) through (3-19) is equivalent to the following problem: Find the control u which minimizes

$$I = \int_0^T dt\,||u(t)||_U^2 - \psi(0,\bar{X},T) \tag{5-1}$$

subject to

$$\dot{y} = Ay - Bu, \quad y(0) = 0 \tag{5-2}$$

where

$$\psi(t_1,\xi,t_2) = \mathrm{Prob}\{x(t) \in S \text{ for some } t \in [t_1,t_2] \,|\, z(t_1)=\xi\} \tag{5-3}$$

$$dz = Az\,dt + C\,dn \tag{5-4}$$

The restrictions and definitions are as stated in Chapter 3.

In Chapter 4, an estimate for ψ was developed. This estimate, $\bar{\psi}$, is given by (4-112) through (4-120), (4-128), and (4-129). The procedure in this chapter is to accept this estimate as the probability portion of the performance index. Moreover, according to (4-115), $\psi_o(0,\overline{X},t_2)$ does not depend on u or y(t), t > 0. Then this term may be dropped without affecting the choice of u. Thus the performance index to be minimized is

$$\hat{I} = \int_0^T \|u(t)\|_U^2 \, dt - \psi_1(0,\overline{X},T) \tag{5-5}$$

where

$$\psi_1(0,\overline{X},T) = \int_0^T dt \int_{\|\tilde{\eta}\|_{\tilde{a}^{-1}} \geq 0} d\eta \, p_{z(t)|z(0)}(\eta|\overline{X}) \sum_{i=1}^{m} \{A[\eta-y(t)]+Bu(t)\}_i \frac{\partial \psi_o}{\partial \eta_i} \tag{5-6}$$

and

$$\nabla_{\tilde{\eta}}\psi_o = -(k-2)r^{k-2}[\hat{\eta}-\hat{y}(t)]\tilde{a}^{-1}\Big\{\frac{\tilde{\eta}-\tilde{y}(t)}{\|\tilde{\eta}-\tilde{y}(t)\|^k_{\tilde{a}^{-1}}}$$

$$- \int_{\|\zeta\|_{\tilde{a}^{-1}} \geq 0} d\zeta \, \frac{\zeta}{\|\zeta\|^k_{\tilde{a}^{-1}}} \, h(t,\tilde{\eta},T,\zeta)\Big\} \tag{5-7}$$

$$\nabla_{\hat{\eta}}\psi_o = (k-2)r^{k-3}[\hat{\eta}-\hat{y}(t)] \Big\{\frac{1}{\|\tilde{\eta}-\tilde{y}(t)\|^{k-2}_{\tilde{a}^{-1}}}$$

$$- \int_{\|\zeta\|_{\tilde{a}^{-1}} \geq 0} d\zeta \, \frac{h(t,\tilde{\eta},T,\zeta)}{\|\zeta\|^{k-2}_{\tilde{a}^{-1}}}\Big\} \, \nabla_{\hat{\eta}} r[\hat{\eta}-\hat{y}(t)]\Big\} \tag{5-8}$$

The problem is now ready for application of the Pontryagin Maximum Principle. The Hamiltonian, H, is

$$H = \Phi'(Ay-Bu)-\|u\|_U^2 + \int_{\|\tilde{\eta}\| \geq 0} d\eta \, p_{z(t)|z(0)}(\eta|\overline{X}) \sum_{i=1}^{m} \{A[\eta-y]+Bu\}_i \frac{\partial \psi_o}{\partial \eta_i} \tag{5-9}$$

where $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix}$ is the adjoint variable. Let

$$H^* = \underset{u}{\text{Max}}\ H = H(u^*) \tag{5-10}$$

where u^* is the optimal control. Then

$$\dot{\xi} = - \nabla y\ H\Big|_{u^*} \tag{5-11}$$

From the transversality condition,

$$\xi(T) = 0 \tag{5-12}$$

Finally, from (5-8)

$$\dot{y} = Ay - Bu^*,\ y(0) = 0 \tag{5-13}$$

Equations (5-9) through (5-13) define the two point boundary value problem.

The algorithm for the solution of the two point boundary value problem will require the computation of $\nabla_u H$ and $\nabla_y H$. The expression for $\nabla_u H$ is

$$\nabla_u H = -2Uu + \int_{||\eta||_{\tilde{a}^{-1}} \geq 0} d\eta\ p_{z(t)|z(0)}(\eta|\overline{X}) B'\ \nabla_\eta \psi_o - B'\xi \tag{5-14}$$

The expression for $\nabla_y H$ is

$$\nabla_y H = A'\xi + \int_{||\eta||_{\tilde{a}^{-1}} \geq 0} d\eta\ p_{z(t)|z(0)}(\eta|\overline{X}) \left\{ -A'\nabla_\eta \psi_o + \nabla_y \nabla_\eta \psi_o [A(\eta - y) + Bu] \right\} \tag{5-15}$$

where, by (5-7) and (5-8),

$$\nabla_y\ \nabla_\eta\ \psi_o = \begin{bmatrix} \nabla_1 & \nabla_2 \\ \nabla_3 & \nabla_4 \end{bmatrix} \tag{5-16}$$

$$\nabla_1 = -(k-2)r^{k-2}[\hat{\eta}-\hat{y}(t)]\left\{\frac{k\,\tilde{a}^{-1}[\tilde{\eta}-\tilde{y}][\tilde{\eta}-\tilde{y}]'\tilde{a}^{-1}}{||\tilde{\eta}-\tilde{y}||^{k+2}_{\tilde{a}^{-1}}}\right.$$

$$\left. - \frac{\tilde{a}^{-1}}{||\tilde{\eta}-\tilde{y}||^{k}_{\tilde{a}^{-1}}}\right\} + \int_{||\tilde{\zeta}||_{\tilde{a}^{-1}}\geq 0} d\tilde{\zeta}\; h(t,\tilde{\eta},T,\tilde{\zeta})\left[\frac{\tilde{a}^{-1}}{||\tilde{\zeta}||^{k}_{\tilde{a}^{-1}}} - k\,\frac{\tilde{a}^{-1}\tilde{\zeta}\tilde{\zeta}'\tilde{a}^{-1}}{||\tilde{\zeta}||^{k+2}_{\tilde{a}^{-1}}}\right] \quad (5\text{-}17)$$

$$\nabla_2 = (k-2)^2 r^{k-3}[\hat{\eta}-\hat{y}(t)]\left\{\frac{\tilde{a}^{-1}[\tilde{\eta}-\tilde{y}][\nabla_{\hat{\eta}} r(\hat{\eta}-\hat{y})]'}{||\tilde{\eta}-\tilde{y}||^{k}_{\tilde{a}^{-1}}}\right\} \quad (5\text{-}18)$$

$$\nabla_3 = -(k-2)^2 r^{k-3}(\hat{\eta}-\hat{y})\nabla_y r[\hat{\eta}-\hat{y}]\left\{\frac{(\tilde{\eta}-\tilde{y})'}{||\tilde{\eta}-\tilde{y}||^{k}_{\tilde{a}^{-1}}}\right.$$

$$\left. - \int_{||\tilde{\zeta}||_{\tilde{a}^{-1}}\geq 0} d\tilde{\zeta}\,\frac{\tilde{\zeta}'}{||\tilde{\zeta}||^{k}_{\tilde{a}^{-1}}}\,h(t,\tilde{\eta},T,\tilde{\zeta})\right\}\tilde{a}^{-1} \quad (5\text{-}19)$$

$$\nabla_4 = (k-2)r^{k-3}(\hat{\eta}-\hat{y})\left\{\frac{1}{||\tilde{\eta}-\tilde{y}||^{k-2}_{\tilde{a}^{-1}}}\right.$$

$$\left. - \int_{||\tilde{\zeta}||_{\tilde{a}^{-1}}\geq 0} d\tilde{\zeta}\,\frac{h(t,\tilde{\eta},T,\tilde{\zeta})}{||\tilde{\zeta}||^{k-2}_{\tilde{a}^{-1}}}\right\}$$

$$\left\{(k-3)r^{-1}(\tilde{\eta}-\tilde{y})\nabla_{\tilde{y}}\, r(\tilde{\eta}-\tilde{y})[\nabla_{\hat{\eta}} r(\hat{\eta}-\hat{y})]'\right.$$

$$\left. + \nabla_{\hat{y}}\,\nabla_{\hat{\eta}} r(\hat{\eta}-\hat{y})\right\} \quad (5\text{-}20)$$

5.3 Algorithm for the Two-Point Boundary Value Problem

5.3.1 The Conjugate Gradient Technique

The basic algorithm to be used for the solution of the two point boundary value problem is the conjugate gradient procedure described by

Lasdon, et al. [13]. This algorithm starts with an initial guessed control, $u^0(t)$, and generates successive improved controls, $u^1(t)$, $u^2(t)$, $\ldots, u^i(t), \ldots$. The conjugate gradient technique differs from the usual "steepest descent" technique in two respects:

(1) The improvement on u^i is not in the "direction" $\nabla_{u^i} H$. Instead, it is in a direction determined by $\nabla_{u^i} H$ and $\nabla_{u^j} H$, $j < i$.

(2) The step size is chosen to be optimum in a certain sense at each improvement.

The conjugate gradient algorithm will now be discussed in more detail.

Figure 5-1 shows a block diagram of the algorithm. The discussion begins with block A of this diagram. This block utilizes the equation

$$\dot{y} = Ay - Bu^i \ , \ y(0) = 0 \tag{5-21}$$

to compute the right end point value of y. This is carried out by Runge-Kutta integration. Since the right end point value of ϕ is specified by (5-12) to be zero, it is possible to compute $\nabla_{u^i} H(T)$, using (5-14). This is done in block B. The next step (Block C) is to compute y, ϕ, and G^i at the next earlier instant of time by applying the Runge-Kutta technique to (5-21) and

$$\dot{\phi} = - \nabla_y H \Big|_{u^i} \tag{5-22}$$

$$\dot{G}^i = [\nabla_{u^i} H]'[\nabla_{u^i} H] \ , \ G^j(0) = 0 \tag{5-23}$$

This procedure continues until values for $\nabla_{u^i} H$ and G^i have been calculated for the range $0 \leq t \leq T$.

The next step in the procedure (Block D) is the computation of the "direction" in which u^i is to be changed. This "direction" is defined by the function

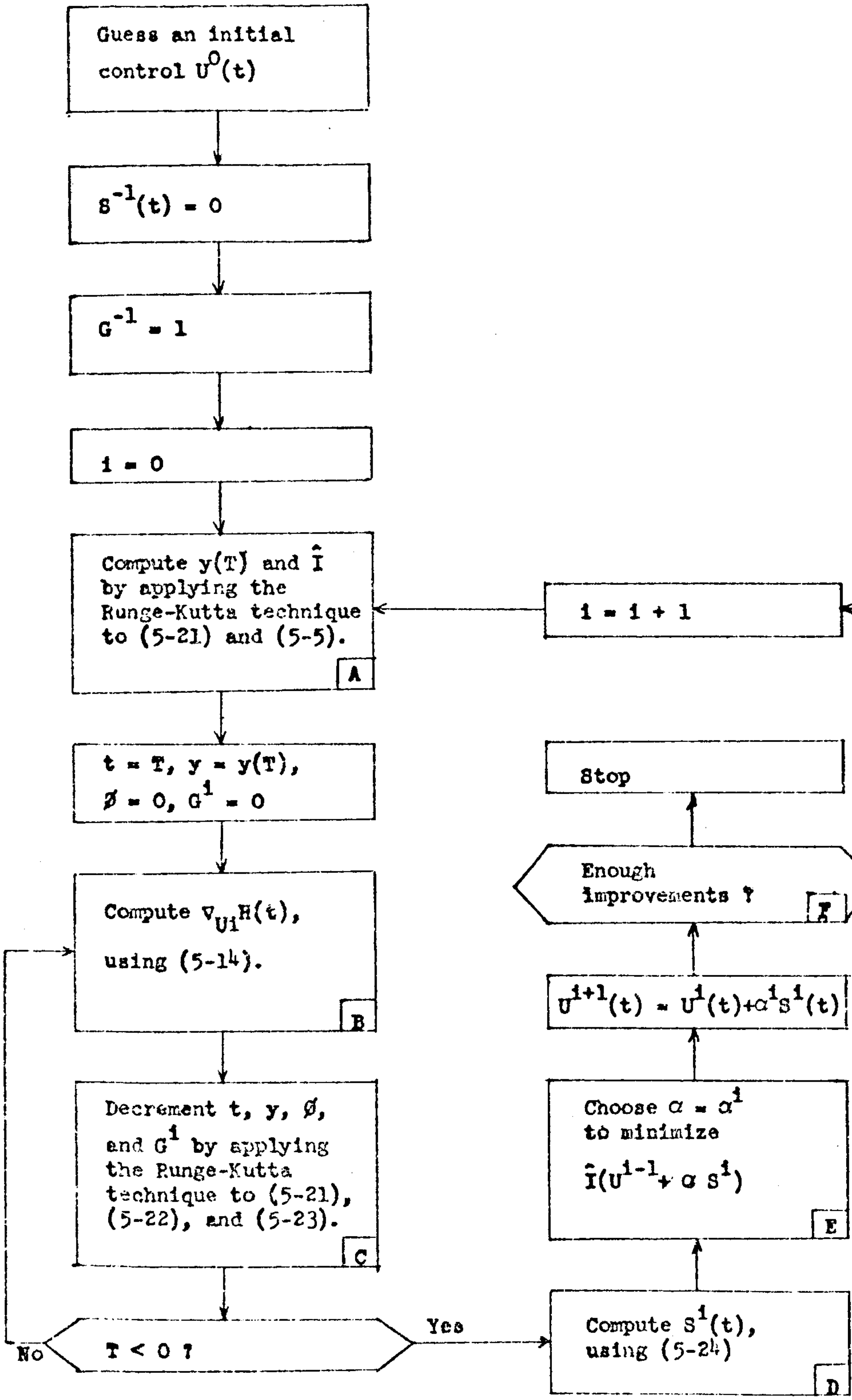


FIGURE 5-1. THE CONJUGATE GRADIENT ALGORITHM

$$s^i(t) = \nabla_{u^i} H + \frac{G^i}{G^{i-1}} s^{i-1}(t) \tag{5-24}$$

The increment in u^i is determined by two quantities: step size, α^i, and "direction," $s^i(t)$. Specifically,

$$u^{i+1}(t) = u^i(t) + \alpha^i s^i(t) \tag{5-25}$$

The step size is chosen to minimize the performance index. That is, α^i is chosen so that

$$\hat{I}(u^i+\alpha^i s^i) = \underset{0 \leq \alpha}{\text{Min}} \ \hat{I}(u^i+\alpha s^i) \tag{5-26}$$

A block diagram of the algorithm for carrying out this minimization is shown in Figure 5-2. The basic idea of this scheme is as follows. The value of $\hat{I}(u^i+\alpha s^i)$ is computed for $\alpha=0$, $\alpha=D$, $\alpha=2D,\ldots,\alpha=\bar{\alpha}$, $\alpha=\bar{\alpha}+D$, where D is a fixed step size and $\bar{\alpha}$ is the first minimum over the sequence $\alpha=jD$, $j=1,2,\ldots.$ Thus $\bar{\alpha}$ may be defined by

$$\bar{\alpha} = \bar{j}\, D \tag{5-27}$$

$$\hat{I}(u^i+\bar{\alpha}s^i) < \hat{I}(u^i+jDs^i),\ 0 \leq j < \bar{j} \tag{5-28}$$

$$\hat{I}(u^i+\bar{\alpha}s^i) \leq \hat{I}[u^i+(\bar{\alpha}+D)s^i] \tag{5-29}$$

(This is illustrated in Figure 5-3.) A parabola is then passed through the point pairs

$$[\alpha = jD,\ \hat{I}(u^i+\alpha s^i)],\ j=\bar{j}-1,\bar{j},\bar{j}+1. \tag{5-30}$$

The minimum point of this parabola is taken to be α^i. The result of this procedure is

$$\alpha^i = -\frac{A_1}{2A_2} \tag{5-31}$$

where

$$A_2 = \frac{\hat{I}[u^i+(\bar{\alpha}+D)s^i]-2\,\hat{I}[u^i+\bar{\alpha}s^i]+\hat{I}[u^i+(\bar{\alpha}-D)s^i]}{2D^2} \tag{5-32}$$

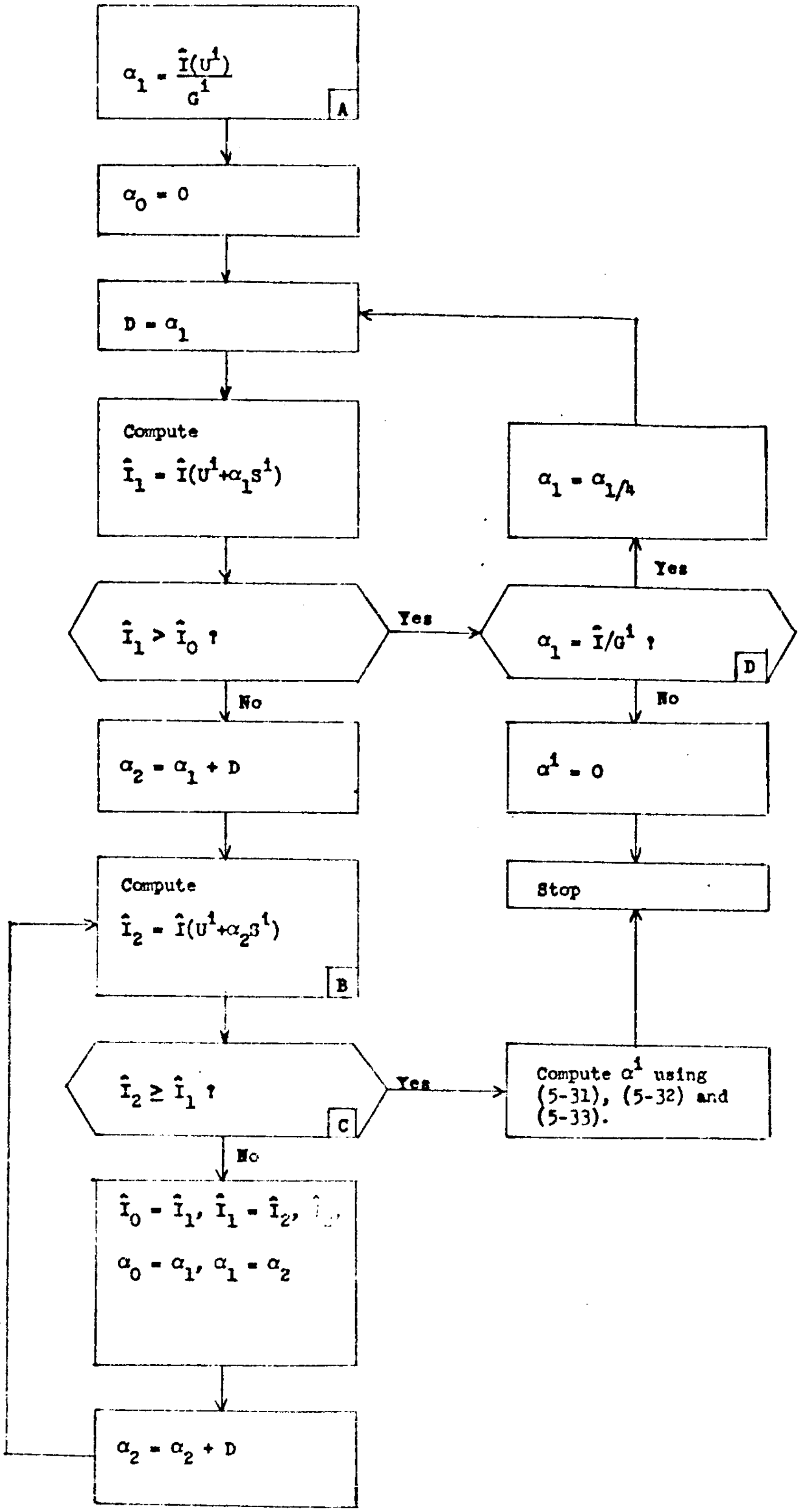


FIGURE 5-2. STEP SIZE DETERMINATION FOR THE CONJUGATE GRADIENT ALGORITHM

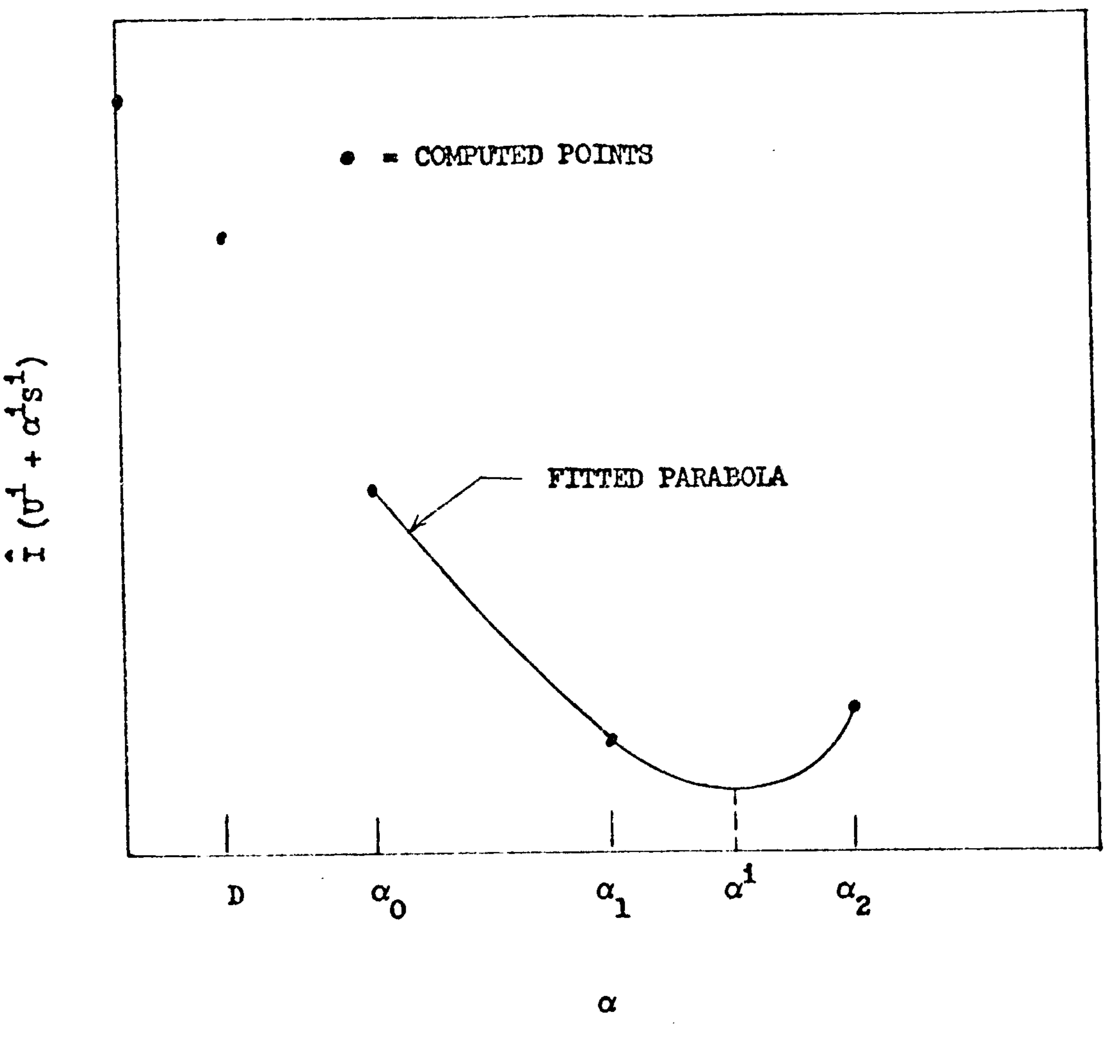


FIGURE 5-3. ILLUSTRATION OF STEP SIZE DETERMINATION

$$A_1 = \frac{1}{D}\left\{\hat{I}[u^i+(\bar{\alpha}+D)s^i]-\hat{I}[u^i+\bar{\alpha}s^i]-A_2[D^2+2\bar{\alpha}D]\right\} \qquad (5\text{-}33)$$

The above discussion covers the main features of the algorithm shown in Figure 5-2. The following two remarks on this algorithm complete the discussion of it:

(1) The initial value given to α_1 (in block A of Figure 5-2) is based on the equation (See [13]):

$$\frac{d}{d\alpha}\,\hat{I}(u^i+\alpha s^i)\Big|_{\alpha=0} = G^i \qquad (5\text{-}34)$$

(The symbols α_o, α_1, and α_2 have the following significance. During each iteration of the loop entered at block B of Figure 5-2 and exited at block C of Figure 5-2, three values of α are retained: α_o, $\alpha_1 = \alpha_o+D$, $\alpha_2=\alpha_1+D$.)

(2) The portion of the algorithm entered at block B of Figure 5-2 serves the following purpose. If the initial value assigned to α_1 is too large, the initial increment in α will not produce a decrease in $\hat{I}$. If this happens, the increments in α are reduced by a factor of four. If the step size $\alpha/4$ is still too large to produce a decrease in $\hat{I}$, the algorithm assumes that the minimization is complete, and assigns the value 0 to α^i.

The discussion returns now to the algorithm of Figure 5-1. Block F represents a stopping condition. It may be one of several types of conditions, e.g., a fixed number of iterations or a threshold on the decrease in $\hat{I}$.

5.3.2 Multidimensional Integral Computation

An examination of equations (5-14), (5-15), and (5-20) reveals that the computations involved in the algorithms shown in Figures 5-1 and 5-2 involve the evaluation of several multidimensional integrals whose integrands contain the factor $p_{z(t)|z(0)}(\eta|\bar{X})$ or the factor $h(t,\tilde{\eta},T,\tilde{\zeta})$. The evaluation of these integrals is based on the fact that these factors are Gaussian probability densities. This leads naturally to the Monte Carlo technique described in detail in Appendix E. The main results of this appendix are the following. Let I_F be an integral of the form

$$I_F = \int d\eta\ p(\eta)\ F(\eta) \tag{5-35}$$

where η is a d dimensional vector,

F is a Baire function whose domain is E^d and whose range is E^c,

$p(\eta)$ is a Gaussian probability density with mean μ and covariance Λ.

Then

$$\underset{N \to \infty}{\text{l.i.m.}}\ \frac{1}{N} \sum_{i=1}^{N} F[P_\Lambda\ \zeta^i + \mu] = I_F \tag{5-36}$$

where

$$P_\Lambda = \left[\sqrt{\lambda_\Lambda^1}\ v_\Lambda^1,\ \ldots\ ,\ \sqrt{\lambda_\Lambda^d}\ v_\Lambda^d\right], \tag{5-37}$$

λ_Λ^1 , ... , λ_Λ^d are the eigenvalues of Λ,

v_Λ^1 , ... , v_Λ^d are the corresponding eigenvectors,

ζ^i, i=1, ..., N, are independent Gaussian random vectors with mean zero and covariance matrix the identity matrix.

This result is applied to the computation of the integrals in equations (5-14), (5-15), and (5-20) as follows. A random number generator is used to generate the ζ^i. A finite sum of the form appearing in (5-36) is used to approximate the desired integral.

The application of this technique to integrals containing $p_{z(t)\,|\,z(0)}(\eta\,|\,\bar{X})$ requires the evaluation of $E[z(t)\,|\,z(0)=\bar{X}]$ and the matrix P (corresponding to P_Λ in equation (5-37)) as a function of t. This is accomplished as follows.

A comparison of equations (5-4) and (2-1) shows that the results stated by Theorems 2-1 and 2-3, apply to z(t). Then $E[z(t)\,|\,z(0)=\bar{X}]$ and $\mathrm{cov}[z(t)\,|\,z(0)]$ may be computed according to

$$\dot{\mu}_z = A\mu_z, \;\; \mu_z(0) = \bar{X} \tag{5-38}$$

$$\dot{Q}_z = AQ_z + Q_z A' + CWC', \; Q(0) = 0 \tag{5-39}$$

where

$$\mu_z(t) = E[z(t)\,|\,z(0) = \bar{X}] \tag{5-40}$$

$$Q_z(t) = \mathrm{cov}[z(t)\,|\,z(0) = \bar{X}] \tag{5-41}$$

5.3.3 Summary and Discussion

The overall procedure for the solution of the two-point boundary value problem is summarized. as follows:

(1) Compute and store the matrix $P_Q(t)$ and the vector $\mu(t)$. (P_Q is defined by (5-37) with Q substituted for Λ.) This is accomplished by using equations (5-38) through (5-41) and some standard method for the computation of eigenvectors and eigenvalues.

(2) Apply the algorithm shown in Figure 5.1, using the Monte Carlo technique described above for the computation of the multi-dimensional integrals which appear during the execution of the algorithm.

The reason for computing and storing P_Q and μ before rather than during the execution of the conjugate gradient algorithm is economy of computer time. The computation of P_Q is carried out by the solution of a matrix differential equation with eigenvalue and eigenvector computation at each step.

The selection of u^o remains to be discussed. One obvious possibility is $u^o=0$. The complexity of the algorithm, however, suggests that it might pay to devote some time to the selection of u^o. This is part of the motivation for the work done in Chapter 6, where two supoptimal problems are solved, thus providing better guesses for u^o.

5.4 Convergence

5.4.1 Scope and Purpose

This section deals with the convergence of the conjugate gradient algorithm as applied to the present problem. What follows is a discussion rather than a proof. This discussion is intended to serve two purposes:

1. It provides a basis for understanding the numerical work described in Chapter 7.
2. It indicates the lines along which a convergence proof might be constructed.

The discussion is in two parts. The first part deals with the convergence of the algorithm if the multidimensional integrals appearing in the expressions for $\hat{I}$, $\nabla_u H$, and $\nabla_y H$ could be computed precisely. The second part deals with the effects of the random errors arising from the Monte Carlo computation of these integrals.

5.4.2 Convergence with Exact Computation of Integrals

Let I^1, I^2, ..., I^i, ... be the sequence of performance indices corresponding to the sequence of controls u^1, u^2, ..., u^i, ... generated by successive iterations of the algorithm. Since the energy term of I^i is positive semidefinite, and the probability term is less than unity, the sequence of performance indices is bounded below. Since at each step the algorithm selects step size (α) to minimize each successive I^i, the sequence of performance indices is monotonic decreasing. Then the sequence of performance indices is monotonic decreasing and bounded below. Therefore there exists a number $\underline{I}$ such that

$$\lim_{i \to \infty} I^i = \underline{I} \tag{5-41a}$$

There is no guarantee that $\underline{I}$ is a global minimum. This provides further motivation to generate more than one initial control.

The convergence of the sequence of controls, u^i, is discussed by Lasden [13].

5.4.3 Convergence with Monte Carlo Computation of Integrals

A second aspect of convergence is connected with the Monte Carlo method of computing the multidimensional integrals which appear in the expressions for $\hat{I}$, $\nabla_u H$, and $\nabla_y H$. This technique (see section 5.3.2 and Appendix E) contributes a random error to the computation at each iteration of the algorithm. The effect of this random error on convergence will now be discussed.

The discussion begins with the rewritting of equations (5-22), (5-23), and (5-24) to include the error term arising from the Monte Carlo computation:

$$\hat{\Phi} = -\nabla_y H\Big|_{\bar{u}^i} + \epsilon^i_{\dot{\Phi}} \tag{5-42}$$

$$\hat{G}^i = [\nabla_{\bar{u}^i} H]'[\nabla_{\bar{u}^i} H] + \epsilon^i_{\dot{G}} \tag{5-43}$$

$$\bar{S}^i(t) = \nabla_{\bar{u}^i} H + \frac{\bar{G}^i}{\bar{G}^{i-1}} \bar{S}^{i-1}(t) + \epsilon^i_S \tag{5-44}$$

where

$\epsilon^i_{\dot{\Phi}}$ is the random error arising from the Monte Carlo computation of $\nabla_y H$

$\epsilon^i_{\dot{G}}$ is the random error arising from the Monte Carlo computation of $[\nabla_u H]'[\nabla_u H]$

ϵ^i_S is the random error arising from the Monte Carlo computation of $\nabla_u H$ and G^i.

and the super bar denotes computed values, e.g. $\bar{S}^i$ is the computed value of S^i.

It was established in Appendix E that the expectations of the ϵ^i are zero, and that their variances are proportional to $1/N^i$, where N^i is the number of terms in the Monte Carlo sum used to approximate the integral on the i^{th} iteration.

Each iteration of the conjugate gradient algorithm may be viewed as a functional transformation from u^i to u^{i+1},

$$u^{i+1} = T^i[u^i] \tag{5-45}$$

where T^i is defined by the block diagram shown in Figure 5-1. Similarly, let $\bar{T}^i$ be the same transformation with equation (5-42), (5-43), and (5-44) substituted for (5-22), (5-23), and (5-24), respectively. Thus the sequence of $\bar{u}^i$ defined by

$$\bar{u}^{i+1} = \bar{T}^i\,[\bar{u}^i] \tag{5-46}$$

is the sequence of controls generated by the conjugate gradient algorithm with Monte Carlo computation of certain of the integrals, while the sequence u^i defined by equation (5-45) is the sequence of controls generated by the conjugate gradient algorithm with precise computation of these integrals. Consider now the difference between $\bar{u}^i$ and u^i.

Suppose that the transformation $\bar{T}^i$ has two properties:

$$\bar{T}^i[u^i] = u^{i+1} + \epsilon^i(N^i) \tag{5-47}$$

where

$$\underset{N^i \to \infty}{\text{l.i.m.}}\ \epsilon^i(N^i) = 0 \tag{5-48}$$

$$\bar{T}^i[u^i + \delta] = u^{i+1} + \epsilon^i(N^i) + \bar{\epsilon}^i(\delta) \tag{5-49}$$

where

$$\underset{\delta \to 0}{\text{l.i.m.}}\ \bar{\epsilon}^i(\delta) = 0 \tag{5-50}$$

Equations (5-47) and (5-48) assert that the Monte Carlo error term on u^{i+1} may be made arbitrarily small by making the $\epsilon^i_{\dot{\Phi}}$, $\epsilon^i_{\dot{G}}$, and ϵ^i_S terms of equations (5-42), (5-43), and (5-44) sufficiently small. Equations (5-49) and (5-50) assert a continuity property for $\bar{T}^i$.

Consider now the difference between the sequence of u^i and the sequence of $\bar{u}^i$.

$$\bar{u}^1 = \bar{T}^o\ [u^o] \tag{5-51}$$

From (5-47), (5-48), and (5-51),

$$\bar{u}^1 = u^o + \epsilon^o(N^o) = 0 \tag{5-52}$$

$$\underset{N^o \to \infty}{\text{l.i.m.}}\ \epsilon^o(N^o) = 0 \tag{5-53}$$

Next,
$$\bar{u}^2 = \bar{\bar{T}}^1\ [\bar{\bar{u}}^1] \tag{5-54}$$

From (5-47) through (5-50) and (5-52) through (5-54),

$$\bar{u}^2 = u^2 + \epsilon^1(N^1) + \bar{\epsilon}^1 [\epsilon^o(N^o)] \tag{5-55}$$

$$\underset{N^1 \to \infty}{\text{l.i.m.}}\ \epsilon^1 = 0 \tag{5-56}$$

$$\underset{N^o \to 0}{\text{l.i.m.}}\ \bar{\epsilon}^1 = 0 \tag{5-57}$$

Let

$$\bar{\bar{\epsilon}}^1 = \epsilon^1(N^1) + \bar{\epsilon}^1 [\epsilon^o(N^o)] \tag{5-58}$$

From (5-56), (5-57), and (5-58),

$$\underset{\substack{N^1 \to \infty \\ N^o \to \infty}}{\text{l.i.m.}}\bar{\bar{\epsilon}}^1 = 0 \tag{5-59}$$

The pattern, them, is

$$\bar{u}^{i+1} = u^{i+1} + \bar{\bar{\epsilon}}^i \tag{5-60}$$

$$\bar{\bar{\epsilon}}^i = \epsilon^i(N^i) + \bar{\epsilon}^i \{\epsilon^{i-1}[\epsilon^{i-2}(\ldots\epsilon^o(N^o))]\} \tag{5-61}$$

$$\underset{\substack{N^i \to \infty \\ N^{i-1} \to \infty \\ N^{i-2} \to \infty \\ \vdots \\ N^o \to \infty}}{\text{l.i.m.}}\ \bar{\bar{\epsilon}}^i = 0 \tag{5-62}$$

Thus it is possible to keep $\bar{u}^i$ arbitrarily close to u^i by choosing N^i, $N^{i-1}, \ldots, N^o$ sufficiently large.

One other aspect of convergence connected with the Monte Carlo computation of the integrals is the termination of the algorithm. Figure 5-2 indicates that if no improvement of the performance index is obtained for step size $\hat{I}(u^i)/G^i$ or for $\frac{1}{4}$ of this step size, the algorithm terminates. Thus if the Monte Carlo error involved in the computation of $\hat{I}$ is large enough, the algorithm may terminate even though a better control has been found. The probability of such an event can be made arbitrarily low by choosing N^i suitably large.

CHAPTER 6

TWO SUBOPTIMAL PROBLEMS

6.1 Introduction

Chapters 4 and 5 developed a technique for the solution of the basic problem posed in Chapter 3. This technique exhibits several unattractive features. First of all it is at best an approximation. Secondly, the computation is extremely involved. The purpose of this chapter is to present an alternative method of attack on the basic problem. This attack involves choosing an appealing single parameter class of controls, and then optimizing over this parameter. The probability portion of the performance index is determined through the use of a digital simulation of the system described by equation (6-6).

The procedure in this chapter will be to define a class of controls and carry out the above technique for a sample problem. The results of this work lead to the definition of a second class of controls. Results for this class are also given.

6.2 Suboptimal Problem P1

The first class of controls to be considered is defined as follows. Let u_{T_C} be the control vector which minimizes

$$I_{D_1} = \int_0^{T_C} ||u(t)||_U^2 \, dt, \tag{6-1}$$

where U is positive definite, subject to

$$\left.\begin{array}{l}\dot{x} = Ax + Bu,\ x(0) = \bar{X} \\ x_i(T_C) = 0,\ i=1,\ldots,k, k \leq n\end{array}\right\} \qquad (6\text{-}2)$$

where x is an n dimensional vector.

The class $\mathcal{F}_1$ is defined as

$$\mathcal{F}_1 = \left\{ u \middle| \begin{array}{ll} u(t)=u_{T_C}(t), & T_C \geq t \geq 0 \\ u(t)=0 & , t > T_C \end{array} \right\} \qquad (6\text{-}3)$$

The first suboptimal problem to be considered, then, is

P1: Find $u^* \in \mathcal{F}_1$ such that

$$I_s[u^*] \geq I_s[u] \text{ for every } u \in \mathcal{F}_1 \qquad (6\text{-}4)$$

where

$$I_s = \text{Prob}\left\{ \underset{0 \leq t \leq T}{\text{Min}} \ ||x(t)||_{\tilde{a}^{-1}} \leq r[\hat{x}(t)] \right\} - \int_0^T ||u(t)||_U^2 \, dt \qquad (6\text{-}5)$$

subject to

$$dx = [Ax + Bu]\, dt + C\, dn,\ x(0) = \bar{x} \qquad (6\text{-}6)$$

where (6-6) is a stochastic differential equation as described following (2-1), with

$$E\{[n(t_2)-n(t_1)][n(t_2)-n(t_1)]'\} = w|t_2-t_1| \qquad (6\text{-}7)$$

The matrix $\tilde{a}$ is defined as follows:

$$CWC' = \begin{bmatrix} \tilde{a} & 0 \\ 0 & 0 \end{bmatrix} \qquad (6\text{-}8)$$

$$\text{rank}\ (\tilde{a}) = k = \text{dimension of } \tilde{a}. \qquad (6\text{-}9)$$

The following discussion is intended to expose the intuitive appeal of the class $\mathcal{F}_1$. The index I_s involves an energy term and a term dependent upon making the first k components of x small. The class $\mathcal{X}_1$ is defined as the class of controls which bring the first k components of x to 0 at time T_C with minimal energy. The parameter of $\mathcal{X}_1$ is T_C. For small T_C, the first k components of x are brought to 0 relatively quickly at a relatively large expenditure of energy. For larger T_C, the first k components of x are brought to 0 relatively slowly, but at a relatively small expenditure of energy. Thus the parameter T_C expresses the tradeoff between small $||\tilde{x}||_{\tilde{a}^{-1}}$ and small energy, to be optimized in terms of I_s.

Consider now the computation of u_{T_C} given T_C. This is carried out by a straightforward application of Pontryagin's Maximum Principle. The Hamiltonian H is

$$H = \phi'(Ax + Bu) - ||u(t)||_U^2 \qquad (6\text{-}10)$$

where ϕ is the adjoint variable corresponding to (6-6). Then

$$u_{T_C}(t) = \frac{1}{2} U^{-1}B'\phi \qquad (6\text{-}11)$$

$$\dot{\phi} = -A'\phi \qquad (6\text{-}12)$$

From the transversality condition,

$$\phi_i(T_C) = 0, \; i=k+1,\dots,m \qquad (6\text{-}13)$$

Then

$$\dot{y} = My, \; y(0) = \begin{bmatrix} \overline{x} \\ \phi(0) \end{bmatrix}$$

$$y(T_C) = \begin{bmatrix} 0 \\ \phi(T_C) \end{bmatrix} \qquad (6\text{-}14)$$

where

$$y = \begin{bmatrix} x \\ \phi \end{bmatrix} \qquad (6\text{-}15)$$

$$M = \begin{bmatrix} A & \frac{1}{2} BU^{-1}B' \\ 0 & -A' \end{bmatrix} \qquad (6\text{-}16)$$

Let

$$\dot{\Phi}_M(t_2,t_1) = M\,\Phi_M(t_2,t_1),\quad \Phi_M(t_1,t_1) = I_{2m} \qquad (6\text{-}17)$$

where I_{2m} = identity matrix of rank 2m and let

$$\Phi_M(T_C,0) = \begin{bmatrix} \tilde{\Phi}_1 & \tilde{\Phi}_2 \\ \bar{\Phi}_1 & \bar{\Phi}_2 \\ \hat{\Phi}_1 & \hat{\Phi}_2 \end{bmatrix} \qquad (6\text{-}18)$$

where $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ are kxm matrices

$\bar{\Phi}_1$ and $\bar{\Phi}_2$ are mxm matrices

$\hat{\Phi}_1$ and $\hat{\Phi}_2$ are (m-k)xm matrices

Then

$$\Phi_M(T_C,0)\begin{bmatrix} \bar{X} \\ \phi(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ X_{k+1}(T) \\ \vdots \\ X_m(T) \\ \phi_1(T) \\ \vdots \\ \phi_k(T) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (6\text{-}19)$$

From (6-18) and (6-19),

$$\phi(0) = -\begin{bmatrix} \tilde{\Phi}_2 \\ \hat{\Phi}_2 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\Phi}_1 \\ \hat{\Phi}_2 \end{bmatrix} \bar{X} \qquad (6\text{-}20)$$

Using (6-20), (6-12), and (6-11), then, it is possible to compute u_{T_C}.

Consider next the algorithm for the computation of $\text{Prob}\left\{ \min_{0 \le t \le T} ||x(t)||_{\tilde{a}^{-1}} \le r[\hat{x}(t)] \right\}$ for a given $u_{T_C} \in \mathcal{S}_1$. The algorithm makes use of the following approximation to (6-6):

$$\dot{x} = Ax + Bu(t) + C\dot{n}^{\ell}(t), \quad x(0) = \bar{x} \tag{6-21}$$

where

$$\dot{n}^{\ell}(t) = \frac{1}{2}\sqrt{\frac{\ell}{T}} \sum_{i=0}^{\ell-1} \{\mathrm{sgn}(t - \frac{iT}{\ell}) - \mathrm{sgn}[t - \frac{(i+1)T}{\ell}]\} N^{i} \tag{6-22}$$

and the N^i are as defined by (2-31) and (2-32). A block diagram of the algorithm is shown in Figure 6-1. A brief description of it is as follows. The time interval [0,T] is divided into ℓ equal subintervals. x(t) is then computed for each of the subinterval endpoints. At each endpoint, $\frac{1}{r(\hat{x})} \|\tilde{x}\|_{\tilde{a}^{-1}}$ is checked against the minimum previous value of $\frac{1}{r(\hat{x})} \|\tilde{x}\|_{\tilde{a}^{-1}}$. If $\frac{1}{r(\hat{x})}\|\tilde{x}\|_{\tilde{a}^{-1}}$ is less than the previous minimum it becomes the new minimum. Thus at t=T, the smallest value of $\frac{1}{r(\hat{x})} \|\tilde{x}\|_{\tilde{a}^{-1}}$ attained during the time interval has been determined (and stored in the XNØRA array). The procedure is then repeated for a new sample function of $\dot{n}^{\ell}$. After repeating the procedure NC times the XNØRL array is sorted so that the largest value appears first, the second largest value appears next, and so forth. Thus a plot of $\frac{N_C - i + 1}{N_C}$ versus XNØRL(i) gives an estimate of the plot of $\mathrm{Prob}\{\min_{0 \le t \le T} \|\tilde{x}(t)\|_{\tilde{a}^{-1}} \le k_r r(\hat{x})\}$ versus k_r. The reasons for choosing a parabolic fit to the points thus obtained will be given in the discussion of the sample problem.

6.3 A Sample Problem

6.3.1 Statement of the Problem

An engineering system typical of the type to which the above techniques apply is shown in functional block diagram form in Figure 6-2, in operational block diagram form in Figure 6-3, and in state variable form in Figure 6-4. That the system chosen is of a representative type is

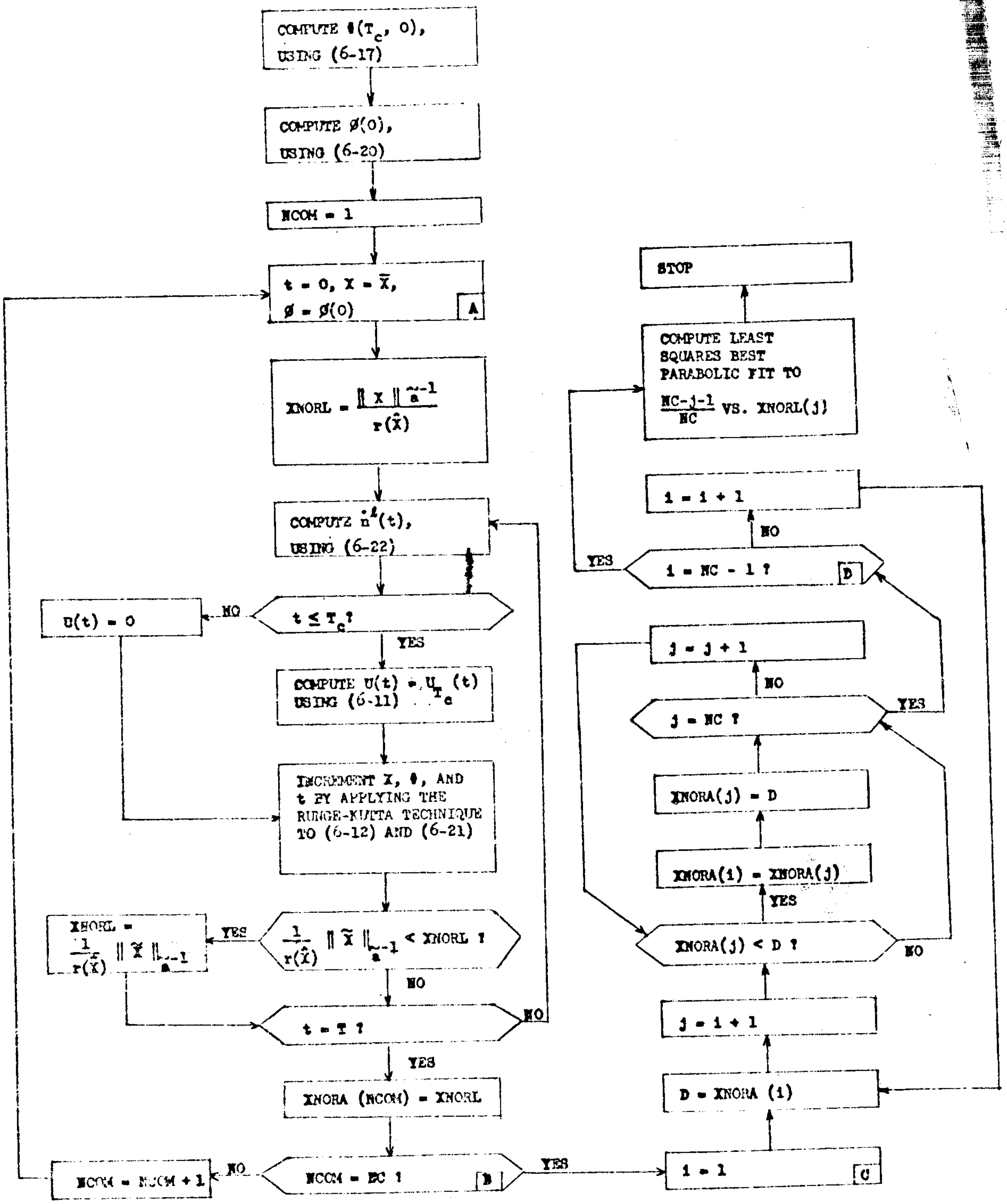


FIGURE 6 - 1. ALGORITHM FOR P1

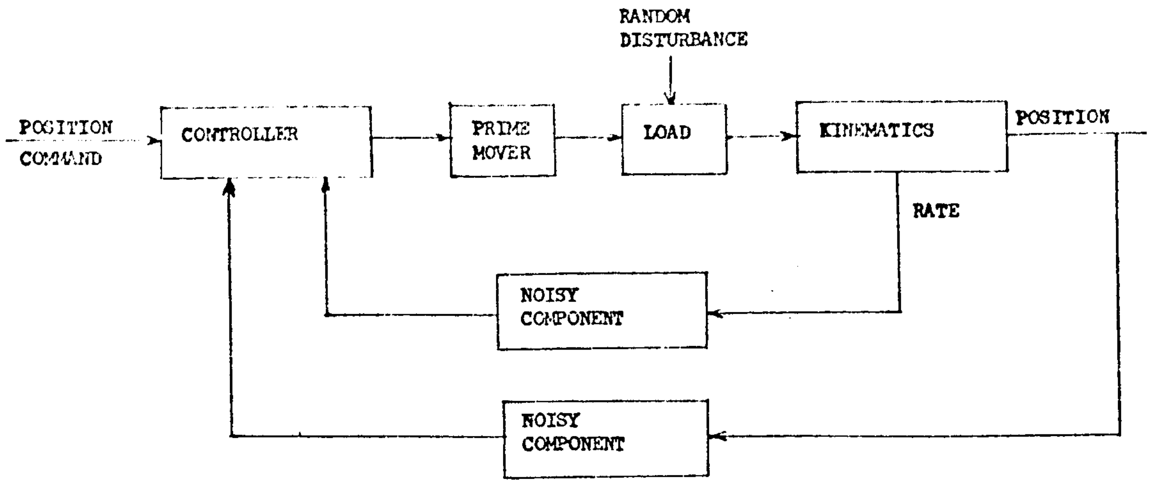


FIGURE 6-2. FUNCTIONAL BLOCK DIAGRAM FOR SAMPLE SYSTEM

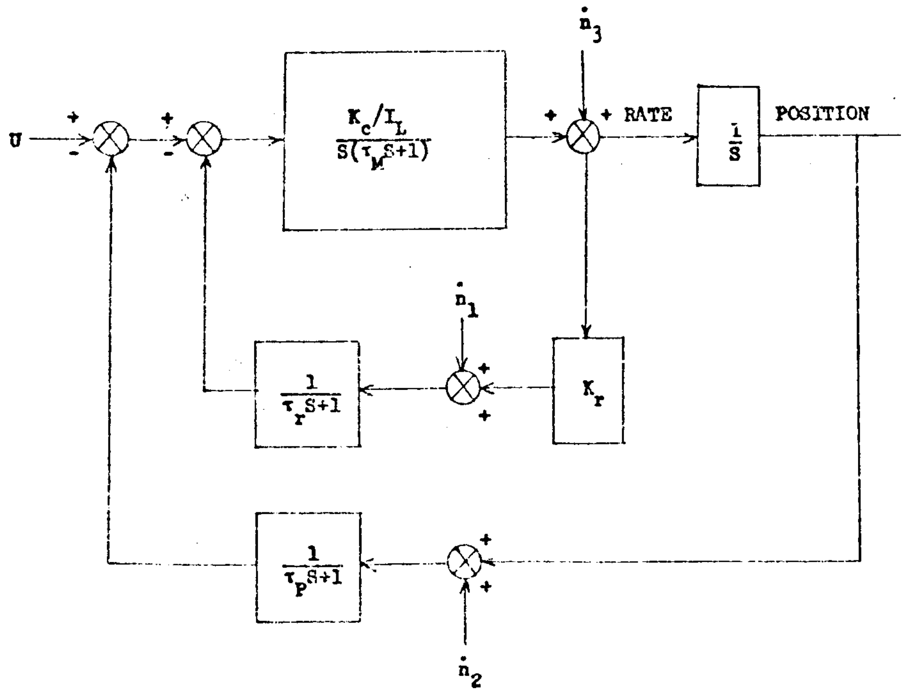


FIGURE 6-3. OPERATIONAL BLOCK DIAGRAM FOR SAMPLE SYSTEM

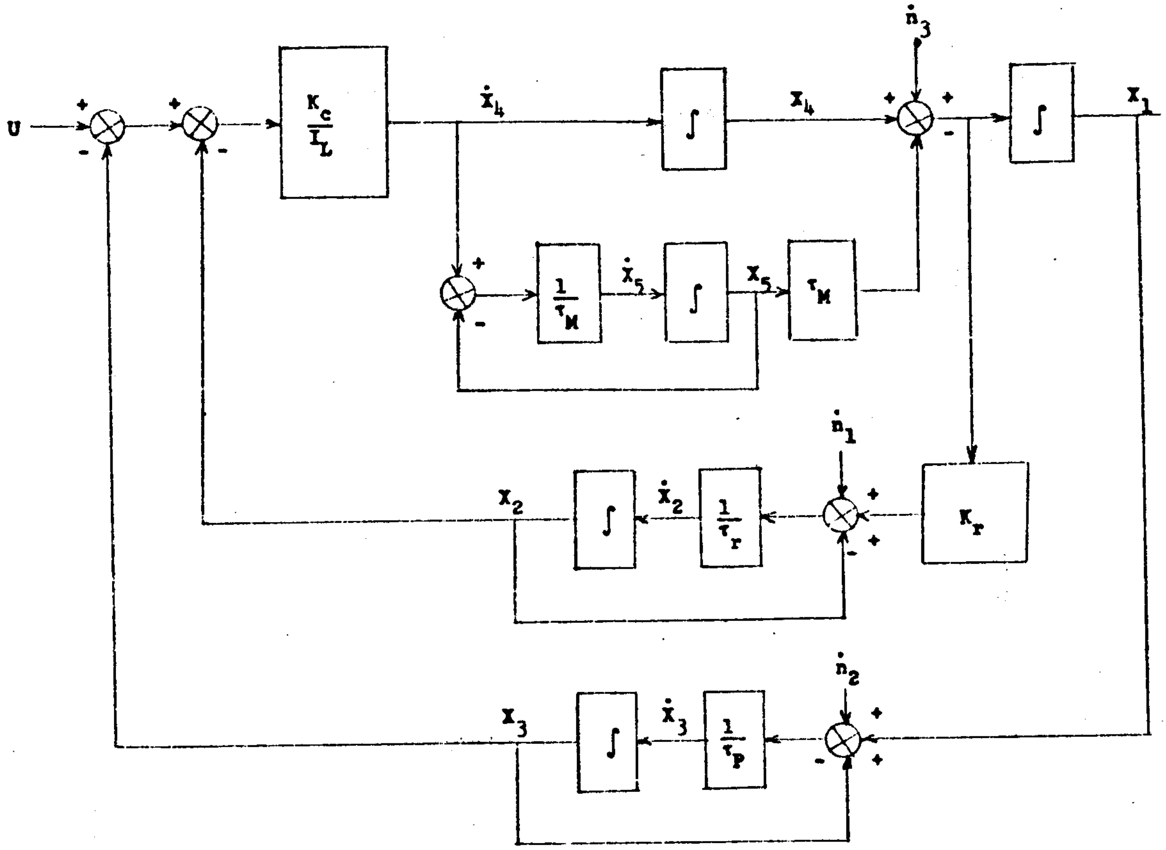


FIGURE 6-4. STATE VARIABLES FOR THE SAMPLE PROBLEM

apparent from Figure 6-2. The notation in Figure 6-3 relates to Figure 6-2 as follows:

K_C represents the product of a controller gain and a prime mover constant.

I_L is the inertia of the load.

τ_m is a time constant associated with the prime mover.

$\dot{n}_3$ is the random disturbance.

K_r is the product of a controller gain and a rate sensor constant.

$\dot{n}_1$ is the component noise.

τ_r is the rate sensor filter time constant.

$\dot{n}_2$ is the component noise.

τ_p is the position sensor filter time constant.

u is the position command.

The state variables defined in Figure 6-4 relate to Figure 6-3 as follows:

x_1 is position.

x_2 is measured rate.

x_3 is measured position.

x_4 and x_5 are employed in the representation of the prime mover and load.

The state equations for this system are of the form of (6-6) with

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & -\tau_m \\ 0 & -1/\tau_r & 0 & K_r/\tau_r & -K_r\tau_m/\tau_r \\ 1/\tau_p & 0 & -1/\tau_p & 0 & 0 \\ 0 & -K_C/I_L & -K_C/I_L & 0 & 0 \\ 0 & -K_C/(I_L\tau_m) & -K_C/(I_L\tau_m) & 0 & -1/\tau_m \end{bmatrix} \quad (6\text{-}23)$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ K_C/I_L \\ K_C/I_L\tau_m \end{bmatrix} \tag{6-24}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1/\tau_r & 0 & K_r/\tau_r \\ 0 & 1/\tau_p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6-25}$$

The system parameter values were chosen to be

$$\tau_r = 1/4 \tag{6-26}$$

$$\tau_p = 1/3 \tag{6-27}$$

$$K_r = 1.2 \tag{6-28}$$

$$K_C/I_L = 1/2 \tag{6-29}$$

$$\tau_m = 1/5 \tag{6-30}$$

$$W = \frac{\mathrm{cov}\{n(t_2)-n(t_1)\}}{|t_2 - t_1|} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6-31}$$

A discussion of the relevance of the form of the system and the choice of parameter values is given in Appdendix F.

The problem to be solved is as follows. The initial state of the system is

$$x(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{6-32}$$

That is, the load is initially at rest in some position, which is known to the controller. It is desired to bring the load to rest at position $x_1=0$. Since the system is subject to random disturbances, provision has been made to lock into place when it gets within a certain distance of $x_1=0$, provided that it is not going too fast. It is also desired to achieve this with the minimum expenditure of energy consistent with an acceptable probability of success in a reasonable time interval. The performance index, I_{s1}, as defined by (6-33), reflects these goals, and is amenable to the optimization techniques developed above:

$$I_{s1} = \text{Prob}\left\{ \min_{0 \le t \le 1} ||\tilde{x}(t)||_{\tilde{a}^{-1}} \le r \right\} - k_u \int_0^1 u^2(\sigma)\, d\sigma \qquad (6\text{-}33)$$

where

$$CWC' = \begin{bmatrix} \tilde{a} & 0 \\ 0 & 0 \end{bmatrix} \qquad (6\text{-}34)$$

$$\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (6\text{-}35)$$

r and k_u are positive scalar constants defining the relative influence of the energy and probability terms of the performance index.

6.3.2 Numerical Results

Numerical results were obtained for the sample problem defined in the preceding section by applying the algorithms described in the section on suboptimal problem P1. Listings of the FORTRAN programs used, together with a discussion of some of the computational details, are contained in Appendix G.

Data on the probability term was obtained by using the algorithm shown in Figure 6-1. The results are shown in Figures 6-5 through 6-10. As was mentioned earlier, the curves shown in Figures 6-5 through 6-9 are least-squares best fit parabolas. The reasons for choosing parabolas rather than higher degree polynomials is that the curves represent probability distributions, and hence are monotonic. The loss of this property by higher degree polynomials is illustrated in Figures 6-11 and 6-12. The curves in these figures are fifth order and eighth order polynomial best fits to the $T_C=2.0$ data (See Figure 6-7.)

The next step is to utilize the data contained in Figures 6-5 through 6-10 to determine the value of T_C which maximizes I_{s1} (see equation (6-33)). This will now be carried out for

$$r = 0.2 \tag{6-36}$$

$$k_u = 10^{-4} \tag{6-37}$$

Figure 6-13 shows a plot of the probability term of I_{s1} versus T_C. The points for this plot were read directly from Figures 6-5 through 6-9. I_{s1} was then computed according to equation (6-33): the values for the probability term were obtained from Figure 6-13 while the values for the energy term were obtained from Figure 6-10. The resulting curve is shown in Figure 6-14. The immediate result is that the optimum control is that corresponding to $T_C=1.5$. The optimal performance turns out to be

$$\text{Prob}\left\{ \min_{0 \leq t \leq 1} ||\tilde{x}(t)||_{\tilde{a}^{-1}} \leq 0.2 \right\} = 0.32 \tag{6-38}$$

$$\int_0^1 [u^*(t)]^2 \, dt = 470 \tag{6-39}$$

A second result is that I_{s1} is not very sensitive to T_C for $T_C > 1.5$. On the other hand, I_{s1} drops off sharply for $T_C < 1.5$. The conclusion to be

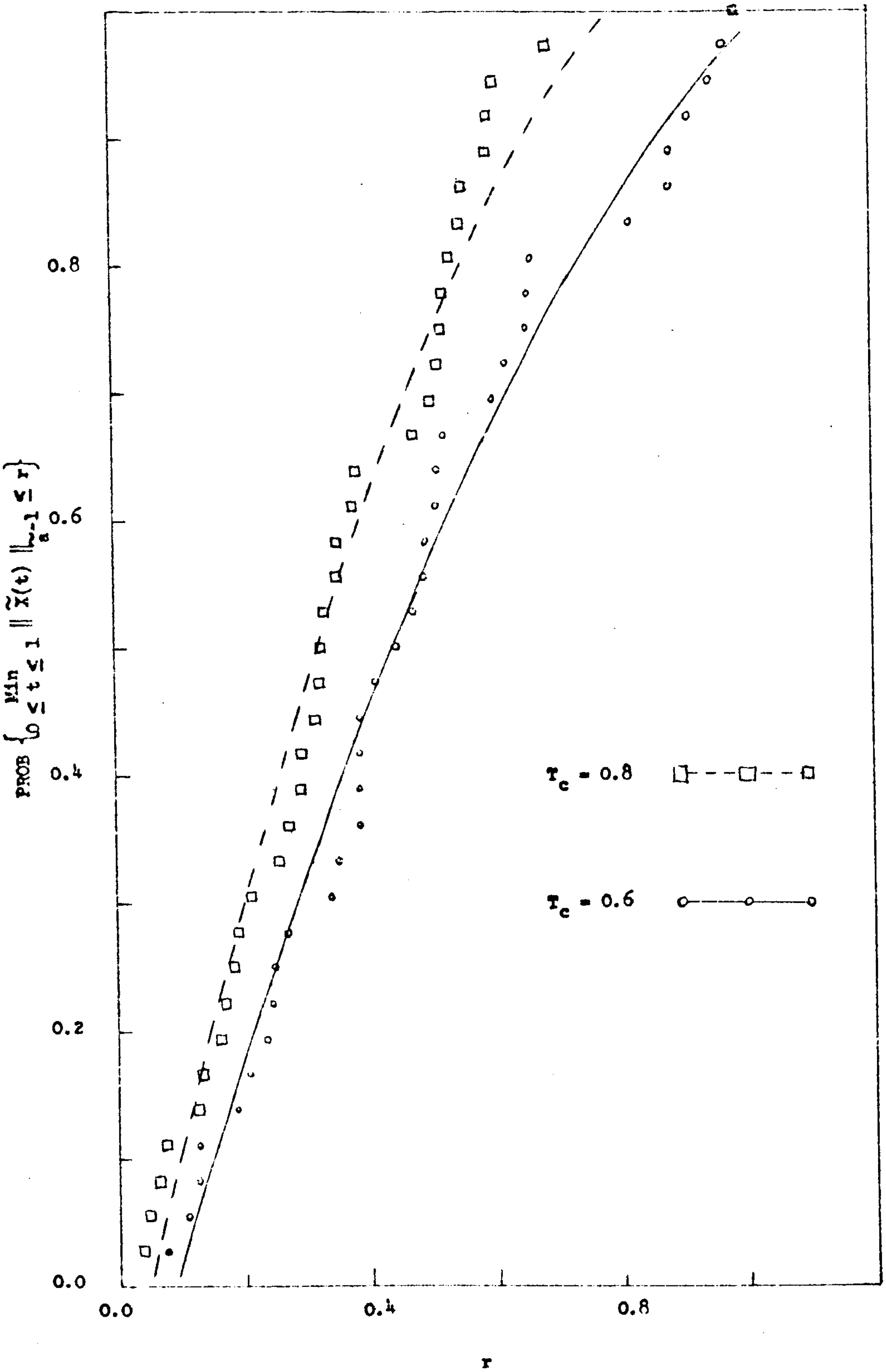


FIGURE 6-5. P1 DISTRIBUTIONS

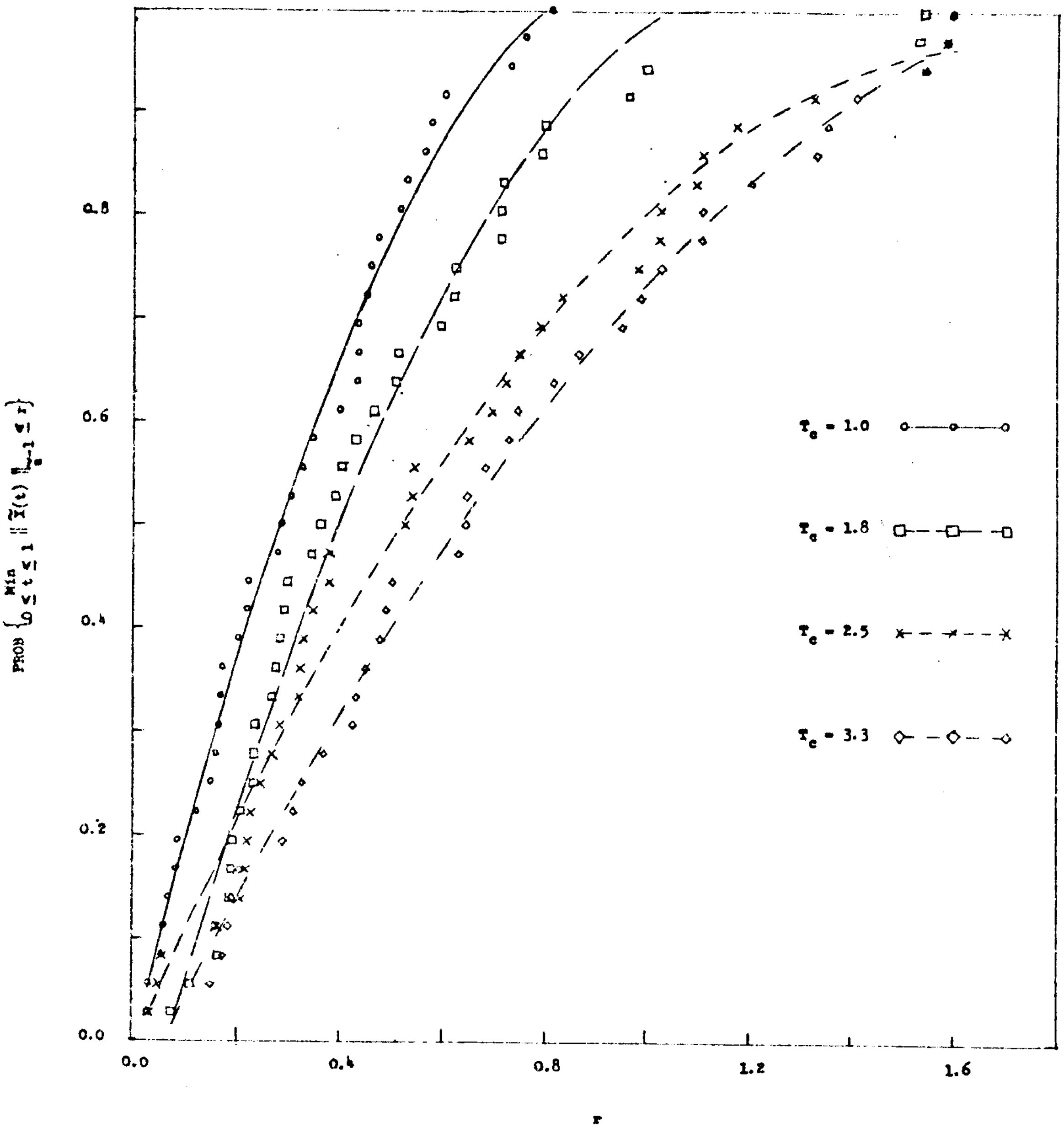


FIGURE 6-6. P1 DISTRIBUTIONS

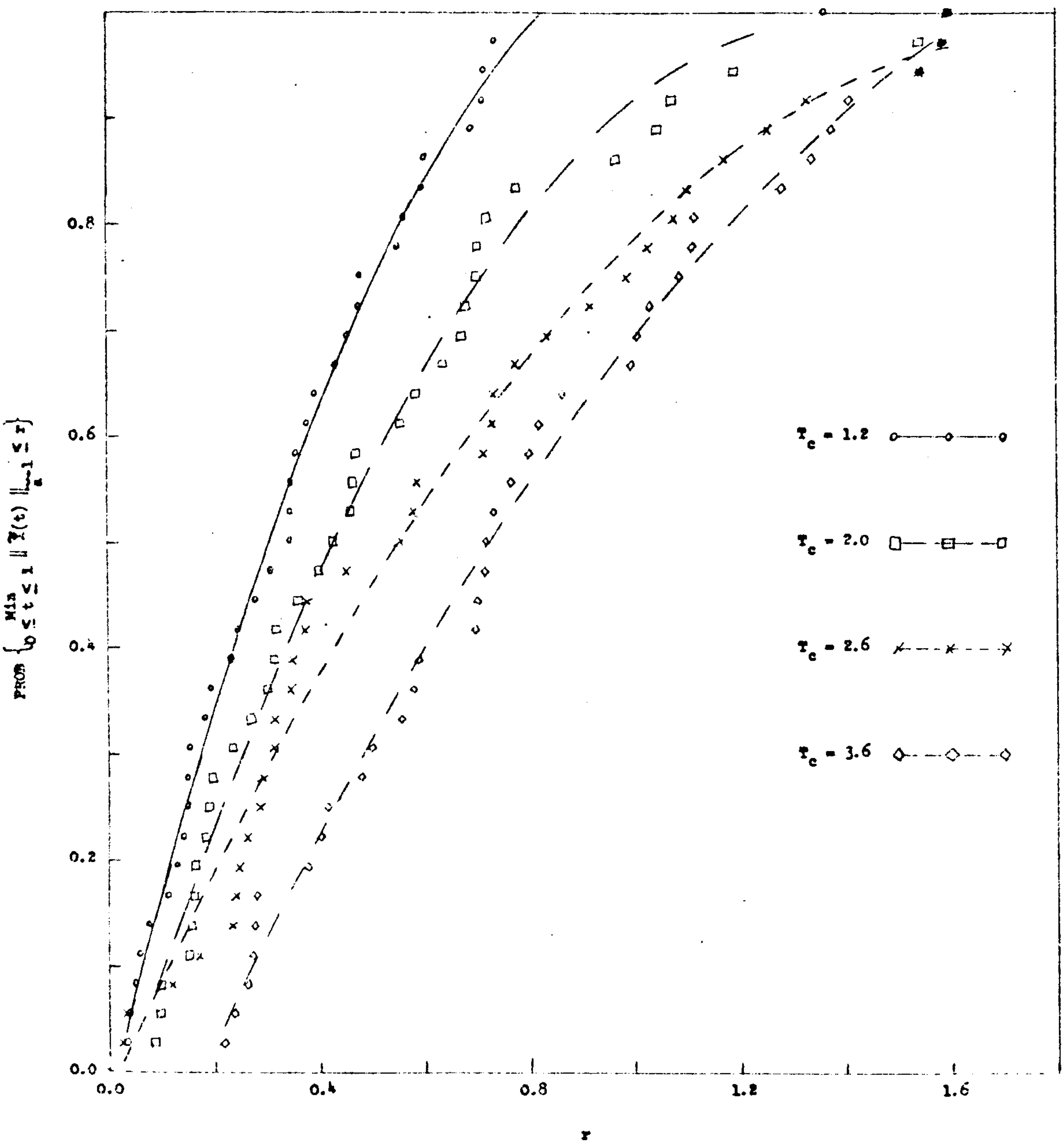


FIGURE 6-7. P1 DISTRIBUTIONS

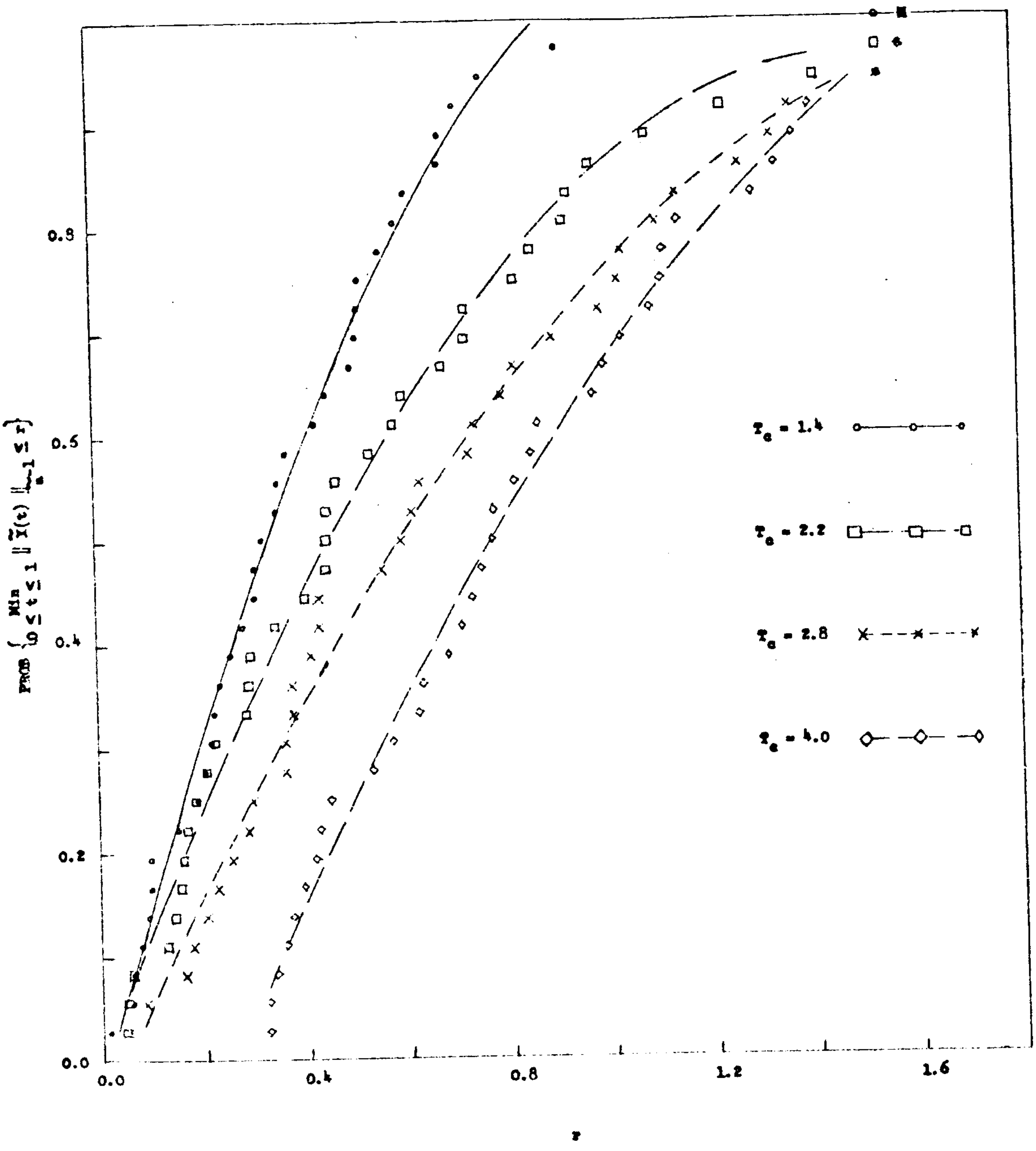


FIGURE 6-8. P1 DISTRIBUTIONS

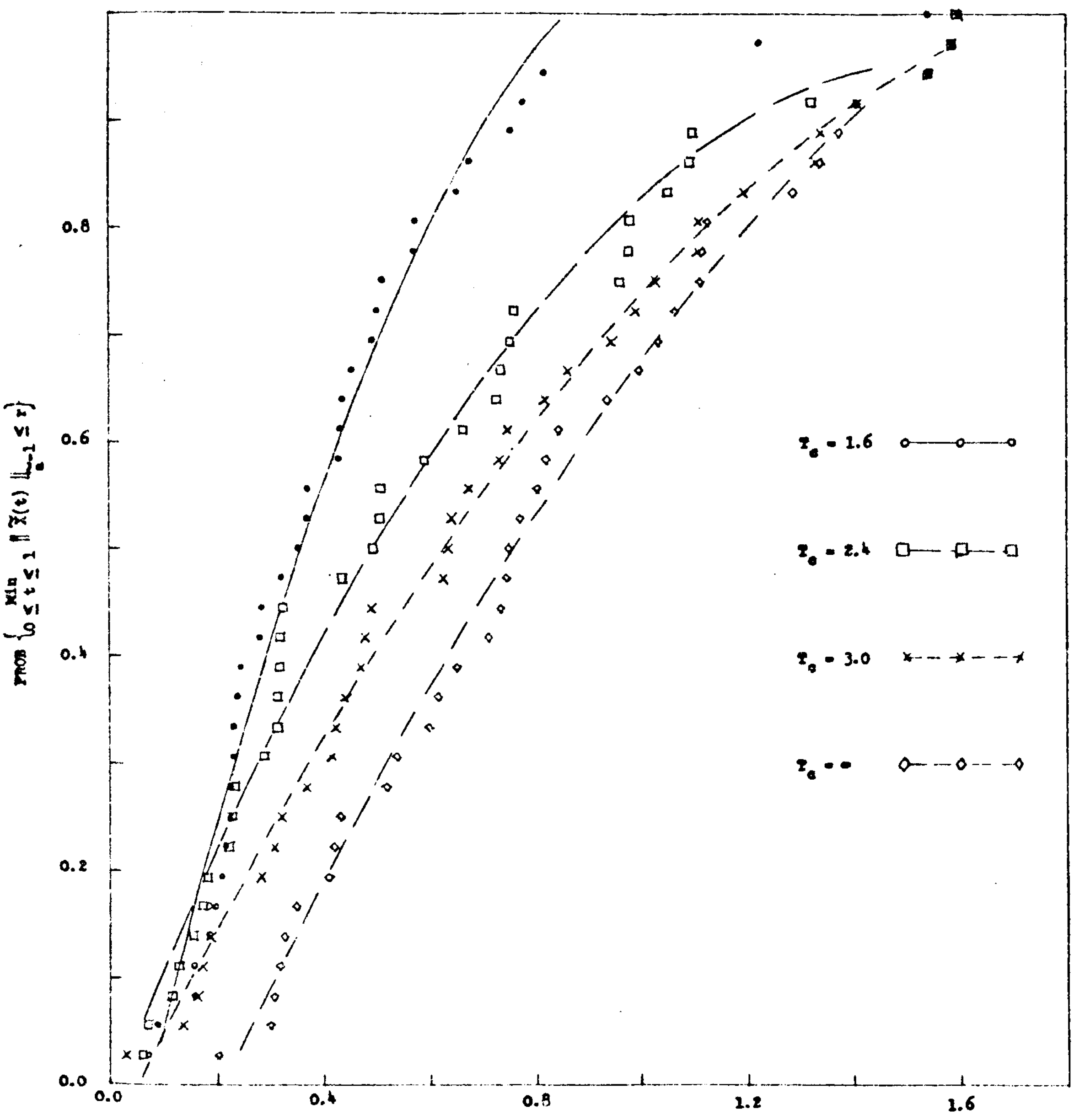


FIGURE 6-9. P1 DISTRIBUTIONS

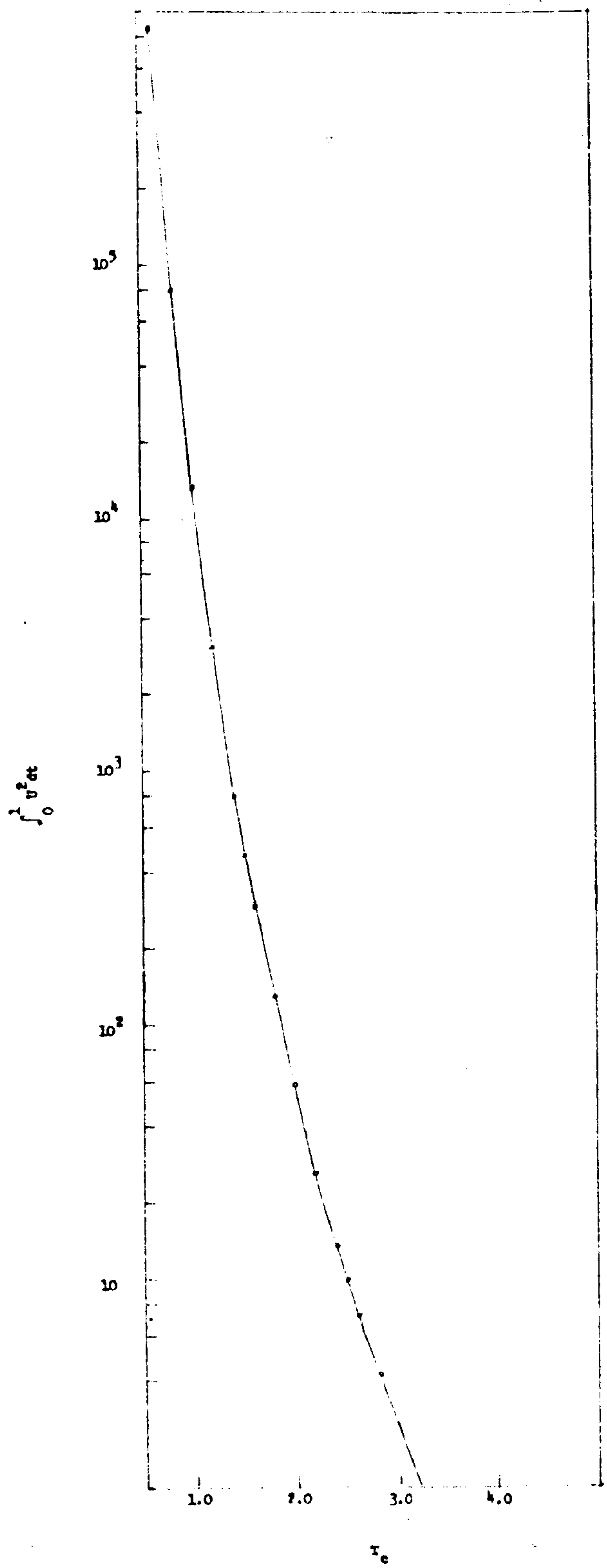


FIGURE 6-10. ENERGY VERSUS T_c FOR P1

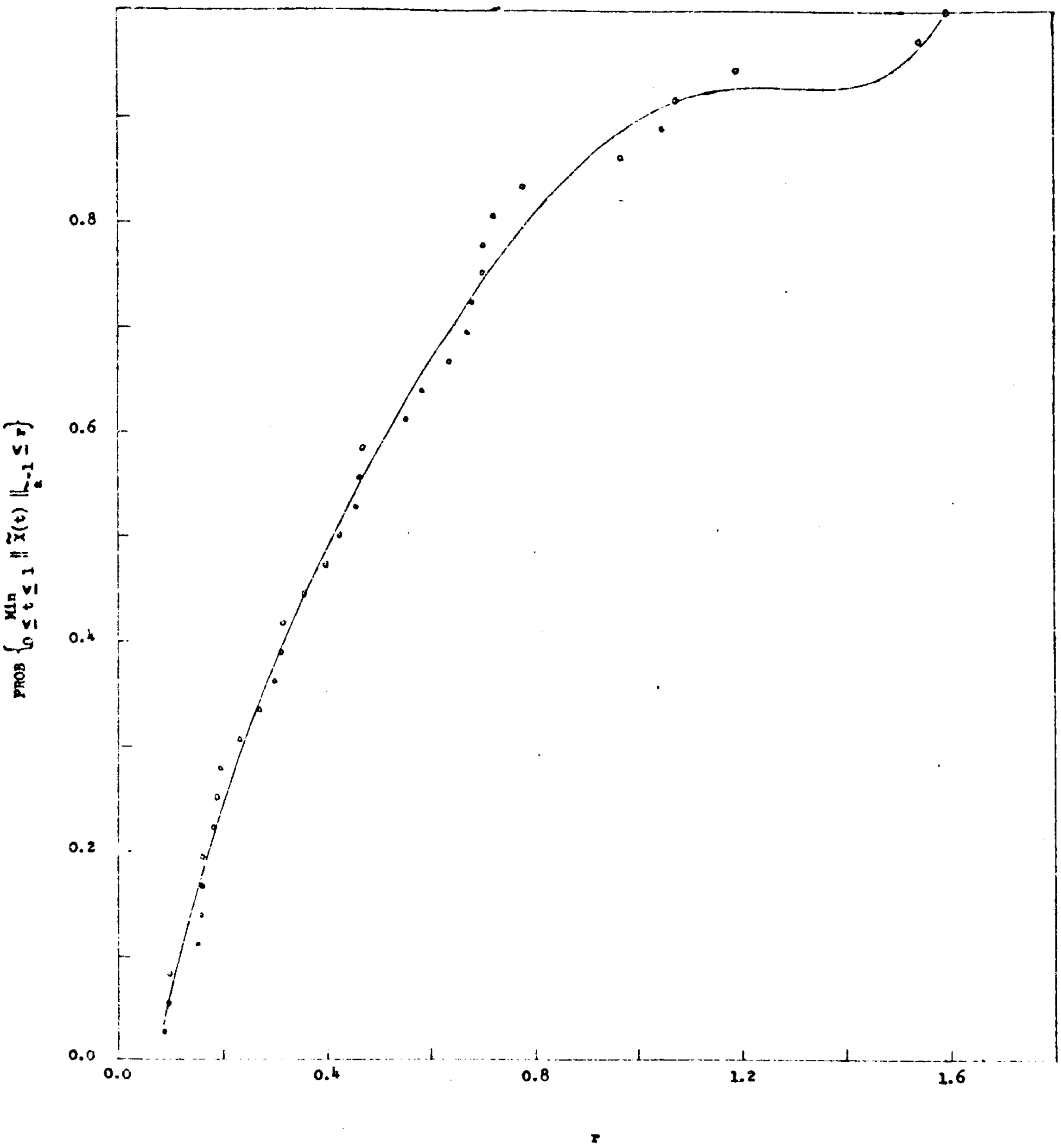


FIGURE 6-11. FIFTH DEGREE POLYNOMIAL FIT TO P1 DISTRIBUTION FOR T_c = 2.0

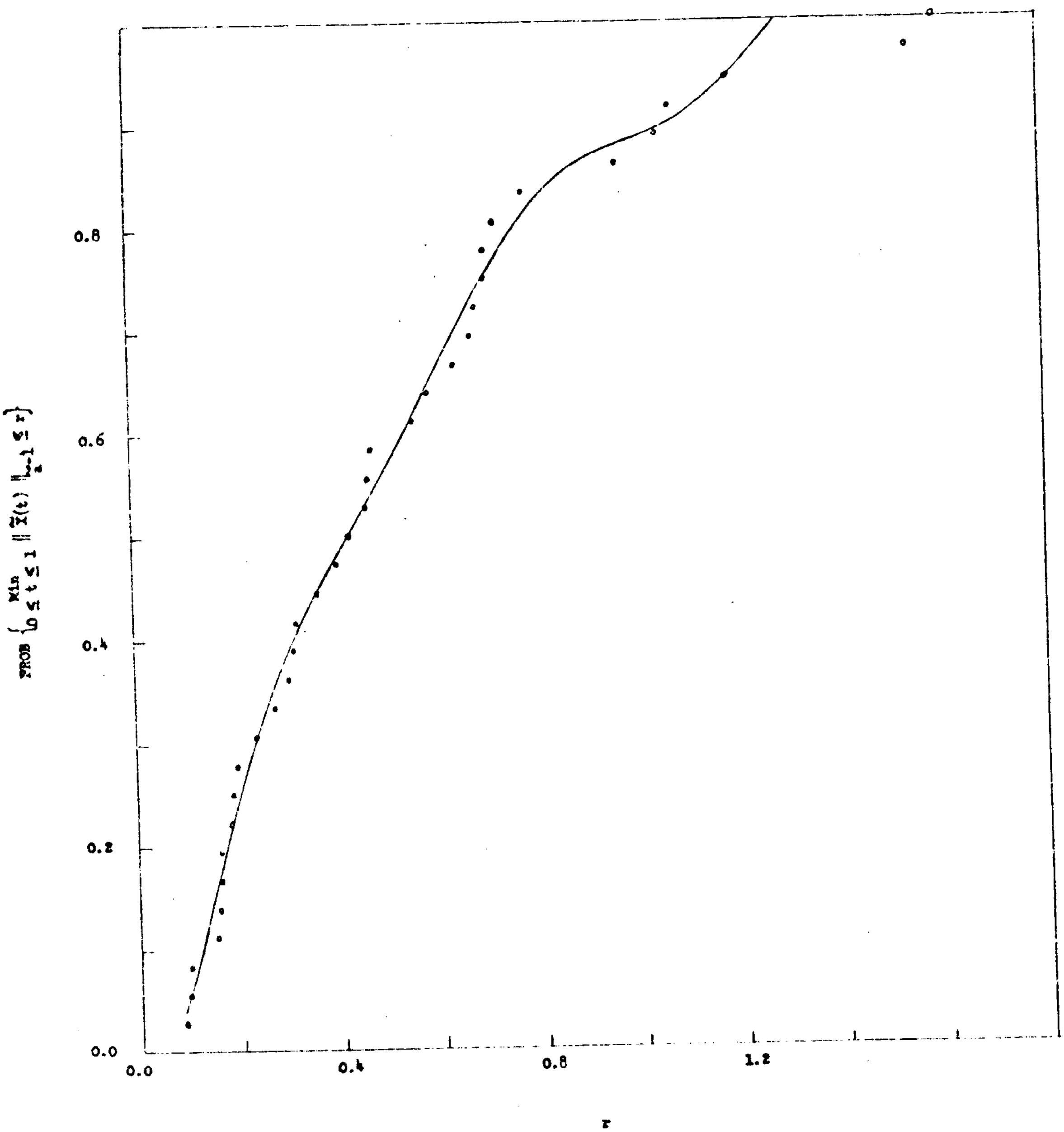


FIGURE 6-12. EIGHTH DEGREE POLYNOMIAL FIT TO P1 DISTRIBUTION FOR $T_c = 2.0$

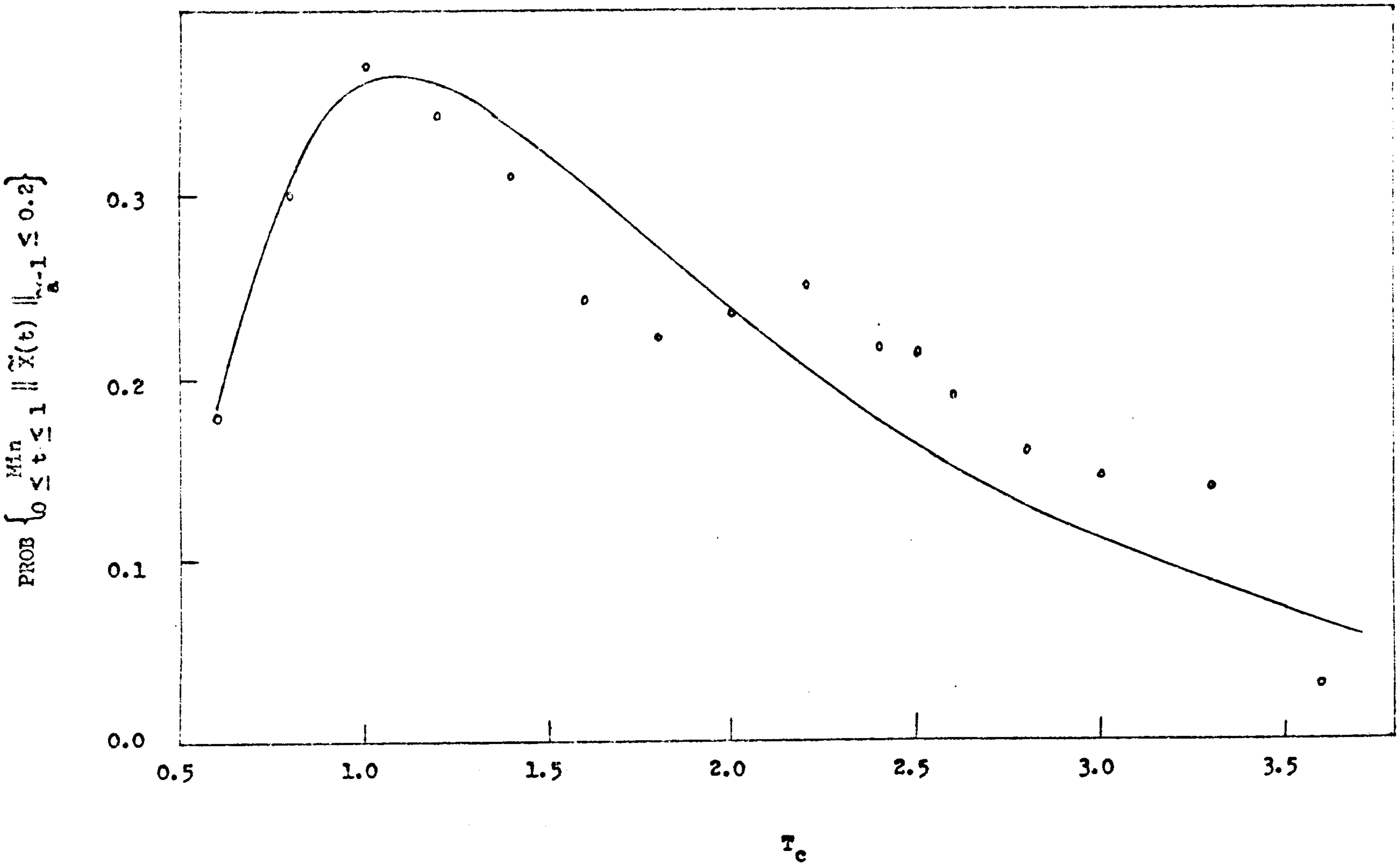


FIGURE 6-13. PROB $\left\{ \min_{0 \le t \le 1} \| \tilde{X} \|_{\tilde{a}^{-1}} \le 0.2 \right\}$ VERSUS T_c

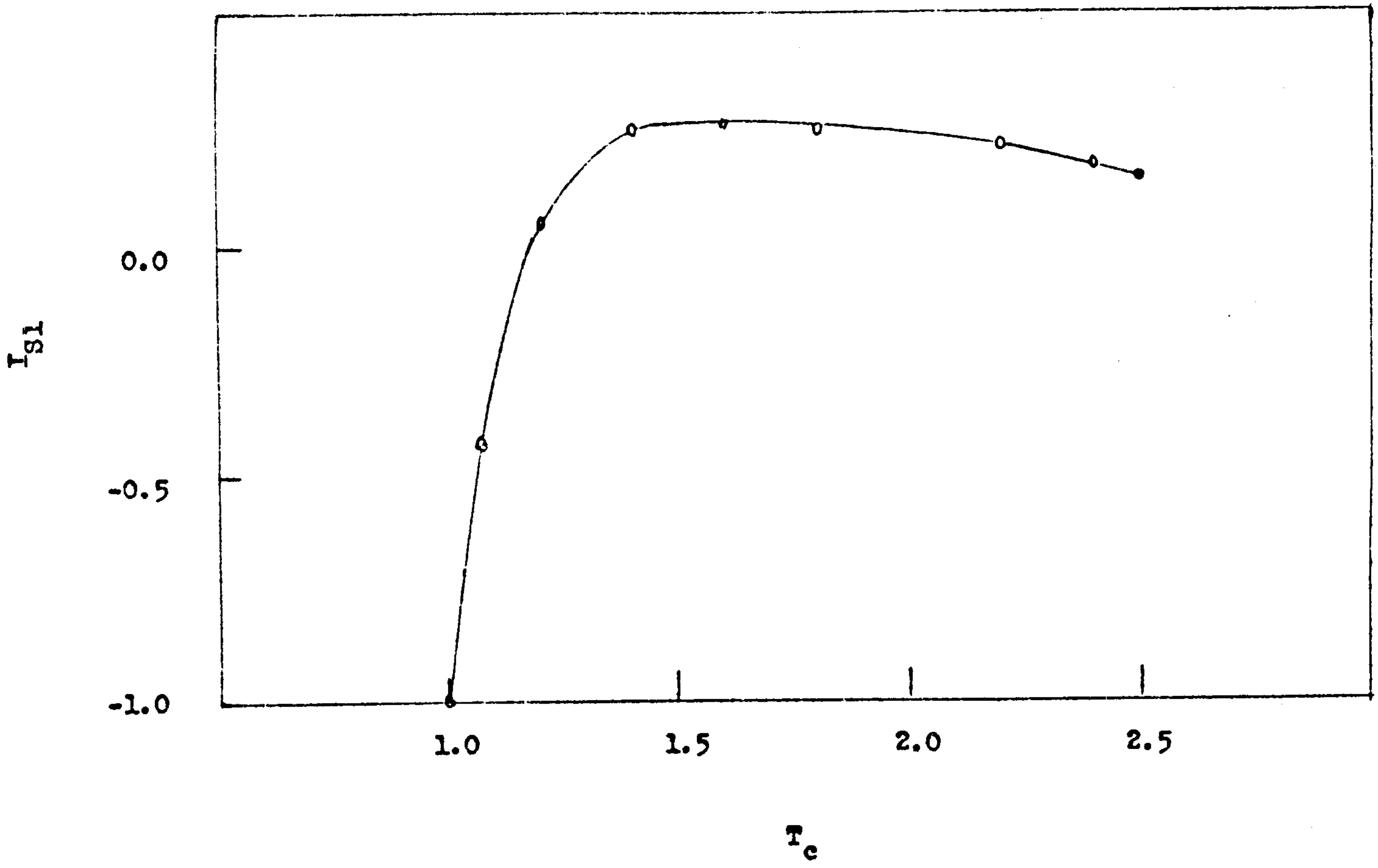


FIGURE 6-14. I_{S1} VERSUS T_c

drawn is that T_C should be chosen somewhat larger than its optimum value to reduce the sensitivity of performance to system parameters.

6.4 Suboptimal Problem P2

6.4.1 Motivation

Figure 6-13 shows that the probability term of I_{s1} has a maximum value at T_C=1. The reason for this is apparent from Figures 6-15 through 6-17. For $T_C < 1$, $||\tilde{x}(t)||_{\tilde{a}^{-1}}$ increases during the period $T_C < t < 1$. This happens because x_4 and x_5 were not brought to zero, as were x_1, x_2, and x_3. As a result, for $T_C < 1$, decreasing T_C also decreases the probability term of I_{s1}, but still increases the energy term. Thus for $T_C < 1$, I_{s1} fails to express the trade off between energy and probability, and thus compromises its value as a useful performance index. A possible remedy would be to redefine the terminal condition on (6-2) to read

$$x_i(T_C) = 0, \ i=1,\ldots,m \tag{6-40}$$

This approach, however, spends energy on zeroing components of x which do not affect the probability term. This objection points to another criticism of $\mathcal{J}_1$: no account is taken of the shape of the target manifold. That is, the same control is applied regardless of the actual values of the elements of $\tilde{a}$.

Suboptimal problem P2 as defined below was conceived to improve the objectionable features of problem P1 mentioned above.

6.4.2 Definition of Suboptimal Problem P2

The second class of controls to be considered is defined as follows. Let u_{Ku} be the control which minimizes

$$I_{D2} = \int_0^T \left\{ ||\tilde{x}(t)||^2_{\tilde{a}^{-1}} + K_u ||u(t)||^2_U \right\} dt \tag{6-41}$$

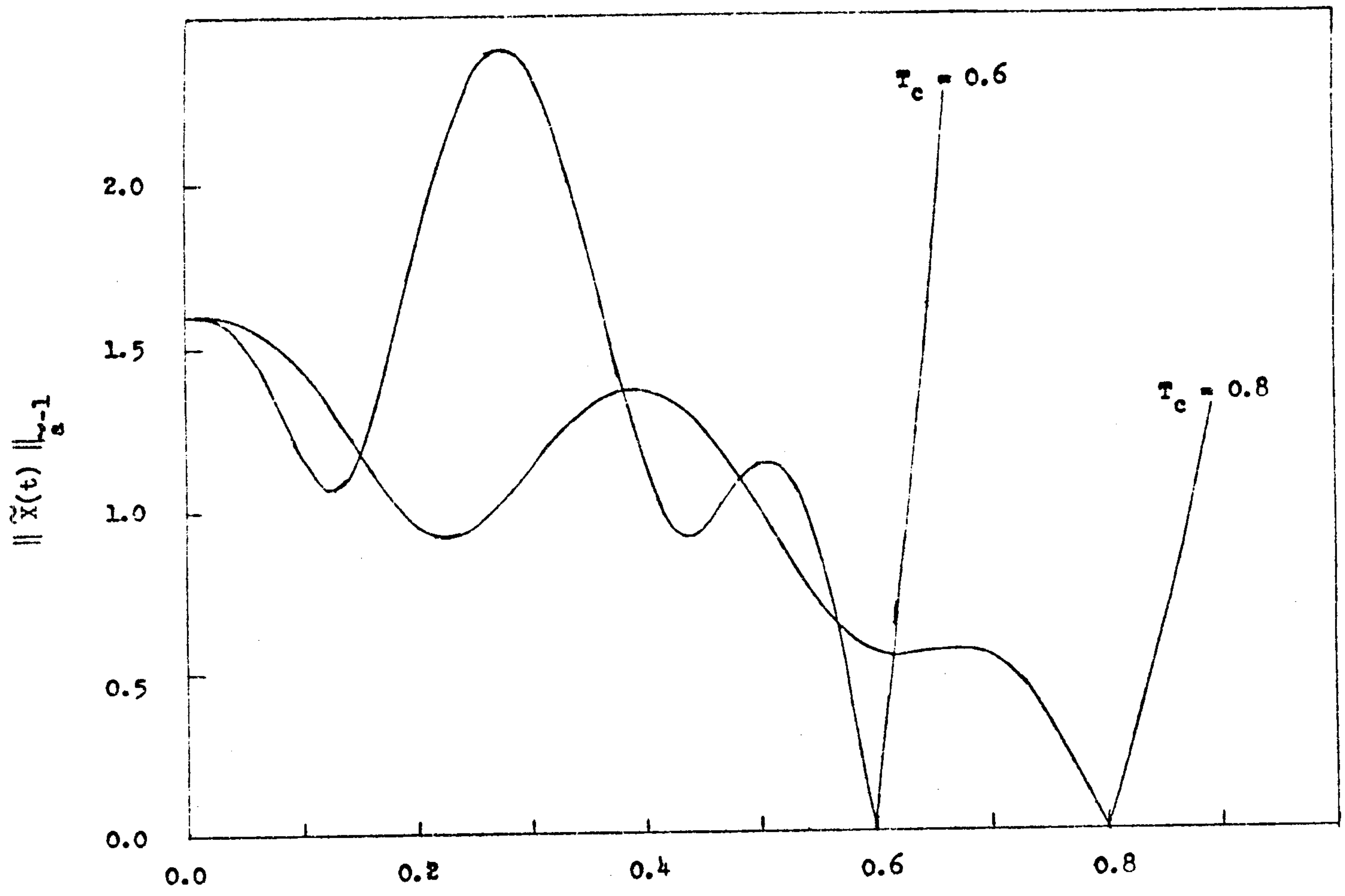


FIGURE 6-15. $\| \tilde{X} \|_{a^{-1}}$ VERSUS t FOR P1

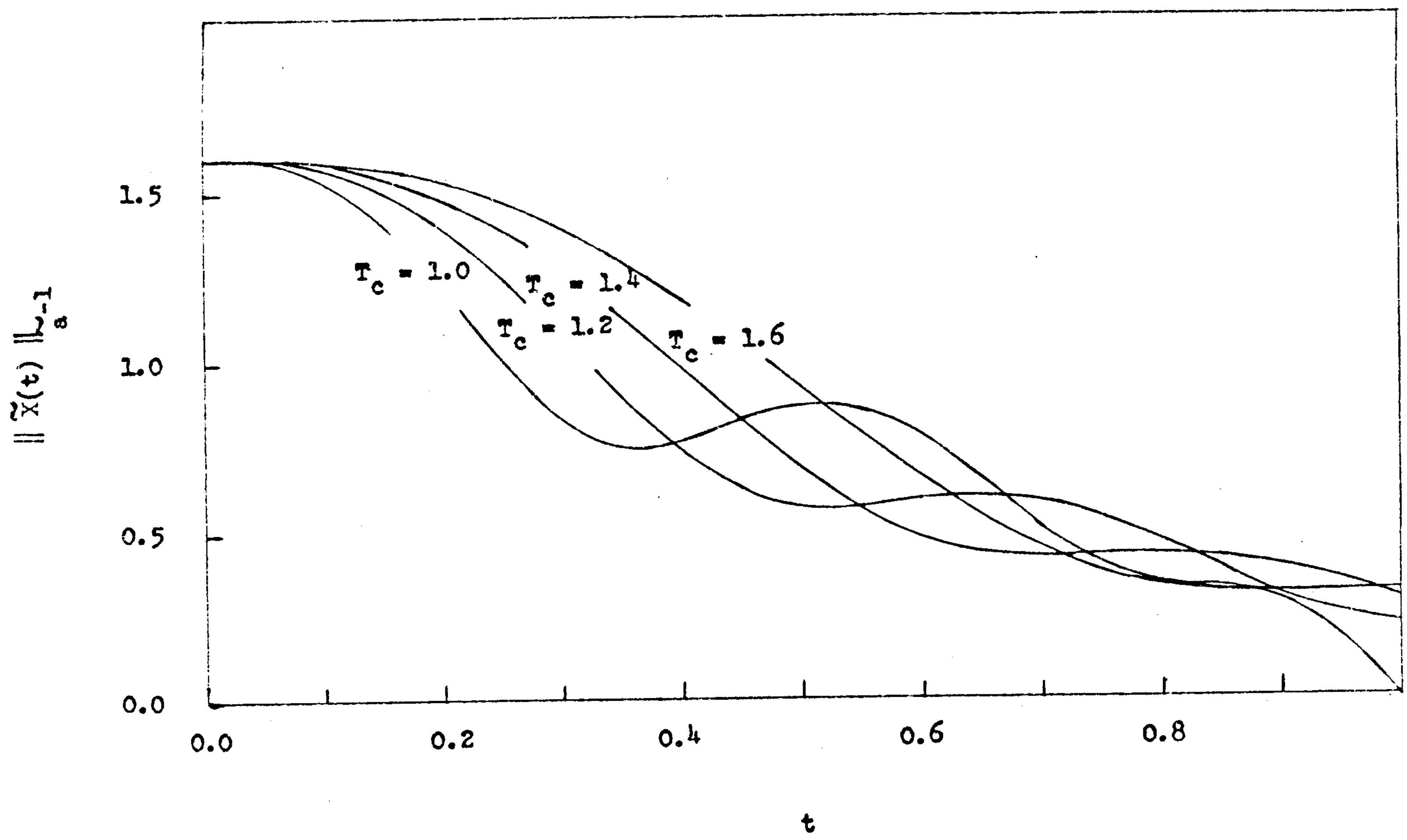


FIGURE 6-16. $\| \tilde{X}(t) \|_{\underset{\sim}{a}^{-1}}$ VERSUS t FOR P1

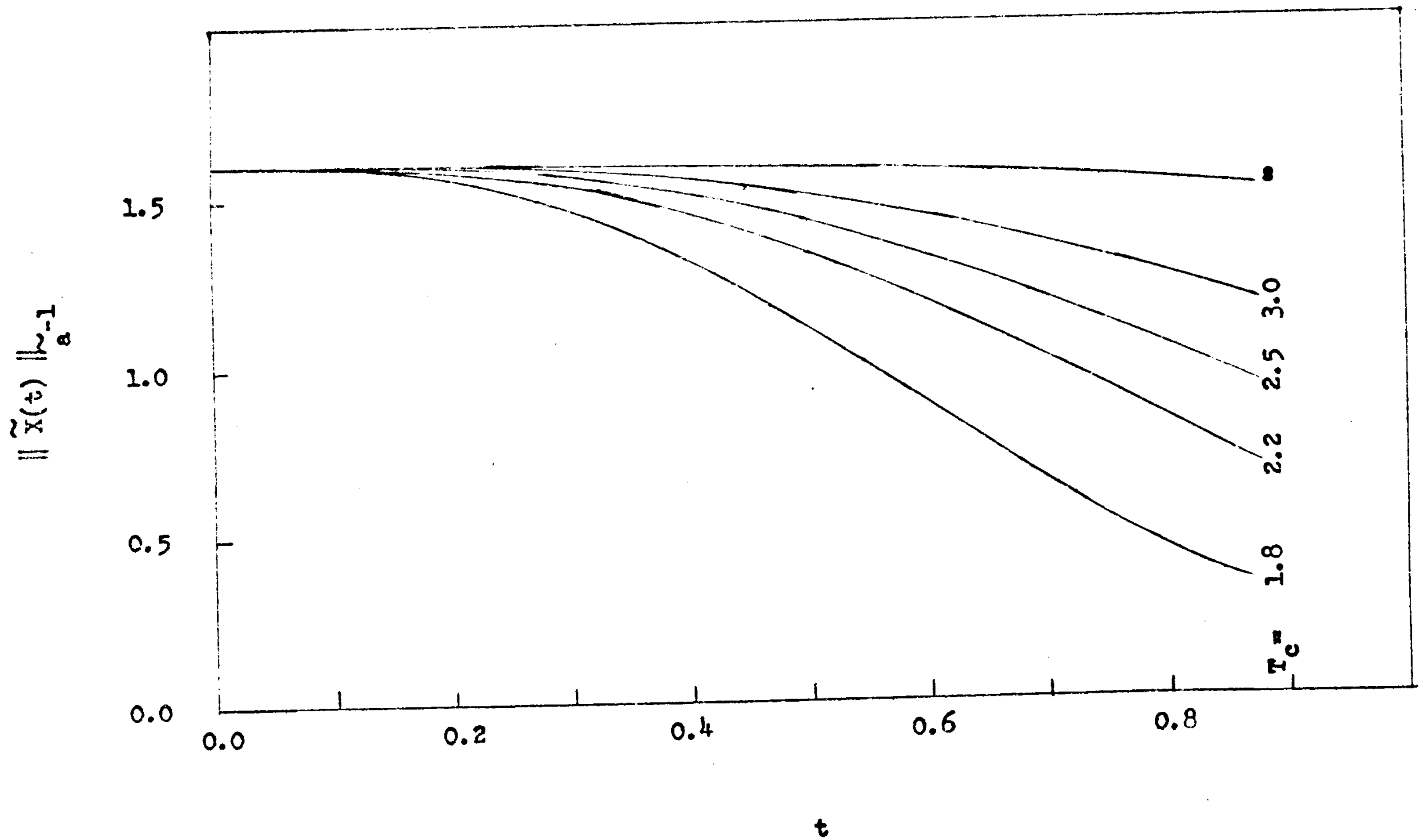


FIGURE 6-17. $\| \tilde{X}(t) \|_{\underset{\sim}{a}^{-1}}$ VERSUS t FOR P1

subject to

$$\dot{x} = Ax + Bu, \quad x(0) = \bar{x} \tag{6-42}$$

Then the class $\mathcal{F}_2$ is defined as

$$\mathcal{F}_2 = \left\{ u \middle| u(t) = u_{K_u}(t) \text{ for some } K_u \geq 0 \right\} \tag{6-43}$$

The second suboptimal problem to be considered, then, is

P2: Find $u^* \in \mathcal{F}_2$ such that

$$I_s[u^*] \geq I_s[u] \text{ for every } u \in \mathcal{F}_2 \tag{6-44}$$

subject to (6-6). (I_s is defined by (6-5)).

Consider now the computation of u_{K_u} given K_u. This is carried out by a straightforward application of Pontryagin's Maximum Principle. The Hamiltonian H is

$$H = \xi'(Ax + Bu) - \|\tilde{x}\|^2_{\tilde{a}^{-1}} - K_u \|u\|^2_U \tag{6-45}$$

where ξ is the adjoint variable. Then

$$u_{K_u} = \frac{1}{2K_u} U^{-1} B'\xi \tag{6-46}$$

$$\dot{\xi} = -A'\xi + 2\,\tilde{a}^{-1}\,\tilde{x} \tag{6-47}$$

From the transversality condition

$$\xi(T) = 0 \tag{6-48}$$

Then

$$\dot{y} = M_2\, y, \quad y(0) = \begin{bmatrix} \bar{x} \\ \xi(0) \end{bmatrix}$$

$$y(T) = \begin{bmatrix} x(T) \\ 0 \end{bmatrix} \tag{6-49}$$

where

$$y = \begin{bmatrix} x \\ \xi \end{bmatrix} \tag{6-50}$$

$$M_2 = \begin{bmatrix} A & \frac{BU^{-1}F'}{2K_u} \\ 2\,\tilde{a}^{-1} & -A' \end{bmatrix} \tag{6-51}$$

Let

$$\dot{\Phi}_{M_2}(t_2,t_1) = M_2\,\Phi(t_2,t_1),\quad \Phi_{M_2}(t_1,t_1) = I_{2m} \tag{6-52}$$

where I_{2m} = identity matrix of rank 2m and let

$$\Phi_{M_2}(T,0) = \begin{bmatrix} \tilde{\Phi}_1 & \tilde{\Phi}_2 \\ \tilde{\Phi}_3 & \tilde{\Phi}_4 \end{bmatrix} \tag{6-53}$$

Then

$$\Phi_M(T,0)\begin{bmatrix} \bar{x} \\ \phi(0) \end{bmatrix} = \begin{bmatrix} X(T) \\ 0 \end{bmatrix} \tag{6-54}$$

From (6-53), and (6-54)

$$\phi(0) = -\tilde{\Phi}_4^{-1}\,\tilde{\Phi}_3\,\bar{x} \tag{6-55}$$

Using (6-42), (6-46), (6-47), and (6-55), then, it is possible to compute u_{K_u}.

6.4.3 Numerical Results

Numerical results were obtained for suboptimal problem P2 using the same sample problem and the same computational techniques which were applied for suboptimal problem P1. Comparison of Figure 6-18 with Figures 6-15 through 6-17 shows that P2 should indeed express the tradeoff between energy and probability better than did P1. The resulting plots of

$\text{Prob}\left\{ \min_{0 \le t \le 1} \|\tilde{x}(t)\|_{\tilde{a}^{-1}} \le r \right\}$ versus r are shown in Figures 6-19 through 6-21, and $\int_0^1 u^2\,dt$ versus K_u is shown in Figure 6-22. Following the same procedures described under suboptimal problem P1, the data contained in

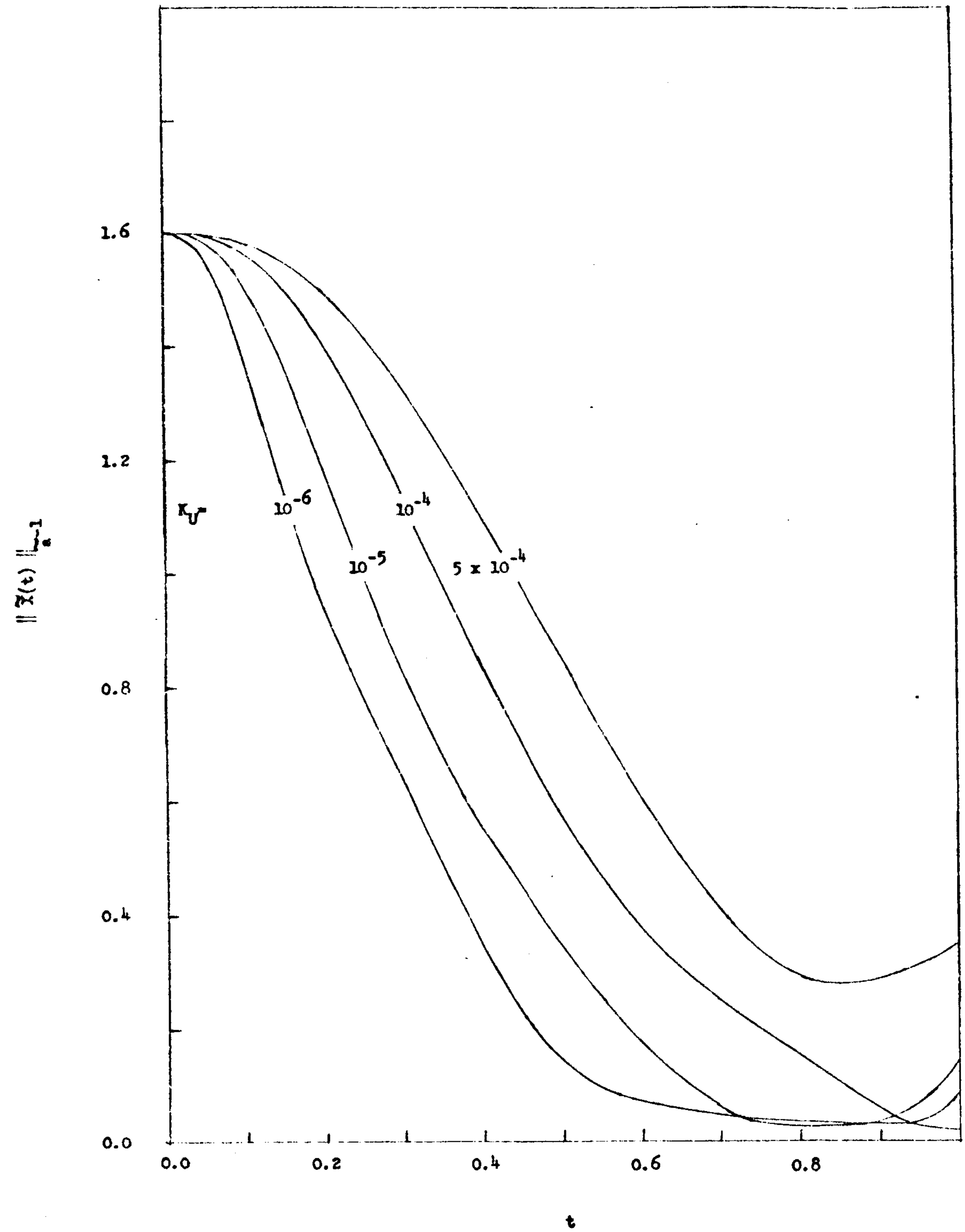


FIGURE 6-18 $\| \tilde{X}(t) \|_{L_a^{-1}}$ VERSUS t FOR P2

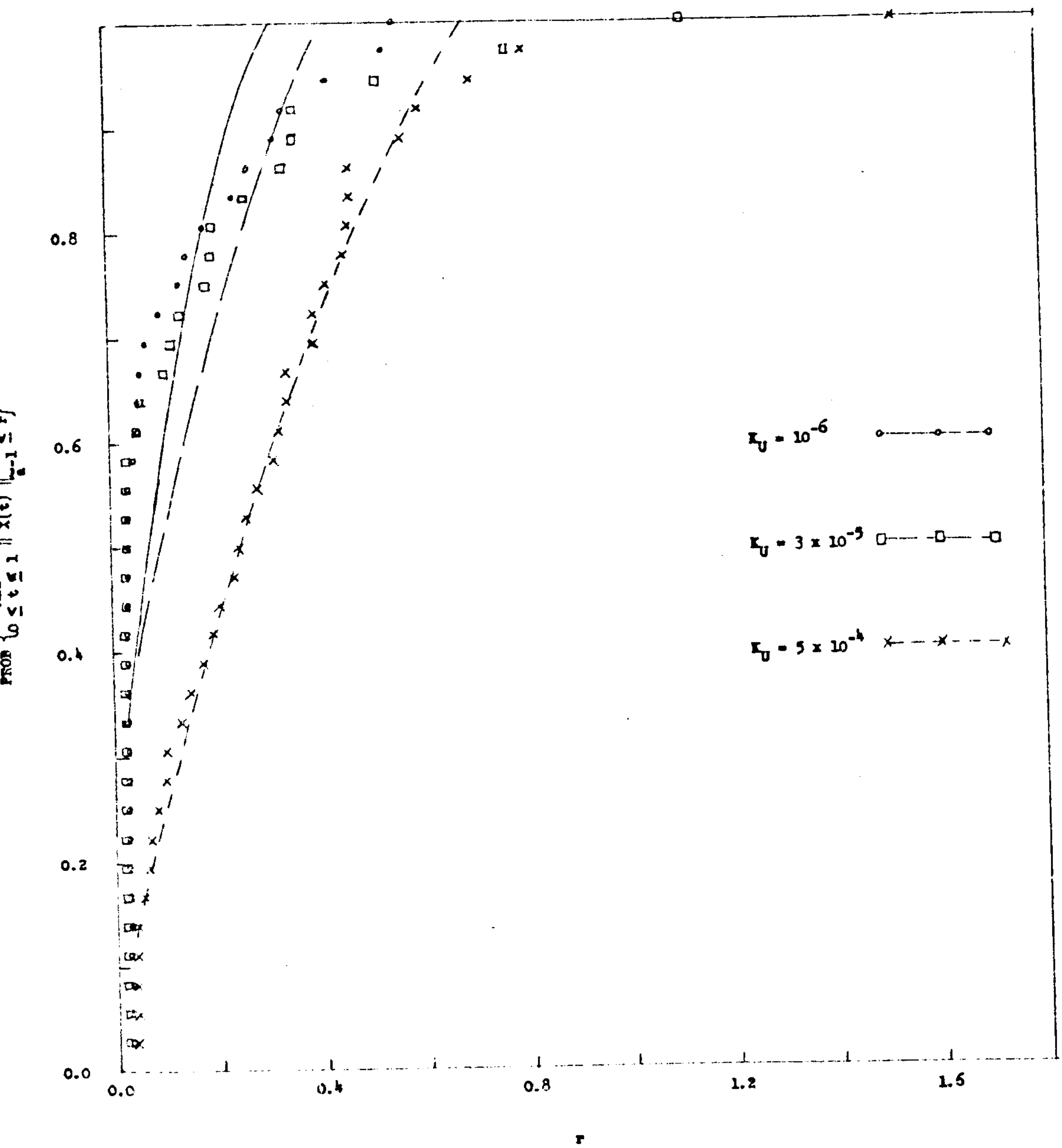
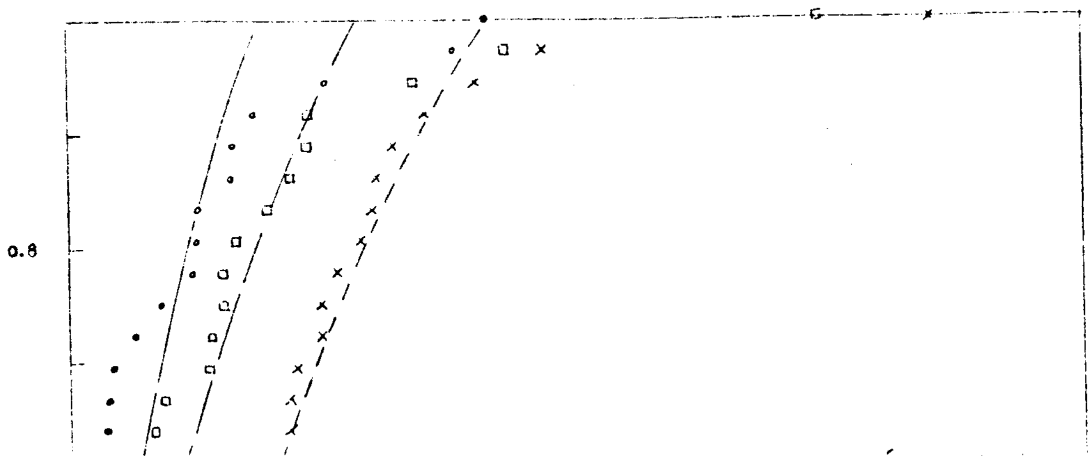


FIGURE 6-19. P2 DISTRIBUTIONS


0.8

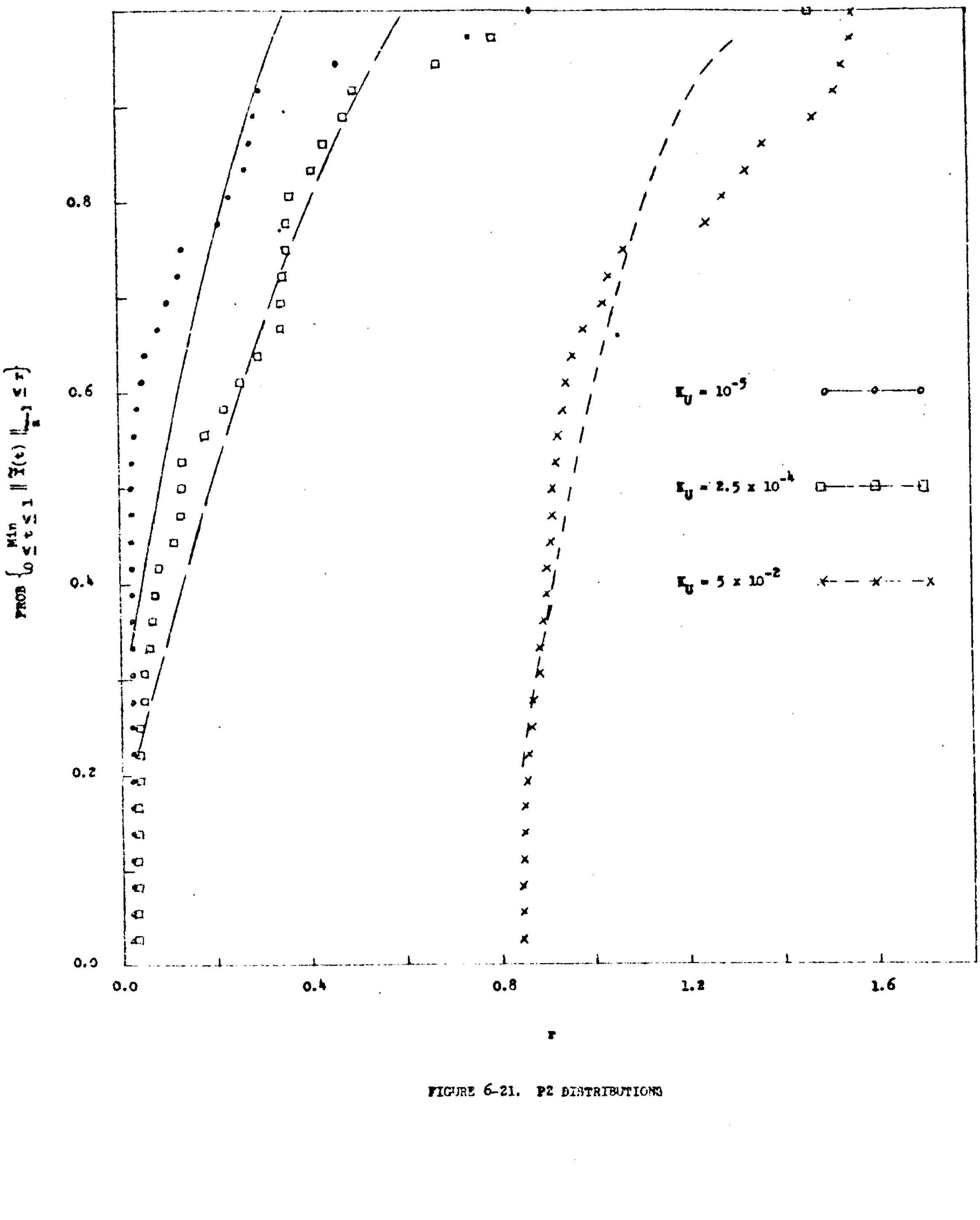


FIGURE 6-21. P2 DISTRIBUTIONS

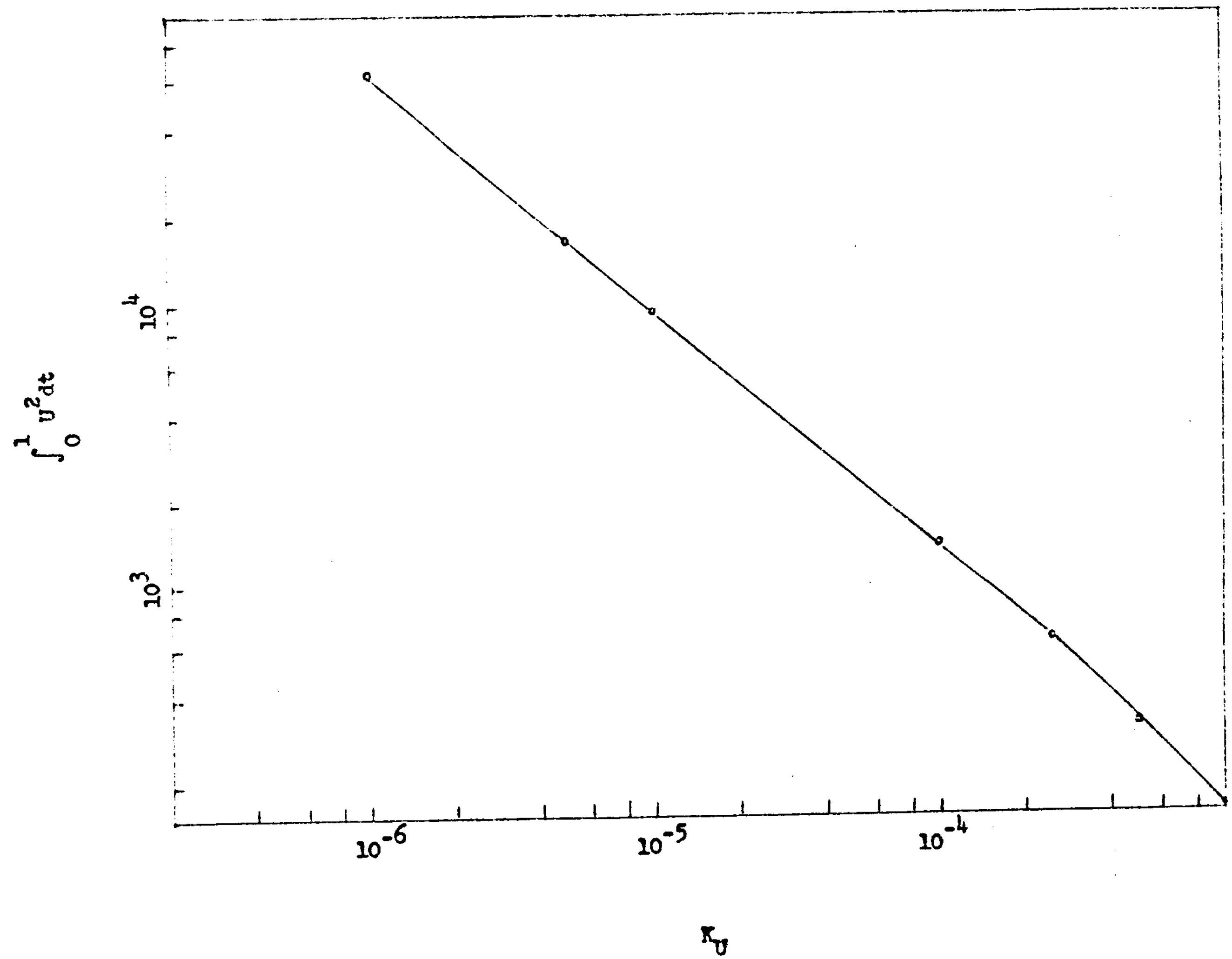


FIGURE 6-22. ENERGY VERSUS K_U

Figures 6-19 through 6-22 was used to maximize I_s for r=0.2 and $k_u=10^{-4}$. Figure 6-23 shows the plot of Prob$\left\{ \min_{0 \le t \le 1} ||\tilde{x}(t)||_{\tilde{a}^{-1}} \le 0.2 \right\}$ versus K_u.

The plot of I_s versus K_u is shown in Figure 6-24. From this plot it is seen that the optimal control is that corresponding to $K_u = 10^{-4}$. The optimal performance turns out to be

$$\text{Prob}\left\{ \min_{0 \le t \le 1} ||\tilde{x}(t)||_{\tilde{a}^{-1}} \le 0.2 \right\} = 0.62 \tag{6-56}$$

$$\int_0^1 [u^*(t)]^2 \, dt = 1400 \tag{6-57}$$

6.5 Comparison of Results

Some comparisons will now be made between suboptimal problems P1 and P2. Comparison of (6-56) and (6-57) to (6-38) and (6-39) shows that the optimal control in $\mathcal{F}_2$ achieves higher probability of a hit at a higher energy cost than the optimal control in $\mathcal{F}_1$. Comparison of Figure 6-24 to Figure 6-14 shows that the optimal control in $\mathcal{F}_2$ achieves a considerably higher I_s than does that in $\mathcal{F}_1$. Thus for this sample problem, at least, $\mathcal{F}_2$ is a better class of controls.

Note that the superiority of $\mathcal{F}_2$ over $\mathcal{F}_1$ has been demonstrated only for r=0.2 and $k_u=10^{-4}$. It turns out, however, that $\mathcal{F}_2$ is superior to $\mathcal{F}_1$ in a broader sense. Figure 6-25 shows that for the range of energies investigated, the controls in $\mathcal{F}_2$ always produce higher probability of hit than those of $\mathcal{F}_1$, for the same energy.

A comparison of Figure 6-13 with 6-23 raises another point. The random error arising from the Monte Carlo nature of the computational technique, i.e., the distance of the experimental points from the fitted curve, seems much higher for the controls in $\mathcal{F}_1$ than for those in $\mathcal{F}_2$. This is

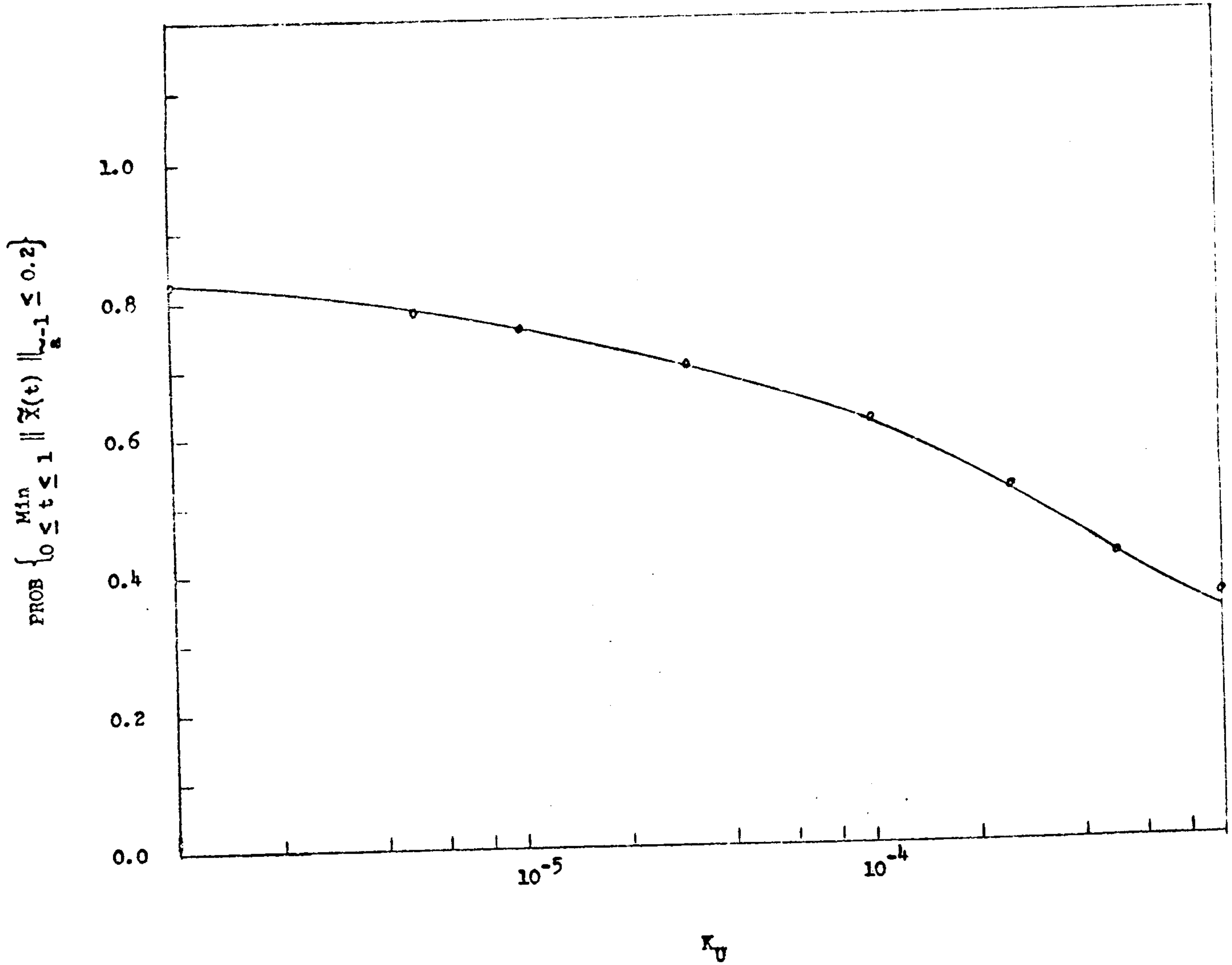


FIGURE 6-23. PROB $\left\{ \min_{0 \le t \le 1} \| \tilde{X}(t) \|_{\underset{\sim}{a}^{-1}} \le 0.2 \right\}$ VERSUS K_U

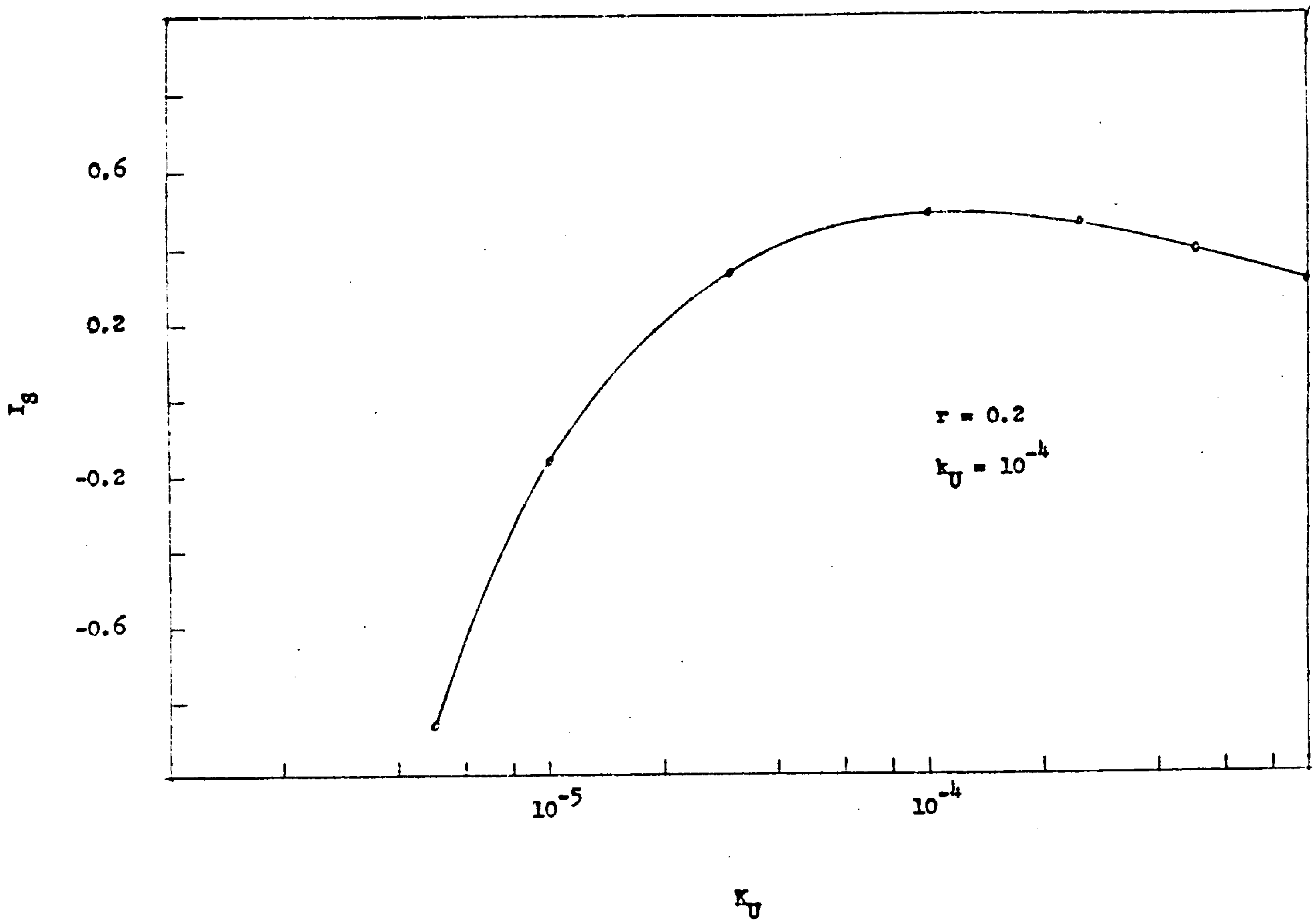


FIGURE 6-24. I_s VERSUS K_U

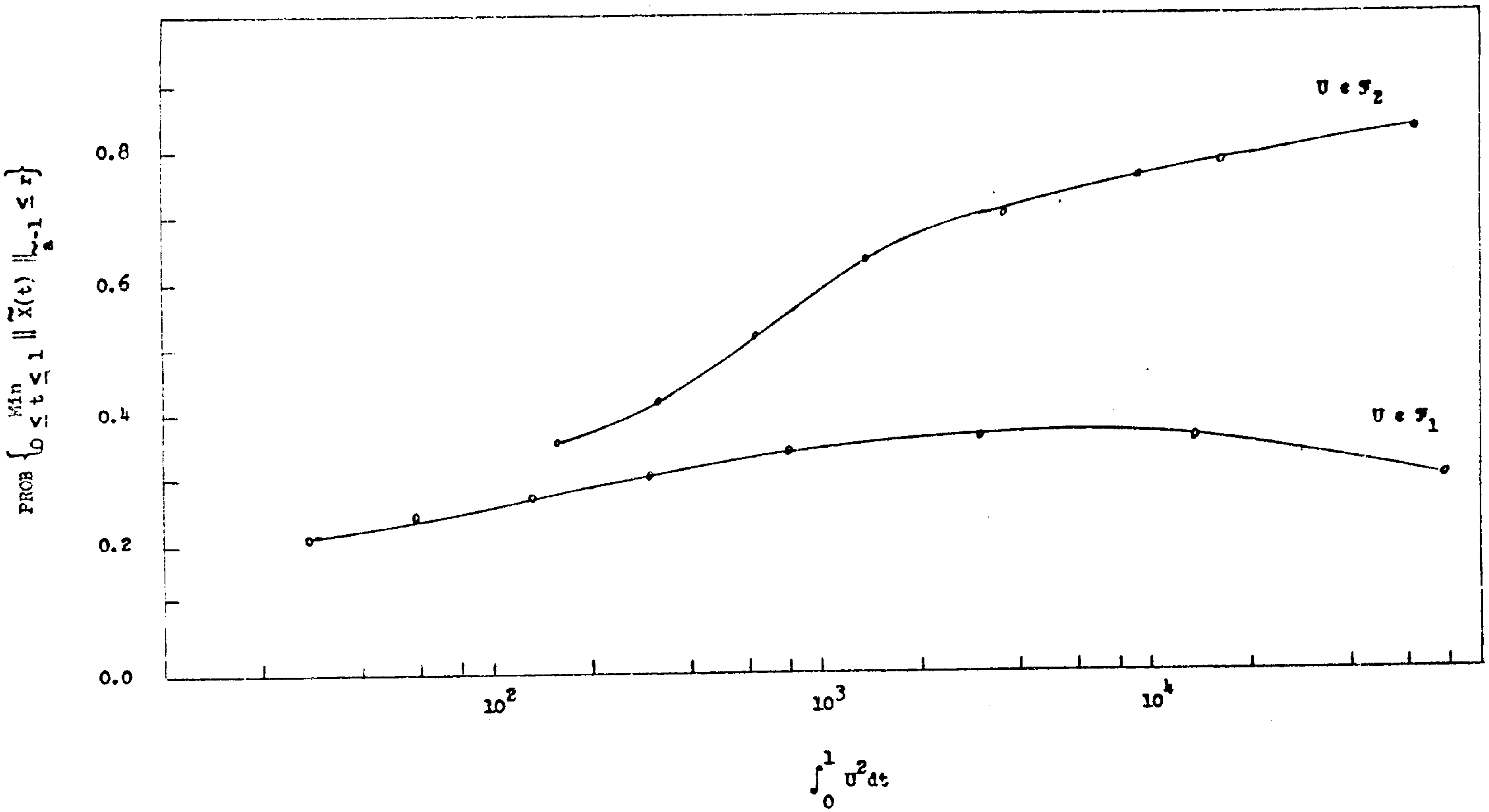


FIGURE 6-25. PROBABILITY VERSUS ENERGY FOR $\mathcal{F}_1$ AND $\mathcal{F}_2$.

attributable to the relatively higher number of experimental points for $r \leq 0.2$.

CHAPTER 7

NUMERICAL RESULTS BASED ON THE $\bar{\psi}$ ESTIMATE

7.1 Introduction

In Chapter 4, an estimate, $\bar{\psi}$, to the probability term of the performance index was developed, while in Chapter 5, an algorithm utilizing the $\bar{\psi}$ estimate for the solution of the basic optimization problem was presented. In the present chapter, numerical results from the application of the analysis in Chapters 4 and 5 to the sample problem stated in Chapter 6 are given.

The accuracy of the $\bar{\psi}$ estimate is investigated by comparison of the $\bar{\psi}$ estimate of the performance index probability term for controls in $\mathcal{F}_1$ and $\mathcal{F}_2$ (the subclasses of controls corresponding to problems P1 and P2 respectively, as discussed in Chapter 6) with the results obtained in Chapter 6 (via observation of a digital simulation of the system).

The conjugate gradient algorithm presented in Chapter 5 is applied to the sample problem stated in Chapter 6, using two different controls as starting points.

The first starting control is the control in $\mathcal{F}_1$ corresponding to $T_c = 1.0$ (see Chapter 6). The reason for choosing this starting point is that Figure 6-14 suggests that reasonably steep gradients appear here.

The second starting control is the control in $\mathcal{F}_2$ corresponding to $K_u = 10^{-4}$. The reason for choosing this starting point is that this represents the best control found in Chapter 6. Thus any further improvement demonstrates the usefulness of the optimization algorithm of Chapter 5.

Some comments are made on the convergence of the conjugate gradient algorithm in light of the numerical results.

Discussion of the computational aspects of the work described in this chapter is contained in Appendix H.

7.2 Accuracy of the $\bar{\psi}$ Estimate

7.2.1 Control-independent Term

The $\bar{\psi}$ estimate, whose accuracy is to be checked, is given by equation (4-114). An examination of equations (4-115), (4-116), and (4-117) shows that ψ_o does not depend on the control. Thus it is necessary to compute this term only once. (This was noted in Chapter 5, and ψ_o was dropped from the performance index.) The computation of $\psi_o(0, \bar{x}, T)$, where $\bar{x}$ and T are the initial state and final time for the sample problem stated in Chapter 6, was carried out by the FORTRAN program shown in Table H-2. The results of this computation are shown in Figure 7-1. In this Figure N_M is the number of terms in the Monte Carlo estimation of the integral appearing in equation (4-115). The value of ψ_o is taken to be 0.02 in the remainder of this chapter. All subsequent $\bar{\psi}$ estimates of the probability term include this number.

7.2.2 $\bar{\psi}$ For Controls in $\mathcal{F}_1$

As stated in section 6.2, the class, $\mathcal{F}_1$, of controls considered in suboptimal problem P1 is parameterized by T_c. Controls were computed for a range of T_c, and the $\bar{\psi}$ estimates corresponding to them were computed. Corresponding probability estimates were made in Chapter 6 by

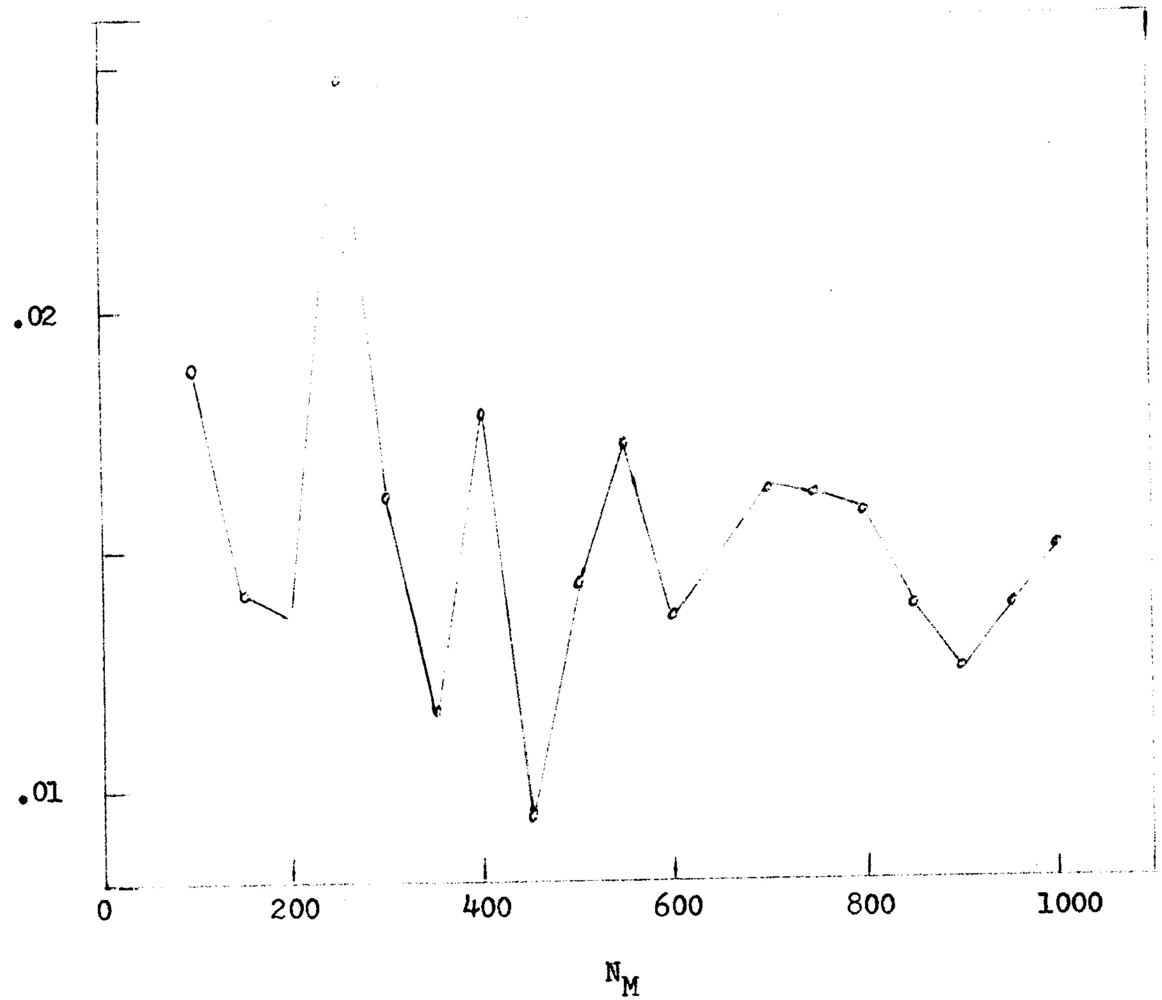


FIGURE 7-1. ESTIMATION OF ψ_o

observation of a digital simulation of the system. The results of these observations are plotted in Figure 6-13. A comparison of these results with the $\bar{\Psi}$ estimates is shown in Figure 7-2. The solid line is the same curve which appears in Figure 6-13. Six terms were used in the Monte Carlo estimate of the integrals appearing in Ψ_1.

An examination of Figure 7-2 indicates that the error in the $\bar{\Psi}$ estimate of probability increases with probability. This is in accordance with the error bound given by equation (4-69). The term $\nabla(\bar{\xi},\underline{r})$ (see equations (A-162) and (A-150) through (A-153)) contains terms with factors of $\bar{R}$ to various negative powers. $\underline{R}$ (see equation(A-56)) is a lower bound on $E[\|\tilde{x}\|_{\tilde{a}^{-1}}]$ over the time interval $[0,T]$. Thus the error in the probability estimate corresponding to controls which bring $\|\tilde{x}\|_{\tilde{a}^{-1}}$ close to zero tends to be large. These controls also tend to yield high probability of hitting the target manifold. Thus an increase in probability estimate error as probability increases might be expected.

7.2.3 $\bar{\Psi}$ for Controls in $\mathcal{F}_2$

Following the same procedure as for $\mathcal{F}_1$ controls, results were obtained for $\mathcal{F}_2$ controls, which are parameterized by K_u (see section 6.4). The results are shown in Figure 7-3. The increase in the error of the estimate of probability as probability increases appears even more clearly here than it did in Figure 7-2. An additional feature which shows up in Figure 7-3 is that the slope of the $\bar{\Psi}$ estimate of probability versus K_u is incorrect for small K_u. This suggests the possibility that the error in the estimate of the probability term could cause the conjugate gradient algorithm to search in the wrong direction.

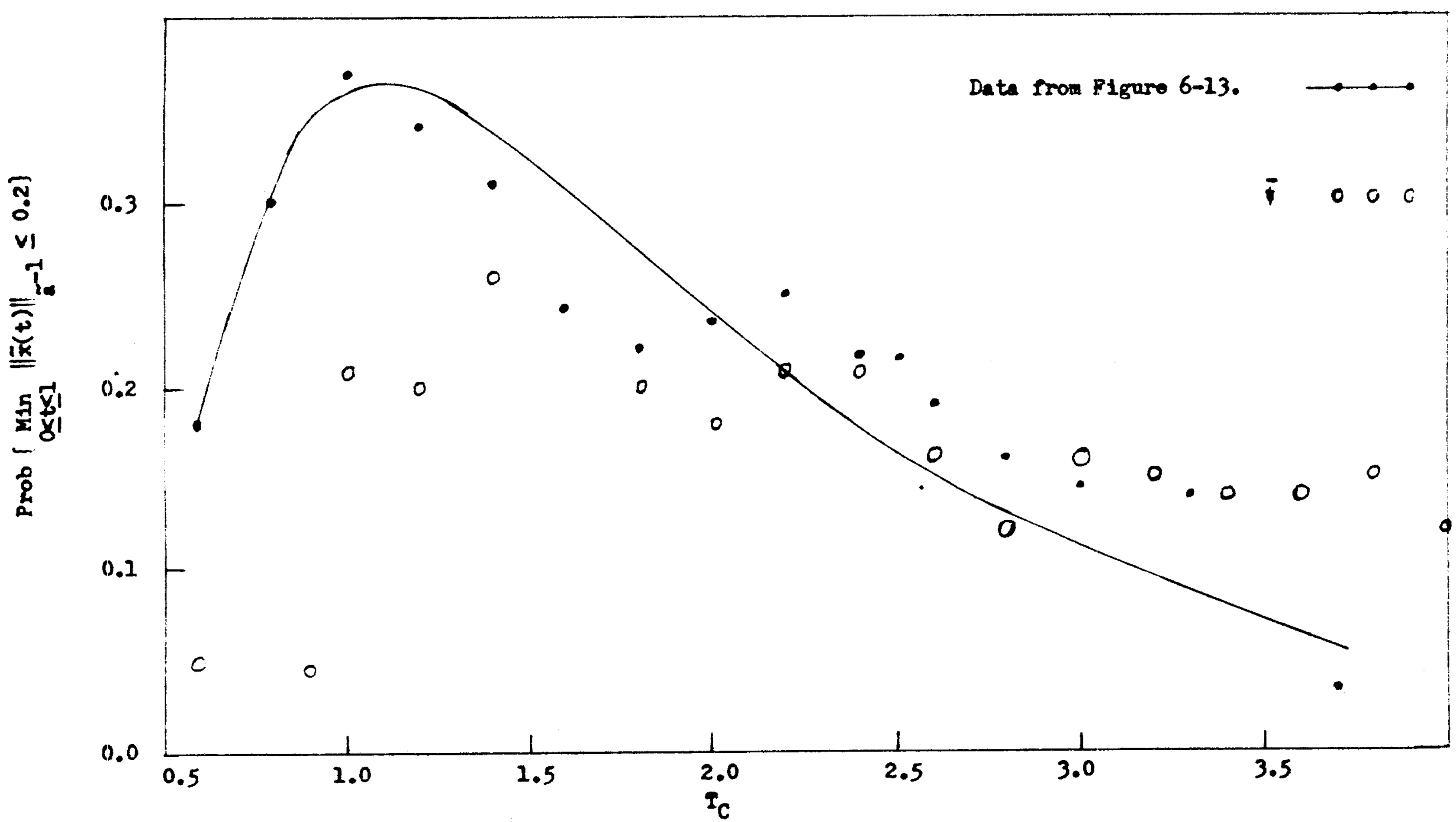


FIGURE 7-2. $\bar{\phi}$ FOR CONTROLS IN $\mathcal{F}_1$

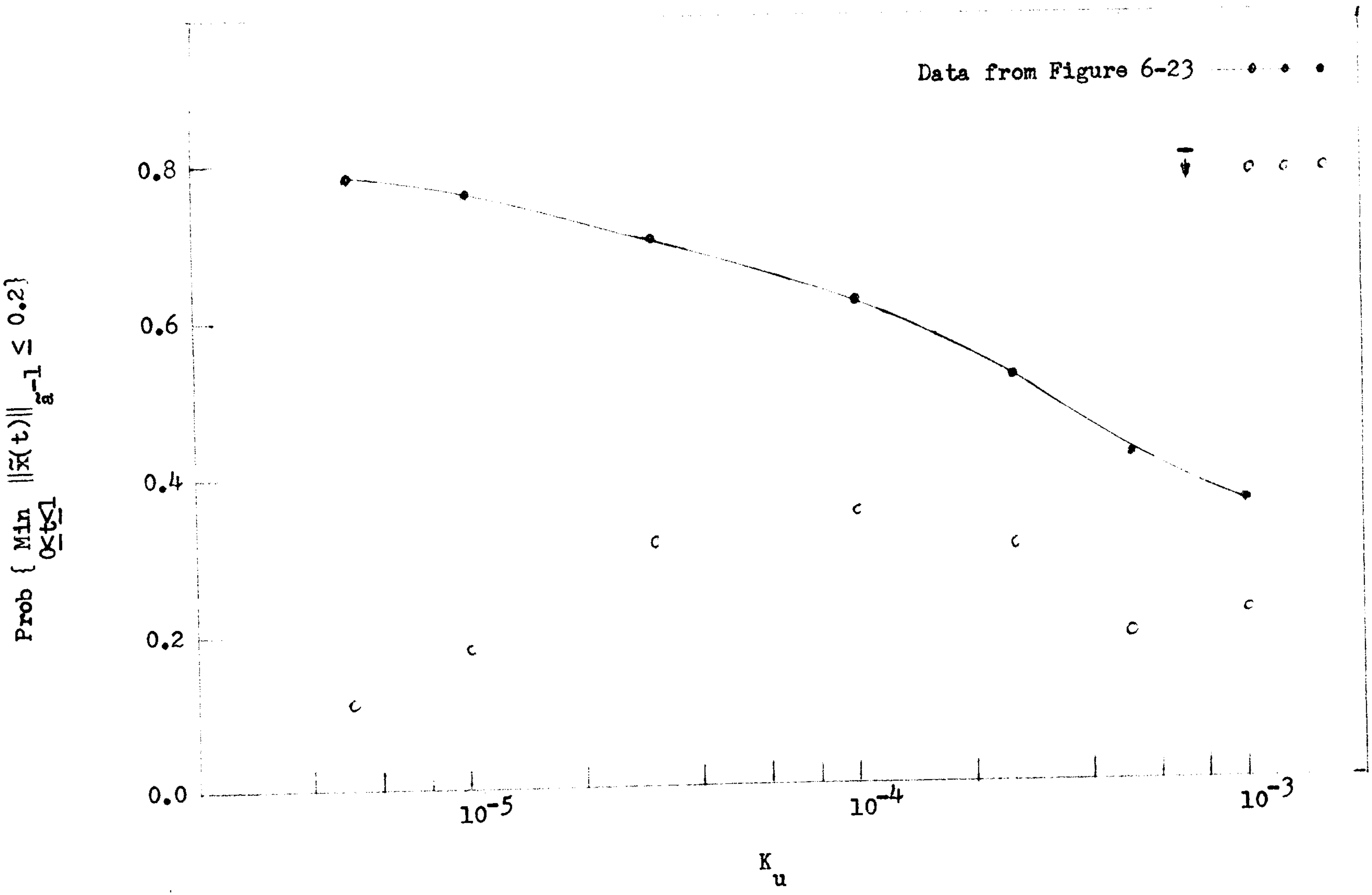


FIGURE 7-3. $\bar{\psi}$ FOR CONTROLS IN $\mathcal{F}_2$

7.3 Optimization Results

7.3.1 Starting Point in $\mathcal{F}_1$

Several iterations of the algorithm shown in Figure 5-1 were carried out for the sample problem stated in Chapter 6, using the control in $\mathcal{F}_1$ corresponding to $T_c = 1.0$ as a starting point. The results are shown in Figure 7-4. The solid lines in this figure connect the indices corresponding to the controls existing at successive passes through block F of Figure 5-1. The dotted lines connect the indices corresponding to the controls existing at successive passes through block B of Figure 5-2. (The dotted lines correspond to the curve shown in Figure 5-3).

The probability term for the control corresponding to the final point of Figure 7-4 was computed from observations on a digital simulation of the system as described in Chapter 6. The results are shown in Figure 7-5. Using these results the final performance index is

$$I = 0.24 \tag{7-1}$$

The apparent error in the computation of the minimum performance index over each iteration is attributable to error in the Monte Carlo estimate of integrals. This also accounts for the "jump" at the third iteration. Each iteration begins by recalculating the performance index corresponding to the control generated by the preceeding index. Except at the third iteration, the two computations were in agreement to two significant figures.

The control corresponding to the final point in Figure 7-4 is shown in Figure 7-6. The $\|\mathbf{x}(t)\|_{\mathbf{a}^{-1}}$ trajectory corresponding to this control and no random disturbance (that is with $n = 0$) is shown in Figure 7-7.

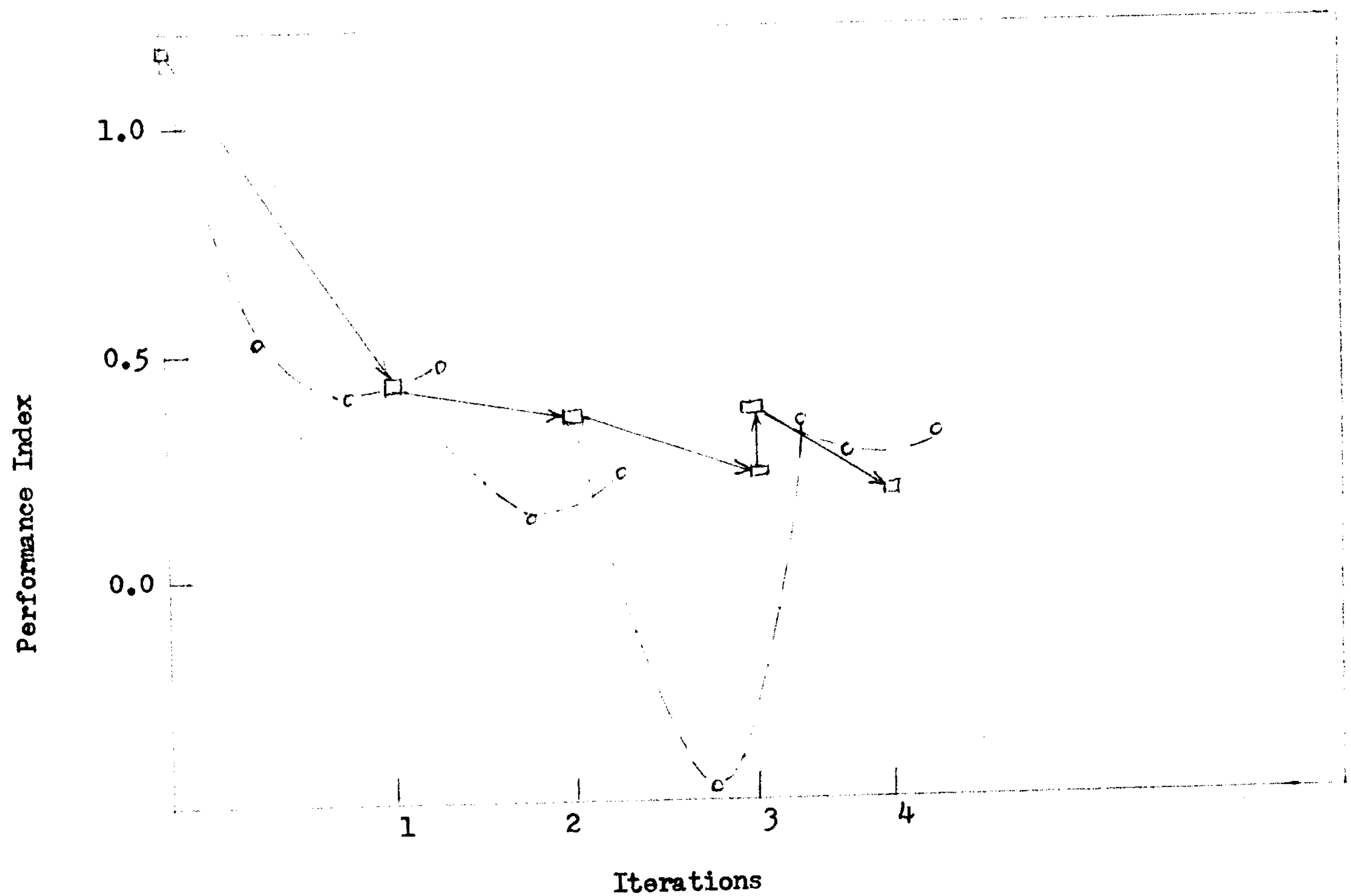


FIGURE 7-4. OPTIMIZATION STARTING FROM $u_{T_C} = 1.0$

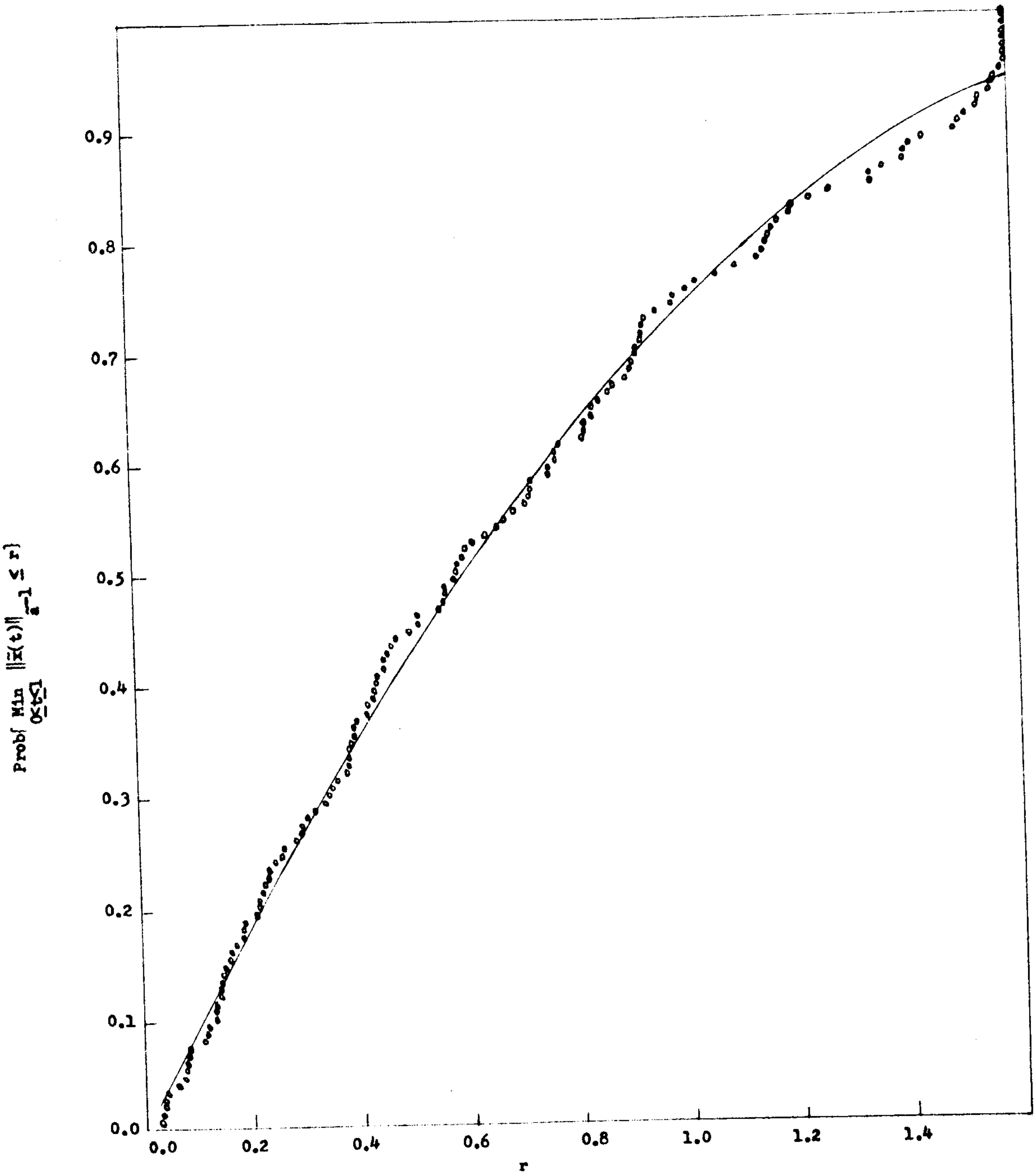


FIGURE 7-5. OBSERVED PROBABILITY TERM FOR FINAL POINT OF FIGURE 7-4.

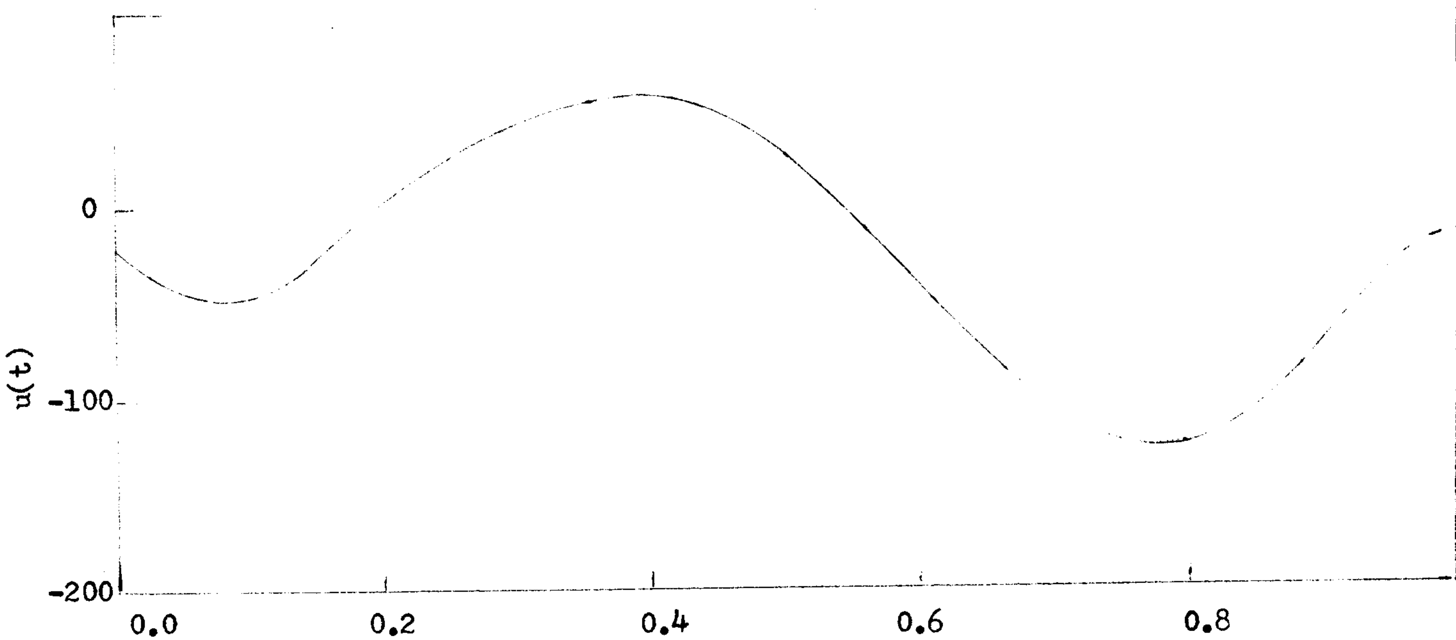


FIGURE 7-6. CONTROL CORRESPONDING TO FINAL POINT OF FIGURE 7-4.

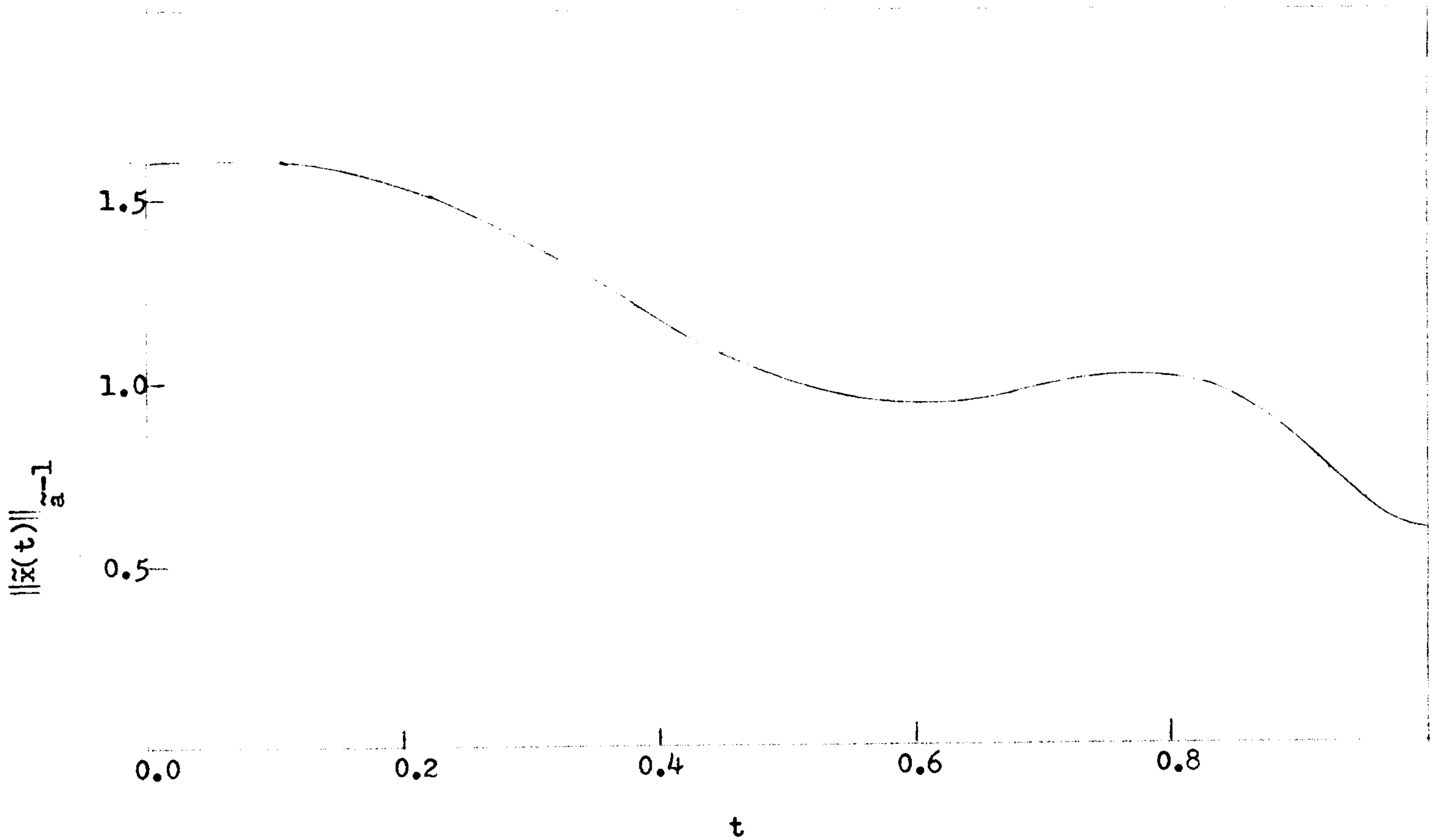


FIGURE 7-7. $\|\tilde{x}(t)\|_{\tilde{a}^{-1}}$ CORRESPONDING TO THE CONTROL IN FIGURE 7-6 AND NO DISTURBANCE.

7.3.2 Starting Point in $\mathcal{F}_2$

Two iterations of the algorithm shown in Figure 5-1 were carried out for the sample problem stated in Chapter 6, using the control in $\mathcal{F}_2$ corresponding to $K_u = 10^{-4}$ (the optimum control for P2). The results are shown in Figure 7-8. (The lines in Figure 7-8 have the same significance as those in Figure 7-4).

The probability term for the control corresponding to the final iteration was computed from observations on a digital simulation of the system as described in Chapter 6. The results are shown in Figure 7-9. Using these results the final performance index is

$$I = -0.08 \qquad (7\text{-}2)$$

7.4 Comments on Convergence

The convergence of the conjugate gradient algorithm appears to have been affected by the following factors:

1. local minima;
2. Monte Carlo error;
3. error in the $\bar{\Psi}$ estimate.

The first two factors were anticipated in section 5.4. The shapes of the curves in Figures 7-4 and 7-8 suggest that local minima were being approached. In both Figures the effect of Monte Carlo error on the minimization over one iteration was apparent. In Figure 7-8 this appears to have been rather critical.

The third factor, error in the $\bar{\Psi}$ estimate, was probably important in the optimization shown in Figure 7-8. Not only was the error in $\bar{\Psi}$ large in this case (see Figure 7-3), but it also caused the shape of the $\bar{\Psi}$ versus K_u curve to be significantly distorted. This suggests the

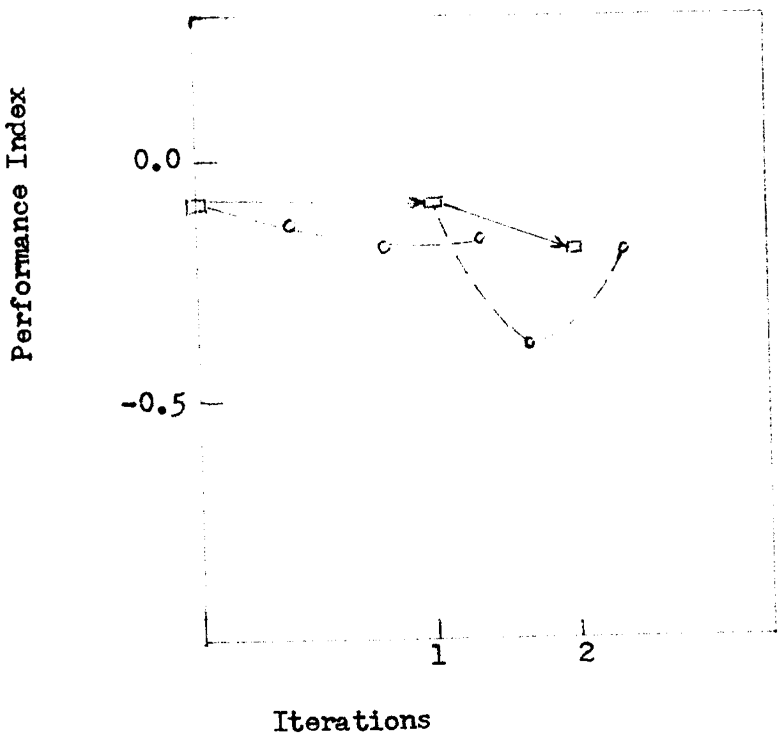


FIGURE 7-8. OPTIMIZATION STARTING FROM $u_{K_u} = 10^{-4}$

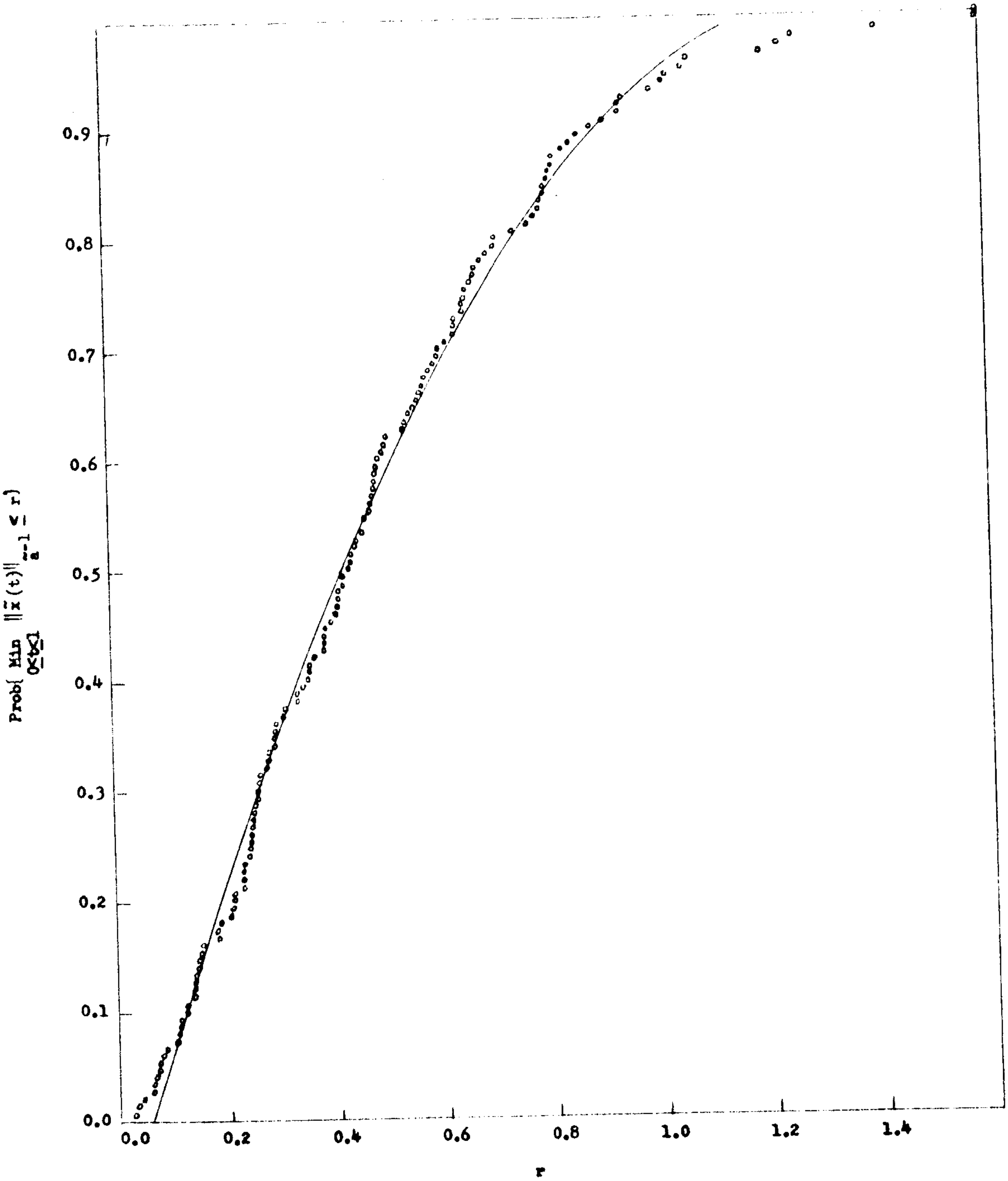


FIGURE 7-9. OBSERVED PROBABILITY TERM FOR FINAL POINT OF FIGURE 7-8

possibility that the error in $\bar{\psi}$ caused the algorithm to search in a significantly incorrect direction.

CHAPTER 8

SUMMARY AND CONCLUSIONS

8.1 Summary

A stochastic optimal control problem whose novel feature is the probability term in the performance index was formulated. The mathematical foundation on which the problem rests was reviewed. Two methods of attack on the problem were investigated:

1. A closed form estimate of the probability term was obtained by constructing an approximate solution to the diffusion equation. Pontryagin's maximum principle was applied to yield a two-point boundary value problem, and the conjugate gradient algorithm was applied for computation.
2. A suboptimal solution was obtained by formulating an appealing single parameter subclass of controls. The probability term was then found by digital simulation, and the class of controls was searched over its parameter for the optimal solution.

A sample problem was formulated and numerical results were obtained by both methods.

8.2 Conclusions

8.2.1 Suboptimal Solution

One of the most appealing features of the suboptimal approach is that it allows the user to exercise ingenuity in the selection of the subclass of controls. In Chapter 6 it was seen that the application of judgement to the results from one class of controls lead to the formulation of a better class. An obvious disadvantage of the method is that it offers no information as to the optimum solution.

It was observed in section 6.5 that the accuracy of the estimate of the probability term increases with probability. Thus the method is most effective if the optimal performance index has a relatively high probability term.

8.2.2 Conjugate Gradient Method

In contrast to the suboptimal method, this method is rather mechanical and does not allow much room for ingenuity. It does offer some information about the optimum solution, subject to the following comments:

1. The accuracy of the estimate of the probability term depends on a number of things, including the problem parameters, the control, and the Monte Carlo estimate of the integrals.
2. The conjugate gradient algorithm may converge to local minima.

It was observed in section 7.2.2 that the accuracy of the estimate of the probability term decreases with increasing probability. Thus, in contrast to the suboptimal method, this method is most effective if the optimal performance index has a relatively low probability term.

8.2.3 General Comments

The efficient use of computation time in both the above methods depends on the tradeoff of the number of Monte Carlo samples against the size of the time increment. The work done in Chapter 6 employed a time increment of .001. For this time increment, thirty six trials (points in Figures 6-5 through 6-9) required approximately twenty minutes of

IBM 7094 time. In the work done in Chapter 7, a time increment of .01 was used. This was apparently justified, since the Q_z matrix (see equation (5-39)) computed with a time increment of .01 agreed with the Q_z matrix computed with a time increment of .001 to at least four significant figures in each element. If this had been recognized during the work of Chapter 6, computing time could have been used much more efficiently.

A general observation on the relationship between the two methods is that they tend to complement each other. While one works best for low probabilities, the other works best for high probabilities. The suboptimal method provides a variety of appealing starting points for the conjugate gradient algorithm. This alleviates the problem of local minima. The conjugate gradient algorithm provides a possibility for improving on the best control in a subclass which has been searched by the suboptimal method. Thus the two methods should be viewed as complementary rather than competitive.

8.3 Avenues for Future Investigation

8.3.1 Linear Systems

The suboptimal method could be continued by seeking new subclasses. For example, a time varying factor might be included in the integrand of equation (6-41). An examination of the distribution of target manifold intercept times should suggest the general form of such a factor.

The effectiveness of the conjugate gradient method as a function of the number of terms in the Monte Carlo estimate of integrals should be investigated further, both theoretically and computationally.

The effect of system characteristics (e.g. pole and zero locations) on the problem should be investigated.

8.3.2 Non-linear Systems

Within the limitations discussed in section 2.4, the suboptimal method applies as easily to non-linear systems as to linear systems, provided that suitable subclasses of controls can be generated. The closed form probability estimate used in the conjugate gradient method, on the other hand, would have to be completely reworked. The formulation in section 3.4 makes immediate use of the linearity of the system. This could probably be avoided, since the system state is still a diffusion process for a broad class of non-linear systems (as mentioned in section 2.4). The construction of an approximate solution for such an equation is carried out in form by Mishchenko [4], [11]. This solution makes use of the transition density of the system state, as did the case considered here. A method for computing the transition density in the non-linear case would have to be found.

8.3.3 Closed Loop Solution

The only subject treated here has been the open loop solution of the problem. The closed loop solution should also be investigated. The application of the dynamic programming approach to the closed loop problem gives rise to some interesting questions. This comes about as follows. Let

$$J(\bar{x}, \tau) = \underset{u}{\mathrm{Min}} \left[\int_{\tau}^{T} \|u(t)\|_U^2 \, dt - \psi(\tau, \bar{x}, T)\right] \tag{8-1}$$

with

$$dx_t = (Ax + Bu)dt + C dn_t, \quad x(\tau) = \bar{x} \tag{8-2}$$

where u, U, ψ, x, A, B, C, and n are defined as in equations (3-1), (3-2), and (4-1).

Thus, the basic problem has been imbedded in a larger problem by making variable the initial time and state. Formal application of the principal of optimality leads to

$$0 = \underset{u}{\text{Min}} \{[\nabla_{\bar{x}} J]'[(Ax + Bu)d\tau + Cdn_\tau]$$

$$+ \frac{\partial J}{\partial \tau} d\tau + \|u(\tau)\|_U^2 d\tau + \frac{\partial \psi}{\partial \tau} d\tau$$

$$+ [\nabla_{\bar{x}} \psi]'[(Ax + Bu)d\tau + Cdn_\tau]\} \qquad (8\text{-}3)$$

Two difficulties are apparent:

1. The presence of dn_τ in (8-3) raises the question of what is really meant by the "optimal closed-loop solution." The meaning must be the best solution conditioned on $x(\tau) = \bar{x}$, since "dn_τ" cannot be measured. The question of how to carry out the minimization in (8-3) still remains. A possibility is to use the best estimate of "dn_τ" which is zero, and proceed from there. Obviously some careful definitions and theoretical work are required.

2. Supposing that the first difficulty can be surmounted, the minimization in (8-3) is far from trivial. The terms $\nabla_{\bar{x}}\psi$ and $\frac{\partial \psi}{\partial \tau}$ are functionals on $u(t)$, $t \in [\tau, T]$. How, then is the minimization to be interpreted? Is $u(\tau)$ alone involved, or is $u(t)$, $t \in [\tau, T]$ to be considered?

8.3.4 Performance Index and Constraints

An interesting variation on the problem would be to maximize the probability term subject to bounded control. A second variety of constraint would be bounded state. A third variety would be bounded energy. All present interesting avenues for future investigation.

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APPENDIX A

SOME ESTIMATES USEFUL FOR THE APPROXIMATE SOLUTION OF THE DIFFUSION EQUATION

The purpose of the appendix is to develop estimates of certain integrals and of certain solutions to the diffusion equation.

Let

$$\omega_j(\sigma,\xi,\tau) = \int_{\|\tilde{\eta}\| \geq 0} d\eta_1 \text{----} d\eta_m \frac{g(\sigma,\xi,\tau,\eta)}{\|\tilde{\eta}\|^j} , \quad j < k, \quad \tau > \sigma \qquad \text{(A-1)}$$

$$\text{where } g(\sigma,\xi,\tau,\eta) = \frac{\exp\left\{-\frac{\|\eta - \xi\|^2}{4(\tau-\sigma)}\right\}}{(2\pi)^{m/2}\ [2(\tau-\sigma)]^{m/2}} \qquad \text{(A-2)}$$

$$\tilde{\eta} = \begin{bmatrix} \eta_1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_k \end{bmatrix} , \qquad \hat{\eta} = \begin{bmatrix} \eta_{k+1} \\ \cdot \\ \cdot \\ \cdot \\ \eta_m \end{bmatrix} \qquad \text{(A-3)}$$

Two estimates for

$$\omega_j(\sigma,\xi,\tau) \Big|_{\xi \in \partial S} \qquad \text{(A-4)}$$

$$\text{where } \partial S = \left\{\xi \,\middle|\, \|\tilde{\xi}\| = r(\hat{\xi})\right\} \qquad \text{(A-5)}$$

will now be developed.

A rotation of coordinates yields

$$\omega_j(\sigma,\xi_1,\text{----},\xi_m, \tau) = \omega_j(\sigma, \|\tilde{\xi}\|, 0,\text{---}, 0, \hat{\xi},\tau) \qquad \text{(A-6)}$$

so that

$$\omega_j(\sigma,\tau,\tau) = \frac{1}{(2\pi)^{n/2}\,[2(\tau-\sigma)]^{n/2}} \int_{||\hat{\eta}||\geq 0} d\,\hat{\eta} \quad \exp\left\{-\frac{||\hat{\eta}-\hat{\xi}||^2}{4(\tau-\sigma)}\right\}$$

$$\int_{||\tilde{\eta}||\geq 0} d\tilde{\eta} \quad \frac{\exp\left\{-\frac{(\eta_1-||\tilde{\xi}||)^2+\cdots+\eta_k^2}{4(\tau-\sigma)}\right\}}{||\tilde{\eta}||^j} \qquad \text{(A-7)}$$

Let the second integral of (A-7) be

$$(2\pi)^{k/2}\,[2(\tau-\sigma)]^{k/2}\, I_j \qquad \text{(A-8)}$$

and let

$$\eta_i = ||\tilde{\xi}||\, x_i,\ i = 1,\ldots,k \qquad \text{(A-9)}$$

$$\tau - \sigma = ||\tilde{\xi}||^2 t \qquad \text{(A-10)}$$

Then

$$I_j = \frac{V(t)}{||\tilde{\xi}||^j} \qquad \text{(A-11)}$$

where

$$V(t) = \frac{1}{(4\pi t)^{k/2}} \int_{||x||\geq 0} dx_1 \cdots dx_k \, \frac{\exp\left\{-\frac{(x_1-1)^2+\cdots+x_k^2}{4t}\right\}}{||x||^j} \qquad \text{(A-12)}$$

Let

$$2\sqrt{t}\, y_i = x_i \quad ,\ i = 1, \cdots, k \qquad \text{(A-13)}$$

then

$$V(t) = \frac{1}{(2\sqrt{t})^j\, \pi^{k/2}} \int_{||y||\geq 0} dy_1 \cdots dy_k \, \frac{\exp\left\{-\left[(y_1-\frac{1}{2\sqrt{t}})^2+\cdots+y_k^2\right]\right\}}{||y||^j} \qquad \text{(A-14)}$$

From (A-12),

$$\lim_{t\to 0} V(t) = 1 \qquad \text{(A-15)}$$

Let

$$I_v(t) = \frac{1}{\pi^{k/2}} \int_{||y|| \geq 0} dy_1 --- dy_k \frac{\exp\left\{-\left[\left(y_1 - \frac{1}{2\sqrt{t}}\right)^2 + --- + y_k^2\right]\right\}}{|| y ||^j} \tag{A-16}$$

then from (A-14) and (A-15),

$$\lim_{t \to 0} I_v(t) = 0 \tag{A-17}$$

From (A-16),

$$\lim_{t \to \infty} I_v(t) = \frac{1}{\pi^{k/2}} \int_{||y|| \geq 0} dy_1 --- dy_k \frac{\exp\left\{- || y ||^2\right\}}{|| y ||^j} < \infty \tag{A-18}$$

Since $I_v(t)$ is continuous in t on $t \in (0, \infty)$, then

$$K_j = \sup_{t \in (0, \infty)} I_v(t) \tag{A-19}$$

exists. Then from (A-14) and (A-16)

$$V(t) \leq \frac{K_j}{(2\sqrt{t})^j} \tag{A-20}$$

Then from (A-14), (A-15), and (A-16)

$$\bar{V} = \max_{t > 0} V(t) \tag{A-21}$$

exists.

From (A-7), (A-8), (A-11), and (A-21),

$$\omega_j(\sigma, \xi, \tau) \leq \frac{\bar{V}}{|| \xi ||^j} \tag{A-22}$$

From (A-7), (A-8), (A-11), (A-20), and (A-10),

$$\omega_j(\sigma, \xi, \tau) \leq \frac{K_j}{2^j(\tau - \sigma)^{j/2}} \tag{A-23}$$

From (A-22) and (A-5),

$$\omega_j(\sigma,\xi,\tau)\Big|_{\xi\in\partial S} \leq \frac{\overline{V}}{r^j(\hat{\xi})} \qquad \text{(A-24)}$$

From (A-23)

$$\omega_j(\sigma,\xi,\tau) \leq \frac{K_j}{2^j K_t^{j/2} r^{j/2}(\hat{\xi})} \text{ for } \tau - \sigma \geq K_t\, r(\hat{\xi}) \qquad \text{(A-25)}$$

where K_t is any positive constant.

Next, let $\overline{\omega}_j$ be defined as

$$\overline{\omega}_j(\sigma,\xi,\tau) = \int_{||\hat{\eta}||\geq 0} d\,\eta \frac{||\hat{\eta}||\, g(\sigma,\xi,\tau,\eta)}{||\hat{\eta}||^j},\ j < k,\ \tau < \sigma \qquad \text{(A-26)}$$

Following the same procedure used in deriving (A-7) yields

$$\overline{\omega}_j(\sigma,\xi,\tau) = \frac{I_j}{(2\pi)^{\frac{m-k}{2}}\,[2(\tau-\sigma)]^{\frac{n-k}{2}}} \int_{||\hat{\eta}||\geq 0} d\hat{\eta}\, ||\hat{\eta}||\, \exp\left\{-\frac{||\hat{\eta}-\hat{\xi}||^2}{4(\tau-\sigma)}\right\} \qquad \text{(A-27)}$$

Let

$$\overline{\overline{\omega}}_i(\sigma,\underline{\xi},\tau) = \frac{1}{(2\pi)^{i/2}\,[2(\tau-\sigma)]^{\frac{i}{2}}} \int_{||\underline{r}||\geq 0} d\,\underline{\eta}\, ||\underline{\eta}||\, \exp\left\{-\frac{||\underline{\eta}-\underline{\xi}||^2}{4(\tau-\sigma)}\right\} \qquad \text{(A-28)}$$

where $\underline{\xi}$ and $\underline{\eta}$ are i-dimensional vectors. Then from (A-27) and (A-28),

$$\overline{\omega}_j(\sigma,\xi,\tau) = I_j\, \overline{\overline{\omega}}_{m-k}(\sigma,\hat{\xi},\tau) \qquad \text{(A-29)}$$

The estimate of $\overline{\overline{\omega}}_i$ proceeds as follows. Let

$$\underline{\eta} = \underline{r} + \underline{\xi} \qquad \text{(A-30)}$$

and

$$\rho = ||\underline{\xi}|| \qquad \text{(A-31)}$$

Then $\overline{\overline{\omega}}_i$ may be written

$$\overline{\overline{\omega}}_i(\sigma,\xi,\tau) = \frac{K_{\rho i}}{(2\pi)^{i/2}\,[2(\tau-\sigma)]^{i/2}} \int_0^\infty d\rho\,\rho^{i-1}\,||\xi+\xi||\,\exp\left\{-\frac{\rho^2}{4(\tau-\sigma)}\right\} \tag{A-32}$$

$$\text{where } K_{\rho i} = \begin{cases} 2^{\frac{i+1}{2}}\,\pi^{\frac{i-1}{2}}\,i \prod_{\ell=1}^{\frac{i+1}{2}} \frac{1}{2\ell-1}, & i \text{ odd} \\[2ex] \frac{i\pi^{i/2}}{(i/2)!}, & i \text{ even} \end{cases} \tag{A-33}$$

From (A-32)

$$\overline{\overline{\omega}}_i(\sigma,\xi,\tau) \le \frac{K_{\rho i}}{(2\pi)^{i/2}\,[2(\tau-\sigma)]^{i/2}} \int_0^\infty d\rho\,[\rho^i + \rho^{i-1}||\xi||]\,\exp\left\{-\frac{\rho^2}{4(\tau-\sigma)}\right\} \tag{A-34}$$

Application of formulas (860.15) and (860.16) of [7] to (A-34) yields

$$\overline{\overline{\omega}}_i(\sigma,\xi,\tau) \le k_{\rho i}\sqrt{\tau-\sigma} + \overline{k}_{\rho_i}\,||\,\xi\,|| \tag{A-35}$$

where

$$k_{\rho i} = \begin{cases} \frac{(\frac{i-1}{2})!\,K_{\rho i}}{2\pi^{i/2}}, & i \text{ odd} \\[2ex] \frac{1.3.5\ldots(i-1)\,K_{\rho i}}{2^{i/2}\,\pi^{\frac{i+1}{2}}}, & i \text{ even} \end{cases} \tag{A-36}$$

$$\bar{k}_{\rho i} = \begin{cases} \dfrac{1.3.5.\cdots.(i-2)\,K_\rho}{(2\pi)^{\frac{i+1}{2}}}, & i \text{ odd} \\[2ex] \dfrac{\left(\frac{i-2}{2}\right)!\,K_\rho}{2\pi^{k/2}}, & i \text{ even} \end{cases} \qquad \text{(A-37)}$$

then from (A-29), (A-11), (A-21), and (A-35),

$$\bar{\omega}_j(\sigma,\xi,\tau) \le \frac{\bar{V}}{\|\hat{\xi}\|^j}\left[k_{\rho,m-k}\sqrt{\tau-\sigma} + \bar{k}_{\rho,m-k}\,\|\hat{\xi}\|\right] \qquad \text{(A-38)}$$

and from (A-29), (A-11), (A-20), (A-10), and (A-35),

$$\bar{\omega}_j(\sigma,\xi,\tau) \le \frac{K_j}{2^j(\tau-\sigma)^{j/2}}\left[k_{\rho,m-k}\sqrt{\tau-\sigma} + \bar{k}_{\rho,m-k}\,\|\hat{\xi}\|\right] \qquad \text{(A-39)}$$

At this point a theorem bounding the solution of the diffusion equation will be proved.

Theorem Let $u(\sigma,\xi,\tau)$ be the solution of the equation.

$$\frac{\partial u}{\partial \sigma} = -\sum_{i,j=1}^{m} a_{ij}\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} - \sum_{i=1}^{m}[A\,\xi + U(\sigma)]_i\frac{\partial u}{\partial \xi_i} = L[u] \qquad \text{(A-40)}$$

with

$$u(\tau,\xi,\tau) = 0 \qquad \text{(A-41)}$$

$$u(\sigma,\xi,\tau)\Big|_{\xi \in \partial S} = w(\sigma,\tau) \qquad \text{(A-42)}$$

where $[a_{ij}]$ is a positive semidefinite real symmetric matrix with rank k and

$a_{ij} = 0$ for $i > k$ or $j > k$ (A-43)

$\| U(\sigma) \| \leq \bar{U}$ for all σ (A-44)

$\bar{U}$ is a positive constant

$$\lambda s = \left\{ \xi \;\middle|\; \| \tilde{\xi} \|_{\tilde{a}^{-1}} = r(\hat{\xi}) \right\} \tag{A-45}$$

$$\left[a_{ij} \right] = \begin{bmatrix} \tilde{a} & 0 \\ 0 & 0 \end{bmatrix} \tag{A-46}$$

ξ is an m dimensional vector

$\tilde{\xi}$ is a k dimensional vector

$\hat{\xi}$ is an m-k dimensional vector

$$\xi = \begin{bmatrix} \tilde{\xi} \\ \hat{\xi} \end{bmatrix} \tag{A-47}$$

$\bar{r} \geq r(\hat{\xi}) \geq \underline{r} > 0$ for all $\hat{\xi}$ (A-48)

$\bar{r}$ and $\underline{r}$ are constants

$\| \nabla_{\hat{\xi}} \, r(\hat{\xi}) \| \leq K_r \, r^{\beta}$

β is some constant such that

$$\beta > +1 \tag{A-49}$$

K_r is a constant

$$0 < w(\sigma,\tau) < \begin{cases} C \text{ for } \tau - \sigma \leq K_t \, r(\hat{\xi}) \\ \delta(r) \text{ for } \tau - \sigma > K_t \, r(\hat{\xi}) \end{cases} \tag{A-50}$$

C is a constant

K_t is a constant satisfying

$$K_t \geq \frac{\bar{\lambda}_Q(\tau,\sigma)}{\underline{r}} \tag{A-51}$$

$\bar{\lambda}_{Q(\tau,\sigma)}$ is the maximum eigenvalue of $Q(\tau,\sigma)$.

$$Q(\tau,\sigma) = 2 \int_{\sigma}^{\tau} d\alpha \left\{\exp \left[A(\tau-\alpha)\right]\right\} \left[a_{ij}\right] \left\{\exp \left[A(\tau-\alpha)\right]\right\}' \qquad \text{(A-52)}$$

$$\lim_{r \to 0} \delta(r) = 0 \qquad \text{(A-53)}$$

Then

$$0 < u(\sigma,\xi,\tau) < \Lambda(\xi,\underline{r}) + \delta\left[r(\hat{\xi})\right] \chi(\sigma,\xi,\tau) \text{ for } \xi \in S_U(\sigma,\underline{R},\tau) \qquad \text{(A-54)}$$

where Λ is defined by (A-159)

$\chi(\sigma,\xi,\tau)$ satisfies (A-40), (A-41), and

$$\chi(\sigma,\xi,\tau)\Big|_{\xi \in \partial S} = 1 \qquad \text{(A-55)}$$

$$S_U(\sigma,\underline{R},\tau) = \left\{\xi \;\Big|\; \min_{S\in[\sigma,\tau]} \|\tilde{\mu}(s)\| \geq \underline{R}\right\} \qquad \text{(A-56)}$$

$$\mu(s) = \left\{\exp\left[A(s-\sigma)\right]\right\} \xi + \int_{\sigma}^{s} d\alpha \left\{\exp\left[A(s-\alpha)\right]\right\} U(\alpha) \qquad \text{(A-56a)}$$

$\tilde{\mu}$ is a k dimensional vector

$\hat{\mu}$ is an m-k dimensional vector

$$\mu = \begin{bmatrix} \tilde{\mu} \\ \hat{\mu} \end{bmatrix} \qquad \text{(A-57)}$$

Proof Let

$$\bar{w}(\sigma,\tau) = \bar{w}_1(\sigma,\tau) + \bar{w}_2(\sigma,\tau) \qquad \text{(A-58)}$$

where

$$\bar{w}_1(\sigma,\tau) = 0 < \bar{C}, \text{ for } \tau - \sigma \leq K_t\, r(\hat{\xi}) \qquad \text{(A-59)}$$

where $\bar{C}$ is some positive constant greater than C.

$$\bar{w}_1(\sigma,\tau) = 0, \text{ for } \tau - \sigma > K_t\, r(\hat{\xi}) \tag{A-60}$$

$$\bar{w}_2(\sigma,\tau) = 0, \text{ for } \tau - \sigma \leq K_t\, r(\hat{\xi}) \tag{A-61}$$

$$\bar{w}_2(\sigma,\tau) = \delta\,[r(\hat{\xi})], \text{ for } \tau - \sigma > K_t\, r(\hat{\xi}) \tag{A-62}$$

Let $\bar{u}(\sigma,\xi,\tau)$, $\bar{u}_1(\sigma,\xi,\tau)$, and $\bar{u}_2(\sigma,\xi,\tau)$ be the solutions of (A-40) which have the value zero for $\sigma = \tau$, and the value $\bar{w}$, $\bar{w}_1$, $\bar{w}_2$, respectively for $\xi \in \partial S$. By Theorem C - 2,

$$0 \leq u(\sigma,\xi,\tau) < \bar{u}(\sigma,\xi,\tau) \tag{A-63}$$

Clearly,

$$\bar{u}(\sigma,\xi,\tau) = \bar{u}_1(\sigma,\xi,\tau) + \bar{u}_2(\sigma,\xi,\tau) \tag{A-64}$$

Again by Theorem C-2

$$0 \leq \bar{u}_2(\sigma,\xi,\tau) \leq \delta\,[r(\hat{\xi})]\,\chi(\sigma,\xi,\tau) \tag{A-65}$$

Let

$$\gamma(\xi) = \frac{\bar{C}\, r^{k-1}(\hat{\xi})}{||\xi||^{k-1}_{\tilde{a}^{-1}}} \tag{A-66}$$

Let v be defined as

$$v(\sigma,\xi,\tau) = \gamma(\xi) + \int_\sigma^\tau ds \int_{||\tilde{\eta}||_{\tilde{a}^{-1}} \geq r(\hat{\eta})} d\eta\; q(\sigma,\xi,s,\eta)\; L[\gamma(\eta)] \tag{A-67}$$

q is the fundamental solution to (A-40) outside ∂S

then $v(\sigma,\xi,\tau)$ is a solution to (A-40) with

$$v(\tau,\xi,\tau) = \frac{\bar{C}\, r^{k-1}(\hat{\xi})}{||\xi||^{k-1}_{\tilde{a}^{-1}}} \tag{A-68}$$

$$v(\sigma,\xi,\tau)\Big|_{\partial S} = \bar{C} \tag{A-69}$$

Then by Theorem C-1

$$v(\sigma, \tau, \tau) > \bar{u}_1(\sigma, \tau, \tau) \tag{A-70}$$

The next step is to estimate $L[\gamma(\xi)]$. Differentiating (A-66) yields

$$\frac{\partial \gamma}{\partial \xi_i} = - \frac{(k-1)\, \bar{C}\, r^{k-1}(\hat{\xi})}{\|\tilde{\xi}\|^{k+1}_{\tilde{a}^{-1}}} \sum_{\nu=1}^{k} \tilde{a}^{-1}_{i\nu}\, \xi_\nu, \quad i=1,\ldots,k \tag{A-71}$$

$$\frac{\partial \gamma}{\partial \xi_i} = \frac{(k-1)\, \bar{C}\, r^{k-2}(\hat{\xi})}{\|\tilde{\xi}\|^{k-1}_{\tilde{a}^{-1}}} \frac{\partial r}{\partial \xi_i}, \quad i = k+1, ---, m \tag{A-72}$$

$$\frac{\partial^2 \gamma}{\partial \xi_i \partial \xi_j} = \frac{(k-1)\,(k+1)\, \bar{C}\, r^{k-1}(\hat{\xi}) \sum_{\nu=1}^{k} \tilde{a}^{-1}_{i\nu}\, \xi_\nu \sum_{\beta=1}^{k} \tilde{a}^{-1}_{j\beta}\, \xi_\beta}{\|\tilde{\xi}\|^{k+3}_{\tilde{a}^{-1}}}$$

$$- \frac{(k-1)\, \bar{C}\, r^{k-1}(\hat{\xi})}{\|\tilde{\xi}\|^{k+1}_{\tilde{a}^{-1}}} \tilde{a}^{-1}_{ij}, \quad i, j = 1, 2, \ldots, k \tag{A-73}$$

Now, because of symmetry,

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \tilde{a}_{ij}\, \tilde{a}^{-1}_{ij} = \sum_{i=1}^{k} \sum_{j=1}^{k} \tilde{a}_{ij}\, \tilde{a}^{-1}_{ji} = \sum_{i=1}^{k} I^k_{ii} \tag{A-74}$$

where I^k is the identity matrix of rank k. Then

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \tilde{a}_{ij}\, \tilde{a}^{-1}_{ij} = k \tag{A-75}$$

Next,

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \tilde{a}_{ij} \sum_{\nu=1}^{k} \tilde{a}^{-1}_{i\nu} \xi_\nu \sum_{\beta=1}^{k} \tilde{a}^{-1}_{j\beta} \xi_\beta = \sum_{j=1}^{k} \sum_{\nu=1}^{k} I^{k}_{j\nu} \xi_\nu$$

$$\cdot \sum_{\beta=1}^{k} \tilde{a}^{-1}_{j\beta} \xi_\beta = \sum_{j=1}^{k} \xi_j (\tilde{a}^{-1} \tilde{\xi})_j = || \tilde{\xi} ||^2_{\tilde{a}-1} \qquad \text{(A-76)}$$

Combining (A-73), (A-75), and (A-76)

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \tilde{a}_{ij} \frac{\partial^2 v}{\partial \xi_i \partial \xi_j} = \frac{(k-1)\, \overline{C}\, r^{k-1}(\hat{\xi})}{||\tilde{\xi}||^{k+1}_{\tilde{a}^{-1}}} \qquad \text{(A-77)}$$

Next, consider $\sum_{i=1}^{m} [A\xi+BU]_i \frac{\partial v}{\partial \xi_i}$.

First,

$$\sum_{i=1}^{k} (A\xi)_i \sum_{\nu=1}^{k} \tilde{a}^{-1}_{i\nu} \xi_\nu \leq || \tilde{A} \tilde{\xi} + \hat{A} \hat{\xi} || \, || \tilde{a}^{-1} \tilde{\xi} || \qquad \text{(A-78)}$$

where

$$\tilde{A} = \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \\ A_{k1} & \cdots & A_{kk} \end{bmatrix} \qquad \text{(A-79)}$$

$$\hat{A} \quad \begin{bmatrix} A_{1,k+1} & \cdots & A_{1m} \\ \vdots & & \\ A_{k,k+1} & \cdots & A_{km} \end{bmatrix} \qquad \text{(A-80)}$$

Then from (B-29) and (B-18)

$$\sum_{i=1}^{k} (\Lambda\xi)_i \sum_{\nu=1}^{k} \tilde{a}^{-1}_{i\nu} \xi_\nu \leq \frac{||\tilde{\xi}||}{\lambda_{\tilde{a}}} \left[k K_{\tilde{\Lambda}} \; ||\tilde{\xi}|| + (m-k) K_{\hat{\Lambda}} \; ||\hat{\xi}|| \right] \tag{A-81}$$

where $\lambda_{\tilde{a}}$ is the smallest eigenvalue of $\tilde{a}$

$$K_{\tilde{\Lambda}} = \underset{1 \leq i \leq k}{\text{Max}} \; || \tilde{\Lambda} \, \tilde{e}_i || \tag{A-82}$$

$$K_{\hat{\Lambda}} = \underset{1 \leq i \leq m-k}{\text{Max}} \; || \hat{\Lambda} \, \hat{e}_i || \tag{A-83}$$

$\tilde{e}_i$ is the k dimensional vector with the i^{th} element equal to one, and all other elements equal to zero.

$\hat{e}_i$ is the m-k dimensional vector with i^{th} element equal to one, and all other elements equal to zero.

From (B-24),

$$\sum_{i=1}^{k} (\Lambda\xi)_i \sum_{\nu=1}^{k} \tilde{a}^{-1}_{i\nu} \xi_\nu \leq \frac{k K_{\tilde{\Lambda}} \lambda_{\tilde{a}}}{\lambda_{\tilde{a}}} ||\tilde{\xi}||^2_{\tilde{a}^{-1}} + \frac{(m-k) K_{\hat{\Lambda}} ||\hat{\xi}|| \sqrt{\lambda_{\tilde{a}}}}{\lambda_{\tilde{a}}} ||\tilde{\xi}||_{\tilde{a}^{-1}} \tag{A-84}$$

Similarly, using the bound on $||U(\sigma)||$ given in (A-44),

$$\sum_{i=1}^{k} U_i(\sigma) \sum_{\nu=1}^{k} \tilde{a}^{-1}_{i\nu} \xi_\nu < \bar{U} \frac{\sqrt{\lambda_{\tilde{a}}}}{\lambda_{\tilde{a}}} || \tilde{\xi} ||_{\tilde{a}^{-1}} \tag{A-85}$$

Next,

$$\sum_{i=k+1}^{m} (\Lambda \xi)_i \frac{\partial r}{\partial \xi_i} \leq || \tilde{\Lambda} \tilde{\xi} + \hat{\Lambda} \hat{\xi} || \; || \nabla_{\hat{\xi}} r || \tag{A-86}$$

where

$$\overline{A} = \begin{bmatrix} A_{k+1,1} & \cdots & A_{k+1,k} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} \end{bmatrix} \tag{A-87}$$

$$\underline{A} = \begin{bmatrix} A_{k+1,k+1} & \cdots & A_{k+1,m} \\ \vdots & & \vdots \\ A_{m,k+1} & \cdots & A_{mm} \end{bmatrix} \tag{A-88}$$

From (B-30), (B-24), and (B-26),

$$\sum_{i=k+1}^{m} (A\ \xi)_i \ \frac{\partial r}{\partial \xi_i} \le \Big[k\ K_{\overline{A}} \sqrt{\lambda_{\tilde{a}}}\ \|\ \tilde{\xi}\ \|_{\tilde{a}^{-1}} + (m-k)\ K_{\underline{A}}\ \|\ \hat{\xi}\ \| \Big] \Big[K_r r^{\beta}(\hat{\xi}) \Big] \tag{A-89}$$

where β is given by (A-49)

$$K_{\overline{A}} = \max_{1 \le i \le k} \|\ \overline{A}\ \tilde{e}_i\ \| \tag{A-90}$$

$$K_{\underline{A}} = \max_{1 \le i \le m-k} \|\ \underline{A}\ \hat{e}_i\ \| \tag{A-91}$$

Also

$$\sum_{i=k+1}^{m} \big[U(\sigma) \big]_i \ \frac{\partial r}{\partial \xi_i} \le \overline{U}\ K_r\ r^{\beta}(\hat{\xi}) \tag{A-92}$$

Combining (A-71), (A-72), (A-84), (A-85), (A-89), and (A-92) yields

$$\sum_{i=1}^{m} \big[A\xi + U(\sigma) \big]_i \ \frac{\partial \gamma}{\partial \xi_i} \le \frac{\Big[\overline{U} + (m-k) K_{\underline{A}} \|\hat{\xi}\| \Big] \frac{\sqrt{\lambda_{\tilde{a}}}}{\lambda_{\tilde{a}}} (k-1)\ \overline{C}\ r^{k-1}(\hat{\xi})}{\|\ \tilde{\xi}\ \|^{k}_{\tilde{a}^{-1}}}$$

$$+ \frac{(k-1)\ \overline{C} \Big\{ \Big[(m-k) K_{\underline{A}} \|\hat{\xi}\| + \overline{U} \Big] K_r r^{k-2+\beta}(\hat{\xi}) \cdot \frac{k K_{\overline{A}}\ \overline{\lambda}_{\tilde{a}}}{\lambda_{\tilde{a}}} r^{k-1}(\tilde{\xi}) \Big\}}{\|\ \tilde{\xi}\ \|^{k-1}_{\tilde{a}^{-1}}} +$$

$$+ \frac{k(k-1)\, K_{\tilde{A}}\, \bar{U} \sqrt{\bar{\lambda}_{\tilde{a}}}\; r^{k-2+\beta}(\hat{\xi}) K_r}{\|\tilde{\xi}\|_{\varepsilon-1}^{k-2}} \tag{A-93}$$

Combining (A-77), (A-93), and (B-24),

$$\left| L\left[v(\xi)\right] \right| \leq \frac{M_{k+1}\, r^{k-1}(\hat{\xi})}{\|\tilde{\xi}\|^{k+1}} + \frac{\left[M_k + \bar{M}_k \|\hat{\xi}\|\right] r^{k-1}(\hat{\xi})}{\|\tilde{\xi}\|^{k}}$$

$$+ \frac{M_{k-1}\, r^{k-1}(\hat{\xi}) + \left[\bar{\bar{M}}_{k-1} + \bar{M}_{k-1} \|\hat{\xi}\|\right] r^{k-2+\beta}(\hat{\xi})}{\|\tilde{\xi}\|^{k-1}}$$

$$+ \frac{M_{k-2}\, r^{k-2+\beta}(\hat{\xi})}{\|\tilde{\xi}\|^{k-2}} \tag{A-94}$$

where

$$M_{k+1} = (k-1)\, \bar{C}\, \bar{\lambda}_{\tilde{a}}^{\frac{k+1}{2}} \tag{A-95}$$

$$M_k = \bar{U} \frac{\sqrt{\bar{\lambda}_{\tilde{a}}}}{\underline{\lambda}_{\tilde{a}}} (k-1)\, \bar{C}\, \bar{\lambda}_{\tilde{a}}^{k/2} \tag{A-96}$$

$$\bar{M}_k = (m-k)\, K_{\hat{A}}\, (k-1)\, \bar{C}\, \frac{\bar{\lambda}_{\tilde{a}}^{\frac{k+1}{2}}}{\underline{\lambda}_{\tilde{a}}} \tag{A-97}$$

$$M_{k-1} = (k-1)\, \bar{C}\, k\, K_{\tilde{A}} \frac{\bar{\lambda}_{\tilde{a}}^{\frac{k+1}{2}}}{\underline{\lambda}_{\tilde{a}}} \tag{A-98}$$

$$\bar{M}_{k-1} = (k-1)\, \bar{C}\, (m-k)\, K_{\underline{A}}\, K_r\, \bar{\lambda}_{\tilde{a}}^{\frac{k-1}{2}} \tag{A-99}$$

$$\bar{\bar{M}}_{k-1} = (k-1)\, \bar{C}\, \bar{U}\, K_r\, \bar{\lambda}_{\tilde{a}}^{\frac{k-1}{2}} \tag{A-100}$$

$$M_{k-2} = k\,(k-1)\, K_{\bar{A}}\, \bar{C}\, \bar{\lambda}_{\tilde{a}}^{\frac{k-1}{2}}\, K_r \tag{A-101}$$

Let

$$p(\sigma,\xi,s,\eta) = \frac{\exp\{-\frac{1}{2}\,\|\eta-\mu(s)\|^2_{Q^{-1}(s,\sigma)}\}}{(2\pi)^{m/2}\,[\det Q(s,\sigma)]^{1/2}} \tag{A-102}$$

where $\mu(s)$ and Q are defined by (A-56a) and (A-52) respectively.

Then from Theorem 2-5,

$p(\sigma,\xi,s,\eta)$ = probability density associated with the event $x(s) = \eta$ given that $x(\sigma) = \xi$, where x is the solution of (2-1) (A-103)

Mishchenko [4] shows that

$q(\sigma,\xi,s,\eta)$ = Probability density associated with the event $x(s) = \eta$ given that $x(\sigma) = \xi$ and $\xi(t) \notin \partial S$ for all $t \in [\sigma,s]$, where x is the solution to (2-1). (A-104)

From (A-103) and (A-104),

$$p(\sigma,\xi,s,\eta) \geq q(\sigma,\xi,s,\eta) \tag{A-105}$$

then from (A-67),

$$v(\sigma,\xi,\tau) \leq \gamma(\xi) + \int_{\sigma}^{\tau} ds \int_{\|\tilde{\eta}\|_{\tilde{a}^{-1}} \geq r(\hat{\eta})} d\eta\, p(\sigma,\xi,s,\eta) L[\gamma(\eta)] \tag{A-106}$$

By the law of the mean,

$$v(\sigma,\xi,\tau) \leq \gamma(\xi) + (\tau-\sigma) \int_{\|\tilde{\eta}\|_{\tilde{a}^{-1}} \geq r(\hat{\eta})} d\eta\; p(\sigma,\xi,\bar{s},\eta) L[\gamma(\eta)]$$

for some $\bar{s} \in [\sigma,\tau]$ (A-107)

From (A-107)

$$v(\sigma,\xi,\tau) \leq \gamma(\xi) + (\tau - \sigma)\, L\,[v(\xi)] \text{ if } \bar{s} = \sigma \tag{A-108}$$

The estimate for $\bar{s} \in (\sigma,\tau]$ is carred out as follows.

By the law of the mean,

$$Q(\bar{s},\sigma) = 2\,(\bar{s} - \sigma)\,\bar{Q}\,(\bar{s},\sigma) \tag{A-109}$$

where
$$\bar{Q}_{ij}\,(\bar{s},\sigma) = \left\{\exp\,[A(\bar{s}-\bar{\sigma}_{ij})]\,[a_{ij}]\,\exp\,[A'(\bar{s}-\bar{\sigma}_{ij})]\right\}_{ij} \tag{A-110}$$

and each $\bar{\sigma}_{ij} \in [\sigma,\bar{s}]$ (A-111)

From (B-24),

$$||\eta - \mu||_{Q^{-1}_{(\bar{s},\sigma)}} \geq \frac{||\eta - \mu||}{\sqrt{2(\bar{s}-\sigma)\,\bar{\lambda}_{\bar{s}\sigma}}} \tag{A-112}$$

where $\bar{\lambda}_{\bar{s}\sigma}$ is the maximum eigenvalue of $\bar{Q}(\bar{s},\sigma)$

and
$$\det Q(\bar{s},\sigma) \geq \left[2\,\underline{\lambda}_{\bar{s}\sigma}\,(\bar{s} - \sigma)\right]^m \tag{A-113}$$

where $\underline{\lambda}_{\bar{s}\sigma}$ is the minimum eigenvalue of $\bar{Q}(\bar{s},\sigma)$. Then from (A-102) and (A-2)

$$p(\sigma,\xi,\bar{s},\eta) \leq \left[\frac{\bar{\lambda}_{\bar{s}\sigma}}{\underline{\lambda}_{\bar{s}\sigma}}\right]^{\frac{m}{2}} g(\bar{\lambda}_{\bar{s}\sigma}\,\sigma,\mu,\bar{\lambda}_{\bar{s}\sigma}\bar{s},\eta) \tag{A-114}$$

From (B-24),

$$\int_{||\tilde{\eta}||_{\tilde{Q}^{-1}} \geq r(\hat{\eta})} d\eta\; p(\sigma,\xi,\bar{s},\eta)\,\Big|\,L[\gamma(\eta)]\Big| \leq \int_{||\tilde{\eta}|| \geq \sqrt{\lambda_{\tilde{Q}}}\, r(\hat{\eta})} d\eta\; p(\sigma,\xi,\bar{s},\eta)\,\Big|\,L[\gamma(\eta)]\Big| \tag{A-115}$$

From (A-114), and (A-115),

$$\int\limits_{\|\tilde{\eta}\|_{\tilde{a}^{-1}} \geq r(\hat{\eta})} d\eta \; p(\sigma,\xi,\bar{s},\eta)\,|L[\gamma(\eta)]| \leq \left[\frac{\bar{\lambda}_{\bar{s}\sigma}}{\underline{\lambda}_{\bar{s}\sigma}}\right]^{\frac{m}{2}} \int\limits_{\|\tilde{\eta}\| \geq \sqrt{\underline{\lambda}_{\tilde{a}}}\,r(\hat{\eta})} d\eta \; g(\bar{\lambda}_{\bar{s}\sigma}\sigma,\mu,\bar{\lambda}_{\bar{s}\sigma}\bar{s},\eta)\,|L[\gamma(\eta)]| \qquad \text{(A-116)}$$

The next step is to estimate the various terms of the right hand side of (A-116) which will result from substitution from (A-94).

From (A-48),

$$[\sqrt{\underline{\lambda}_{\tilde{a}}}\;\underline{r}]^{-(1+\nu)} \geq \|\tilde{\eta}\|^{-(1+\nu)} \text{ for } \|\tilde{\eta}\| > \sqrt{\underline{\lambda}_{\tilde{a}}}\; r(\hat{\eta}) \text{ and } 0 < \nu < 1 \qquad \text{(A-117)}$$

Then

$$\int\limits_{\|\tilde{\eta}\| > \sqrt{\underline{\lambda}_{\tilde{a}}}\,r(\hat{\eta})} d\eta \; \frac{g(\delta,\mu,t,\eta)}{\|\tilde{\eta}\|^{k-1}} \leq [\sqrt{\underline{\lambda}_{\tilde{a}}}\;\underline{r}]^{-(1+\nu)} \int\limits_{\|\tilde{\eta}\| > \sqrt{\underline{\lambda}_{\tilde{a}}}\,r(\hat{\eta})} d\eta \; \frac{g(s,\mu,t,\eta)}{\|\tilde{\eta}\|^{k-\nu}} \qquad \text{(A-118)}$$

Then from (A-22) and (A-1),

$$\int\limits_{\|\tilde{\eta}\| > \sqrt{\underline{\lambda}_{\tilde{a}}}\,r(\hat{\eta})} d\eta \qquad g(\delta,\mu,t,\eta)\,\frac{1}{\|\tilde{\eta}\|^{k+1}} \leq \frac{[\sqrt{\underline{\lambda}_{\tilde{a}}}\;\underline{r}]^{-(1+\nu)}\;\bar{V}}{\|\tilde{\mu}\|^{k-\nu}} \qquad \text{(A-119)}$$

where $\tilde{\mu}$ is defined by (A-57).

Similarly,

$$\int\limits_{\|\tilde{\eta}\| > \sqrt{\lambda_g}\, r(\hat{\eta})} d\eta \quad g(\delta,\mu,t,\eta)\, \frac{1}{\|\tilde{\eta}\|^{k}} \leq \frac{\left[\sqrt{\lambda_g}\, \underline{r}\right]^{-\nu} \bar{V}}{\|\tilde{\mu}\|^{k-\nu}} \tag{A-120}$$

$$\int\limits_{\|\tilde{\eta}\| > \sqrt{\lambda_g}\, r(\hat{\eta})} d\eta \quad g(\delta,\mu,t,\eta)\, \frac{1}{\|\tilde{\eta}\|^{k-1}} \leq \frac{\bar{V}}{\|\tilde{\mu}\|^{k-1}} \tag{A-121}$$

$$\int\limits_{\|\tilde{\eta}\| > \sqrt{\lambda_g}\, r(\hat{\eta})} d\eta \quad g(\delta,\mu,t,\eta)\, \frac{1}{\|\tilde{\eta}\|^{k-2}} \leq \frac{\bar{V}}{\|\tilde{\mu}\|^{k-2}} \tag{A-122}$$

From (A-26) and (A-117),

$$\int\limits_{\|\tilde{\eta}\| > \sqrt{\lambda_g}\, r(\hat{\eta})} d\eta\; g(\delta,\mu,t,\eta)\, \frac{\|\hat{\eta}\|}{\|\tilde{\eta}\|^{k}} \leq \left[\sqrt{\lambda_g}\, \underline{r}\right]^{-\nu} \bar{\omega}_{k-\nu}(\delta,\mu,t) \tag{A-123}$$

then from (A-38),

$$\int\limits_{\|\tilde{\eta}\| > \sqrt{\lambda_g}\, r(\hat{\eta})} d\eta\; g(\delta,\mu,t,\eta)\, \frac{\|\hat{\eta}\|}{\|\tilde{\eta}\|^{k}} \leq \left[\sqrt{\lambda_g}\, \underline{r}\right]^{-\nu} \left\{ \frac{\bar{V}\, \bar{k}_{\rho,m-k}\, \|\hat{\mu}\|}{\|\tilde{\mu}\|^{k-\nu}} + \frac{k_{\rho,m-k} \sqrt{K_t\, \underline{r}}\, \bar{V}}{\|\tilde{\mu}\|^{k-\nu}} \right\} \tag{A-124}$$

for $t - \delta \leq K_t\, \underline{r}$ where $\hat{\mu}$ is defined by (A-57).

Similarly,

$$\int\limits_{\|\tilde{\eta}\| > \sqrt{\lambda_g}\, r(\hat{\eta})} d\eta\; g(\delta,\mu,t,\eta)\, \frac{\|\hat{\eta}\|}{\|\tilde{\eta}\|^{k-1}} = \frac{\bar{V}\, \bar{k}_{\rho,m-k}\, \|\hat{\mu}\|}{\|\tilde{\mu}\|^{k-1}} + \frac{k_{\rho,m-k} \sqrt{K_t\, \underline{r}}\, \bar{V}}{\|\tilde{\mu}\|^{k-1}}, \quad t-\delta \leq K_t\, \underline{r} \tag{A-125}$$

Combining (A-94), (A-119), (A-120), (A-121), (A-122), (A-124), and (A-125)

$$\int\limits_{||\tilde{\eta}|| > \sqrt{\lambda_{\tilde{\xi}}}\, r(\hat{\eta})} d\eta\; g(s,\mu,t,\eta) \left| L[\gamma(\eta)] \right| \leq \frac{M_{k+1} \bar{r}^{k-1} [\sqrt{\lambda_{\tilde{\xi}}}\, \underline{r}]^{-(1+\nu)}\, \bar{V} + M_k \bar{r}^{k-1} [\sqrt{\lambda_{\tilde{\xi}}}\, \underline{r}]^{-\nu}\, \bar{V}}{||\tilde{\mu}||^{k-\nu}}$$

$$+ \frac{\bar{M}_k\, \bar{r}^{k-1}\, \bar{V}\, [\sqrt{\lambda_{\tilde{\xi}}}\, \underline{r}]^{-\nu}\, [\bar{K}_{\rho,m-k} ||\hat{\mu}|| + k_{\rho,m-k} \sqrt{K_t\, \underline{r}}]}{||\tilde{\mu}||^{k-\nu}}$$

$$+ \frac{M_{k-1} \bar{r}^{k-1}\, \bar{V} + \bar{\bar{M}}_{k-1} \bar{V} \bar{r}^{k-2+\beta} + \bar{M}_{k-1}\, \bar{r}^{k-2+\beta}\, \bar{V} [\bar{K}_{\rho,m-k} ||\hat{\mu}|| + k_{\rho,m-k} \sqrt{K_t\, \underline{r}}]}{||\tilde{\mu}||^{k-1}}$$

$$+ \frac{M_{k-2}\, \bar{r}^{k-2+\beta}\, \bar{V}}{||\tilde{\mu}||^{k-2}}, \quad t - s \leq K_t\, \underline{r} \qquad \text{(A-126)}$$

From (A-116) and (A-126)

$$\int\limits_{||\tilde{\eta}||_{\xi-1} \geq r(\hat{\eta})} d\eta\; p(\sigma,\xi,s,\eta) \left| L[\gamma(\eta)] \right| \leq \left[\frac{\bar{\lambda}_{s\sigma}}{\underline{\lambda}_{s\sigma}} \right] \left\{ \frac{M_{k+1} \bar{r}^{k-1} [\sqrt{\lambda_{\tilde{\xi}}}\, \underline{r}]^{-(1+\nu)} \bar{V} + M_k \bar{r}^{k-1} [\sqrt{\lambda_{\tilde{\xi}}}\, \underline{r}]^{-\nu} \bar{V}}{||\tilde{\mu}||^{k-\nu}} \right.$$

$$+ \frac{\bar{M}_k\, \bar{r}^{k-1}\, \bar{V} [\sqrt{\lambda_{\tilde{\xi}}}\, \underline{r}]^{-\nu} [\bar{K}_{\rho,m-k} ||\hat{\mu}|| + k_{\rho,m-k} \sqrt{K_t\, \underline{r}}]}{||\tilde{\mu}||^{k-\nu}}$$

$$+ \frac{M_{k-1}\, \bar{r}^{k-1}\, \bar{V} + \bar{\bar{M}}_{k-1} \bar{V}\, \bar{r}^{k-2+\beta} + \bar{M}_{k-1} \bar{r}^{k-2+\beta}\, \bar{V} [\bar{K}_{\rho,m-k} ||\hat{\mu}|| + k_{\rho,m-k} \sqrt{K_t\, \underline{r}}]}{||\tilde{\mu}||^{k-1}}$$

$$\left. + \frac{M_{k-2}\, \bar{r}^{k-2+\beta}\, \bar{V}}{||\tilde{\mu}||^{k-2}} \right\}, \quad \bar{s} - \sigma \leq \frac{K_t}{\underline{\lambda}_{s\sigma}}\, \underline{r} \qquad \text{(A-127)}$$

where K_t is defined in (A-51)

The requirement placed on $\bar{s} - \sigma$ in (A-127) requires some discussion. It will now be shown that (A-51) guarantees that condition. Wonham [2] defines the concept of a monotonic matrix and shows that $Q(s,\sigma)$ is monotonic increasing in s. Then, for $\bar{s} \in [\sigma,\tau]$,

$$\bar{\lambda}_{Q(\tau,\sigma)} \geq \bar{\lambda}_{Q(\bar{s},\sigma)} \tag{A-128}$$

Then from (A-51),

$$K_t \geq \frac{\bar{\lambda}_{Q(\bar{s},\sigma)}}{\underline{r}} \tag{A-129}$$

From (A-109)

$$\bar{\lambda}_{Q(\bar{s},\sigma)} = 2(\bar{s} - \sigma)\, \bar{\lambda}_{\bar{s}\sigma} \tag{A-130}$$

Then from (A-129) and (A-130),

$$\bar{s} - \sigma \leq \frac{K_t}{2\bar{\lambda}_{\bar{s}\sigma}}\, \underline{r} < \frac{K_t}{\bar{\lambda}_{\bar{s}\sigma}}\, \underline{r} \tag{A-131}$$

which is the requirement placed on $\bar{s} - \sigma$ by (A-127). Thus (A-127) holds for all $\bar{s} \in [\sigma,\tau]$.

The estimate of the left hand side of (A-127) will now be put in final form. Let

$$k_r = \bar{r} \,/\, \underline{r} \tag{A-132}$$

Then (A-127) may be written

$$\int d\eta\, K(\sigma,\tau,s,\eta)\ \| L[Y(\eta)] \| \leq \frac{N_{1+\nu}\, \underline{r}^{k-2-\nu} + N_{\nu-1/2}\, \underline{r}^{k-1/2-\nu} + N_{\nu}\, \underline{r}^{k-1-\nu}}{\|\tilde{u}\|^{k-\nu}}$$

$\|\tilde{\eta}\|_{\tilde{a}-1} \geq r(\hat{\eta})$

$$+ \frac{\bar{N}_{\nu}\, \|\hat{u}\|\, \underline{r}^{k-1-\nu}}{\|\tilde{u}\|^{k-\nu}} + \frac{N_{1/2-\beta}\, \underline{r}^{k-3/2+\beta} + N_{o}\, \underline{r}^{k-1} + N_{1-\beta}\, \underline{r}^{k-2+\beta}}{\|\tilde{u}\|^{k-1}}$$

$$+ \frac{\bar{N}_{1-\beta}\, \|\hat{u}\|\, \underline{r}^{k-2+\beta}}{\|\tilde{u}\|^{k-1}} + \frac{N_{1}\, \underline{r}^{k-2+\beta}}{\|\tilde{u}\|^{k-2}}\ , \quad \bar{s} \in (\sigma,\tau] \qquad \text{(A-133)}$$

where

$$N_{1+\nu} = M_{k+1} k_r^{k-1}\, \bar{V}\, \underline{\lambda}_{\tilde{a}}^{-\frac{1+\nu}{2}} \left[\frac{\bar{\lambda}_{\bar{s}\sigma}}{\underline{\lambda}_{\bar{s}\sigma}}\right]^{m/2} \qquad \text{(A-134)}$$

$$N_{\nu-1/2} = \bar{M}_k\, k_r^{k-1}\, \bar{V}\, \underline{\lambda}_{\tilde{a}}^{-\nu/2}\ k_{\rho,m-k} \sqrt{K_t} \left[\frac{\bar{\lambda}_{\bar{s}\sigma}}{\underline{\lambda}_{\bar{s}\sigma}}\right]^{m/2} \qquad \text{(A-135)}$$

$$N_{\nu} = M_k\, k_r^{k-1}\ \bar{V}\ \underline{\lambda}_{\tilde{a}}^{-\nu/2} \left[\frac{\bar{\lambda}_{\bar{s}\sigma}}{\underline{\lambda}_{\bar{s}\sigma}}\right]^{m/2} \qquad \text{(A-136)}$$

$$\bar{N}_{\nu} = \bar{M}_k\, k_r^{k-1}\, \bar{V}\ \underline{\lambda}_{\tilde{a}}^{-\nu/2}\, \bar{k}_{\rho,m-k} \left[\frac{\bar{\lambda}_{\bar{s}\sigma}}{\underline{\lambda}_{\bar{s}\sigma}}\right]^{m/2} \qquad \text{(A-137)}$$

$$N_{1/2-\beta} = M_{k-1}\, k_r^{k-2+\beta}\, \bar{V}\, k_{\rho,m-k} \sqrt{K_t} \left[\frac{\bar{\lambda}_{\bar{s}\sigma}}{\underline{\lambda}_{\bar{s}\sigma}}\right]^{m/2} \qquad \text{(A-138)}$$

$$N_{o} = M_{k-1}\, \bar{V}\, k_r^{k-1} \left[\frac{\bar{\lambda}_{\bar{s}\sigma}}{\underline{\lambda}_{\bar{s}\sigma}}\right]^{m/2} \qquad \text{(A-139)}$$

$$\bar{N}_{1-\beta} = \bar{\bar{N}}_{k-1} \bar{V} k_r^{k-2+\beta} \left[\frac{\bar{\lambda}_{\bar{s}\sigma}}{\underline{\lambda}_{\bar{s}\sigma}}\right]^{m/2} \tag{A-140}$$

$$\bar{N}_{1-\beta} = \bar{N}_{k-1} \bar{V} k_r^{k-2+\beta} k_{\rho,m-k} \left[\frac{\bar{\lambda}_{\bar{s}\sigma}}{\underline{\lambda}_{\bar{s}\sigma}}\right]^{m/2} \tag{A-141}$$

$$N_1 = M_{k-2} \bar{V} k_r^{k-2+\beta} \left[\frac{\bar{\lambda}_{\bar{s}\sigma}}{\underline{\lambda}_{\bar{s}\sigma}}\right]^{m/2} \tag{A-142}$$

From (A-70), (A-107), (B-24), (A-66), and (A-133),

$$\bar{u}_1(\sigma,\xi,\tau) \le \frac{\bar{N}_1 \underline{r}^{k-1}}{||\xi||^{k-1}} + \frac{K_t (N_{1+\nu} \underline{r}^{k-1-\nu} + N_{\nu-1/2} \underline{r}^{k+1/2-\nu} + N_\nu \underline{r}^{k-\nu}}{||\tilde{u}||^{k-\nu}}$$

$$+ \frac{K_t \bar{N}_\nu ||\hat{u}|| \underline{r}^{k-\nu}}{||\tilde{u}||^{k-\nu}} + \frac{K_t(N_{1/2-\beta} \underline{r}^{k-1/2+\beta} + N_o \underline{r}^k + N_{1-\beta} \underline{r}^{k-1+\beta})}{||\tilde{u}||^{k-1}}$$

$$+ \frac{K_t \bar{N}_{1-\beta} ||\hat{u}|| \underline{r}^{k-1+\beta}}{||\tilde{u}||^{k-1}} + \frac{K_t N_1 \underline{r}^{k-1+\beta}}{||\tilde{u}||^{k-2}} \quad \text{for } \bar{s} \in (\sigma,\tau],\ \tau-\sigma \le K_t \underline{r} \tag{A-143}$$

where

$$\bar{N}_1 = \bar{C} k_r^{k-1} \bar{\lambda}_{\xi}^{(k-1)/2} \tag{A-144}$$

From (A-70), (A-109), (A-94), (A-66), and (B-24),

$$\bar{u}_1(\sigma,\tau,\tau) \leq \frac{\bar{M}_1 \; \underline{r}^{k-1}}{||\xi||^{k-1}} + \frac{k_r \; K_t \; \underline{r}^k}{||\xi||^{k-1}}$$

$$+ \frac{k_r \; K_t \left[M_k + \bar{M}_k \; ||\hat{\xi}||\right] \underline{r}^{k-2}}{||\xi||^k}$$

$$+ \frac{K_t \; k_r \left\{M_{k-1} \; \underline{r}^k + \left[\bar{\bar{M}}_{k-1} + \bar{M}_{k-1} \; ||\hat{\xi}||\right] \underline{r}^{k-1+\beta}\right.}{||\xi||^{k-1}}$$

$$+ \frac{K_t \; k_r \; \underline{r}^{k-1+\beta}}{||\xi||^{k-2}} \quad \text{for } \bar{s} = \sigma, \; \tau-\sigma \leq K_t \; \underline{r} \tag{A-145}$$

From (A-56a) and (B-30)

$$||u|| \leq K_{exp} \left[||\xi|| + (\tau - \sigma) \; \bar{U}\right] \tag{A-146}$$

where

$$K_{exp} = m \max_{s \in [\tau,\sigma]} \; \max_{1 \leq i \leq m} \; || \left\{\exp\left[A(s-\sigma)\right]\right\} e_i || \tag{A-147}$$

e_i is the m dimensional vector whose i^{th} element is one and whose other elements are zero.

From (A-56a),

$$u = \xi \text{ for } \bar{s} = \sigma \tag{A-148}$$

Then from (A-56), (A-143), (A-146), and (A-145)

$$\bar{u}_1(\sigma,\tau,\tau) \leq D_1^{\bar{s}}(\underline{r}) + D_2^{\bar{s}}(\underline{r}) \; ||\xi|| \; , \; \tau \in S_U(\sigma,\underline{R},\tau), \; \tau-\sigma \leq K_t \; \underline{r} \tag{A-149}$$

where

$$D_1^{\bar{s}}(\underline{r}) = \frac{\bar{M}_1}{\underline{R}^{k-1}} + \frac{K_t M_1 \underline{r}^{\beta}}{\underline{R}^{k-2}} \underline{r}^{k-1} + \frac{K_t (M_{1+\nu} \underline{r}^{k-1-\nu} + M_{\nu-1/2} \underline{r}^{k+1/2-\nu} + M_{\nu} \underline{r}^{k-\nu}}{\underline{R}^{k-\nu}}$$

$$+ \frac{K_t (M_{1/2-\beta} \underline{r}^{k-1/2+\beta} + M_{\sigma} \underline{r}^{k} + M_{1-\beta} \underline{r}^{k-1+\beta})}{\underline{R}^{k-1}} + \frac{K_t^2 \bar{M}_{\nu} \bar{U} \underline{r}^{k-\nu+1}}{\underline{R}^{k-\nu}}$$

$$+ \frac{K_t^2 \bar{M}_{1-\beta} \bar{U} \underline{r}^{k-\beta}}{\underline{R}^{k-1}}, \quad \bar{s} \in (\sigma, \tau], \quad \tau - \sigma \leq K_t \underline{r} \tag{A-150}$$

$$D_2^{\bar{s}}(\underline{r}) = \frac{K_t \bar{M}_{\nu}}{\underline{R}^{k-\nu}} \underline{r}^{k-\nu} + \frac{K_t \bar{M}_{1-\beta}}{\underline{R}^{k-1}} \underline{r}^{k-1+\beta}, \quad \bar{s} \in (\sigma, \tau], \quad \tau - \sigma \leq K_t \underline{r} \tag{A-151}$$

and for $\bar{s} = \sigma$, $\tau - \sigma \leq K_t \underline{r}$:

$$D_1^{\bar{s}}(\underline{r}) = \frac{\bar{M}_1}{\underline{R}^{k-1}} \underline{r}^{k-1} + \frac{k_r K_t}{\underline{R}^{k+1}} \underline{r}^{k} + \frac{k_r K_t}{\underline{R}^{k}} M_k \underline{r}^{k-2}$$

$$+ \frac{K_t k_r \{M_{k-1} \underline{r}^{k} + \bar{M}_{k+1} \underline{r}^{k-1+\beta}\}}{\underline{R}^{k-1}} + \frac{K_t k_r \underline{r}^{k-1+\beta}}{\underline{R}^{k-2}} + \frac{\bar{M}_{k-1} k_r K_t^2 \bar{U} \underline{r}^{k+\beta}}{\underline{R}^{k-1}}$$

$$+ \frac{k_r K_t^2 \bar{M}_k \bar{U} \underline{r}^{k-1}}{\underline{R}^{k}}, \tag{A-152}$$

$$D_2^{\bar{s}}(\underline{r}) = \frac{k_r K_t \bar{M}_k \underline{r}^{k-2}}{\underline{R}^{k}} + \frac{K_t k_r \bar{M}_{k-1} \underline{r}^{k-1+\beta}}{\underline{R}^{k-1}} \tag{A-153}$$

Finally, consider $\bar{u}_1$ for $\tau - \sigma > K_t \underline{r}$. Let

$$\bar{u}(\sigma, r, \tau) = \int dy \, g(\sigma + K_t \underline{r}, r, \tau, y) \left[D_1^{\bar{s}}(\underline{r}) + D_2^{\bar{s}}(\underline{r}) \, \| y \| \right] \tag{A-154}$$

$$\| \bar{y} \|_{\tilde{a}^{-1}} \geq r(y)$$

where $\tilde{y}$ is a k dimensional vector

$\hat{y}$ is an m-k dimensional vector

$$y = \begin{bmatrix} \tilde{y} \\ \hat{y} \end{bmatrix} \qquad \text{(A-155)}$$

then $\bar{\bar{u}}$ is a solution to (A-40) with

$$\lim_{\sigma \to \tau - K_t \underline{r}} \bar{\bar{u}}\ (\sigma,\xi,\tau) = D_1^{\bar{s}}\ (\underline{r}) + D_2^{\bar{s}}\ (\underline{r})\|\xi\| \qquad \text{(A-156)}$$

$$\bar{\bar{u}}(\sigma,\xi,\tau)\Big|_{\xi \in \partial s} = 0 \qquad \text{(A-157)}$$

From (A-149), (A-156), (A-157), and theorem C-2,

$$\bar{\bar{u}}(\sigma,\xi,\tau) \geq \bar{u}_1(\sigma,\xi,\tau) \qquad \text{(A-158)}$$

Application to (A-154) of the same reasoning used in arriving at (A-116) yields

$$\bar{\bar{u}}(\sigma,\xi,\tau) \leq \left[\frac{\bar{\lambda}_{\sigma+K_t\underline{r},\tau}}{\underline{\lambda}_{\sigma+K_t\underline{r},\tau}}\right]^{m/2} \int_{\|y\| \geq 0} dy\ g[\bar{\lambda}_{\sigma+K_t\underline{r},\tau}(\sigma+K_t\underline{r}),\mu,\bar{\lambda}_{\sigma+K_t\underline{r},\tau}\tau,y]$$

$$\cdot\ [D_1^{\bar{s}}\ (\underline{r}) + D_2^{\bar{s}}\ (\underline{r})\ \|y\|] \qquad \text{(A-159)}$$

then from (A-28), (A-35), (A-130), and (A-159),

$$\bar{\bar{u}}(\sigma,\xi,\tau) \leq \left[\frac{\bar{\lambda}_{\sigma}+K_t\underline{r},\tau}{\underline{\lambda}_{\sigma+K_t\underline{r},\tau}}\right] \{D_1^{\bar{s}}(\underline{r}) + D_2^{\bar{s}}(\underline{r})[k_{\rho,m}\sqrt{\tfrac{1}{2}\ \bar{\lambda}_{Q(\tau,\sigma+K_t\underline{r})}} + k_{\rho,m}\|\mu\|]\}$$

(A-160)

From (A-146), (A-[illegible]), (A-159), and (A-160)

$$\bar{u}_1(\epsilon,\xi,\tau) \le \bar{D}_1^{\xi}(\underline{r}) + \bar{D}_2^{\xi}(\underline{r}) \Big[k_{\rho,m} \sqrt{\lambda}_Q(\tau,\sigma + K_t \underline{r})/2 + (\tau-\sigma)\,\bar{U}\,k_{\rho,m} K_{exp}$$

$$+ \bar{K}_{\rho,m} K_{exp} \,||\, \xi \,|| \Big] \qquad \text{(A-161)}$$

Let

$$A(\xi,\underline{r}) = \bar{D}_1^{\xi}(\underline{r}) + \bar{D}_2^{\xi}(\underline{r}) \Big[k_{\rho,m} \sqrt{\lambda}_Q(\tau,\sigma + K_t \underline{r}) + (\tau-\sigma)\,\bar{U}\,k_{\rho,m} K_{exp} \Big]$$

$$+ \operatorname{Max}\Big[\bar{D}_2^{\xi}(\underline{r}),\ \bar{D}_2^{\xi}(\underline{r})\,\bar{k}_{\rho,m} K_{exp} \Big] \,||\, \xi \,|| \qquad \text{(A-162)}$$

A combination of (A-63), (A-64), (A-65), (A-149), (A-161), and (A-162) completes the proof.

Appendix B

SOME BOUNDS ON MATRIX TRANSFORMATIONS

The purpose of this appendix is to develop bounds on matrix transformations. The method employed is that of Halmos [8], except that best estimates are made for real symmetric matrices.

Let A be a positive definite real symmetric matrix of dimension and rank ρ. Let

$$v_1 = \begin{bmatrix} v_1^1 \\ v_1^2 \\ \cdot \\ \cdot \\ \cdot \\ v_1^\rho \end{bmatrix}, \ldots, v_\rho = \begin{bmatrix} v_\rho^1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_\rho^\rho \end{bmatrix} \tag{B-1}$$

be the set of orthonormal eigenvectors of A with corresponding eigenvalues $\lambda_1, \ldots, \lambda_\rho$. Let X be an arbitrary ρ-vector. Then X may be expressed as

$$X = \sum_{i=1}^{\rho} (X' v_i) v_i \tag{B-2}$$

where X' is the transpose of X. Then

$$A X = \sum_{i=1}^{\rho} (X' v_i) \lambda_i v_i \tag{B-3}$$

and

$$||AX||^2 = \sum_{i=1}^{\rho} \lambda_i^2 (X' v_i)^2 \tag{B-4}$$

Let

$$\bar{\lambda}^2 = \underset{1 \leq i \leq \rho}{\text{Max}} (\lambda_i^2) = \lambda_j^2 \qquad \text{(B-5)}$$

Then

$$\frac{||AX||^2}{\bar{\lambda}^2} = \sum_{i=1}^{\rho} \frac{\lambda_i^2}{\bar{\lambda}^2} (X' \; v_i)^2 \qquad \text{(B-6)}$$

Let

$$\ddot{X} = k \; v_j \text{ for some scalar } k \qquad \text{(B-7)}$$

Then

$$\frac{||A\ddot{X}||^2}{||\ddot{X}||^2} = \bar{\lambda}^2 \qquad \text{(B-8)}$$

Now let

$$X \neq k \; v_j \text{ for all } k \qquad \text{(B-9)}$$

From (B-6),

$$\frac{||AX||^2}{\bar{\lambda}^2} - (X' \; v_j)^2 = \sum_{i \neq j} \frac{\lambda_i^2}{\bar{\lambda}^2} (X' \; v_i)^2 \qquad \text{(B-10)}$$

From (B-5)

$$\frac{\lambda_i^2}{\bar{\lambda}^2} \leq 1, \; i \neq j \qquad \text{(B-11)}$$

Then from (B-10) and (B-4)

$$\frac{||AX||^2}{\bar{\lambda}^2} \leq ||X||^2 \qquad \text{(B-12)}$$

or

$$\frac{||AX||^2}{||X||^2} \leq \bar{\lambda}^2 \qquad \text{(B-13)}$$

Then from (B-8) and (B-13),

$$\frac{||Ax||}{||x||} \leq \bar{\lambda} \quad \text{for all } x \tag{B-14}$$

and $\bar{\lambda}$ is the l.u.b. of $\frac{||Ax||}{||x||}$

Let
$$\bar{\nu} = \max_{1 \leq i \leq \rho} (\nu_i) \tag{B-15}$$

where $\nu_1, \ldots, \nu_\rho$ are the eigenvalues of A^{-1}. Then

$$\bar{\nu} = \frac{1}{\underline{\lambda}} \tag{B-16}$$

where

$$\underline{\lambda} = \min_{1 \leq i \leq \rho} (\lambda_i) \tag{B-17}$$

Then

$$\frac{||A^{-1}y||^2}{||y||^2} \leq \frac{1}{\underline{\lambda}^2} \quad \text{for all } y \tag{B-18}$$

Let

$$y = Ax \tag{B-19}$$

Then

$$\frac{||Ax||}{||x||} \geq \underline{\lambda} \quad \text{for all } x \tag{B-20}$$

and $\underline{\lambda}$ is the g.l.b. of $\frac{||Ax||}{||x||}$.

Next let a be the symmetric matrix of rank ρ whose eigenvectors are v_i, and whose corresponding eigenvalues are $\sqrt{\lambda_i}$. Then

$$A = a\ a \tag{B-21}$$

and

$$||x||_A^2 = ||ax||^2 \tag{B-22}$$

Then from (B-14) and (B-20),

$$\sqrt{\underline{\lambda}} \leq \frac{||x||_\Lambda}{||x||} \leq \sqrt{\bar{\lambda}} \tag{B-23}$$

and $\sqrt{\underline{\lambda}}$ and $\sqrt{\bar{\lambda}}$ are respectively the g.l.b. and l.u.b. of $\dfrac{||x||_\Lambda}{||x||}$.

Similarly,

$$\frac{1}{\sqrt{\bar{\lambda}}} \leq \frac{||x||_{\Lambda^{-1}}}{||x||} \leq \frac{1}{\sqrt{\underline{\lambda}}} \tag{B-24}$$

Finally, consider $||Bx||$ where B is restricted only to be real. Let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \; e_\rho = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{B-25}$$

Let

$$K_B = \max_{1 \leq i \leq \delta} \left\{ ||Be_i|| \right\} \tag{B-26}$$

An arbitrary ρ vector, x, can be expressed as

$$x = \sum_{i=1}^{\rho} (x'e_i)e_i \tag{B-27}$$

Then

$$Bx = \sum_{i=1}^{\rho} (x'e_i)\, Be_i \tag{B-28}$$

and

$$||Bx|| \leq \sum_{i=1}^{\rho} \left|(x'e_i)\right| \, ||Be_i|| \tag{B-29}$$

Then from (B-26),

$$\frac{||Bx||}{||x||} \leq \rho\, K_B \tag{B-30}$$

Suppose that B has an inverse, B^{-1}, and let

$$K_{B^{-1}} = \max_{1 \le i \le \rho} \left\{ ||B^{-1} e_i|| \right\} \tag{B-31}$$

Let

$$y = B^{-1} x \tag{B-32}$$

From (B-30), (B-31) and (B-32),

$$\frac{||B^{-1}x||}{||x||} \le \rho \; K_{B^{-1}} \tag{B-33}$$

and

$$\frac{||y||}{||By||} \le \rho \; K_{B^{-1}} \tag{B-34}$$

or

$$\frac{||By||}{||y||} \ge \frac{1}{\rho K_{B^{-1}}} \tag{B-35}$$

APPENDIX C

COMPARISON THEOREMS FOR SOLUTION TO THE DIFFUSION EQUATION

The purpose of this appendix is to develop a theorem similar to Theorem 16 of Chapter 2 of Friedman [9] the argument follows that of Friedman.

Before starting the theorem some notation must be defined. Let R be the m + 1 dimensional domain (x,t). Let $B_\tau \subset R$ be the hyperplane (x,τ). Let

$$S = \left\{ x \,\middle|\, \| \tilde{x} \|_{\tilde{a}^{-1}} < r(\hat{x}) \right\} \tag{C-1}$$

where $\tilde{x}$, $\hat{x}$, $\tilde{a}$ are used as in Chapters 3 and 4. Let

$$D_\tau = \bigcup_{0 \le t < \tau} B_t - S \tag{C-2}$$

Closure and boundary will be denoted as follows:

$\overline{D}_\tau$ = closure of D_τ

∂S = boundary of S.

Let

$$L = -\sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_i b_i \frac{\partial}{\partial x_i} \tag{C-3}$$

where a is positive semidefinite.

Theorem C-1 Let v(x,t) be a solution to

$$\frac{\partial v}{\partial t} = Lv \tag{C-4}$$

in D_T, with

$$v > 0 \text{ on } B_T \cup (\partial S \cap D_T) \qquad \text{(C-5)}$$

then $v > 0$ everywhere in D_T

Proof Let M be the set of points σ on $(0,T)$ such that $v(x,t) > 0$ everywhere in $D_T - \bar{D}_\sigma$. Let t_o be the g. l. b. of σ. From (C-5),

$$t_o < T \qquad \text{(C-6)}$$

Suppose

$$t_o > 0 \qquad \text{(C-7)}$$

then

$$v > 0 \text{ in } D_T - \bar{D}_{to} \qquad \text{(C-8)}$$

and

$$v = 0 \text{ at some } (x_o, t_o) \in B_{t_o} \qquad \text{(C-9)}$$

From (C-5), and (C-9), $(x_o,t_o) \notin \partial S$. Then from (C-8) and (C-9), (x_o,t_o) is a minimum point for v over B_{to}.

Then

$$\left.\frac{\partial v}{\partial x_i}\right|_{(x_o,t_o)} = 0 \qquad \text{(C-10)}$$

Since a is positive semidefinite,

$$\sum_{i,j} a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} \geq 0 \qquad \text{(C-11)}$$

Then from (C-3), (C-4), (C-10), and (C-11),

$$\left.\frac{\partial v}{\partial t}\right|_{(x_o,t_o)} \leq 0 \qquad \text{(C-12)}$$

But from (C-8) and (C-9),

$$\left.\frac{\partial v}{\partial t}\right|_{(x_o,t_o)} > 0 \qquad \text{(C-13)}$$

Since (C-12) contradicts (C-13), supposition (C-7) must be false, and the theorem is proved.

Theorem C-2 Let $v(x,t)$ be a solution to

$$\frac{\partial v}{dt} = L\, v \qquad \text{(C-14)}$$

in D_T with

$$v = V(x,t) \geq 0 \text{ on } \partial S \cap D_T \cup B_T \qquad \text{(C-15)}$$

then $v \geq 0$ everywhere in D_T.

Proof Let $g(\sigma, x, \tau, y)$ be the fundamental solution of (C-14). Let

$$v_\epsilon(t,x,T) = \epsilon + \bar{v}(x,t) + \int dy\, g(t,x,T,y) \left[\epsilon + V\,(y,T)\right] \qquad \text{(C-16)}$$

then v_ϵ is a solution to (C-14) with

$$v_\epsilon = 2\epsilon + V(x,T) \text{ on } B_T \qquad \text{(C-17)}$$

$$v_\epsilon = V(x,t) + \epsilon \text{ on } \partial S \cap D_T \qquad \text{(C-18)}$$

where ϵ is independent of x and t, $\bar{v}(y,t)$ is a solution of

$$L\,\bar{v} = \frac{\partial \bar{v}}{dt} \text{ in } D_T \qquad \text{(C-19)}$$

$$\bar{v}(y,t) = V(y,t) \text{ on } \partial S \cap D_T \qquad \text{(C-20)}$$

$$\bar{v}(y,t) = 0 \text{ on } B_T \qquad \text{(C-21)}$$

v_ϵ is clearly continuous in ϵ at $\epsilon = 0$.

By Theorem C-1,

$$v_\epsilon > 0 \text{ everywhere in } D_T \text{ for } \epsilon > 0 \qquad \text{(C-22)}$$

Then by continuity,

$$v_o \geq 0 \qquad \text{(C-23)}$$

Since v_o is a solution of (C-14) and (C-15), the theorem is proved.

APPENDIX D

AN EXPRESSION AND A BOUND FOR $\nabla \bar{\Phi}_o$

The purpose of this appendix is to develop an expression and a bound for $\nabla_{\bar{\xi}} \bar{\Phi}_o(t_1,\bar{\xi},t_2)$, where $\bar{\Phi}_o$ is defined by (4-40). The expression will be developed in two parts, $\nabla_{\tilde{\bar{\xi}}} \bar{\Phi}_o$ and $\nabla_{\hat{\bar{\xi}}} \bar{\Phi}_o$.

Working directly from (4-42), (4-43), and (4-44),

$$\frac{\partial \bar{\Phi}_o}{\partial \bar{\xi}_i} = \frac{-(k-2)r^{k-2}(\hat{\bar{\xi}})}{\|\bar{\xi}\|^k}\,\bar{\bar{\xi}}_i$$

$$- \int_{\|\tilde{\bar{\eta}}\| \geq 0} d\tilde{\bar{\eta}}\, \bar{\Phi}_o(\tilde{\bar{\eta}},\hat{\bar{\xi}}) \left[\frac{\partial}{\partial \bar{\bar{\xi}}_i} \bar{h}(t_1,\tilde{\bar{\xi}},t_2,\tilde{\bar{\eta}})\right]$$

$$t_2 > t_1 \qquad , \; i=1,2,\ldots,k \tag{D-1}$$

The integral in (D-1) will now be put into a more convenient form.

$$\int_{\|\tilde{\bar{\eta}}\| \geq 0} d\tilde{\bar{\eta}}\, \bar{\Phi}_o\,(\tilde{\bar{\eta}},\hat{\bar{\xi}}) \left[\frac{\partial}{\partial \bar{\xi}_i} \bar{h}(t_1,\tilde{\bar{\xi}},t_2,\tilde{\bar{\eta}})\right] =$$

$$\int_{\|\approx{\eta}^i\| \geq 0} d\approx{\eta}^i\, \approx{h}(t_1,\approx{\xi}^i,t_2,\approx{\eta}^i) \int_{-\infty}^{\infty} d\bar{\eta}_i\, \bar{\Phi}_o(\tilde{\bar{\eta}},\hat{\bar{\xi}}) \left[\frac{\partial}{\partial \bar{\bar{\xi}}_i} \tilde{h}(t_1,\tilde{\bar{\xi}}_i,t_2,\tilde{\bar{\eta}}_i)\right] \tag{D-2}$$

where

$\approx{\eta}^i$ is the k-1 dimensional vector, i=1,...,k,

with

$$\tilde{\tilde\eta}^{i}_{j} = \begin{cases} \tilde{\tilde\eta}_{j} \;, & \text{if } j < i \\ \tilde{\tilde\eta}_{j+1}, & \text{if } j \geq i \end{cases} \;, \; 1 \leq j \leq k-1 \qquad \text{(D-3)}$$

$\tilde{\tilde\xi}^{i}_{j}$ = is the k-1 dimensional vector, i=1,...,k, with

$$\tilde{\tilde\xi}^{i}_{j} = \begin{cases} \tilde{\tilde\xi}_{j} \;, & \text{if } j < i \\ \tilde{\tilde\xi}_{j+1}, & \text{if } j \geq i \end{cases} \;, \; 1 \leq j \leq k-1 \qquad \text{(D-4)}$$

$$\tilde{\tilde h}(t_1,\tilde{\tilde\xi}^{i},t_2,\tilde{\tilde\eta}^{i}) = \frac{\exp\left\{-\dfrac{\|\tilde{\tilde\eta}^{i}-\tilde{\tilde\xi}^{i}\|^2}{4(t_2-t_1)}\right\}}{[4\pi(t_2-t_1)]^{\frac{k-1}{2}}} \qquad \text{(D-5)}$$

$$\tilde h(t_1,\bar{\bar\xi}_i,t_2,\bar{\bar\eta}_i) = \frac{\exp\left\{-\dfrac{(\bar{\bar\eta}_i-\bar{\bar\xi}_i)^2}{4(t_2-t_1)}\right\}}{[4\pi(t_2-t_1)]^{1/2}} \qquad \text{(D-6)}$$

From (D-6),

$$\frac{\partial}{\partial\bar{\bar\xi}_i}\tilde h(t_1,\bar{\bar\xi}_i,t_2,\bar{\bar\eta}_i) = -\frac{\partial}{\partial\bar{\bar\eta}_i}\tilde h(t_1,\bar{\bar\xi}_i,t_2,\bar{\bar\eta}_i) \qquad \text{(D-7)}$$

Then the inner integral of (D-2) may be evaluated through integration by parts to yield

$$\int_{-\infty}^{\infty} d\bar{\bar\eta}_i \;\bar{\bar\xi}_0(\tilde{\bar\eta},\hat{\bar\xi})\left[\frac{\partial}{\partial\bar{\bar\xi}_i}\tilde h(t_1,\bar{\bar\xi}_i,t_2,\bar{\bar\eta}_i)\right]$$

$$= -\,\tilde h(t_1,\bar{\bar\xi}_i,t_2,\bar{\bar\eta}_i)\;\bar{\bar\xi}_0(\tilde{\bar\eta},\hat{\bar\xi})\Big|_{\eta_i=-\infty}^{\infty}$$

$$+\int_{-\infty}^{\infty} d\bar{\bar\eta}_i\;\tilde h(t_1,\bar{\bar\xi}_i,t_2,\bar{\bar\eta}_i)\,\frac{\partial}{\partial\bar{\bar\eta}_i}\bar{\bar\xi}_0(\tilde{\bar\eta},\hat{\bar\xi}) \qquad \text{(D-8)}$$

From (4-42) and (D-6), the first term of (D-8) vanishes. Then from (D-1), (D-2), (D-8), and the differential of (4-42),

$$\nabla_{\tilde{\bar{\xi}}} \bar{\Phi}_o = -(k-2)r^{k-2}(\hat{\bar{\xi}}) \left[\frac{\tilde{\bar{\xi}}}{\|\bar{\xi}\|^k} - \int_{\|\tilde{\bar{\eta}}\|\geq 0} d\tilde{\bar{\eta}} \frac{\tilde{\bar{\eta}}}{\|\bar{\eta}\|^k} \bar{h}(t_1,\bar{\xi},t_2,\bar{\eta})\right] \quad \text{(D-9)}$$

Working directly from (4-42), (4-43) and (4-44),

$$\frac{\partial \bar{\Phi}_o}{\partial \bar{\xi}_i} = (k-2)r^{k-3}(\hat{\bar{\xi}}) \frac{\partial r}{\partial \bar{\xi}_i} \left\{ \frac{1}{\|\bar{\xi}\|^{k-2}} - \int_{\|\tilde{\bar{\eta}}\| \geq 0} d\tilde{\bar{\eta}}\, \bar{h}(t_1,\bar{\xi},t_2,\bar{\eta}) \frac{1}{\|\bar{\eta}\|^{k-2}}\right\}$$

$$, \ t_2 \geq t_1, \ i=k+1,\ldots,m \quad \text{(D-10)}$$

Equations (D-9) and (D-10) will now be used to estimate $\|\nabla_{\bar{\xi}} \bar{\Phi}_o\|$.

Note first that

$$\|\nabla_{\bar{\xi}} \bar{\Phi}_o\| \leq \|\nabla_{\tilde{\bar{\xi}}} \bar{\Phi}_o\| + \|\nabla_{\hat{\bar{\xi}}} \bar{\Phi}_o\| \quad \text{(D-11)}$$

From (D-9),

$$\|\nabla_{\tilde{\bar{\xi}}} \bar{\Phi}_o\| \leq (k-2)r^{k-2}(\hat{\bar{\xi}}) \left\{\frac{1}{\|\bar{\xi}\|^{k-1}} + \int_{\|\tilde{\bar{\eta}}\| \geq 0} d\tilde{\bar{\eta}} \frac{\bar{h}(t_1,\bar{\xi},t_2,\bar{\eta})}{\|\bar{\eta}\|^{k-1}}\right\}$$

$$, \ t_2 > t_1. \quad \text{(D-12)}$$

From (D-12), (A-1), and (A-22),

$$||\nabla_{\tilde{\xi}} \bar{\bar{e}}_o|| \le \frac{(k-2)(\bar{V}+1)r^{k-2}(\hat{\xi})}{||\tilde{\xi}||^{k-1}} \tag{D-13}$$

From (D-10)

$$\nabla_{\hat{\xi}} \bar{\bar{e}}_o = (k-2)r^{k-3}(\hat{\xi}) \, (\nabla_{\hat{\xi}} r) \left[\frac{1}{||\tilde{\xi}||^{k-2}} - \int_{||\tilde{\eta}|| \ge 0} d\tilde{\eta} \, \frac{\bar{\bar{h}}(t_1, \tilde{\xi}, t_2, \tilde{\eta})}{||\tilde{\eta}||^{k-2}}\right]$$

$$, \; t_2 > t_1 \tag{D-14}$$

From (D-14), (3-16), (A-1), and (A-22),

$$||\nabla_{\hat{\xi}} \bar{\bar{e}}_o|| \le \frac{(k-2)(\bar{V}+1)r^{k-3+\beta}(\hat{\xi})}{||\tilde{\xi}||^{k-2}} \tag{D-15}$$

From (D-11), (D-13) and (D-3),

$$||\nabla_{\xi} \bar{\bar{e}}_o|| \le \frac{(k-2)(\bar{V}+1)r^{k-2}(\hat{\xi})}{||\tilde{\xi}||^{k-1}}$$

$$+ \frac{(k-2)(\bar{V}+1)r^{k-3+\beta}(\hat{\xi})}{||\tilde{\xi}||^{k-2}} \tag{D-16}$$

where $\beta > 1$ is defined in (A-49).

APPENDIX E

A MONTE CARLO TECHNIQUE FOR THE COMPUTATION OF INTEGRALS

E.1 Introduction

The purpose of this appendix is to state some of the details of the technique for computing the integrals involved in equations (4-115) through (4-129). This subject is dealt with in two parts. First, some theoretical considerations are discussed. Second, some simpler types of integrals are computed in order to demonstrate the quality of the random number generator, and to give some feeling for the rapidity of convergence of the algorithm.

E.2 Theoretical Considerations

The class of integrals considered here is defined as follows:

$$J = \int_V d\,\zeta\; p(\zeta)\; F(\zeta) \tag{E-1}$$

where

$$\zeta = \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_\ell \end{bmatrix} \tag{E-2}$$

$$p(\zeta) = \frac{1}{(2\pi)^{\ell/2}(\det\ \Lambda)^{1/2}}\ \exp\left\{-\frac{\|\ \zeta - \mu\ \|^2_{\Lambda^{-1}}}{2}\right\} \tag{E-3}$$

V is a measurable set in E^ℓ.

F is a Baire function whose domain is in E^ℓ and whose range is in E^m.

Λ is a covariance matrix and μ a mean

Let $\xi_1, \xi_2, \xi_3, \cdots, \xi_j, \cdots$ be a sequence of random scalars with properties

$$p_{\xi_i}(\nu) = \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}\nu^2} \tag{E-4}$$

where $p_{\xi_i}(\nu)$ is the probability density associated with the event $\xi_i = \nu$,

$$\xi_i \perp\!\!\!\perp \xi_j,\ i \neq j \tag{E-5}$$

($x \perp\!\!\!\perp y$ denotes that x and y are statistically independent of each other.)

Let

$$\eta^i = P \begin{bmatrix} \xi_{\ell(i-1)+1} \\ \vdots \\ \xi_{\ell(i-1)+\ell} \end{bmatrix} + \mu \tag{E-6}$$

where $P = [\sqrt{\lambda_1}\, v_1, \ldots, \sqrt{\lambda_\ell}\, v_\ell]$ (E-7)

$v_1, \ldots, v_\ell$ are the 1 orthonormal eigenvectors of Λ, and

$\lambda_1, \ldots, \lambda_\ell$ are the corresponding eigenvalues.

Then

$$E(\eta^i) = \mu \tag{E-8}$$

$$E\{(\eta^i - \mu)(\eta^i - \mu)'\} = EP \left\{ \begin{bmatrix} \xi_{\ell(i-1)+1} \\ \vdots \\ \xi_{\ell(i-1)+\ell} \end{bmatrix} [\xi_{\ell(i-1)+1}, \ldots, \xi_{\ell(i-1)+\ell}] \right\} P' \tag{E-9}$$

From (E-4) and (E-5),

$$E\{(\eta^i - \mu)(\eta^i - \mu)'\} = PP' \tag{E-10}$$

Reasoning as in (4-49),

$$E\{(\eta^i - \mu)(\eta^i - \mu)'\} = \Lambda \tag{E-11}$$

Then η^i is a Gaussian random vector with mean μ and covariance Λ. Also,

$$E\{(\eta^i - \mu)(\eta^j - \mu)'\} = PE \left\{ \begin{bmatrix} \xi_{(i-1)\ell+1} \\ \vdots \\ \xi_{(i-1)\ell+\ell} \end{bmatrix} [\xi_{(j-1)\ell+1}, \ldots, \xi_{(j-1)\ell+\ell}] \right\} P' \tag{E-12}$$

then from (E-5),

$$\eta^i \perp\!\!\!\perp \eta^j \tag{E-13}$$

From (E-1) through (E-5),

$$J = E\left[G(\eta)\right] \tag{E-14}$$

where

$$G(\eta) = \begin{cases} F(\eta) & , \eta \in V \\ 0 & , \eta \notin V \end{cases} \tag{E-15}$$

η is an ℓ-dimensional Gaussian random vector with mean μ and covariance Λ.

Let $\tilde{J}_N$ be defined by

$$\tilde{J}_N = \frac{1}{N} \sum_{i=1}^{N} G(\eta^i) \tag{E-16}$$

From (E-15), (E-8), (E-1), and (E-3),

$$E\, G(\eta^i) = J \tag{E-17}$$

Then from (E-16) and (E-17),

$$E(J - \tilde{J}_N) = 0 \tag{E-18}$$

The covariance of the error between J and $\tilde{J}_N$ is computed as follows.

$$E\left\{(J - \tilde{J}_N)(J - \tilde{J}_N)'\right\} = E\,[JJ'] - 2E\,[J\tilde{J}_N'] + E\,[\tilde{J}_N \tilde{J}_N'] \tag{E-19}$$

From (E-1), (E-18), and (E-19),

$$E\left\{(J - \tilde{J}_N)(J - \tilde{J}_N)'\right\} = E\,[\tilde{J}_N \tilde{J}_N'] - JJ' \tag{E-20}$$

From (E-16),

$$E(\tilde{J}_N \tilde{J}_N') = \frac{1}{N^2} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} E[G(\eta^i)\, G'(\eta^j)] \right\} \tag{E-21}$$

Since (see pages 110 and 224 of [illegible]) [illegible] are functions of independent random variables are independent,

$$E\left[G(\eta^i)\, G'(\eta^j)\right] = JJ' \quad , \quad \text{for } i \neq j \tag{E-22}$$

then from (E-21) and (E-22),

$$E(\tilde{J}_N \tilde{J}_N') = \frac{1}{N}\; E\left[G(\eta)\, G'(\eta)\right] + \frac{N^2-N}{N^2}\; JJ' \tag{E-23}$$

From (E-20) and (E-23),

$$E\left\{(J-\tilde{J}_N)\;(J-\tilde{J}_N)'\right\} = \frac{1}{N}\left\{E\left[G(\eta)\, G'(\eta)\right] - JJ'\right\} \tag{E-24}$$

Stated in words, (E-24) says that the covariance of the error of the discretized integral is $1/N$ times the covariance of G.

E.3 Two Examples

The following examples were constructed to demonstrate the effectiveness of the random number generator used to produce the sequence $\xi_1, \xi_2, \cdots$, and to given some feeling for the rate of covergence of $\tilde{J}_N$ to J.

The first example is defined as follows. Let

$$J = \int\limits_{\|\zeta\| \geq 0} d\,\zeta\; p(\zeta)\, \zeta \tag{E-25}$$

$$\zeta = \begin{bmatrix} \zeta^1 \\ \zeta^2 \\ \zeta^3 \end{bmatrix} \quad , \quad J = \begin{bmatrix} J^1 \\ J^2 \\ J^3 \end{bmatrix} \tag{E-26}$$

$$\Lambda = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \tag{E-27}$$

$$r = \begin{bmatrix} 7 \\ 10 \\ 20 \end{bmatrix} \qquad \text{(E-28)}$$

then $$\tilde{J}_N = \frac{1}{N} \sum_{i=1}^{N} \eta^i \qquad \text{(E-29)}$$

It is clear from (E-3) and (E-25) that

$$J = \mu \qquad \text{(E-30)}$$

The second example is defined as follows. Let

$$K^{ij} = \int d\, \zeta\, p(\zeta)\, \zeta_i\, \zeta_j \qquad \text{(E-31)}$$

$$\|d\| \geq 0$$

with (E-26) through (E-28) remaining valid, and

$$\tilde{K}^{ij}_N = \frac{1}{N} \sum_{k=1}^{N} \eta^k_i\, \eta^k_j \qquad \text{(E-32)}$$

It is clear from (E-3) and (E-31) that

$$K^{ij} = \Lambda_{ij} \qquad \text{(E-33)}$$

These examples were computed using the algorithm shown in block diagram form in Figure E-1. A listing of the FORTRAN program is shown in Table E-1. Results are shown in Figures E-2 through E-10. The random number generator used here and throughout this work is RANDPK (SHARE Library number 745). An examination of Figures E-2 through E-10 shows that the statistical properties of the random number generator output are good enough to give apparent covergence for the examples defined above.

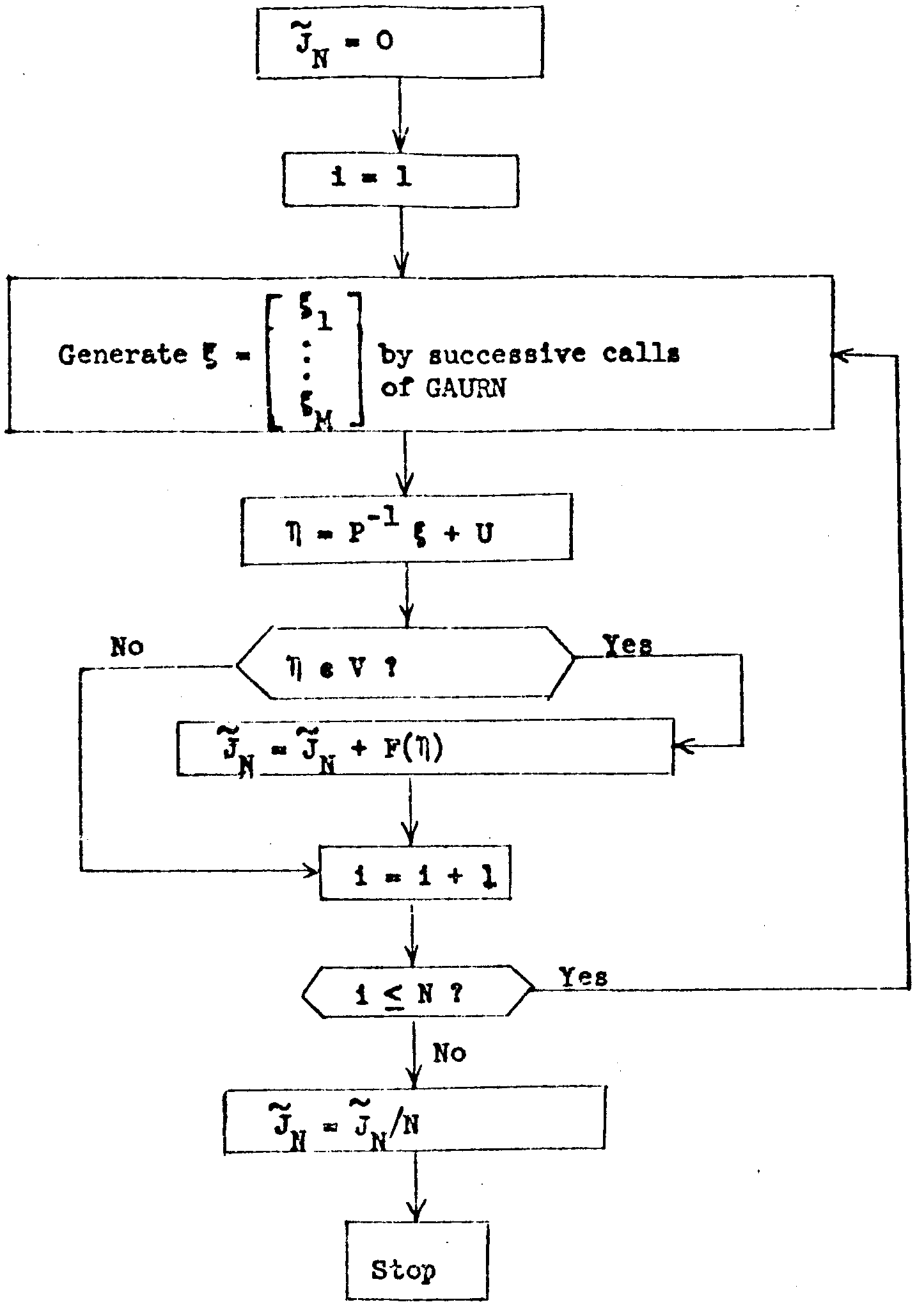

FIGURE E-1. ALGORITHM FOR COMPUTING $\tilde{J}_N$

TABLE E-1. LISTING OF PROGRAM FOR THE COMPUTATION OF $\tilde{J}_N$ AND $\tilde{K}_N$

```
      DIMENSION XMU(3,1),XLAM(3,3),Q(3,3),FMU(3,3),P(3,3),YLAM(3),PINV(3
     1,3),DUM(3,3),TAR(3,3),TES(3,1),TEST(1,3),FMUD(3,3),D1(3,1),FFMU(3,
     21),FFMUD(3,1)
      COMMON XXMU(3)
      EXTERNAL F,FF
      XMU(1,1)=7.
      XMU(2,1)=10.
      XMU(3,1)=20.
      DO 1 I=1,3
    1 XXMU(I)=XMU(I,1)
      XLAM(1,1)=2.
      XLAM(1,2)=-1.
      XLAM(1,3)=0.
      XLAM(2,1)=-1.
      XLAM(2,2)=2.
      XLAM(2,3)=0.
      XLAM(3,1)=0.
      XLAM(3,2)=0.
      XLAM(3,3)=2.
      DO 2 I=1,3
      DO2 J=1,3
      Q(I,J)=0.
      IF(I.EQ.J) Q(I,J)=1.
    2 CONTINUE
      R=0.
      MM=1
      M=3
      N=3
      L=3
      DO 3 I=100,1000,100
      IT=I
      CALL GAUPEC(XMU,XLAM,Q,R,F,FMU,M,N,L,IT,P,YLAM,PINV,DUM,TAR,TES,TE
     1ST,FMUD,D1)
      CALL GAUPEC(XMU,XLAM,Q,R,FF,FFMU,MM,N,L,IT,P,YLAM,PINV,DUM,TAR,TES
     1,TEST,FFMUD,D1)
    3 WRITE(6,4) IT,FMU,FFMU
    4 FORMAT(///1X110,3E20.8/11X3E20.8/11X3E20.8//11X3E20.8)
      STOP
      END
```

TABLE E-1 (Cont'd)

```
      SUBROUTINE CAUPEC(XMU,XLAM,C,R,F,FMU,M,N,L,IT,P,YLAM,PINV,DUM,TAR,
     1TES,TEST,FMUD,D1)
      DIMENSION XMU(N,1),XLAM(N,N),C(N,N),FMU(L,M),P(N,N),YLAM(N),PINV(N
     1,N),DUM(N,N),TAR(N,N),TES(N,1),TEST(1,N),XMAG(1,1),D1(N,1),FMUD(L,
     2M),X(20,20),Y(20)
      EXTERNAL F
      DO1 I=1,L
      DO 1 J=1,M
    1 FMU(I,J)=0.
      DO 2 I=1,N
      DO 2 J=1,N
    2 X (I,J)=XLAM(I,J)
      MX=1
      CALL EIGEN(X,Y,N,MX)
      DO 12 I=1,N
      YLAM(I)=Y(I)
      DO 12 J=1,N
   12 P(I,J)=X(I,J)
      DO 3I=1,N
      DO 3 J=1,N
      II=I
      JJ=J
      DUM(II,JJ)=0.
      IF(II.EQ.JJ) DUM(II,JJ)=1./YLAM(II)**.5
    3 CONTINUE
      CALL MATPLY(P,DUM,PINV,N,N,N)
      I=1
    7 DO 4 J=1,N
    4 TES(J,1)=GAURN(Z)
      NN=1
      CALL MATPLY(PINV,TES,D1,N,N,NN)
      DO 15 J=1,N
   15 TES(J,1)=D1(J,1)+XMU(J,1)
      CALL NORM(C,TES,TEST,XMAG,D1,N,NN)
      IF(XMAG(1,1).LE.R) GO TO 5
      DO13 J=1,N
   13 YLAM(J)=TES(J,1)
      CALL F(YLAM,N,FMUD,L,M)
      DO 6 J=1,L
      DO 6 K=1,M
    6 FMU(J,K)=FMU(J,K)+FMUD(J,K)
    5 I=I+1
      IF(I.LE.IT) GO TO 7
      DO 8 J=1,L
      DO 8 K=1,M
    8 FMU(J,K)=FMU(J,K)/FLOAT(IT)
      RETURN
      END
```

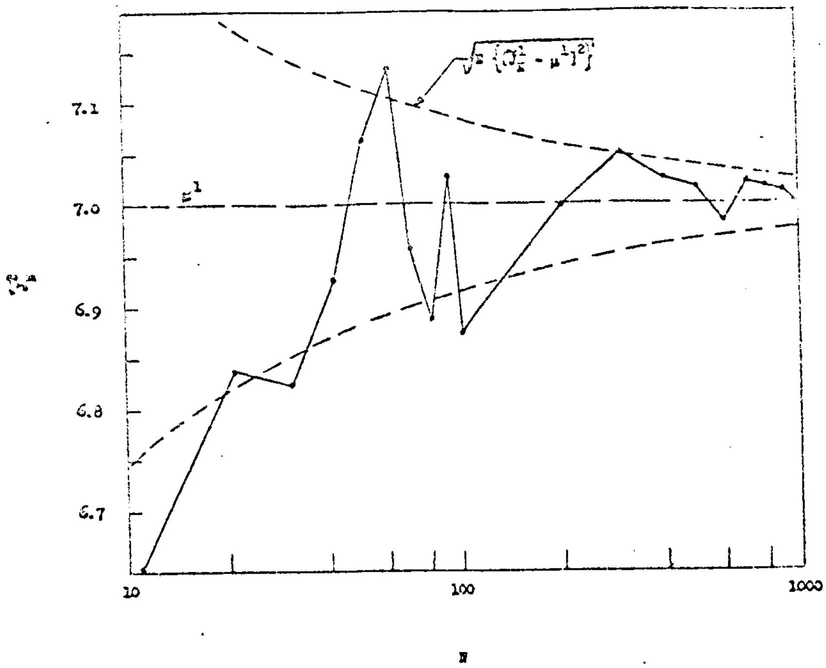

FIGURE E-2. $\bar{J}_N^1$ VERSUS N

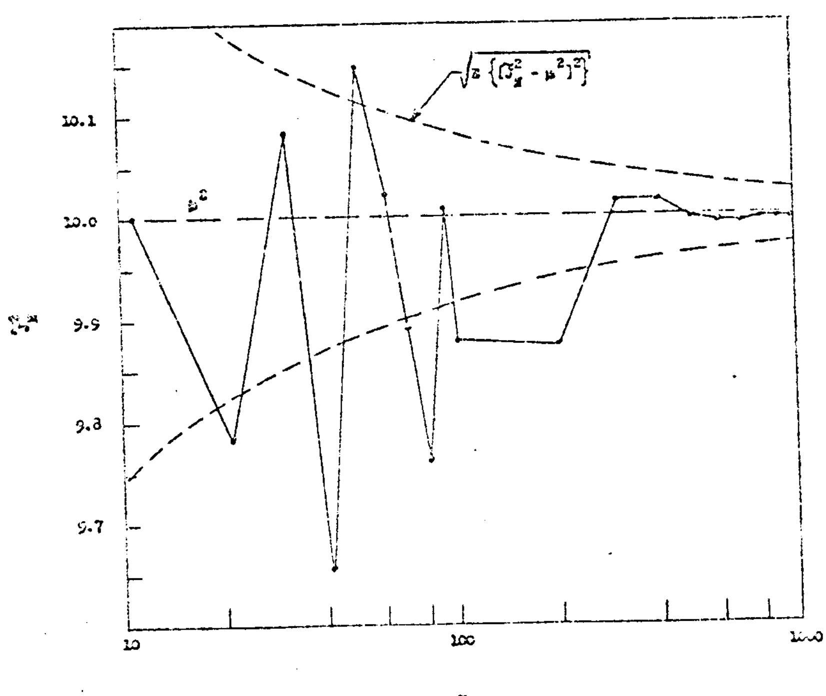

FIGURE E-3. $\bar{J}_N^2$ VERSUS N

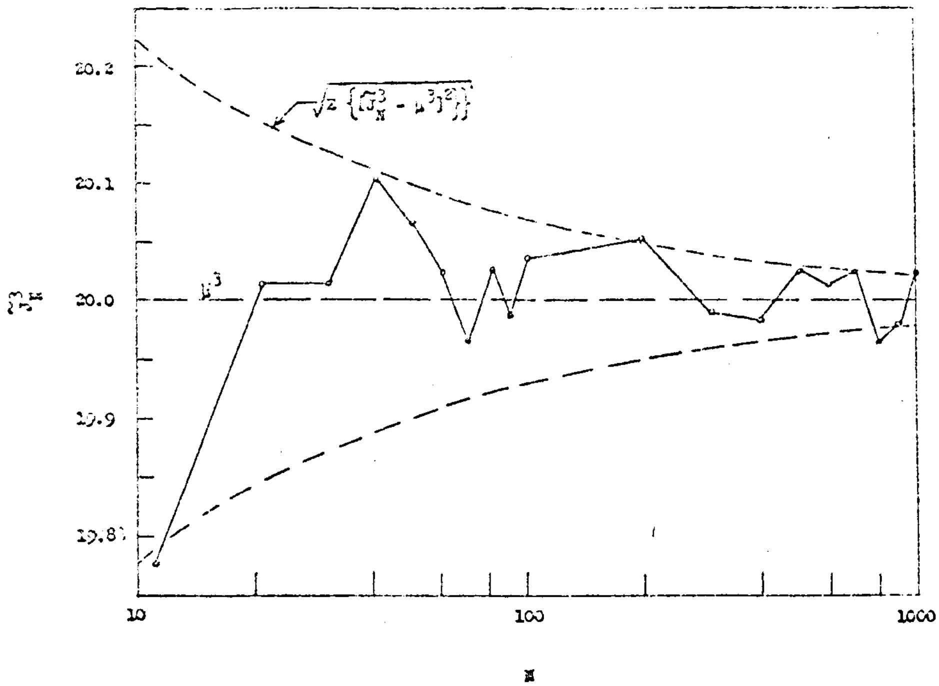

FIGURE E-4. $\tilde{J}^3_N$ VERSUS N

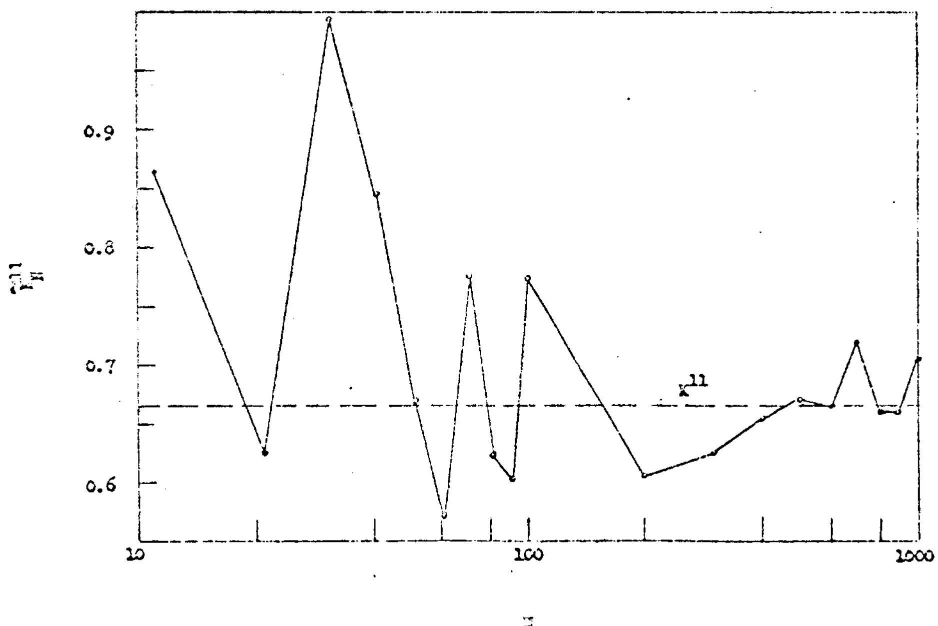

FIGURE E-5. $\tilde{K}^{11}_N$ VERSUS N

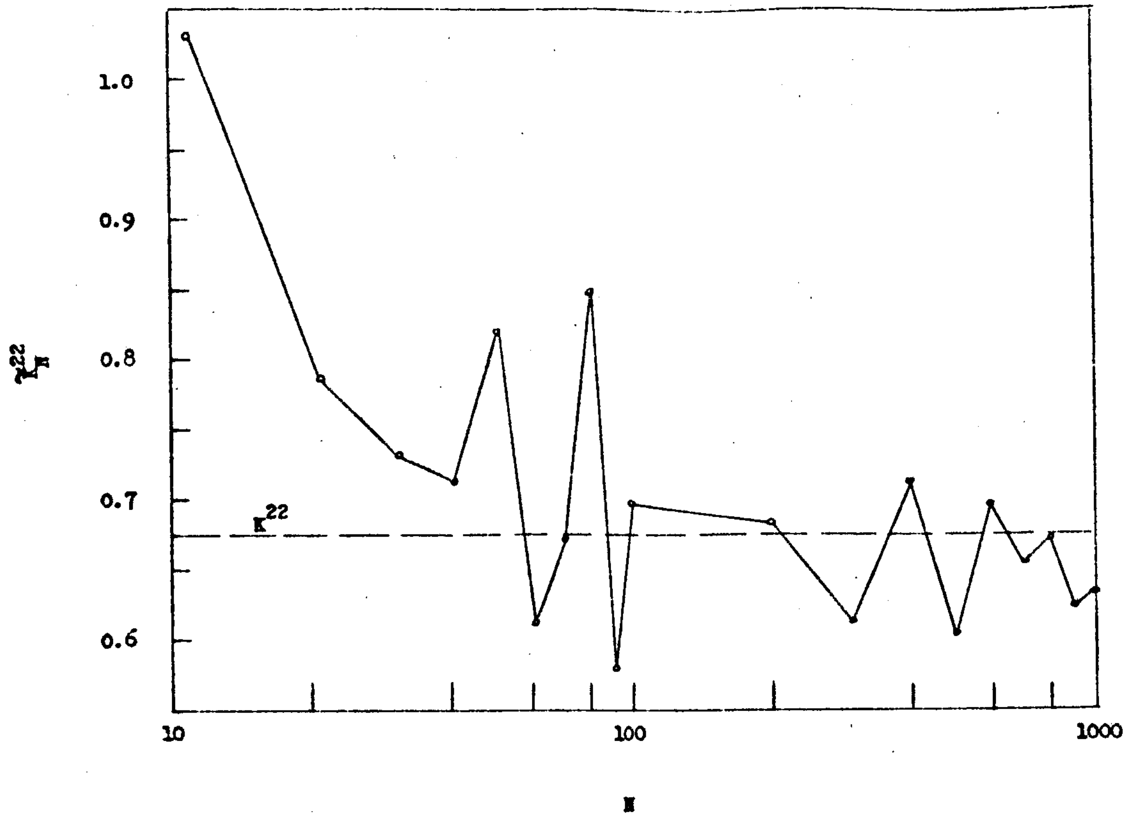

FIGURE E-6. $\tilde{K}_N^{22}$ VERSUS N

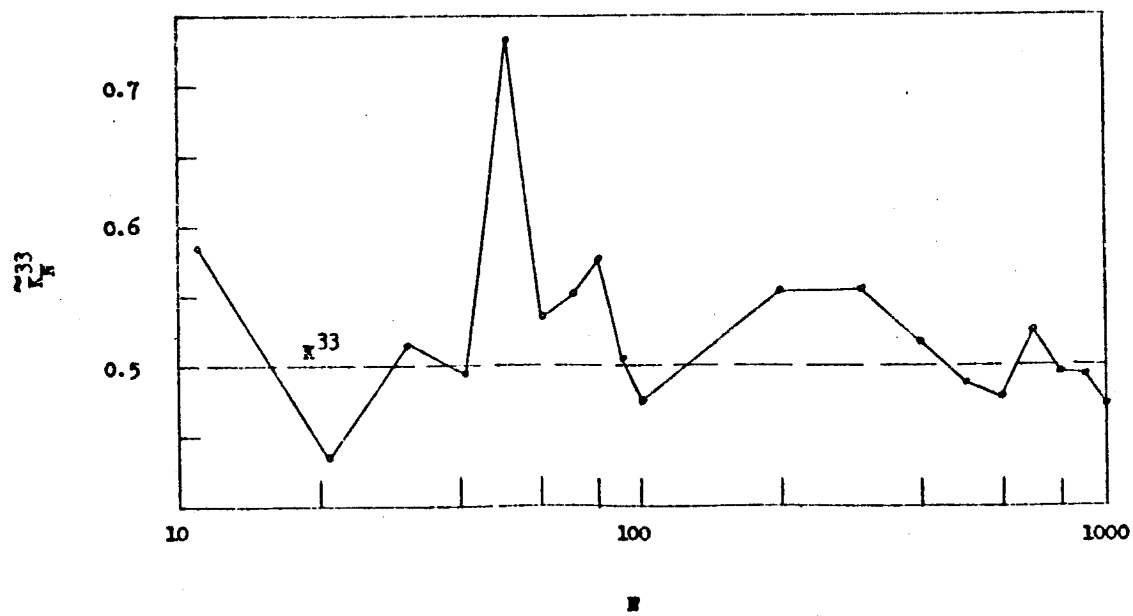

FIGURE E-7. $\tilde{K}_N^{33}$ VERSUS N

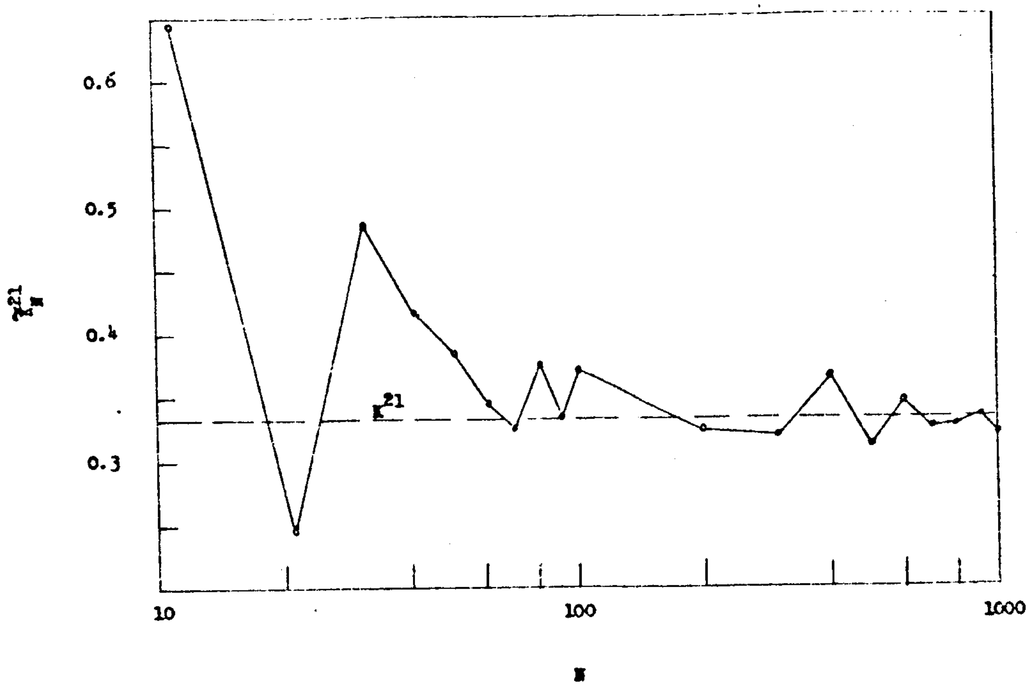

FIGURE E-8. $\tilde{K}^{21}_N$ VERSUS N

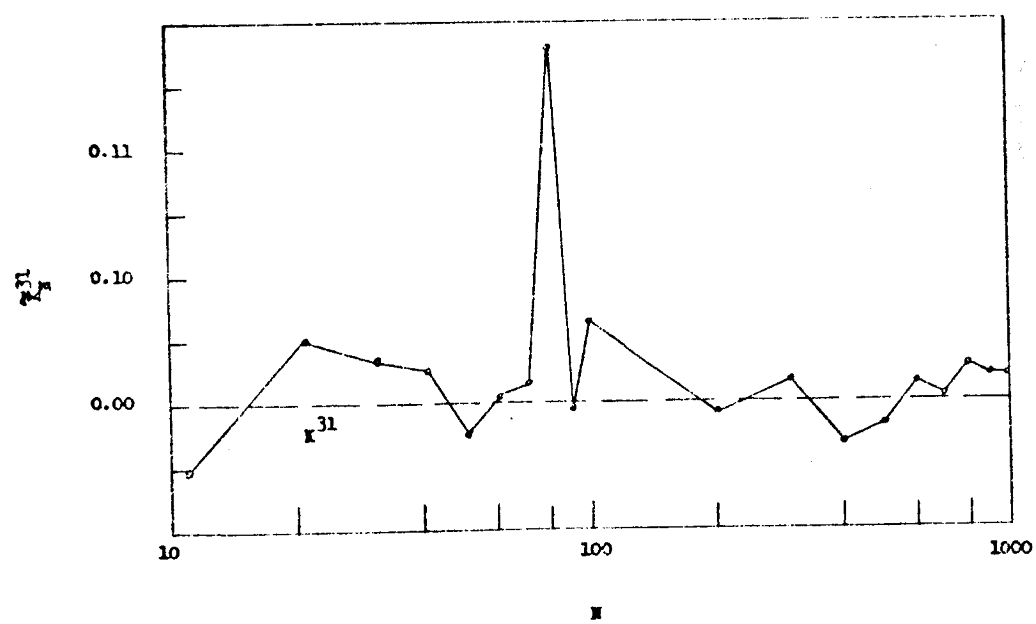

FIGURE E-9. $\tilde{K}^{31}_N$ VERSUS N

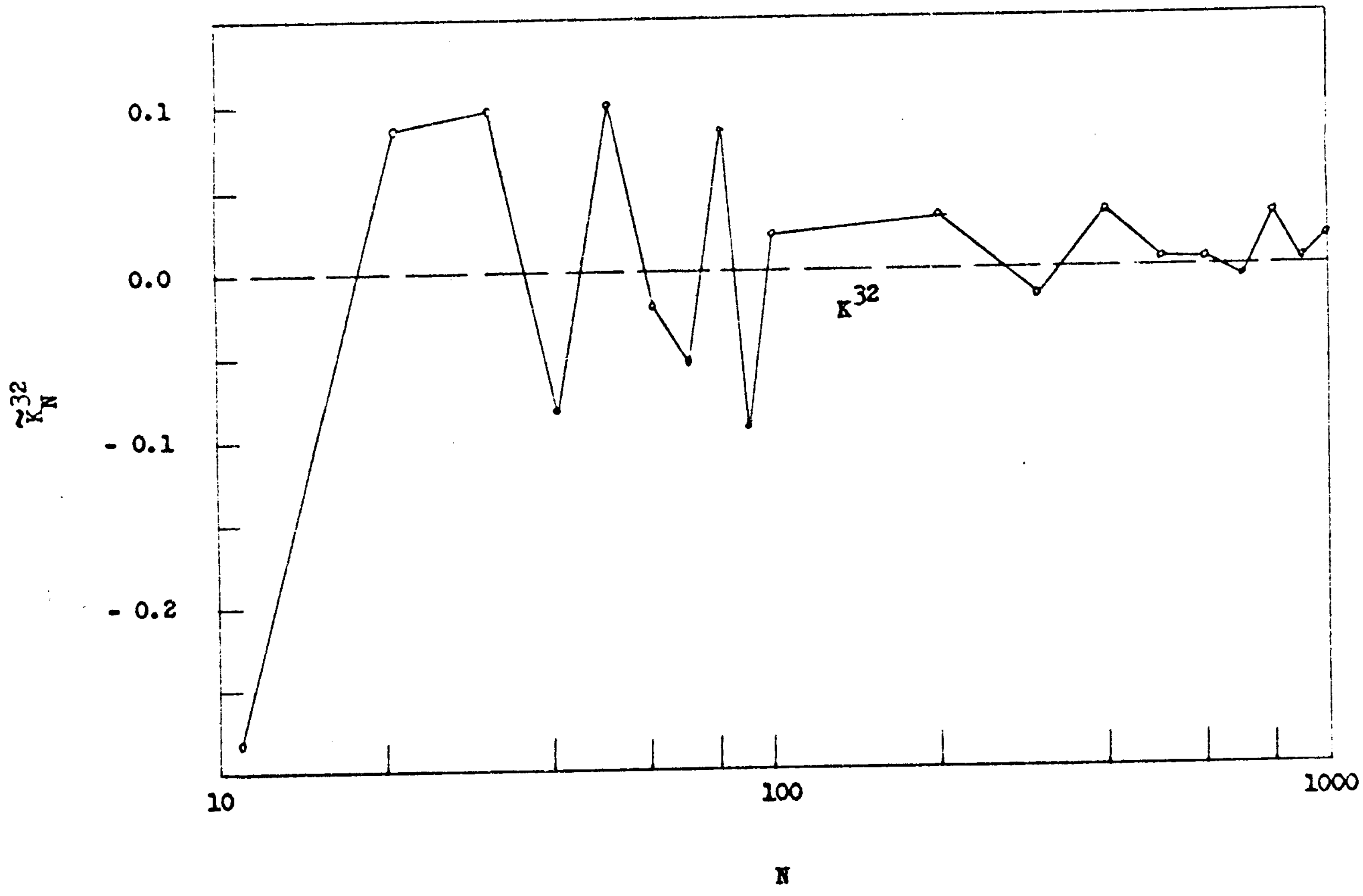

FIGURE E-10. $\tilde{K}_N^{32}$ VERSUS N

APPENDIX F

SAMPLE PROBLEM DESIGN

F.1 Introduction

The choice of a sample problem for the demonstration of a design technique requires careful consideration. The sample problem should indicate the practicality of the design technique. It should be complex enough to expose the critical computational features of the technique, and yet not so complex as to hide the basic thread of the algorithm in a tangle of atypical computer programming detail. Finally it should give some indication of the scope of the technique by revealing the features of the problem which are essential to the technique.

The purpose of this appendix is to describe in some detail the characteristics of the sample problem used in Chapters 6 and 7, and to discuss the application of the above considerations to it.

F.2 Design and Characteristics of the Sample System

The system shown in Figures 6-2, 6-3, and 6-4 embodies a number of commonly encountered features. The prime mover is modelled as a first order lag and an integration. The sensors provide rate and position information. Because the sensors are noisy, first order low pass filters are provided.

The numerical value (see equation (6-30)) for the prime mover time constant was chosen arbitrarily. This in itself costs no generality since changing all time constants proportionately amounts only to time scaling. The numerical values (see equations (6-26) and (6-27)) for the sensor filter time constants were chosen with the idea of providing as much filtering as possible without imposing too severe a penalty on system speed over and above the limitation imposed by the prime mover.

The design of the sample system proceeded as follows. The root locus of the inner loop (see Figures (6-2) and (6-3)) is shown in Figure F-1. The closed loop transfer function of the inner loop will have either three real poles or one real pole and a complex pair, depending on the loop gain. An example of each of the two types was considered. Figure F-2 shows the outter loop root locus for an inner loop transfer function with three real poles, specifically

$$\frac{K_R K_C}{I_L} = 0.6 \qquad \text{(F-1)}$$

Figure F-3 shows the outter loop root locus for an inner loop transfer function with one real pole and a complex pair, specifically,

$$\frac{K_R K_C}{I_L} = 1.0 \qquad \text{(F-2)}$$

The only qualitative difference between the loci in Figures F-2 and F-3 is that the locus of Figure F-2 has a range of gains for which all closed loop poles are real, whereas the locus of Figure F-3 always exhibits at least one complex pair. Since the comparison of different types of systems is a possible subject for future research, the locus of Figure F-2 was selected as preferable to that of F-3. On this basis the value of 0.6 was chosen for $\frac{K_R K_C}{I_L}$.

The only system parameter remaining to be chosen is $\frac{K_C}{I_L}$. This choice was made on the basis of its effect on system performance as judged by its closed loop pole position. The number selected was

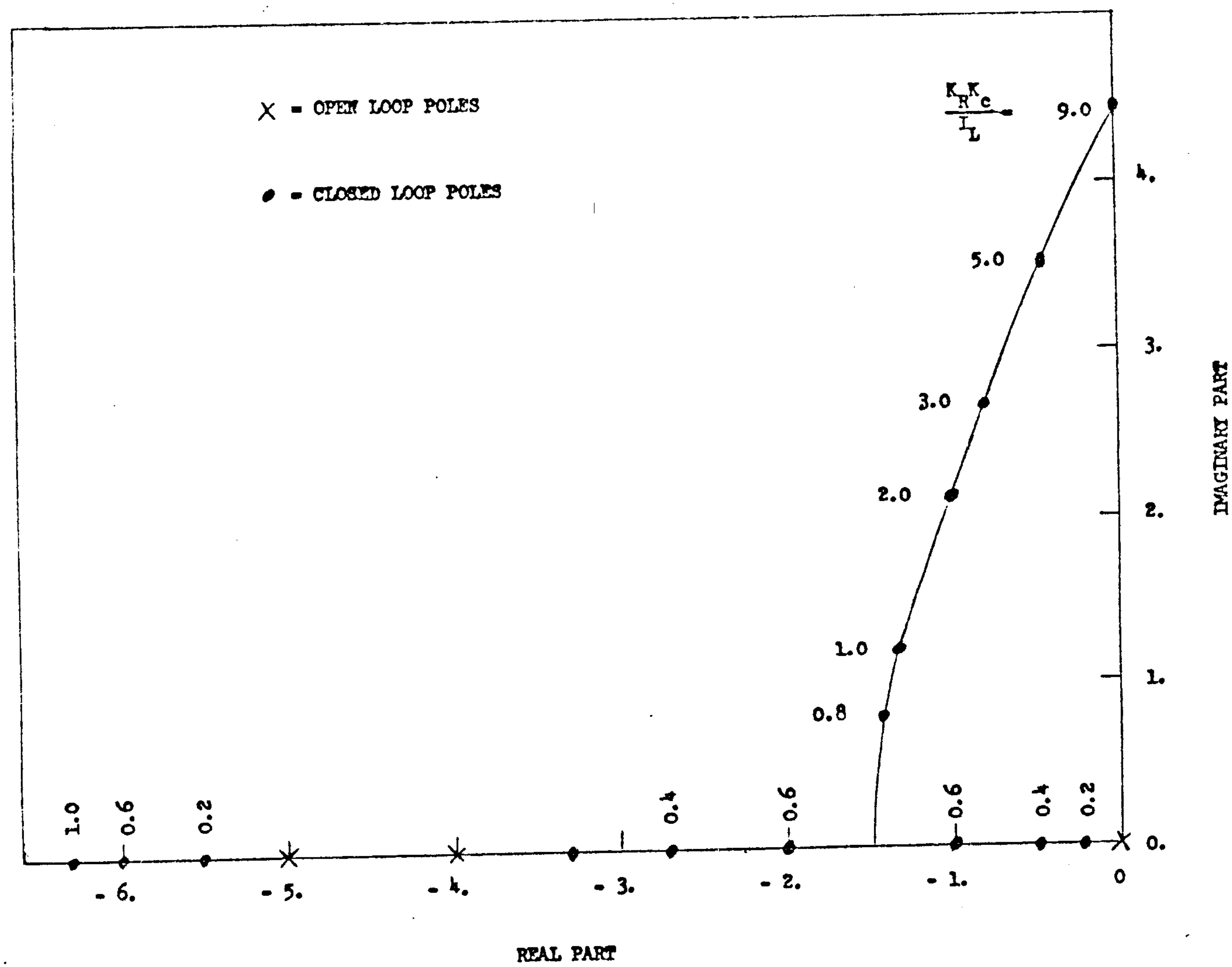

FIGURE F-1. INNER LOOP ROOT LOCUS

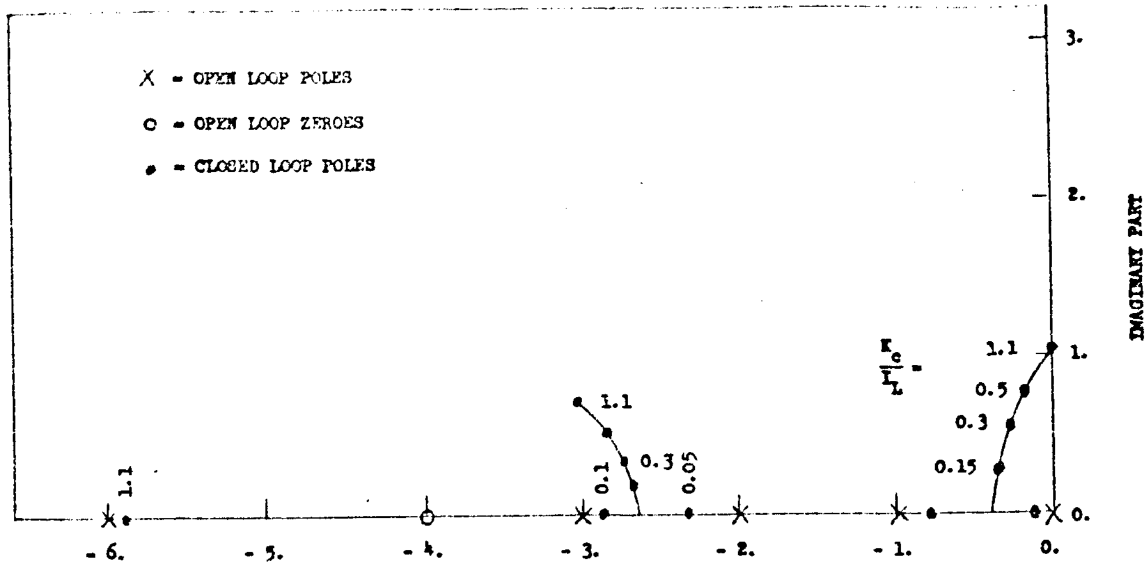

FIGURE F-2. OUTTER LOOP ROOT LOCUS FOR $\frac{K_R K_c}{I_L} = 0.6$

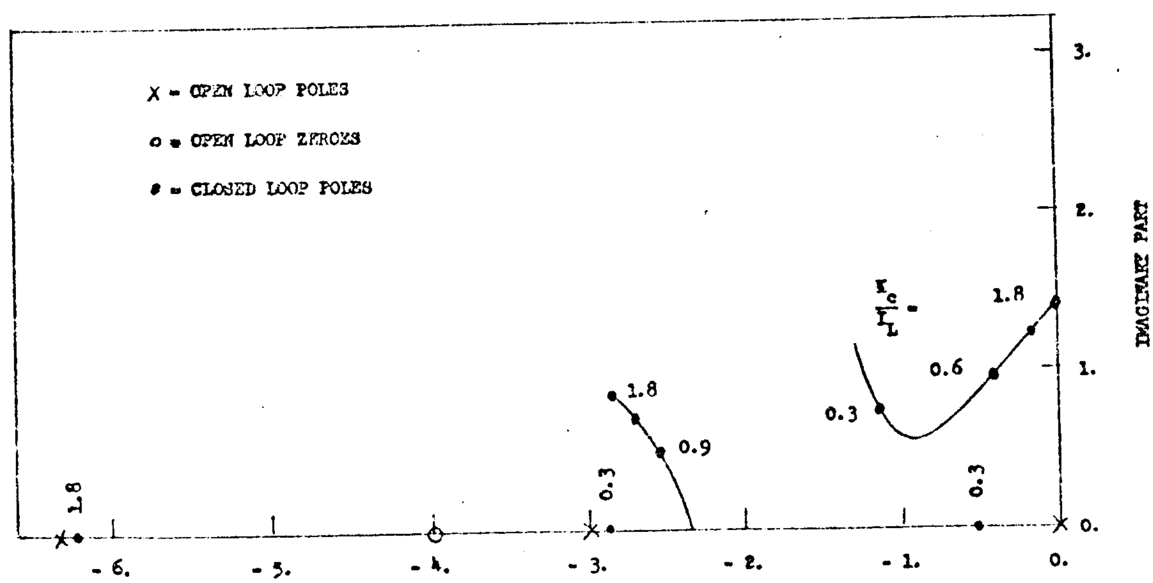

FIGURE F-3. OUTTER LOOP ROOT LOCUS FOR $\frac{K_R K_c}{I_L} = 1.0$

$$\frac{K_C}{I_L} = 0.5 \qquad (F-3)$$

As may be seen from Figure F-2, this means that the system transient response will be determined principally by the complex pair near the origin, and that this pair is lightly damped. This is satisfactory for cases where the system is to perform its maneuver in a time period which is relatively short compared to one cycle of its natural frequency. Since the sample problem time was chosen to be one (see equation (6-33)), this condition holds.

F.3 Random Disturbance Magnitude

It was desired to choose the random disturbance magnitudes (see equation (6-31)) so that each have the same order of magnitude of effect on system performance. The following is apparent from an examination of Figure 6-4:

(a) $\dot{n}_2$ and $\dot{n}_3$ add directly with position (x_1) and rate ($\dot{x}_1$). Thus to have the same order of magnitude of effect, they should have the same order of magnitude.

(b) $\dot{n}_1$ adds directly with $K_r \dot{x}_1$. Since $K_r = 1.2$, $\dot{n}_1$ should also have the same order of magnitude as $\dot{n}_3$.

The conclusion so far is that $\dot{n}_1$, $\dot{n}_2$, and $\dot{n}_3$ should all be of the same order of magnitude. The magnitude of the three was chosen so that, for the given initial condition and target size, the probability term of the performance index was neither nearly one nor nearly zero. The values given in equation (6-31) satisfy this condition, as is evident from the results of the sample problem calculations.

APPENDIX G

COMPUTATIONAL DETAILS ASSOCIATED WITH THE SUBOPTIMAL PROBLEMS

G.1 Introduction

The purpose of this appendix is to present some of the details of the computation associated with problems P1 and P2 of Chapter 6. The FORTRAN listings of the programs used are included. Some additional numerical results which support equation (6-21) as an approximation to (6-6) are presented.

G.2 Computational Details Associated with P1

Figure 6-1 shows the algorithm for the solution of problem P1. In practice, this algorithm was divided into three distinct programs:

(1) The programs SCAL1 was used to compute Φ (0) versus T_c.

(2) The program PRCB was used to execute the remainder of the algorithm, with the exception of the least squares parabolic fit.

These programs will be discussed separately.

G.2.1 SCAL1

A FORTRAN listing of SCAL1 and its associated subprograms is shown in Table G-1. The correspondence between the A matrix in the program and that given by (6-23) is as follows:

SCAL1 A MATRIX SUBSCRIPT	Equation (6-23) A Matrix Subscript
1	4
2	5
3	1
4	2
5	3

TABLE G-1. SCAL1 LISTING

```
$IBFTC SCAL1
      DIMENSION A(5,5),B(5,1),PHI(5),XBAR(5),EU(10),EL(10),TEM(35)
     1,XT(5),PHIT(5)
      COMMON /BSC1/V(11),DV(11),AD(5,5),BD(5,1),U
      EXTERNAL RITE5,DSCAL
      NR=100
      DO 1 I=1,5
      DO 1 J=1,5
  1   A(I,J)=0.
      A(3,1)=1.
      A(4,1)=4.8
      A(2,2)=-5.
      A(3,2)=-.2
      A(4,2)=-.96
      A(5,3)=3.
      A(1,4)=-.5
      A(2,4)=-2.5
      A(4,4)=-4.
      A(1,5)=-.5
      A(2,5)=-2.5
      A(5,5)=-3.
      DO 2 I=1,5
  2   B(I,1)=0.
      B(1,1)=.5
      B(2,1)=2.5
      TF=4.
      DO 3 I=1,5
      XT(I)=0.
      PHIT(I)=0.
    3 XBAR(I)=0.
      XBAR(3)=1.
      XBAR(5)=1.
      DT=.001
      CALL PHIO(A,B,TF,XBAR,5,1,10,DT,RITE5,PHI,T,3)
      DO 4 I=1,5
      II=I
      I1=II+1
      I6=II+6
      V(I1)=XBAR(II)
  4   V(I6)=PHI(II)
      DO 5 I=1,5
      BD(I,1)=B(I,1)
      DO 5 J=1,5
  5   AD(I,J)=A(I,J)
      V(1)=0.
      CALL DSCAL
      DV(1)=DT
      CALL AMRKS(V,DV,DSCAL,10,1,EU,EL,H,H,TEM,0)
      N=IFIX (T/DT)+5
      WRITE(6,6) T
  6   FORMAT(1H1/1X2HT=,E20.8)
      DO 7 I=1,N
      IF(NR.EQ.100)
```

TABLE G-1 (Cont'd)

```
      1WRITE(6,8) (V(L),L=1,6)
       IF(NR.EQ.100)
      1WRITE(6,10)U,(V(L),L=7,11)
       IF(NR.EQ.100)            NR=0
  8    FORMAT(1XE19.8,5E20.8)
 10    FORMAT(1XE16.8,5E20.8)
       NR=NR+1
  7    CALL AMRK
       STOP
       END
$IBFTC DSCALN
       SUBROUTINE DSCAL
       COMMON /BSC1/ V(11),DV(11),AD(5,5),BD(5,1),U
       U=0.
       DO 1 I=1,5
       II=I
       II=II+6
  1    U=U+BD(I,1)*V(II)*.5
       DO 2 I=1,5
       II=I
       II=II+1
       DV(II)=BD(I,1)*U
       DO 2 J=1,5
       JJ=J
       JJ=JJ+1
  2    DV(II)=DV(II)+AD(I,J)*V(JJ)
       DO 3 I=1,5
       II=I
       II=II+6
       DV(II)=0.
       DO 3 J=1,5
       JJ=J
       JJ=JJ+6
  3    DV(II)=DV(II)-AD(J,I)*V(JJ)
       RETURN
       END
$IBFTC PHION
       SUBROUTINE PHID(A,B,TF,XBAR,NS,NU,NRITE,DT,RITE,PHI,T,NO)
       COMMON /BDPH/PHID(20,20),PHIC(20,20),V(401),DV(401),NS2
       DIMENSION A(NS,NS),B(NS,NU),                        EU(400),EL(400
      1),TEM(1205),XBAR(NS),PHI(NS)
       EXTERNAL DPHIC,RITE
       DO 8 I=1,1205
     8 TEM(I)=0.
       N1=1
       DO 1 I=1,NS
       II=I
       INS=II+NS
       DO 1 J=1,NS
       JJ=J
       JNS=JJ+NS
       PHID(II,JJ)=A(II,JJ)
```

TABLE G-1 (Cont'd)

```
      PHID(INS,JJ)=0.
      PHID(INS,JNS)=-A(JJ,II)
      PHIC(II,JJ)=0.
      IF(II.EQ.JJ) PHIC(II,JJ)=1.
      PHIC(INS,JJ)=0.
      PHIC(II,JNS)=0.
      PHIC(INS,JNS)=0.
      IF(INS.EQ.JNS) PHIC(INS,JNS)=1.
      PHID(II,JNS)=0.
      DO 1 K=1,NU
 1    PHID(II,JNS)=PHID(II,JNS)+.5*B(II,K)*B(JJ,K)
      K=1
      NS2=2*NS
      DO 2 I=1,NS2
      DO 2 J=1,NS2
      K=K+1
  2   V(K)=PHIC(I,J)
      V(1)=0.
      READ(5,5) IR
    5 FORMAT(I10)
    6 FORMAT(E14.8/19(5E14.8/),5E14.8)
      IF(IR.EQ.1) WRITE(6,100) (V(I),I=1,101)
      CALL DPHIC
      DV(1)=DT
      CALL AMRKS(V,DV,DPHIC,400,1,EU,EL,H,H,TEM,0)
      NR=0
      NCAL= IFIX(TF/DT)+1
      IF(IR.EQ.1) READ(5,7) (V(I),I=1,101),(DV(I),I=1,101),(TEM(I),I=1,3
     105)
      DO 3 I=1,NCAL
      CALL AMRK
      K=1
      DO 4 J=1,NS2
      DO4 L=1,NS2
      K=K+1
  4   PHIC(J,L)=V(K)
      T=V(1)
      NR=NR+1
      IF(NR.EQ.NRITE) CALL PHSUB(PHI,NS,YBAR,NO)
      IF(NR.EQ.NRITE) CALL RITE(PHI,T,NS,N1)
  3   IF(NR.EQ.NRITE) NR=0
      WRITE(6,100) (V(I),I=1,101)
      PUNCH 7,  (V(I),I=1,101),(DV(I),I=1,101),(TEM(I),I=1,305)
    7 FORMAT(E17.8/24(4E17.8/),4E17.8/101(4E17.8/),2E17.8)
  100 FORMAT(///1XE19.8/20(1XE19.8,4E20.8/))
      RETURN
      END
$IBFTC DPHICN
      SUBROUTINE DPHIC
      COMMON/BDPH/ PHID(20,20),PHIC(20,20),V(401),DV(401),NS2
      K=1
      DO 1 I=1,NS2
      DO 1 J=1,NS2
```

TABLE G-1 (Cont'd)

```
      K=K+1
      DV(K)=0.
      DO 1 L=1,NS2
    1 DV(K)=DV(K)+PHID(I,L)*PHIC(L,J)
      RETURN
      END
$IBFTC PHSUBN
      SUBROUTINE PHSUB(PHI,NS,XBAR,NO)
      DIMENSION R(20,20),XBAR(NS),PHI(NS)
      COMMON /BDPH/PHID(20,20),PHIC(20,20),V(401),DV(401),NS2
      NNS=NS+1
      NE=NO-1
      DO 1 I=1,NS
      R(I,NNS)=0.
      NE=NE+1
      DO 1 J=1,NS
      JJ=J
      JJ=JJ+NS
      R(I,NNS)=R(I,NNS)-PHIC(NE,J)*XBAR(J)
    1 R(I,J)=PHIC(NE,JJ)
      CALL RLMTX(R,NS,1,M,D,-1)
      DO 2 I=1,NS
    2 PHI(I)=R(I,NNS)
      RETURN
      END
$IBFTC RITEN
      SUBROUTINERITE5(PHI,T,NS,N1)
      DIMENSION PHI(NS)
      IF(N1.EQ.1) WRITE(6,1)
    1 FORMAT(1H1/16X4HTIME,15X5HPHI 1,15X5HPHI 2,15X5HPHI 3,15X5HPHI 4,1
     1 5X5HPHI 5)
      N1=0
      WRITE(6,2)T,(PHI(I),I=1,5)
    2 FORMAT(1XE19.8,5E20.8)
      RETURN
      END
```

The reason for this correspondence is that at the time P1 was being developed, the state vector components were inadvertently numbered so that the components corresponding to the target set were 3, 4, and 5, whereas the analysis contained in Chapters 2, 3, 4, and 5 assume that the target set is given in terms of the first k components of the state vector.

SCAL1 operates as follows:

(1) The numerical values of the system parameters are assigned.

(2) Subroutine PHIO is called. A block diagram of this subroutine is shown in Figure G-1. The output from this subroutine is $\xi(0)$ versus T_c for $0 \leq T_c \leq T_F$.

(3) Using the $\xi(0)$ values computed by PHIO for $T_c = T_F$, $x(t)$ versus t is computed as a check on the accuracy of the computation.

SCAL 1 and all other programs requiring differential equation solution used in this work carry out the Runge-Kutta incrementation by means of subroutine AMRK (Purdue University Computer Science Center Library Number D2.01.1). The subroutine for the computation of the derivatives of the elements of PHIC is DPHIC. Because AMRK is written for the solution of vector rather than matrix differential equations, it is necessary to stack the columns of PHIC into a one dimensional array, V. The corresponding derivatives are contained in the array DV.

The computation based on equation (6-20) is carried by subroutine PHSUB, which in turn utilizes the subroutine KLMTX (Purdue University Computer Science Center Library Number F4.01.1) DSCAL is the subroutine used for computing the derivative of X(t).

The values of $\xi(0)$ versus T_c which were computed by SCAL1 and used in the computation of Figures 6-5 through 6-10 are listed in Table G-2.

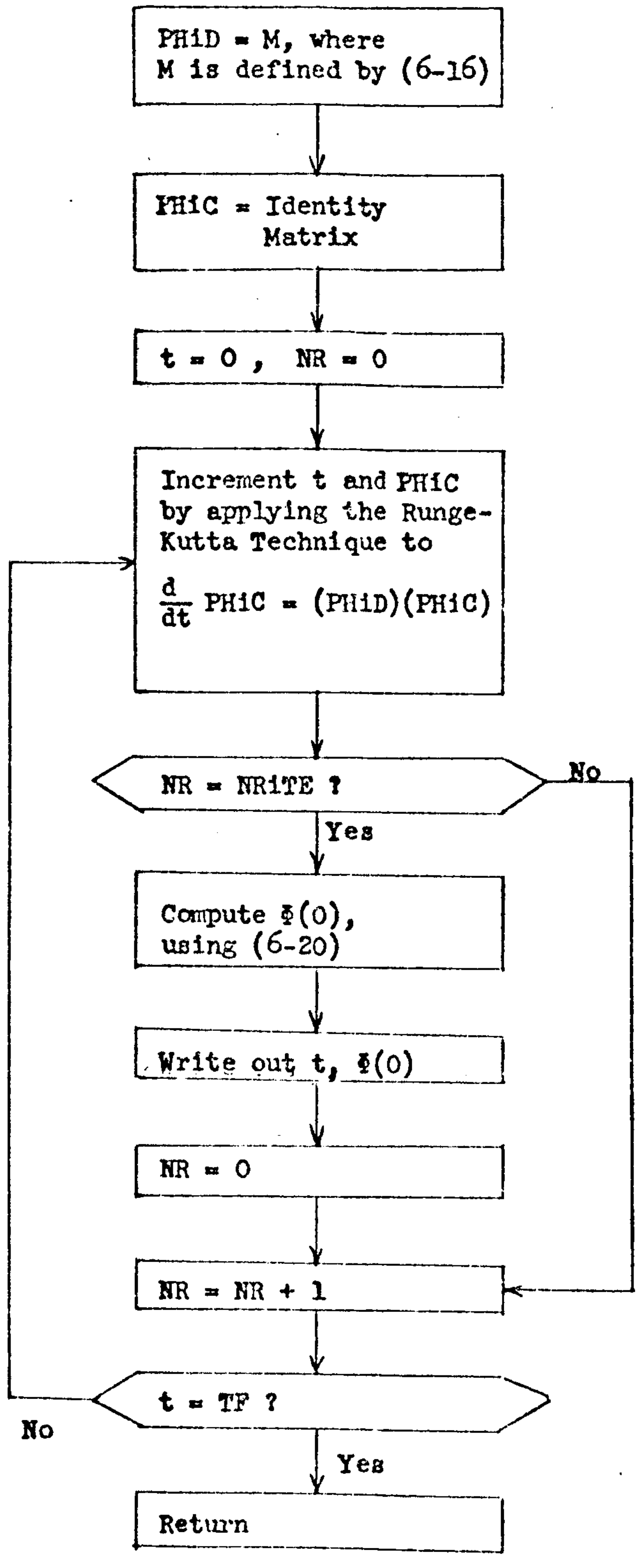

FIGURE G-1. BLOCK DIAGRAM OF PHIO

TABLE G-2. Φ(0) VERSUS T_c FOR P1.

T_c	$Φ_1(0)$	$Φ_2(0)$	$Φ_3(0)$	$Φ_4(0)$	$Φ_5(0)$
0.6	$.51911921 \times 10^6$	$-.23815522 \times 10^6$	$-.22938656 \times 10^7$	$-.65388632 \times 10^5$	$.10297084 \times 10^5$
0.8	$-.20796070 \times 10^5$	$-.12845327 \times 10^5$	$-.14443341 \times 10^6$	$-.12439529 \times 10^5$	$.17663534 \times 10^4$
1.0	$-.11609808 \times 10^5$	$-.12275599 \times 10^4$	$-.16397535 \times 10^5$	$-.35697004 \times 10^4$	$.45220708 \times 10^3$
1.2	$-.42123092 \times 10^4$	$-.15858177 \times 10^3$	$-.26606623 \times 10^4$	$-.13202107 \times 10^4$	$.14778479 \times 10^3$
1.4	$-.16287730 \times 10^4$	$-.20517629 \times 10^2$	$-.54640530 \times 10^3$	$-.51899033 \times 10^3$	$.57112560 \times 10^2$
1.6	$-.69752924 \times 10^3$	$.76157502$	$-.13062334 \times 10^3$	$-.29162383 \times 10^3$	$.24935065 \times 10^2$
1.8	$-.32872724 \times 10^3$	$.37303358 \times 10$	$-.33476790 \times 10^2$	$-.16087124 \times 10^3$	$.11933620 \times 10^2$
2.0	$-.16837026 \times 10^3$	$.35239139 \times 10$	$-.79865829 \times 10$	$-.95584446 \times 10^2$	$.61364505 \times 10$
2.2	$-.92414835 \times 10^2$	$.28522589 \times 10$	$-.88113624$	$-.60118195 \times 10^2$	$.33369158 \times 10$
2.4	$-.54206645 \times 10^2$	$.22400233 \times 10$	$.10317026 \times 10$	$-.39835703 \times 10^2$	$.19152695 \times 10$
2.5	$-.42267818 \times 10^2$	$.19770494 \times 10$	$.13195650 \times 10$	$-.32856771 \times 10^2$	$.14710444 \times 10$
2.6	$-.32599135 \times 10^2$	$.17147862 \times 10$	$.13972642 \times 10$	$-.26782741 \times 10^2$	$.11135555 \times 10$
2.8	$-.21019237 \times 10^2$	$.13402212 \times 10$	$.13207220 \times 10$	$-.18997664 \times 10^2$	$.68902510$
3.0	$-.14235581 \times 10^2$	$.10650501 \times 10$	$.11474189 \times 10$	$-.13986541 \times 10^2$	$.44511757$
3.3	$-.11534449 \times 10^2$	$.10135249 \times 10$	$.11556018 \times 10$	$-.12371893 \times 10^2$	$.33855105$
3.6	$-.28270747 \times 10$	$.27132489$	$.31699324$	$-.31983252 \times 10$	$.79848812 \times 10^{-1}$
4.0	$.89927064$	$-.14283326$	$-.17784476$	$.13675197 \times 10$	$-.15590134 \times 10^{-1}$
∞	0.0	0.0	0.0	0.0	0.0

6.4.2 PROB

A FORTRAN listing of PROB and its associated subprograms is shown in Table C-5. The correspondence between array subscripts in PROB and subscripts in (3-23) is the same as in SCAL1.

The principal task performed by PROB is the execution of the loop shown in Figure 6-1 which is entered at block "A" and exited at block "B". In addition to the computation shown in Figure 6-1, PROB also computes $\int u^2\, dt$ during the execution of the loop. The values of t, ξ, x, $\int u^2\, dt$, and $\|\tilde{x}\|^2_{g^{-1}}$ are printed out every NRIT iterations of the loop. The subroutine used to generate the derivatives is DXCIS. The random number generator is RANDEK (see Appendix B).

In addition to the above, PROB also executes the loop entered at "C" and exited at "D". This loop sorts the contents of XNORA so that the maximum value of $\|\tilde{x}\|^2_{\xi^{-1}}$ appears first. This is done by subroutine SORT. The array XNORA is also sorted by the time at which the minimum value of $\|\tilde{x}\|_{g^{-1}}$ occurs.

Because of certain administrative policies in the Department of Computer Sciences, the turn around time for PROB was minimized for NC = 1 9. This did not appear to be an adequate number of points for the plots in Chapter 6, so two runs (19 points per run) were made for each value of T_c. This required that the random number generator be reset at the beginning of the second run. This was accomplished as follows:

(1) At the end of the first run the statement "CALL CLOSE (NRB)" appears. This returns an eleven digit integer (NRB) which specifies the "position" of the random number generator.

(2) At the beginning of the second run the statement "CALL STORE (NRB)" appears. This sets the random number generator at the "position"

TABLE G-3. PROB LISTING FOR P1

```
$IBFTC PROB
      DIMENSION E1(11),E2(11),TE(38),XNORA(500,2)
      COMMON /BL/B(5),C(5,3),A(5,5),V(12),XN(3),CCPINV(5,5),DV(12),KN
      EXTERNAL DNOIS
      KN=1
      NCM=37
      NCON=1000
      DO 1 I=1,5
      B(I)=0.
      C(I,1)=0.
      C(I,2)=0.
      C(I,3)=0.
      DO 1 J=1,5
1     A(I,J)=0.
      A(3,1)=1.
      A(4,1)=4.8
      A(2,2)=-5.
      A(3,2)=-.2
      A(4,2)=-.96
      A(5,3)=3.
      A(1,4)=-.5
      A(2,4)=-2.5
      A(5,5)=-3.
      A(1,5)=-.5
      A(4,4)=-4.
      A(2,5)=-2.5
      B(1)=.5
      B(2)=2.5
      C(3,3)=1.
      C(4,1)=4.
      C(4,3)=4.8
      C(5,2)=3.
      NC=1
11    DO 2 I=1,12
2     V(I)=0.
      DO 15 I=1,38
   15 TE(I)=0.
      V(4)=1.
      V(6)=1.
      TF=1.
      NT=1000
      DV(1)=TF/FLOAT(NT)
      XNF=DV(1)**.5
      DO 3 I=1,3
3     XN(I)=GAURN(Z)/XNF
      V(7)=.13675197       E01
      V(8)=-.15590134E-01
      V(9)=.89927064E00
      V(10)=-.14283326E00
      V(11)=-.17784476E00
      CALL DNOIS
      DO 4 I=1,5
      DO 4 J=1,5
4     CCPINV(I,J)=0.
      CCPINV(3,3)=39.04/16.
      CCPINV(3,4)=-.3
      CCPINV(4,3)=-.3
      CCPINV(4,4)=.0625
      CCPINV(5,5)=1./9.
```

TABLE G-3 (Cont'd)

```
      XNOR=0.
      DO 5 I=1,5
      II=I
      II=II+1
      DO 5 J=1,5
      JJ=J
      JJ=JJ+1
5     XNOR=XNOR+V(II)*CCPINV(I,J)*V(JJ)
      XNORL=XNOR
      CALL AMRKS(V,DV,DNOIS,11,1,E1,E2,H,H,TE,0)
      NRIT=100
      DO 6 I=1,1000
      IF(NRIT.EQ.100) WRITE(6,100)  (V(K),K=1,11),XNOR,V(12)
      IF(NRIT.EQ.100) NRIT=0
      NRIT=NRIT+1
100   FORMAT(1XE19.8,5E20.8/1XE18.8,5E20.8/1XE19.8)
      IF(I.LE.NCON) GO TO 13
      DO 14 ICO=7,11
   14 V(ICO)=0.
   13 CALL AMRK
      XNOR=0.
      DO 7 J=1,5
      JJ=J
      JJ=JJ+1
      DO 7 K=1,5
      KK=K
      KK=KK+1
7     XNOR=XNOR+V(JJ)*CCPINV(J,K)*V(KK)
      IF(XNOR.LT.XNORL)TL=V(1)
      IF(XNOR.LT.XNORL)XNORL=XNOR
      DO 6 J=1,3
6     XN(J)=GAURN(Z)/XNF
      WRITE(6,100) (V(K),K=1,11),XNOR,V(12)
      XNORA(NC,2)=TL
      XNORA(NC,1)=XNORL
      KN=2
      IF(NC.LT.NCM) GO TO 8
      DO 12 I=1,NCM
   12 WRITE(6,10)(XNORA(I,J),J=1, 2)
      CALL SORT(XNORA,NCM)
      DO 9 I=1,NCM
9     WRITE(6,10) (XNORA(I,J),J=1,2)
10    FORMAT (1X2E20.8)
      CALL GETNM(NMB)
      WRITE(6,101) NMB
  101 FORMAT(1X4HNMB=,I13)
      STOP
8     NC=NC+1
      GO TO 11
      END
$IBFTC DNOISN
      SUBROUTINE DNOIS
      COMMON /BL/B(5),C(5,3),A(5,5),V(12),XN(3),CCPINV(5,5),DV(12) ,KN
      DO 1 I=1,5
      II=I
      II=II+1
      I6=II+6
      DV(II)=0.
      DV(I6)=0.
      DO 1 J=1,5
      JJ=J
```

TABLE G-3 (Cont'd)

```
      J1=JJ+1
      J6=JJ+6
      DV(I1)=DV(I1)+A(II,JJ)*V(J1)+.5*B(II)*B(JJ)*V(J6)
1     DV(I6)=-A(JJ,II)*V(J6)+DV(I6)
      IF(KN.EQ.1) GO TO 3
      DO 2 I=1,5
      II=I
      I1=II+1
      DO 2 J=1,3
2     DV(I1)=DV(I1)+C(II,J)*XN(J)
      U=0.
      DO 4 I=1,5
      II=I
      II=II+6
    4 U=U+.5*B(I)*V(II)
      DV(12)=U**2
3     RETURN
      END
$IBFTC SORTN
      SUBROUTINE SORT(A,N)
      DIMENSION A(500,2)
      DO 1 I=1,N
      X=0.
      II=I
      DO 2 J=II,N
      IF(A(J,1).GT.X) Y=A(J,2)
      IF(A(J,1).GT.X)  K=J
    2 IF(A(J,1).GT.X)  X=A(J,1)
      A(K,1)=A(II,1)
      A(K,2)=A(II,2)
      A(II,1)=X
    1 A(II,2)=Y
      RETURN
      END
```

corresponding to [illegible].

G 2.3 PLOPRO

The least squares best parabolic fit referred to in block "F" of Figure 6-1 was made by subroutine PLOPRO. A FORTRAN listing of this program is shown in Table G-4. The main computation was done by the ESSO system. PLOPRO merely reads in the data supplied by PROB, and transfer it to a data tape to be processed by the ESSO system. The ESSO system determines the parabola which best fits (in a least squares sense) the data points, and generates a tape to operate the CALCOMP plotter. Using the resulting tapes, the CALCOMP plotter generated the curves shown in Figures 6-5 through 6-9.

G.3 Computational Details Associated with P2

The computation for P2 was carried out in a manner exactly parallel to that for P1. The comments made about the computation for P1 apply also to the computation for P2. The computation for P2 was carried out as follows.

(1) The program NDT2 was used to compute [illegible](o) versus K_u. The FORTRAN listing of NDT2 is shown in Table G-5. The results of this part of the computation are given in Table G-6.

(2) The program PROB was modified slightly to reflect the difference in the equations of P1 and P2, and use in the same manner as before. The FORTRAN listing of PROB as it was used for P2 is shown in Table G-7.

(3) The program PLOPRO was used exactly as for P2, using the output from PROB to generate Figures 6-19 through 6-22.

G.4 Equation (6-21) as an Approximation to (6-6).

G.4.1 Purpose

The purpose of this section is to present some numerical evidence that equation (6-21) is an adequate approximation to equation (6-6). The

TABLE G-4. PLOPRO LISTING

```
$IBFTC PLOPRO
      DIMENSION X(36),PROB(500,2),NO(6)
      DO 1 I=1,36
      II=I
    1 X(I)=(37.-FLOAT(II))/36.
      READ(5,2)ISETS,(NO(I),I=1,6)
    2 FORMAT(7I10)
      DO 3 I=1,ISETS
      READ(5,4)(PROB(J,1),J=1,36)
    4 FORMAT(8(4E15.8/),4E15.8)
      CALL SORT(PROB,36)
      WRITE(6,5) (PROB(J,1),X(J),J=1,36)
      DO 7 J=1,36
    7 PROB(J,1)=PROB(J,1)**.5
      K=NO(I)+1
      WRITE(3)(PROB(J,1),X(J),J=K,36)
      ENDFILE 3
      WRITE(6,5) (PROB(J,1),X(J),J=K,36)
    5 FORMAT(1H1///36(1XE19.8,E20.8/))
    3 CONTINUE
      WRITE(3)(PROB(J,1),X(J),J=K,36)
      ENDFILE 3
      WRITE(3)(PROB(J,1),X(J),J=K,36)
      ENDFILE 3
      STOP
      END
$IBFTC SORTN
      SUBROUTINE SORT(A,N)
      DIMENSION A(500,2)
      DO 1 I=1,N
      X=0.
      II=I
      DO 2 J=II,N
      IF(A(J,1).GT.X) Y=A(J,2)
      IF(A(J,1).GT.X)  K=J
    2 IF(A(J,1).GT.X)  X=A(J,1)
      A(K,1)=A(II,1)
      A(K,2)=A(II,2)
      A(II,1)=X
    1 A(II,2)=Y
      RETURN
      END
```

TABLE G-5. DET2 LISTING

```
$IBFTC DET2
      DIMENSION E1(100),E2(100),TEM(305),R(20,20),E3(13),E4(13),TE2(44)
      COMMON /SYS/B(5),A(5,5),CCPINV(5,5),XBAR(5),UK  /AM1/PHIC(10,10),P
     1HID(10,10),VP(101),DVP(101)  /AM2/V(14),DV(14)
      EXTERNAL DPH1,DA2
      UK=5.E-06
      KU=1
 1000 DO 1 I=1,5
      XBAR(I)=0.
      B(I)=0.
      DO 1 J=1,5
      A(I,J)=0.
    1 CCPINV(I,J)=0.
      CCPINV(3,3)=39.04/16.
      CCPINV(3,4)=-.3
      CCPINV(4,3)=-.3
      CCPINV(4,4)=.0625
      CCPINV(5,5)=1./9.
      A(3,1)=1.
      A(4,1)=4.8
      A(2,2)=-5.
      A(3,2)=-.2
      A(4,2)=-.96
      A(5,3)=3.
      A(1,4)=-.5
      A(2,4)=-2.5
      A(4,4)=-4.
      A(1,5)=-.5
      A(2,5)=-2.5
      A(5,5)=-3.
      B(1)=.5
      B(2)=2.5
      XBAR(3)=1.
      XBAR(5)=1.
      DT=.001
      TF=1.
      NCAL=IFIX(TF/DT)+1
      DC 2 I=1,5
      I1=I
      I5 =I1+5
      DO 2 J=1,5
      JJ=J
      J5=JJ+5
      PHIC(I,J)=0.
      PHIC(I5,J)=0.
      PHIC(I5,J5)=0.
      PHIC(I,J5)=0.
      IF(I.EQ.J) PHIC(I,J)=1.
      IF(I5.EQ.J5) PHIC(I5,J5)=1.
      PHID(I,J)=A(I,J)
      PHID(I,J5)=.5*   B(I)*B(J)/UK
      PHID(I5,J)=2.*CCPINV(I,J)
    2 PHID(I5,J5)=-A(J,I)
      VP(1)=0.
      K=1
      DO 3 I=1,10
      DO 3 J=1,10
      K=K+1
    3 VP(K)=PHIC(I,J)
```

TABLE G-5 (Cont'd)

```
      CALL DPHI
      DVP(1)=DT
      CALL AMRKS(VP,DVP,DPHI,100,1,E1,E2,H,H,TEM,0)
      DO 4 I=1,305
    4 TEM(I)=0.
      DO 5 I=1,NCAL
      CALL AMRK
      K=1
      DO 5 J=1,10
      DO 5 L=1,10
      K=K+1
    5 PHIC(J,L)=VP(K)
      DO 6 I=1,5
      II=I
      I5=II+5
      R(I,6)=0.
      DO 6 J=1,5
      JJ=J
      J5=JJ+5
      R(I,J)=PHIC(I5,J5)
    6 R(I,6)=R(I,6)-PHIC(I5,J)*XBAR(J)
      CALL RLMTX(R,5,1,M,D,-1)
      WRITE(6,7) UK,(R(I,6),I=1,5)
    7 FORMAT(1H1//1XE19.8,5E20.8)
      V(1)=0.
      DO 8 I=1,5
      II=I
      I1=II+1
      I6=II+6
      V(I1)=XBAR(I)
    8 V(I6)=R(I,6)
      V(12)=0.
      V(13)=0.
      V(14)=0.
      DV(1)=DT
      CALL DA2
      CALL AMRKS(V,DV,DA2,13,1,E3,E4,H,H,TE2,0)
      NR=100
      NN=NCAL+1
      DO 9 I=1,NN
      IF(NR.EQ.100)WRITE(6,10) (V(J),J=1,14)
   10 FORMAT(1XE19.8,5E20.8/20X5E20.8/20X3E20.8)
      IF(NR.EQ.100)NR=0
      CALL AMRK
    9 NR=NR+1
      KU=KU+1
      IF(KU.EQ.2) UK=5.E-02
      IF(KU.EQ.3) UK=1.E-02
      IF (KU.LT.4) GO TO 1000
      STOP
      END
$IBFTC DPHIN
      SUBROUTINE DPHI
      COMMON/AM1/PHIC(10,10),PHID(10,10),VP(101),DVP(101)
      K=1
      DO 1 I=1,10
      DO 1 J=1,10
      K=K+1
      DVP(K)=0.
      DO 1 L=1,10
    1 DVP(K)=DVP(K)+PHID(I,L)*PHIC(L,J)
```

TABLE G-5 (Cont'd)

```
      RETURN
      END
$IBFTC DA2N
      SUBROUTINE DA2
      COMMON/SYS/B(5),A(5,5),CCPINV(5,5),XBAR(5),UK/AM2/V(14),DV(14)
  100 FORMAT(50XE20.8)
      U=0.
      DO 1 I=1,5
      II=I
      I6 =II+6
    1 U=U+.5*   B(I)*V(I6)/UK
      DO 2 I=1,5
      II=I
      I1=II+1
      DV(I1)=B(I)*U
      DO 2 J=1,5
      JJ=J
      J1=JJ+1
    2 DV(I1)=DV(I1)+A(I,J)*V(J1)
  101 FORMAT(20X5E20.8)
      DO 3 I=1,5
      II=I
      I1=II+1
      I6=II+6
      DV(I6)=0.
      DO 3 J=1,5
      JJ=J
      J1=JJ+1
      J6=JJ+6
    3 DV(I6)=DV(I6)+2.*CCPINV(I,J)*V(J1)-A(J,I)*V(J6)
      DV(12)=0.
      DO 4 I=1,5
      II=I
      I1=II+1
      DO 4 J=1,5
      JJ=J
      J1=JJ+1
    4 DV(12)=DV(12)+V(I1)*CCPINV(I,J)*V(J1)
      DV(13)=UK*U**2
      DV(14)=DV(12)+DV(13)
      RETURN
      END
```

TABLE G-6. $\Phi(0)$ VERSUS K_U FOR P2.

K_U	$\Phi_1(0)$	$\Phi_2(0)$	$\Phi_3(0)$	$\Phi_4(0)$	$\Phi_5(0)$
10^{-6}	$-.10129945 \times 10$	$.98359052 \times 10^{-1}$	$-.45501560 \times 10^{-1}$	$-.37718435 \times 10^{-1}$	$.62667440 \times 10^{-2}$
5×10^{-6}	$-.66115499 \times 10^{-1}$	$.10370311 \times 10^{-1}$	$-.12416788 \times 10^{1}$	$.10662140$	$-.48970312 \times 10^{-1}$
10^{-5}	$-.13612062 \times 10^{1}$	$.11036362$	$-.50550851 \times 10^{-1}$	$-.84376331 \times 10^{-1}$	$.12842781 \times 10^{-1}$
3×10^{-5}	$-.15808358 \times 10^{1}$	$.11643922$	$-.53055505 \times 10^{-1}$	$-.12426937$	$.17895837 \times 10^{-1}$
10^{-4}	$-.18715569 \times 10^{1}$	$.12323603$	$-.55493969 \times 10^{-1}$	$-.19101224$	$.25515453 \times 10^{-1}$
2.5×10^{-4}	$-.21416055 \times 10^{1}$	$.12901094$	$-.56760166 \times 10^{-1}$	$-.26568242$	$.33215750 \times 10^{-1}$
5×10^{-4}	$-.23609425 \times 10^{1}$	$.13395314$	$-.56724220 \times 10^{-1}$	$-.33598436$	$.39692757 \times 10^{-1}$
10^{-3}	$-.25799853 \times 10^{1}$	$.13969206$	$-.54908562 \times 10^{-1}$	$-.42048188$	$.46327014 \times 10^{-1}$
5×10^{-2}	$-.43589614 \times 10^{1}$	$.20932049$	$.58894593 \times 10^{-2}$	$-.14655693 \times 10^{1}$	$.10329224$

TABLE G-7. PROB LISTING FOR P2

```
$IBFTC PROB
      DIMENSION E1(16),E2(16),TE(53),XNORA(500,2),TLAP(500,2)
      COMMON /BL/B(5),C(5,3),A(5,5),V(17),XN(3),CCPINV(5,5),DV(17),KN
     1,UK
      EXTERNAL DNOIS
      KN=1
      NCM=19
      NCON=1000
      DO 1 I=1,5
      B(I)=0.
      C(I,1)=0.
      C(I,2)=0.
      C(I,3)=0.
      DO 1 J=1,5
1     A(I,J)=0.
      A(3,1)=1.
      A(4,1)=4.8
      A(2,2)=-5.
      A(3,2)=-.2
      A(4,2)=-.96
      A(5,3)=3.
      A(1,4)=-.5
      A(2,4)=-2.5
      A(5,5)=-3.
      A(1,5)=-.5
      A(4,4)=-4.
      A(2,5)=-2.5
      B(1)=.5
      B(2)=2.5
      C(3,3)=1.
      C(4,1)=4.
      C(4,3)=4.8
      C(5,2)=3.
      NC=1
11    DO 2 I=1,17
2     V(I)=0.
      DO 15 I=1,53
   15 TE(I)=0.
      V(4)=1.
      V(6)=1.
      TF=1.
      NT=1000
      DV(1)=.001
      XNF=DV(1)**.5
      DO 3 I=1,3
3     XN(I)=GAURN(Z)/XNF
      UK=5.E-02
      V(7)=-.14655693E01
      V(8)= .10329224E00
      V(9)=-.43589614E01
      V(10)=.20932049E00
      V(11)=  .58894593E-02
      DO 17 I=2,6
      II=I
      II=II+11
   17 V(II)=V(I)
      DO 4 I=1,5
      DO 4 J=1,5
```

TABLE G-7 (Cont'd)

```
    4 CCPINV(I,J)=0.
      CCPINV(3,3)=39.04/16.
      CCPINV(3,4)=-.3
      CCPINV(4,3)=-.3
      CCPINV(4,4)=.0525
      CCPINV(5,5)=1./9.
      XNOR=0.
      DO 5 I=1,5
      II=I
      II=II+12
      DO 5 J=1,5
      JJ=J
      JJ=JJ+12
    5 XNOR=XNOR+V(II)*CCPINV(I,J)*V(JJ)
      XNORL=XNOR
      CALL DNOIS
      CALL AMRKS(V,DV,DNOIS,16,1,E1,E2,H,H,TE,0)
      NRIT=100
      DO 6 I=1,1000
      IF(NRIT.EQ.100) WRITE(6,100)V(1),(V(K),K=13,17),(V(K),K=7,11),XNOR
     1,V(12)
      IF(NRIT.EQ.100) NRIT=0
      NRIT=NRIT+1
  100 FORMAT(1XE19.8,5E20.8/1XE18.8,5E20.8/1XE19.8)
      IF(I.LE.NCON) GO TO 13
      DO 14 ICO=7,11
   14 V(ICO)=0.
   13 CALL AMRK
      XNOR=0.
      DO 7 J=1,5
      JJ=J
      JJ=JJ+12
      DO 7 K=1,5
      KK=K
      KK=KK+12
    7 XNOR=XNOR+V(JJ)*CCPINV(J,K)*V(KK)
      IF(XNOR.LT.XNORL)TL=V(1)
      IF(XNOR.LT.XNORL)XNORL=XNOR
      DO 6 J=1,3
    6 XN(J)=GAURN(Z)/XNF
                          WRITE(6,100)V(1),(V(K),K=13,17),(V(K),K=7,11),XNOR
     1,V(12)
      XNORA(NC,2)=TL
      XNORA(NC,1)=XNORL
      TLAR(NC,1)=TL
      TLAR(NC,2)=XNORL
      KN=2
      IF(NC.LT.NCM) GO TO 8
      DO 12 I=1,NCM
   12 WRITE(6,10)(XNORA(I,J),J=1, 2)
      CALL SORT(XNORA,NCM)
      CALL SORT(TLAR,NCM)
      DO 9 I=1,NCM
    9 WRITE(6,10) (XNORA(I,J),J=1,2)
      DO 16 I=1,NCM
   16 WRITE(6,10) (TLAR(I,J),J=1,2)
   10 FORMAT (1X2E20.8)
      CALL GETNM(NMB)
      WRITE(6,101) NMB
  101 FORMAT(1X4HNMB=,I13)
      STOP
```

TABLE G-7 (Cont'd)

```
8     NC=NC+1
      GO TO 11
      END
$IBFTC DA2NOI
      SUBROUTINE DNOIS
      COMMON /BL/B(5),C(5,3),A(5,5),V(17),XN(3),CCPINV(5,5),DV(17),KN
     1,UK
      U=0.
      DO 1 I=1,5
      II=I
      I6 =II+6
    1 U=U+.5*    B(I)*V(I6)/UK
      DO 2 I=1,5
      II=I
      I1=II+1
      DV(I1)=B(I)*U
      DO 2 J=1,5
      JJ=J
      J1=JJ+1
    2 DV(I1)=DV(I1)+A(I,J)*V(J1)
      DO 3 I=1,5
      II=I
      I1=II+1
      I6=II+6
      I12=II+12
      DV(I6)=0.
      DV(I12)=DV(I1)
      DO 3 J=1,5
      JJ=J
      J1=JJ+1
      J6=JJ+6
    3 DV(I6)=DV(I6)+2.*CCPINV(I,J)*V(J1)-A(J,I)*V(J6)
      IF(KN.EQ.1) GO TO 4
      DO 5 I=1,5
      II=I
      I1=II+1
      I2=II+12
      DO 5 J=1,3
    5 DV(I2)=DV(I1)+C(II,J)*XN(J)
      DV(12)=U**2
    4 RETURN
      END
$IBFTC SORTN
      SUBROUTINE SORT(A,N)
      DIMENSION A(500,2)
      DO 1 I=1,N
      X=0.
      II=I
      DO 2 J=II,N
      IF(A(J,1).GT.X) Y=A(J,2)
      IF(A(J,1).GT.X)  K=J
    2 IF(A(J,1).GT.X)  X=A(J,1)
      A(K,1)=A(II,1)
      A(K,2)=A(II,2)
      A(II,1)=X
    1 A(II,2)=Y
      RETURN
      END
```

theoretical convergence of the solution of (6-2) to the solution of (6-6) is established by theorem 1-7. The questions remain as to whether the discretization of (6-21) is sufficiently fine, and as to whether the statistical properties of the random number generator outputs are good enough to make the computed values of the solution of (6-21) a good representation of (6-6). The results presented below give evidence that this is in fact the case. The numerical results were obtained using the numerical values for system parameters given in Chapter 6.

G.4.2 Theoretical Development

The basic idea behind the following computation is to choose some statistical parameter of $x(t)$ (as defined by (6-6)) which can be computed from theory, and to compare its valve with the same parameter computed from observations on $x^{\ell}(t)$ (as defined by (6-21)).

Let

$$R(t,\tau) = E\left[x(t)x'(t+\tau) \mid x(0)\right] \tag{G-1}$$

where $x(t)$ is defined by (6-6), with $u = 0$. Let

$$Q(t,\tau) = E\left[x(t)x'(t) \mid x(\tau)\right] \tag{G-2}$$

Then

$$R(t,\tau) = Q(t,0)\,\phi_A'(\tau,t) \tag{G-3}$$

where ϕ_A is the transition matrix associated with A. Wonham [2] shows that for stable systems, Q is asymptotic. Then R is asymptotic also. Thus, for time invariant A, and for t much larger than the system time constants,

$$R(t,\tau) \approx \tilde{R}(\tau) \tag{G-4}$$

where $\tilde{R}(\tau) = \lim_{t\to\infty} R(t,\tau)$.

That is, after a settling period on the order of magnitude of the system time constants, the statistics of $x(t)$ conditioned on $x(0)$ become almost stationary. Suppose that for large t, $x(t)$ becomes ergodic as well as stationary, then

$$R(t,\tau) \approx \tilde{R}(\tau) \approx \hat{R}(\tau) \qquad \text{(G-5)}$$

where

$$\hat{R}(\tau) = \lim_{T_2 \to \infty} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} dt[x(t)x'(t+\tau)] \qquad \text{(G-6)}$$

The next section will show the procedure and results of the computation of $\tilde{R}(\tau)$ from (G-3) and (G-4), the procedure and results of the computation of $\hat{R}(\tau)$ from (6-21), and how the results compare.

G.4.3 Numerical Results

The computation of $\tilde{R}(\tau)$ was carried out in two steps:

(1) Equation (2-14) was used to compute $Q(t,0)$ versus t. The FORTRAN listing of the program used for this is shown in Table G-8. The result of this computation is that the asymptotic value of $Q(t,0)$ is

$$Q(t,0)=\begin{bmatrix} 2.1420426 & .40025407 & -.91239833 & 1.6087804 & -1.6087804 \\ .40025407 & 2.0012703 & 2.0619918 & -1.9235772 & -2.0789634 \\ -.91239833 & -2.0619918 & 3.4232723 & 1.0983740 & 3.0256096 \\ 1.6087804 & -1.9235772 & 1.0983740 & 7.2721950 & -.34731703 \\ -1.6087804 & -2.0789634 & 3.0256096 & -.34731703 & 4.5256096 \end{bmatrix} \qquad \text{(G-7)}$$

(2) Using the values given by (G-7), and computing $\Phi_A'(\tau,t)$ by using (2-5), $\tilde{R}(\tau)$ was computed according to (G-3) and (G-4). A FORTRAN listing of the program used for this computation is shown in Table G-9. The results of this computation were used to obtain the plots of $\tilde{R}(\tau)$ versus τ shown in Figures G-2, G-3, and G-4.

TABLE G-8. LISTING OF PROGRAM FOR THE COMPUTATION OF Q

```
$IBFTC THI
      SUBROUTINE INPUT
      DIMENSION DUM(20,20),VDUM(20)
      COMMON/INPMU/Z0(5),INDP,XC(11,5)
      COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),CWCINV(5,5),UI(1001),S(1001),GG
     1I1,UK,R
      COMMON/RANB/PA(3,3),P(5,5),XMU(5),JLOG,Y(5),RJ,T,RY(5),RU(5),UR
      LOGICAL JLOG
      UK=1.E-04
      R=.1
      DO 1 I=1,5
      Z0(I)=0.
      B(I)=0.
      DO 1 J=1,5
      A(I,J)=0.
      CWC(I,J)=0.
    1 CWCINV=0.
      A(3,1)=1.
      A(4,1)=4.8
      A(2,2)=-5.
      A(3,2)=-.2
      A(4,2)=-.96
      A(5,3)=3.
      A(1,4)=-.5
      A(2,4)=-2.5
      A(5,5)=-3.
      A(1,5)=-.5
      A(4,4)=-4.
      A(2,5)=-2.5
      B(1)=.5
      B(2)=2.5
      CWCINV(3,3)=39.04/16.
      CWCINV(3,4)=-.3
      CWCINV(4,3)=-.3
      CWCINV(4,4)=.0625
      CWCINV(5,5)=1./9.
      Z0(3)=1.
      Z0(5)=1.
      CWC(3,3)=1.
      CWC(3,4)=4.8
      CWC(4,3)=4.8
      CWC(5,5)=9.
      CWC(4,4)=39.04
    5 FORMAT(3I10)
      CALL PMUCAL
      STOP
      END
$IBFTC PMUCAN
      SUBROUTINE PMUCAL
      DIMENSION TEM(65),E1(20),E2(20),Q(20,20),XLAM(20)
      COMMON/INPMU/Z0(5),INDP,XC(11,5)
      COMMON/DPMUB/V(21),DV(21)
      EXTERNAL DPMU
      NRIT=100
      DO 1 I=1,16
    1 V(I)=0.
      DO 2 I=17,21
      II=I
      II=II-16
```

TABLE G-8 (Cont'd)

```
  2  V(I)=ZO(II)
     DO 3 I=1,65
  3  TEM(I)=0.
     DV(1)=.01
     CALL DPMU
     CALL AMRKS(V,DV,DPMU,20,1,E1,E2,H,H,TEM,0)
     DO 4 I=1,5000
200  FORMAT(50XE16.8)
     CALL AMRK
     L=1
     DO 5 J=1,5
     JJ=J
     DO 5 K=1,JJ
     L=L+1
     Q(J,K)=V(L)
  5  Q(K,J)=V(L)
201  FORMAT(5(1XE19.8,4E20.8/))
     L=1
     DO 8 J=1,5
     JJ=J
     DO 8 K=1,JJ
     L=L+1
     Q(J,K)=V(L)
  8  Q(K,J)=V(L)
     IF(NRIT.NE.100) GC TO 4
     WRITE(6,200) V(1)
     WRITE(6,201)((Q(K,J),J=1,5),K=1,5)
     NRIT=0
100  FORMAT(/////50XE20.8//5(1XE19.8,4E20.8/))
202  FORMAT(5E15.8/5E15.8/5E15.8)
  4  NRIT=NRIT+1
     PUNCH 202,(V(L),L=2,16)
     RETURN
     END
$IBFTC DPMUN
      SUBROUTINE DPMU
      DIMENSION Q(5,5)
      COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),CWCINV(5,5),UI(100),S(100),GG
     1I1,UK,R
      COMMON/DPMUB/V(21),DV(21)
      L=1
      DO 1 J=1,5
      JJ=J
      DO 1 K=1,JJ
      L=L+1
      Q(J,K)=V(L)
   1  Q(K,J)=V(L)
      L=1
      DO 2 J=1,5
      JJ=J
      DO 2 K=1,JJ
      L=L+1
      DV(L)=CWC(J,K)
      DO 2 M=1,5
    2 DV(L)=DV(L)+A(J,M)*Q(M,K)+Q(J,M)*A(K,M)
      DO 3 I=1,5
      II=I
      II=II+16
      DV(II)=0.
      DO 3 J=1,5
      JJ=J
      JJ=JJ+16
   3  DV(II)=A(I,J)*V(JJ)+DV(II)
      RETURN
      END
```

TABLE G-9. LISTING OF PROGRAM FOR THE COMPUTATION OF $\widetilde{R}(\tau)$

```
$IBFTC THCO5
      DIMENSION EU(25),EL(25),TEM(80),V11(1002),V22(1002),V33(1002),TAR(
     21002),V1(16)
     1,WORK(1024),R(5,5),DUM(5,5)
      COMMON /ORTH/A(5,5),PHI(5,5),V(26),DV(26)
      EXTERNAL DTHR
      K=1
      DO 1 I=1,5
      DO 1 J=1,5
      A(I,J)=0.
      PHI(I,J)=0.
      IF(I.EQ.J) PHI(I,J)=1.
      K=K+1
    1 V(K)=PHI(I,J)
      READ(5,10)(V1(L),L=2,16)
   10 FORMAT(5E15.8/5E15.8/5E15.8)
      L=1
      DO 2 I=1,5
      I1=I
      DO 2 J=1,I1
      L=L+1
      R(I,J)=V1(L)
    2 R(I1,J)=R(J,I1)
      A(3,1)=1.
      A(4,1)=4.8
      A(2,2)=-5.
      A(3,2)=-.2
      A(4,2)=-.96
      A(5,3)=3.
      A(1,4)=-.5
      A(2,4)=-2.5
      A(5,5)=-3.
      A(1,5)=-.5
      A(4,4)=-4.
      A(2,5)=-2.5
      V(1)=0.
      DV(1)=.01
      DO 3 I=1,80
    3 TEM(I)=0.
      CALL DTHR
      CALL AMRKS(V,DV,DTHR,25,1,EU,EL,H,H,TEM,0)
      DO 4 I=1,1000
      TAR(I)=V(1)
      DO 6 II=1,5
      DO 6 JJ=1,5
      DUM(II,JJ)=0.
      DO 6 LL=1,5
    6 DUM(II,JJ)=DUM(II,JJ)+R(II,LL)*PHI(JJ,LL)
      V33(I)=DUM(5,5)
      CALL AMRK
      K=1
      DO 4 J=1,5
      DO 4 L=1,5
      K=K+1
    4 PHI(J,L)=V(K)
      DO 5 I=1,1000,2
    5 WRITE(3) TAR(I),V33(I)
      ENDFILE 3
      STOP
      END
$IBFTC DTHRN
      SUBROUTINE DTHR
      COMMON/ORTH/A(5,5),PHI(5,5),V(26),DV(26)
      K=1
      DO 1 I=1,5
      DO 1 J=1,5
      K=K+1
      DV(K)=0.
      DO 1 L=1,5
    1 DV(K)=DV(K)+A(I,L)*PHI(L,J)
      RETURN
      END
```

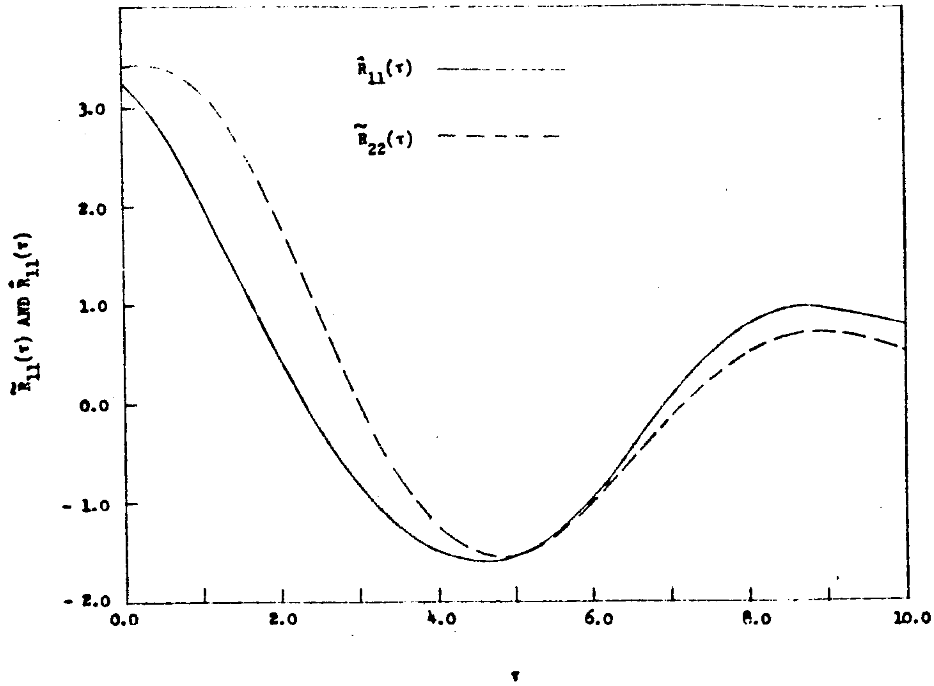

FIGURE G-2. $\tilde{R}_{11}(\tau)$ AND $\hat{R}_{11}(\tau)$ VERSUS τ

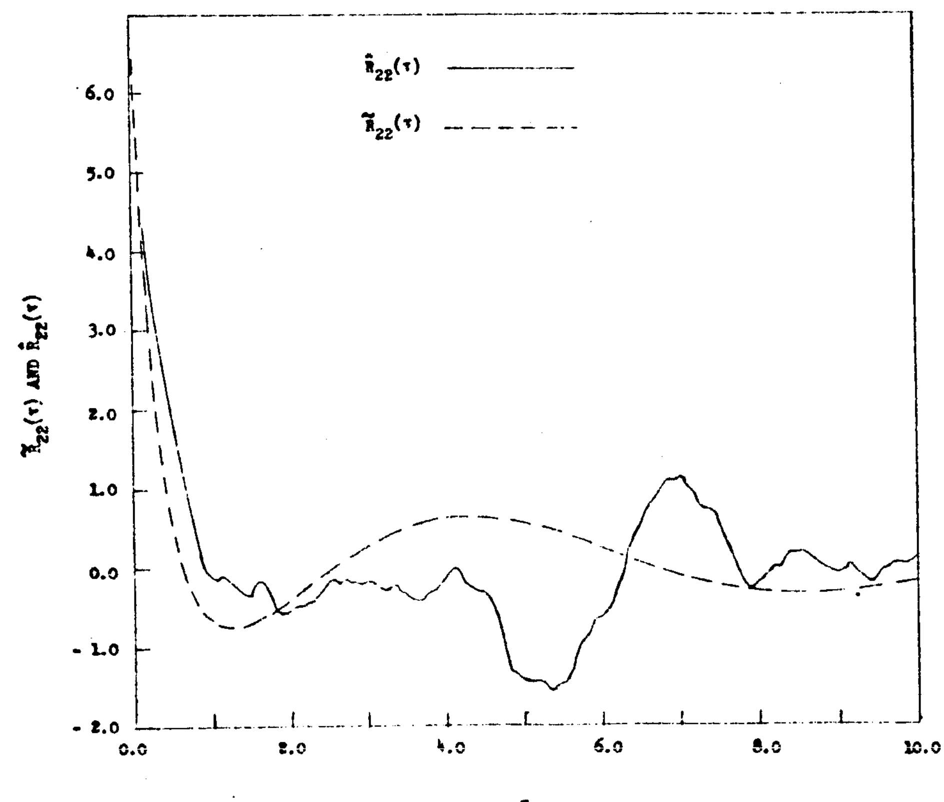

FIGURE G-3. $\tilde{R}_{22}(\tau)$ AND $\hat{R}_{22}(\tau)$ VERSUS τ

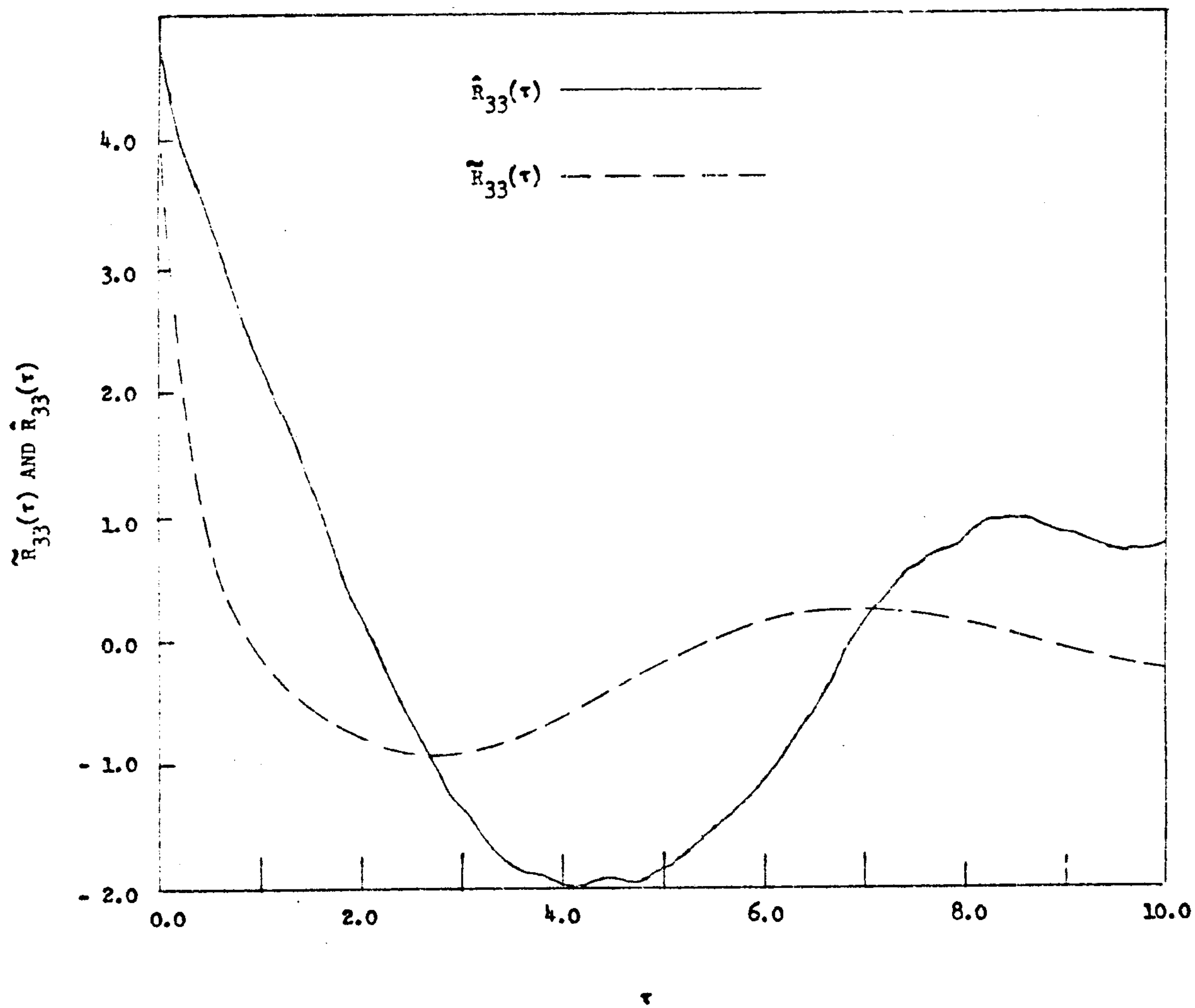

FIGURE G-4. $\tilde{R}_{33}(\tau)$ AND $\hat{R}_{33}(\tau)$ VERSUS τ

The computation of $\hat{R}(\tau)$ was carried out by the program whose [illegible] listing is shown in Table G-10. A block diagram of the algorithm is shown in Figure G-5. The values used for T_1 and T_2 in (G-6) were

$$T_1 = 10 \qquad (G-8)$$

$$T_2 = 70 \qquad (G-9)$$

The results of the computation were used to obtain the plots of $\hat{R}(\tau)$ versus τ shown in Figures G-2, G-3, and G-4.

In summary, $\tilde{R}(\tau)$ and $\hat{R}(\tau)$ were computed on the basic of (6-6) and (6-21) respectively. It was shown that if (6-21) is a good representation of (6-6), $\tilde{R}(\tau)$ should approximate $\hat{R}(\tau)$. Figures G-2, G-3, and G-4 give some feeling for the quality of the approximation.

TABLE G-10. LISTING OF PROGRAM FOR THE COMPUTATION OF $\hat{R}(\tau)$

```
$IBFTC EXPC
      DIMENSION EU(5),EL(5),TEM(20),X3 AR(4500),EUU(1),ELL(1),TEM1(8),W(
     11024),RAR(402),TAUR(402)
      COMMON /DR/V(6),DV(6),A(5,5),C(5,3),XN(3)  /DRB/X(2),DX(2),ITAU,IN
     2T,X3AR
      EXTERNAL DNO,DRS
      DO 1 I=1,5
      DO 2 J=1,3
    2 C(I,J)=0.
      DO 1 J=1,5
    1 A(I,J)=0.
      A(3,1)=1.
      A(4,1)=4.8
      A(2,2)=-5.
      A(3,2)=-.2
      A(4,2)=-.96
      A(5,3)=3.
      A(1,4)=-.5
      A(2,4)=-2.5
      A(5,5)=-3.
      A(1,5)=-.5
      A(4,4)=-4.
      A(2,5)=-2.5
      C(3,3)=1.
      C(4,1)=4.
      C(4,3)=4.8
      C(5,3)=3.
      DO 3 I=1,6
    3 V(I)=0.
      DT=.025
      DTT=DT**.5
      DV(1)=DT
      DO 4 I=1,3
    4 XN(I)=GAURN(Z)/DTT
      CALL DNO
      DO 5 I=1,20
    5 TEM(I)=0.
      CALL AMRKS(V,DV,DNO,5,5,EU,EL,H,H,TEM,0)
      NR=100
      DO 6 I=1,3300
      X3AR(I)=V(6)
      IF(NR.EQ.100) WRITE(6,100)NR,X3AR(I)
  100 FORMAT(1XI10,E20.8)
      IF(NR.EQ.100) NR=0
      NR=NR+1
      CALL AMRK
      DO 6 J=1,3
    6 XN(J)=GAURN(Z)/DTT
      CALL AMRKS(X,DX,DRS,1,1,EUU,ELL,H,H,TEM1)
      DO 7 I=1,400
      DO 8 J=1,8
    8 TEM1(J)=0.
      X(1)=0.
      X(2)=0.
      DX(1)=.025
      ITAU=I
      ITAU=ITAU-1
      TAUR(I)=DX(1)*FLOAT(ITAU)
```

TABLE G-10 (Cont'd)

```
      DO 9 J=400,2800
      INT=J
      CALL DRS
    9 X(2)=X(2)+DX(1)*DX(2)
      RAR(I)=X(2)/60.
    7 WRITE(6,100)ITAU,RAR(I)
      CALL PLOTS(W(1),1024,0)
      CALL FACTOR(.7)
      CALL SYMBOL(0.,0.,.2,12HOCONNOR,1553,0.,12)
      CALL PLOT(2.5,0.0,-3)
      CALL  SCALE(TAUR,10.,400,1,20.)
      CALL SCALE(RAR,10.,400,1,20.)
      CALL AXIS(0.,0.,8HR33(TAU),8,10.,90.,RAR(401),RAR(402),20.)
      CALL AXIS(0.,0.,3HTAU,-3,10.,0.,TAUR(401),TAUR(402),20.)
      CALL SYMBOL(3.5,10.    ,.2,6HTAV=60,0.,6)
      CALL LINE(TAUR,RAR,400,1,0,0)
      CALL PLOT(0,0,999)
      STOP
      END
$IBFTC DNON
      SUBROUTINE DNO
      COMMON /DR/V(6),DV(6),A(5,5),C(5,3),XN(3)
      DO 1 I=1,5
      II=I
      II=II+1
      DV(II)=0.
      DO 2 J=1,3
    2 DV(II)=DV(II)+C(I,J)*XN(J)
      DO 1 J=1,5
      JJ=J
      JJ=JJ+1
    1 DV(II)=DV(II)+A(I,J)*V(JJ)
      RETURN
      END
$IBFTC DRSN
      SUBROUTINE DRS
      COMMON /DRB/X(2),DX(2),ITAU,INT,X3AR(4500)
      I=+NT+ITAU
      DX(2)=X3AR(I)*X3AR(INT)
      RETURN
      END
```

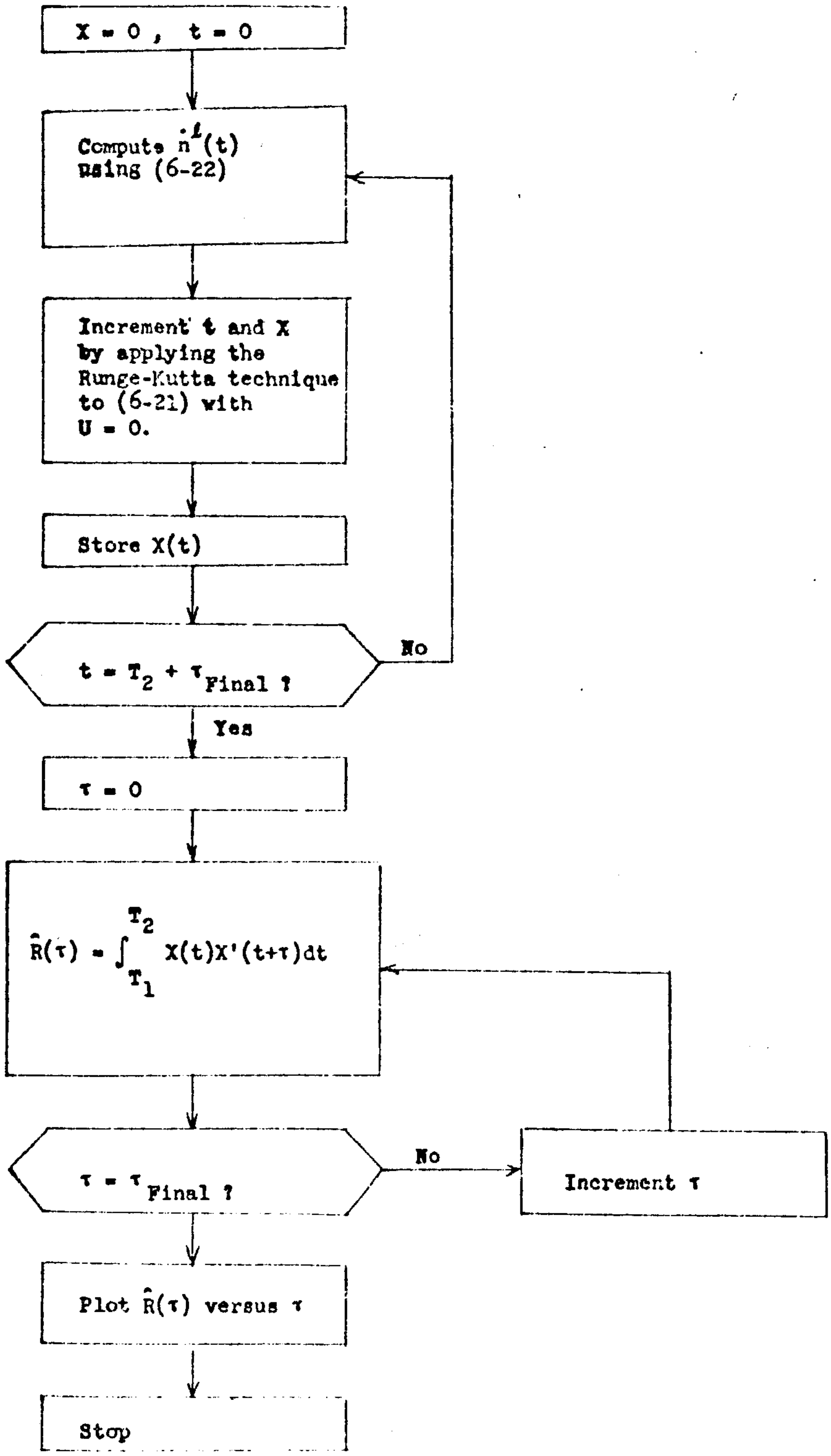

FIGURE G-5. ALGORITHM FOR THE COMPUTATION OF $\hat{R}(\tau)$

APPENDIX H

COMPUTATIONAL DETAILS ASSOCIATED WITH CHAPTER 7

H.1 Introduction

The purpose of this appendix is to record some of the computational details associated with the work reported in Chapter 7. FORTRAN listings are included.

H.2 Conjugate Gradient Program

A FORTRAN listing of the program corresponding to the block diagrams of Figures 5-1 and 5-2 is shown in Table H-1. The algorithm shown in Figure 5-1 is contained in the main program (whose deck name is CONGRA). The subroutines function as follows.

Subroutines USGEN and DGEN generate the initial control (u^o). They use the initial values of the adjoint variables tabulated in Tables G-2 and G-6 to generate the controls corresponding to P1 or P2 respectively (see Chapter 6 and Appendix G).

Subroutines JCAL and DJCAL solve equations (5-21) and (5-5).

Subroutine DCON computes the derivatives required in the execution of the loop including blocks B and C of Figure 5-1.

Subroutine ALSTAR computes equations (5-31), (5-32), and (5-33).

Subroutine INPUT inputs the system parameters, uses FMUCAL to compute μ_z and Q_z (see equations (5-38) through (5-41)), and initializes the random number generator.

TABLE H-1. CONJUGATE GRADIENT PROGRAM

```
$IBFTC CONGRA
      DIMENSION    YO(5),TEM1(26),GI( 401),UC( 401),YT(5),TE(41),E1(12),
     1E2(12)
      COMMON/INPMU/Z0(5),INDP,XC(11,5)
      COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),CWCINV(5,5),UI( 401),S( 401),GG
     1I1,UK,R
      COMMON/RANB/PA(3,3),P(5,5),XMU(5),JLOG,Y(5),RJ,T,RY(5),RU    ,UR
     1,PMUAR(401,5,6)
      COMMON/DCONB/VF(13),DVF(13),UCON,DELH
      LOGICAL JLOG
      EXTERNAL DCON
      LOGICAL DEC,ALDEC
      CALL INPUT
  100 FORMAT(///5(1XE19.8,5E20.8/)///5(1XE19.8,5E20.8/)///E60.8)
      DO 1 I=1,5
    1 YO(I)=0.
   14 DO 2 I=1,101
    2 UC(I)=UI(I)
      GGI=GGI1
      CALL JCAL(UC,YO,XJ,YT)
      WRITE(6,101)(YT(I),I=1,5),XJ
  101 FORMAT(//1XE19.8,5E20.8)
      VF(1)=1.
      VF(7)=XJ
      DO 3 I=1,5
      I2=I
      I2=I2+1
      VF(I2)=YT(I)
      I8=I2+6
    3 VF(I8)=0.
      VF(13)=0.
      DVF(1)=-.01
      DO 4 I=1,41
    4 TE(I)=0.
      UCON=UI(100)
      DO 18 I=1,5
      XMU(I)=PMUAR( 100,I,6)
      DO 18 J=1,5
   18 P(I,J)=PMUAR(100,I,J)
   19 FORMAT(E15.8)
      CALL DCON
      WRITE(6,102)
  102 FORMAT(///)
      CALL AMRKS(VF,DVF,DCON,12,1,E1,E2,H,H,TE,0)
      NRIT=10
      NSTOR=0
      ISTOR=10
      DO 5 I=1,100
      II=I
      IT=101-II
      GI(IT)=DELH
      UCON=UI(IT)
      DO 21 J=1,5
      XMU(J)=PMUAR(IT,J,6)
      DO 21 K=1,5
   21 P(K,J)=PMUAR(IT,K,J)
      IF(NSTOR.NE.10) GO TO 16
      DO 17 K=1,5
      KK=K
      KK=KK+1
   17 VF(KK)=XC(ISTOR,K)
```

TABLE H-1. (Cont'd)

```
      ISTOR=ISTOR-1
      NSTOR=0
   16 IF(NRIT.EQ.10 ) WRITE(6,103) (VF(K),K=1,13)
      IF(NRIT.EQ.10 ) NRIT=0
      NSTOR=NSTOR+1
      NRIT=NRIT+1
  103 FORMAT(2(1XE19.8,3E20.8/)3XE20.8)
    5 CALL AMRK
      WRITE(6,103)(VF(I),I=1,13)
      GI(101)=DELH
      GGI1=-VF(13)
      BETA=GGI1/GGI
      DO 6 I=1,101
6     S(I)= GI(I)+BETA*S(I)
      DEC=.FALSE.
      XJ0=XJ
      ALDEC=.FALSE.
      AL1=ABS(XJ0)/CGI1
      AL1=12.*AL1
      AL0=0.
   11 D=AL1
      DO 7 I=1,101
    7 UC(I)=UI(I)+AL1*S(I)
      CALL JCAL(UC,Y0,XJ1,YT)
      WRITE(6,104) XJ0,XJ1,XJ2,AL0,AL1,AL2
  104 FORMAT(///20X3E20.8/20X3E20.8)
      IF(XJ1.LE.XJ0) GO TO 8
      IF(ALDEC) STOP
      AL1=AL1/4.
      ALDEC=.TRUE.
      GO TO 11
    8 AL2=AL1+D
   12 DO 9 I=1,101
    9 UC(I)=UI(I)+AL2*S(I)
      CALL JCAL(UC,Y0,XJ2,YT)
      WRITE(6,104) XJ0,XJ1,XJ2,AL0,AL1,AL2
      IF(XJ2.GE.XJ1) GO TO 10
      XJ0=XJ1
      XJ1=XJ2
      AL0=AL1
      AL1=AL2
      AL2=AL2+D
      GO TO 12
   10 CALL ALSTAR(AL,XJ0,XJ1,XJ2,AL0,D)
      WRITE(6,105) AL
      DO 13 I=1,101
  105 FORMAT(1X7HALSTAR=,E20.8)
   13 UC(I)=UI(I)+AL*S(I)
      CALL JCAL(UC,Y0,XJST,YT)
      WRITE(6,106)XJST
  106 FORMAT(1X5HXJST=,E16.8)
      CALL PUNPRO(UC)
      CALL PUNPRO(S)
      CALL GETNM(NBR)
      PUNCH 27,GGI1,NBR
   27 FORMAT(E15.8,I20)
      DO 30 I=1,101,10
   30 WRITE(6,31) UC(I)
   31 FORMAT(1XE20.8)
      STOP
      END
```

TABLE H-1. (Cont'd)

```
$IBFTC UGENI
      SUBROUTINE USGEN
      DIMENSION E1(6),E2(6),TE(23)
      COMMON/GENB/V(7),DV(7)
      COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),CWCINV(5,5),UI( 401),S( 401),GG
     1I1,UK,R
      EXTERNAL DGEN
      GGI1=1.
      V(1)=0.
      V(4)=-.11609808E05
      V(5)=-.12275599E04
      V(6)=-.16397535E05
      V(2)=-.35697004E04
      V(3)=.45220708E03
      V(7)=0.
      DV(1)=.01
      CALL DGEN
      CALL AMRKS(V,DV,DGEN,6,1,E1,E2,H,H,TE,0)
      NRIT=10
      DO 1 I=1,101
      S(I)=0.
      UI(I)=0.
      DO 2 J=1,5
      JJ=J
      J1=JJ+1
    2 UI(I)=UI(I)+B(J)*V(J1)
      UI(I)=UI(I)/2.
      IF(NRIT.NE.10) GO TO 4
      WRITE(6,3) (V(J),J=1,7),UI(I)
    3 FORMAT(1XE19.8,5E20.8/20X2E20.8)
      NRIT=0
    4 NRIT=NRIT+1
    1 CALL AMRK
      RETURN
      END
$IBFTC DGENN
      SUBROUTINE DGEN
      COMMON/GENB/V(7),DV(7)
      COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),CWCINV(5,5),UI( 401),S( 401),GG
     1I1,UK,R
      DV(7)=0.
      DO 1 I=1,5
      II=I
      I1=II+1
      DV(I1)=0.
      DV(7)=DV(7)+B(I)*V(I1)
      DO 1 J=1,5
      JJ=J
      J1=JJ+1
    1 DV(I1)=DV(I1)-A(J,I)*V(J1)
      DV(7)=DV(7)**2
      DV(7)=DV(7)/4.
      RETURN
      END
$IBFTC JCALN
      SUBROUTINE JCAL(UC,Y0,XJ,YI)
      DIMENSION UC( 401),Y0(5),YI(5),TE(26),E1(7),E2(7)
      COMMON/INPMU/Z0(5),INDP,XC(11,5)
      COMMON/RANB/PA(3,3),P(5,5),XMU(5),LOG,Y(5),RJ,T,RY(5),RU    ,UR
     1,PMUAR(401,5,6)
      COMMON/DJCALB/V(8),DV(8),U
```

TABLE H-1. (Cont'd)

```
      LOGICAL JLOG
      EXTERNAL DJCAL
      DV(1)=.01
      V(1)=0.
      V(7)=0.
      V(8)=0.
      NSTOR=10
      ISTOR=1
      NRIT=10
      DO 1 I=1,28
1     TE(I)=0.
101   FORMAT(////)
      DO 2 I=1,5
      II=I
      II=II+1
2     V(II)=YO(I)
      U=UC(1)
      DO 7 I=1,5
      XMU(I)=PMUAR(1,I,6)
      DO 7 J=1,5
    7 P(I,J)=PMUAR(1,I,J)
    8 FORMAT(E15.8)
      CALL DJCAL
      CALL AMRKS(V,DV,DJCAL,7,1,E1,E2,H,H,TE,0)
      DO 3 I=1,100
      IF(NSTOR.NE.10) GO TO 5
      DO 6 K=1,5
      KK=K
      KK=KK+1
    6 XC(ISTOR,K)=V(KK)
      ISTOR=ISTOR+1
      NSTOR=0
    5 IF(NRIT.EQ.10  ) WRITE(6,100)(V(K),K=1,8)
      IF(NRIT.EQ.10  ) NRIT=0
      NRIT=NRIT+1
      NSTOR=NSTOR+1
      IT=I
      U=UC(IT)
      DO 10 J=1,5
      XMU(J)=PMUAR(IT,J,6)
      DO 10 K=1,5
   10 P(K,J)=PMUAR(IT,K,J)
    3 CALL AMRK
      XJ=V(7)
      WRITE(6,100)  (V(I),I=1,8)
  100 FORMAT(1XE19.8,5E20.8/2X2E20.8)
      DO 4 I=1,5
      II=I
      II=II+1
  4   YT(I)=V(II)
      RETURN
      END
$IBFTC DJCALN
       SUBROUTINE DJCAL
       COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),CWCINV(5,5),UI( 401),S( 401),GG
      1I1,UK,R
       COMMON/RANB/PA(3,3),P(5,5),XMU(5),JLOG,Y(5),RJ,T,RY(5),RU    ,UR
      1,PMUAR(401,5,6)
       COMMON/DJCALB/V(8),DV(8),U
       LOGICAL JLOG
       JLOG=.TRUE.
```

TABLE H-1. (Cont'd)

```
      UR=U
      DO 1 I=1,5
      II=I
      II=II+1
    1 Y(I)=V(II)
      T=V(1)
      CALL RANCAL
 1000 FORMAT(1X8HDJCAL RJ/1XE19.8)
      DO 2 I=1,5
      II=I
      II=II+1
      DV(II)=-B(I)*U
      DO 2 J=1,5
    2 DV(II)=DV(II)+A(I,J)*Y(J)
      DV(7)=UK*U**2-RJ
      DV(8)=RJ
      RETURN
      END
$IBFTC DCONN
      SUBROUTINE DCON
      COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),CWCINV(5,5),UI( 401),S( 401),GG
     1II,UK,R
      COMMON/RANB/PA(3,3),P(5,5),XMU(5),JLOG,Y(5),RJ,T,RY(5),RU    ,UR
     1,PMUAR(401,5,6)
      COMMON/DCONB/VF(13),DVF(13),UCON,DELH
      LOGICAL JLOG
      JLOG=.FALSE.
      DO 1 I=1,5
      II=I
      II=II+1
      I7=II+7
    1 Y(I)=VF(II)
      UR=UCON
      T=VF(1)+DVF(1)
      CALL RANCAL
      DO 2 I=1,5
      II=I
      II=II+1
      I7=II+7
      DVF(II)=-B(I)*UCON
      DVF(I7)=-RY(I)
      DO 2 J=1,5
      JJ=J
      J7=JJ+7
      DVF(II)=DVF(II)+A(I,J)*Y(J)
    2 DVF(I7)=DVF(I7)-A(J,I)*VF(J7)
      DELH=-2.*UK*UCON-RU
      DO 3 I=1,5
      II=I
      II=II+7
    3 DELH=DELH-B(I)*VF(II)
      DVF(7)=UK*UCON**2-RJ
      DVF(13)=DELH**2
      RETURN
      END
$IBFTC ALSTAN
      SUBROUTINE ALSTAR(AL,XJ0,XJ1,XJ2,AL0,D)
      A2=2.*(XJ2-2.*XJ1+XJ0)/D**2
      A1=2.*(XJ2-XJ1)/D-A2*(2.*AL0+1.5*D)
      AL=-A1/(2.*A2)
      RETURN
```

TABLE H-1. (Cont'd)

```
      END
$IBFTC NEWIN
      SUBROUTINE INPUT
      DIMENSION DUM(20,20),VDUM(20)
      COMMON/INPMU/Z0(5),INDP,XC(11,5)
      COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),CWCINV(5,5),UI( 401),S( 401),GG
     111,UK,R
      COMMON/RANB/PA(3,3),P(5,5),XMU(5),JLOG,Y(5),RJ,T,RY(5),RU     ,UR
     1,PMUAR(401,5,6)
      COMMON/NUMB/N1F,N2F
      LOGICAL JLOG
      UK=1.E-04
      R=.2
      DO 1 I=1,5
      Z0(I)=0.
      B(I)=0.
      DO 1 J=1,5
      A(I,J)=0.
      CWC(I,J)=0.
    1 CWCINV(I,J)=0.
      A(3,1)=1.
      A(4,1)=4.8
      A(2,2)=-5.
      A(3,2)=-.2
      A(4,2)=-.96
      A(5,3)=3.
      A(1,4)=-.5
      A(2,4)=-2.5
      A(5,5)=-3.
      A(1,5)=-.5
      A(4,4)=-4.
      A(2,5)=-2.5
      B(1)=.5
      B(2)=2.5
      CWCINV(3,3)=39.04/16.
      CWCINV(3,4)=-.3
      CWCINV(4,3)=-.3
      CWCINV(4,4)=.0625
      CWCINV(5,5)=1./9.
      Z0(3)=1.
      Z0(5)=1.
      CWC(3,3)=1.
      CWC(3,4)=4.8
      CWC(4,3)=4.8
      CWC(5,5)=9.
      CWC(4,4)=39.04
      DO 8 I=1,3
      I1=I
      I2=I1+2
      DO 8 J=1,3
      JJ=J
      J2=JJ+2
    8 DUM(I,J)=CWC(I2,J2)
      CALL EIGEN(DUM,VDUM,3,1)
      WRITE(6,1000) ((DUM(I,J),J=1,3),VDUM(I),I=1,3)
 1000 FORMAT(1X9HINPUT DUM/3(1X4E20.8/))
      DO 9 I=1,3
      DO 9 J=1,3
    9 PA(I,J)=DUM(I,J)*VDUM(J)**.5
      READ(5,5)IND,INDU,INDP,N1F,N2F
    5 FORMAT(5I10)
```

TABLE H-1. (Cont'd)

```
      CALL PMUCAL
      WRITE(6,1001) ((PMUAR(101,I,J),J=1,6),I=1,5)
1001  FORMAT(1X11HINPUT PMUAR/5(1XE19.8,5E20.8/))
    2 IF(INDU.NE.1) GO TO 3
      CALL USGEN
      RETURN
    3 NRIT=10
      CALL REAPRO(UI)
      CALL REAPRO(S)
      READ(5,6) GGI1,NBR
      DO 4 I=1,101
    6 FORMAT(E15.8,I20)
      I1=I
      T=.01   *FLOAT(I1-1)
      IF(NRIT.NE.10 ) GO TO 4
      NRIT=0
      WRITE(6,7) T    ,UI(I)
    4 NRIT=NRIT+1
    7 FORMAT(1XE19.8,E20.8)
      WRITE(6,10) GGI1
   10 FORMAT(1XE20.8)
      CALL STORNM(N BR)
      RETURN
      END
$IBFTC PMUCAN
      SUBROUTINE PMUCAL
      DIMENSION TEM(65),E1(20),E2(20),Q(20,20),XLAM(20)
      COMMON/RANB/PA(3,3),P(5,5),XMU(5),JLOG,Y(5),RJ,T,RY(5),RU    ,UR
     1,PMUAR(401,5,6)
      COMMON/INPMU/Z0(5),INDP,XC(11,5)
      COMMON/DPMUB/V(21),DV(21)
      LOGICAL JLOG
      EXTERNAL DPMU
      DO 1 I=1,16
    1 V(I)=0.
      DO 2 I=17,21
      I1=I
      I1=I1-16
    2 V(I)=Z0(I1)
      DO 3 I=1,65
    3 TEM(I)=0.
      DV(1)=.01
      CALL DPMU
      CALL AMRKS(V,DV,DPMU,20,1,E1,E2,H,H,TEM,0)
      DO 10 I=1,5
      PMUAR(1,I,6)=Z0(I)
      DO 10 J=1,5
   10 PMUAR(1,I,J)=0.
      DO 4 I=1,100
      I1=I
      I1=I1+1
      CALL AMRK
      L=1
      DO 5 J=1,5
      JJ=J
      DO 5 K=1,JJ
      L=L+1
      Q(J,K)=V(L)
    5 Q(K,J)=V(L)
      IF(I1.EQ.401) WRITE(6,1000) ((Q(J,K),K=1,5),J=1,5)
1000  FORMAT(1X8HPMUCAL Q/5(1XE19.8,4E20.8/))
```

TABLE H-1. (Cont'd)

```
 100 FORMAT(/////50XE20.8//5(1XE19.8,4E20.8/))
     CALL EIGEN(Q,XLAM,5,1)
 101 FORMAT(/5(1XE19.8,5E20.8/))
     DO 6 L=1,5
     DO 6 K=1,5
     Q(K,L)=Q(K,L)*XLAM(L)**.5
   6 PMUAR(I1,K,L)=Q(K,L)
   7 FORMAT(E15.8)
     DO 4 K=1,5
     KK=K
     KK=KK+16
   4 PMUAR(I1,K,6)=V(KK)
     RETURN
     END
$IBFTC DPMUN
      SUBROUTINE DPMU
      DIMENSION Q(5,5)
      COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),      V(5,5),UI( 401),S( 401),GG
     111,UK,R
      COMMON/DPMUB/V(21),DV(21)
      L=1
      DO 1 J=1,5
      JJ=J
      DO 1 K=1,JJ
      L=L+1
      Q(J,K)=V(L)
    1 Q(K,J)=V(L)
      L=1
      DO 2 J=1,5
      JJ=J
      DO 2 K=1,JJ
      L=L+1
      DV(L)=CWC(J,K)
      DO 2 M=1,5
    2 DV(L)=DV(L)+A(J,M)*Q(M,K)+Q(J,M)*A(K,M
      DO 3 I=1,5
      II=I
      II=II+16
      DV(II)=0.
      DO 3 J=1,5
      JJ=J
      JJ=JJ+16
    3 DV(II)=A(I,J)*V(JJ)+DV(II)
      RETURN
      END
$IBFTC RANCAN
      SUBROUTINE RANCAL
      DIMENSION XI(5),ZETA(3),ETA(5),X8(5),X3(5),X4(5),DELPSI(5),X5(5,5)
      DIMENSION XD1(5,5),XD2(5),XD3(5,5)
      DIMENSION X1(5),XD(5),X6(5,5)
      COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),CWCINV(5,5),UI( 401),S( 401),GG
     111,UK,R
      COMMON/RANB/PA(3,3),P(5,5),XMU(5),JLOG,Y(5),RJ,T,RY(5),RU    ,UR
     1,PMUAR(401,5,6)
      COMMON/NUMB/N1F,N2F
      LOGICAL JLOG
      X3(2)=0.
      X3(1)=0.
      RJ=0.
      X7=1.-T
      X1(1)=0.
```

TABLE H-1. (Cont'd)

```
      X1(2)=0.
      RU=0.
      X9=(2.*X7)**.5
      DO 1 I=1,5
 1    RY(I)=0.
      N1=1
 20   DO 2 I=1,5
      XI(I)=GAURN(Z)
1000  FORMAT(1X11HRANCAL XIET/1XE19.8)
 2    ETA(I)=XMU(I)
      DO 3 I=1,5
              DO 3 J=1,5
    3 ETA(I)=ETA(I)+P(I,J)*XI(J)
1001  FORMAT(1X10HRANCAL ETA/1X5E19.8)
      DO 12 I=1,5
 12   X8(I)=ETA(I)-Y(I)
      N2=1
      DO 27 I=1,5
      XD(I)=0.
      DO 27 J=1,5
   27 XD3(I,J)=0.
 11   DO 6 I=1,3
      II=I
      II=II+2
      XI(I)=GAURN(Z)
1002  FORMAT(1X13HRANCAL ETAZET/1XE19.8)
    6 ZETA(I)=X8(II)
      DO 7 I=1,3
      DO 7 J=1,3
 7    ZETA(I)=ZETA(I)+X9*PA   (I,J)*XI(J)
      ZETNOR=0.
      DO 8 I=1,3
      II=I
      II=II+2
      DO 8 J=1,3
      JJ=J
      JJ=JJ+2
 8    ZETNOR=ZETNOR+ZETA(I)*CWCINV(II,JJ)*ZETA(J)
      ZETNOR=ZETNOR**.5
      DO 22 I=1,3
      II=I
      K=II+2
   22 XD(K)=XD(K)+ZETA(I)/ZETNOR**3
      IF(JLOG) GO TO 9
      DO 28 I=1,5
      DO 28 J=1,5
   28 XD1(I,J)=CWCINV(I,J)/ZETNOR**3
      DO 29 I=1,5
      XD2(I)=0.
      DO 29 J=3,5
      JJ=J
      K=JJ-2
   29 XD2(I)=XD2(I)+CWCINV(I,K)*ZETA(K)
      DO 30 I=1,5
      DO 30 J=1,5
   30 XD3(I,J)=XD1(I,J)-3.*XD2(I)*XD2(J)/ZETNOR**5+XD3(I,J)
 9    IF(N2.EQ.N2F) GO TO 10
      N2=N2+1
      GO TO 11
   10 DO 23 I=3,5
      X1(I)=0.
```

TABLE H-1. (Cont'd)

```
      DO 23 J=3,5
   23 X1(I)=X1(I)+CWCINV(I,J)*XD(J)
      DO 24 I=3,5
   24 X1(I)=X1(I)/FLOAT(N2)
      X2=0.
      DO 13 I=3,5
      X3(I)=0.
      DO 13 J=3,5
   13 X3(I)=X3(I)+CWCINV(I,J)*XB(J)
      DO 25 I=3,5
   25 X2=XB(I)*X3(I)+X2
      X2=X2**.5
      DO 14 I=1,5
      X4(I)=B(I)*UR
      DO 14 J=1,5
   14 X4(I)=X4(I)+A(I,J)*XB(J)
      DO 15 I=1,5
   15 DELPSI(I)=R*(X1(I)-X3(I)/X2**3)
 1003 FORMAT(1X18HRANCAL X1 X3 X4 X2/5(1X4E20.8/),1XE19.8)
      DO 16 I=1,5
   16 RJ=RJ+X4(I)*DELPSI(I)
      IF(JLOG) GO TO 5
      DO 31 I=1,5
      DO 31 J=1,5
   31 XD3(I,J)=XD3(I,J)/FLOAT(N2)
      DO 17 I=1,5
      RU =RU+B(I)*DELPSI(I)
      DO 17 J=1,5
   17 X5(I,J)=        X3(I   )*X3(J   )
      DO 26 I=1,5
      DO 26 J=1,5
   26 X6(I,J)=CWCINV(I,J)/X2**3
      DO 18 I=1,5
      DO 18 J=1,5
   18 RY(I)=RY(I)-A(J,I)*DELPSI(J)+R*(X6(I,J)-3.*X5(I,J)/X2**5-XD3(I,J))
     1*X4(J)
    5 IF(N1.EQ.N1F) GO TO 19
      N1=N1+1
      GO TO 20
   19 RJ=RJ/FLOAT(N1)
      RU=RU/FLOAT(N1)
      DO 21 I=1,5
   21 RY(I)=RY(I)/FLOAT(N1)
      RETURN
      END
$IBFTC PUNPRN
      SUBROUTINE PUNPRO(X)
      DIMENSION X(401)
      N=1
    2 N1=N
      N2=N+1
      N3=N+2
      N4=N+3
      N5=N+4
      IF(N5.GT.100) GO TO 3
      PUNCH 1,X(N1),X(N2),X(N3),X(N4),X(N5)
    1 FORMAT(5E15.8)
      N=N+5
      GO TO 2
    3 PUNCH 4,X(101)
    4 FORMAT(E15.8)
      RETURN
      END
$IBFTC REAPRN
      SUBROUTINE REAPRO(X)
      DIMENSION X(401)
      N=1
    2 N1=N
      N2=N+1
      N3=N+2
      N4=N+3
      N5=N+4
      IF(N5.GT.100) GO TO 3
      READ(5,1) X(N1),X(N2),X(N3),X(N4),X(N5)
    1 FORMAT(5E15.8)
      N=N+5
      GO TO 2
    3 READ(5,4) X(101)
    4 FORMAT(E15.8)
      RETURN
      END
```

Subroutines PMUCAL and DPMU solve equations (5-38) and (5-39).

Subroutine RANCAL carries out the Monte Carlo evaluation of the multidimensional integrals involved in equations (5-6) through (5-8) and (5-14) through (5-20).

Subroutines PUNPRO and PUNPRN handle the output and input of the punched cards used to transfer data from one computer run to the next.

Subroutines RANDPK (the random number generator) and AMRK (the differential equation solver) are referenced in Appendices E and G, respectively. Subroutine EIGEN (SHARE Library number ANF202) computes eigenvalues and eigenvectors of real symmetric matrices.

The subscript correspondence given in section G.2.1 applies here also.

The computation time depends on the time increment and the number of terms in the Monte Carlo evaluation of the integrals. For the problem solved here, a time increment of .01 and six terms in the evaluation of each integral corresponds to about ten minutes of IBM 7094 time per iteration of the algorithm shown in Figure 5-1.

The iterations were computed by individual computer runs. At the end of each iteration, the results, i.e. u^i, S^i, G^i, and the random number generator "position" were punched out on cards. At the beginning of the following iteration, these cards were read. The random number generator was "reset" at the beginning of each iteration to its "position" at the end of the preceding iteration.

H.3 $\overline{\psi}$ Estimates for $\mathcal{F}_1$ and $\mathcal{F}_2$ Controls

H.3.1 Control-Independent Term

A listing of the FORTRAN program used for the computation of ψ_o is shown in Table H-2. This computation is based on equation (4-115).

H.3.2 Control-dependent Term

The control dependent term, ψ_1, was computed as follows:

1. The appropriate initial values of the adjoint variable and the corresponding class parameter value were read in.
2. The corresponding control was computed by the appropriate USGEN subroutine. In the case of $\mathcal{F}_2$ controls the deck name of this subroutine is UGEN1 (see Table H-1). In the case of $\mathcal{F}_2$ controls the deck name of this subroutine is UGEN2 (see Table H-3).
3. Subroutine JCAL (see Table H-1) was used to compute ψ_1.

H.4 Computation for Figures 7-5 and 7-9.

A FORTRAN listing of the program used for the computation of Figures 7-5 and 7-9 is shown in Table H-4. This algorithm operates according to Figure 6-1, except that instead of computing the control through the solution of an adjoint equation, it reads the control from the punched-card output from the conjugate gradient program.

H.5 Computation for Figures 7-6 and 7-7.

A FORTRAN listing of the program used for the computation of Figures 7-6 and 7-7 is shown in Table H-5. This program reads in the control from the punched-card output of the conjugate gradient algorithm, computes the $\|x(t)\|$ trajectory using equation (2-1) with n_t=0, and uses the CALCOMP plotter to produce the curves shown in Figures 7-6 and 7-7.

TABLE H-2. PROGRAM FOR COMPUTATION OF Ψ_0.

```
$IBFTC MAIN
      DIMENSION Q(20,20),V(20)
      COMMON/PZB/PA(3,3),CWCINV(5,5),R,ZO(5)
      DO 1 I=1,5
      ZO(I)=0.
      DO 1 J=1,5
      Q(I,J)=0.
    1 CWCINV(I,J)=0.
      CWCINV(3,3)=39.04/16.
      CWCINV(3,4)=-.3
      CWCINV(4,3)=-.3
      CWCINV(5,5)=1./9.
      CWCINV(4,4)=.0625
      Q(1,1)=1.
      Q(1,2)=4.8
      Q(2,1)=4.8
      Q(3,3)=9.
      Q(2,2)=39.04
      CALL EIGEN(Q,V,3,1)
      DO 2 I=1,3
      DO 2 J=1,3
    2 PA(I,J)=Q(I,J)*V(J)**.5
      ZO(3)=1.
      ZO(5)=1.
      R=.2
      DO 4 I=100,1000,50
      II=I
      CALL PSIO(II,X)
    4 WRITE(6,5) II,X
    5 FORMAT(1XI20,E20.8)
    3 FORMAT(1XE20.8)
      STOP
      END
$IBFTC PSIOCA
      SUBROUTINE PSIO(N3F,PSIZER)
      DIMENSION XI(3),ETA(3)
      COMMON/PZB/PA(3,3),CWCINV(5,5),R,ZO(5)
      N3=0
      X1=0.
    5 DO 1 I=1,3
    1 XI(I)=GAURN(Z)
      DO 2 I=1,3
      II=I
      I2=II+2
      ETA(I)=ZO(I2)
      DO 2 J=1,3
    2 ETA(I)=ETA(I)+PA   (I,J)*XI(J)
      ETANOR=0.
      DO 3 I=1,3
      II=I
      I2=II+2
      DO 3 J=1,3
      JJ=J
      J2=JJ+2
    3 ETANOR=ETANOR+ETA(I)*CWCINV(I2,J2)*ETA(J)
      ETANOR=ETANOR**.5
      X1=X1+1./ETANOR
    4 N3=N3+1
      IF(N3.LT.N3F) GO TO 5
      X1=X1/FLOAT(N3)
      XNOR=0.
      DO 6 I=1,3
      II=I
      I2=II+2
      DO 6 J=1,3
      JJ=J
      J2=JJ+2
    6 XNOR=XNOR+ZO(I2)*CWCINV(I2,J2)*ZO(J2)
      XNOR=XNOR**.5
      PSIZER=R*(X1-1./XNOR)
      RETURN
      END
```

TABLE H-3. LISTING FOR UGEN2

```
$IBFTC UGEN2
      SUBROUTINE USGEN
      DIMENSION TE(35),E1(10),E2(10)
      COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),CWCINV(5,5),UI( 401),S( 401),GG
     1I1,UK,R
      COMMON/GENB2/V(11),DV(11),XKU
      EXTERNAL DGEN2
      GGI1=1.
      DO 5 I=1,11
    5 V(I)=0.
      V(4)=1.
      V(6)=1.
      V(7)=-.19101224E00
      V(8)=.28510453E-01
      V(9)=-.18715569E01
      V(10)=.12123603E00
      V(11)=-.55493969E-01
      DV(1)=.01
      XKU=1.E-04
      DO 6 I=1,35
    6 TE(I)=0.
      CALL DGEN2
      CALL AMRKS(V,DV,DGEN2,10,1,E1,E2,H,H,TE,0)
      NRIT=10
      DO 1 I=1,101
      S(I)=0.
      UI(I)=0.
      DO 2 J=1,5
      JJ=J
      JJ=JJ+6
    2 UI(I)=UI(I)+B(J)*V(JJ)*.5/XKU
      IF(NRIT.NE.10) GO TO 4
      WRITE(6,3) (V(J),J=1,11),UI(I)
    3 FORMAT(1XE19.8,5E20.8/1XE19.8,5E20.8)
      NRIT=0
    4 NRIT=NRIT+1
    1 CALL AMRK
      RETURN
      END
$IBFTC DGENN2
      SUBROUTINE DGEN2
      COMMON/SYSPAR/A(5,5),B(5),CWC(5,5),CWCINV(5,5),UI( 401),S( 401),GG
     1I1,UK,R
      COMMON/GENB2/V(11),DV(11),XKU
      U=0.
      DO 1 I=1,5
      II=I
      II=II+6
    1 U=U+.5*B(I)*V(II)/XKU
      DO 2 I=1,5
      II=I
      II=II+1
      DV(II)=B(I)*U
      DO 2 J=1,5
      JJ=J
      JJ=JJ+1
    2 DV(II)=DV(II)+A(I,J)*V(JJ)
      DO 3 I=1,5
      II=I
      I6=II+6
      II=II+1
      DV(I6)=0.
      DO 3 J=1,5
      JJ=J
      J6=JJ+6
      JJ=JJ+1
    3 DV(I6)=DV(I6)+2.*CWCINV(I,J)*V(JJ)-A(J,I)*V(J6)
      RETURN
      END
```

TABLE H-4. DIGITAL SIMULATION PROGRAM

```
$IBFTC CROEX
      DIMENSION TE(23),E1(5),E2(5),XNORA(500,2),XX(36),U(401)
      COMMON/BL/A(5,5),B(5),CWCINV(5,5),C(5,3),UD     ,X(6),DX(6),XN(3)
      EXTERNAL DPRO
      LOGICAL L1
      NCM=150
      NC=1
      DO 1 I=1,5
      B(I)=0.
      X(I)=0.
      DO 1 J=1,5
      A(I,J)=0.
    1 CWCINV(I,J)=0.
      DO 2 I=1,5
      DO 2 J=1,3
    2 C(I,J)=0.
      B(1)=.5
      B(2)=2.5
      CWCINV(3,3)=39.04/16.
      CWCINV(3,4)=-.3
      CWCINV(4,3)=-.3
      CWCINV(4,4)=.0625
      CWCINV(5,5)=1./9.
      A(3,1)=1.
      A(4,1)=4.8
      A(2,2)=-5.
      A(3,2)=-.2
      A(4,2)=-.96
      A(5,3)=3.
      A(1,4)=-.5
      A(2,4)=-2.5
      A(5,5)=-3.
      A(1,5)=-.5
      A(4,4)=-4.
      A(2,5)=-2.5
      C(3,3)=1.
      C(4,1)=4.
      C(4,3)=4.8
      C(5,2)=3.
   14 CALL REAPRO(U)
      NCOM=1
   11 DO 15 I=1,6
   15 X(I)=0.
      X(4)=1.
      X(6)=1.
      L1=.TRUE.
      XNORL=0.
      DO 3 I=3,5
      II=I
      II=II+1
      DO 3 J=3,5
      JJ=J
      JJ=JJ+1
    3 XNORL=XNORL+X(II)*CWCINV(I,J)*X(JJ)
      XNOR=XNORL
      DX(1)=.01
      DO 4 I=1,23
    4 TE(I)=0.
      UD=U(1)
      CALL DPRO
      CALL AMRKS(X,DX,DPRO,5,1,E1,E2,H,H,TE,U)
```

TABLE H-4. (Cont'd)

```
      NRIT=10
      DO 5 IX=1,100
      UD=U(IX)
      IF(NRIT.EQ.10.AND.L1) WRITE(6,100) (X(K),K=1,6),XNOR
      IF(NRIT.EQ.10)
     1L1=.NOT.L1
 100  FORMAT(1XE19.8,5E20.8/20XE20.8)
      IF(NRIT.EQ.10) NRIT=0
      NRIT=NRIT+1
      CALL AMRK
      XNOR=0.
      DO 6 I=3,5
      II=I
      II=II+1
      DO 6 J=3,5
      JJ=J
      JJ=JJ+1
    6 XNOR=XNOR+X(II)*CWCINV(I,J)*X(JJ)
      IF(XNOR.LT.XNORL) TL=X(1)
      IF(XNOR.LT.XNORL) XNORL=XNOR
      DO 5 I=1,3
    5 XN(I)=GAURN(Z)/(DX(1))**.5
      WRITE(6,100) (X(K),K=1,6),XNOR
      XNORA(NC,2)=TL
      XNORA(NC,1)=XNORL
    9 IF(NC.LT.NCM) GO TO 7
      DO 8 I=1,NCM
    8 WRITE(6,91)(XNORA(I,J),J=1,2)
   91 FORMAT(1X 2E20.8)
      CALL SORT(XNORA,NCM)
      DO 10 I=1,NCM
   10 WRITE(6,91)(XNORA(I,J),J=1,2)
      GO TO 12
    7 NC=NC+1
      GO TO 11
   12 DO 13 I=1,NCM
      II=I
      XX(I)=(151.-FLOAT(II))/150.
      XNORA(I,1)=XNORA(I,1)**.5
   13 WRITE(3) XNORA(I,1),XX(I)
      ENDFILE 3
      GO TO 14
      END
$IBFTC DPRON
      SUBROUTINE DPRO
      COMMON/BL/A(5,5),B(5),CWCINV(5,5),C(5,3),UD     ,X(6),DX(6),XN(3)
      DO 1 I=1,5
      II=I
      II=II+1
      DX(II)=B(I)*UD
      DO 2 J=3,5
    2 DX(II)=DX(II)+C(I,J)*XN(J)
      DO 1 J=1,5
      JJ=J
      JJ=JJ+1
    1 DX(II)=DX(II)+A(I,J)*X(JJ)
      RETURN
      END
$IBFTC SORTN
      SUBROUTINE SORT(A,N)
      DIMENSION A(500,2)
```

TABLE H-4. (Cont'd)

```
      DO 1 I=1,N
      X=0.
      II=I
      DO 2 J=II,N
      IF(A(J,1).GT.X) Y=A(J,2)
      IF(A(J,1).GT.X)  K=J
    2 IF(A(J,1).GT.X)  X=A(J,1)
      A(K,1)=A(II,1)
      A(K,2)=A(II,2)
      A(II,1)=X
    1 A(II,2)=Y
      RETURN
      END
$IBFTC PUNPRN
      SUBROUTINE PUNPRO(X)
      DIMENSION X(401)
      N=1
    2 N1=N
      N2=N+1
      N3=N+2
      N4=N+3
      N5=N+4
      IF(N5.GT.100) GO TO 3
      PUNCH 1,X(N1),X(N2),X(N3),X(N4),X(N5)
    1 FORMAT(5E15.8)
      N=N+5
      GO TO 2
    3 PUNCH 4,X(101)
    4 FORMAT(E15.8)
      RETURN
      END
$IBFTC REAPRN
      SUBROUTINE REAPRO(X)
      DIMENSION X(401)
      N=1
    2 N1=N
      N2=N+1
      N3=N+2
      N4=N+3
      N5=N+4
      IF(N5.GT.100) GO TO 3
      READ(5,1) X(N1),X(N2),X(N3),X(N4),X(N5)
    1 FORMAT(5E15.8)
      N=N+5
      GO TO 2
    3 READ(5,4) X(101)
    4 FORMAT(E15.8)
      RETURN
      END
```

TABLE H-5. PROGRAM FOR PLOT OF $\|\bar{x}(t)\|_{\bar{a}^{-1}}$ and u(t)

```
$IBFTC NORPLO
      DIMENSION UD(401),TE(20),E1(5),E2(5)
      COMMON/BL/A(5,5),B(5),CWCINV(5,5),U,X(6),DX(6)
      EXTERNAL DP
      DO 1 I=1,5
      B(I)=0.
      DO 1 J=1,5
      A(I,J)=0.
    1 CWCINV(I,J)=0.
      A(3,1)=1.
      A(4,1)=4.8
      A(2,2)=-5.
      A(3,2)=-.2
      A(4,2)=-.96
      A(5,3)=3.
      A(1,4)=-.5
      A(2,4)=-2.5
      A(5,5)=-3.
      A(1,5)=-.5
      A(4,4)=-4.
      A(2,5)=-2.5
      B(1)=.5
      B(2)=2.5
      CWCINV(3,3)=39.04/16.
      CWCINV(3,4)=-.3
      CWCINV(4,3)=-.3
      CWCINV(4,4)=.0625
      CWCINV(5,5)=1./9.
    8 DO 2 I=1,6
    2 X(I)=0.
      X(4)=1.
      X(6)=1.
      CALL REAPRO(UD)
      DO 3 I=1,20
    3 TE(I)=0.
      U=UD(1)
      CALL AMRKS(X,DX,DP,5,1,E1,E2,H,H,TE,0)
      DX(1)=.01
      CALL DP
      DO 4 M=1,101
      XNOR=0.
      DO 5 I=1,5
      II=I
      II=II+1
      DO 5 J=1,5
      JJ=J
      JJ=JJ+1
    5 XNOR=XNOR+X(II)*CWCINV(I,J)*X(JJ)
      XNOR=XNOR**.5
      WRITE(6,6) (X(I),I=1,6),XNOR
    6 FORMAT(1XE19.8,5E20.8/20XE20.8)
      WRITE(3) X(1),XNOR
      U=UD(M)
    4 CALL AMRK
      ENDFILE 3
      DO 7 I=1,101
      II=I
      T=DX(1)*FLOAT(II-1)
    7 WRITE(3) T,UD(I)
      ENDFILE 3
      GO TO 8
```

TABLE H-5. (Cont'd)

```
      END
$IBFTC DPN
      SUBROUTINE DP
      COMMON/BL/A(5,5),B(5),CWCINV(5,5),U,X(6),DX(6)
      DO 1 I=1,5
      II=I
      II=II+1
      DX(II)=B(I)*U
      DO 1 J=1,5
      JJ=J
      JJ=JJ+1
    1 DX(II)=DX(II)+A(I,J)*X(JJ)
      RETURN
      END
$IBFTC REAPRN
      SUBROUTINE REAPRO(X)
      DIMENSION X(401)
      N=1
    2 N1=N
      N2=N+1
      N3=N+2
      N4=N+3
      N5=N+4
      IF(N5.GT.100) GO TO 3
      READ(5,1) X(N1),X(N2),X(N3),X(N4),X(N5)
    1 FORMAT(5E15.8)
      N=N+5
      GO TO 2
    3 READ(5,4) X(101)
    4 FORMAT(E15.8)
      RETURN
      END
```

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