A UNIFIED APPROACH TO SUBOPTIMUM CONTROL

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ABSTRACT

A class of suboptimum control techniques is developed on the basis of the linear relation $\psi = M\xi$ between a change ψ in the adjoint vector p to a change ξ in the state x in the vicinity of the optimum trajectory. The matrix M relating ψ to ξ can be obtained by solving a linear two-point boundary-value problem or from the matrix Riccati equation $-\dot{M} = MH_{XP} + H_{PX}M + MH_{PP}M + H_{XX}$, where $H_{XP} = H'_{PX}$, H_{PP}, H_{XX} are matrices of second partial derivatives of the Hamiltonian $H(p,x)$ evaluated on an optimum trajectory. One application of this technique uses the solution of a simplified problem as the nominal trajectory; a second application is based on a precomputed nominal trajectory, and a third makes use of the relationship $\dot{p} = M\dot{x}$. Instantaneous constraints on the control variable are permissible with these methods, although formal treatment of impulse terms may be required. Two examples show the performance of the techniques.

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Introduction

The major impediment to the application of optimum control theory to the design of practical feedback control systems is the necessity for solving a two-point boundary-value problem in real-time. Since precise optimality at the expense of costly implementation is rarely desirable, there is a need for techniques which are based on the mathematical framework of optimum control theory, which can be readily implemented, and which give near-optimum performance in the presence of the actual constraints. Techniques of this type can be termed "suboptimum" or "quasi-optimum."

A number of techniques based on the second-variation in the calculus of variations have been proposed recently.[1,2,3,4] These techniques, however, are not capable of dealing with "hard" control variable constraints and require the computation of a nominal trajectory with which the actual trajectory followed by the process is compared.

The techniques presented are related to those of References 1 through 4 except that the linearization is performed with respect to the adjoint vector rather than the control. This linearization has the advantage of permitting the treatment (at least formally) of hard constraints. In one technique, moreover, the linearization is about the solution to a simplified problem rather than about a nominal trajectory. As a consequence an explicit feedback control law, independent of a nominal trajectory, is obtained.

Development of Suboptimum Control Equations

Consider the problem of minimization of $x_0(T)$ for the system

$$\dot{x}(t) = f(x(t), u(t)), \quad x = \{x_0, \ldots x_n\} \tag{1}$$

where $x(0)$ is known, $x_i(T) = c_i$ is required for $i = 1, \ldots m<n$, $x_i(T)$ is free for $i \geq m+1$, T is free and $u(t)$ must be a member of a given set Ω. It is well known that if an optimal u^* exists then u^* maximizes the Hamiltonian

$$H(x, u^*, p) = \max_{u \in \Omega} H(x, u, p) = \max_{u \in \Omega} p' f(x, u) \tag{2}$$

where $H(x, u^*, p) = 0$. Consequently, u^* can be obtained as a function of the adjoint state $p = \{p_0, \ldots, p_n\}$ and the process state x:

$$u^* = \sigma(p, x) \tag{3}$$

The adjoint vector $p(t)$ is governed by

$$\dot{p}(t) = -H_x \tag{4}$$

with $p_0(T) = -1$

$p_i(T) = 0 \qquad i = m+1, \ldots, n$

where H_x is the gradient of the Hamiltonian with respect to x.

Let $X(t)$, $P(t)$ be solutions of the two-point boundary-value problem (1)-(4) and let

$$x(t) = X(t) + \xi(t) \tag{5}$$

be the state in an altered problem (e.g., with different initial or terminal conditions or different dynamics), and suppose that ξ is a small quantity. As a result of the change ξ in x, the adjoint vector will change by an amount ψ, i.e.,

$$p(t) = P(t) + \psi(t) \tag{6}$$

Substitution of (5) and (6) into (1) and (4) results in

$$\begin{aligned} \dot{X} + \dot{\xi} &= H_p = H_P + H_{XP}\xi + H_{PP}\psi + O(\xi^2) \\ \dot{P} + \dot{\psi} &= -H_x = -H_X - H_{XX}\xi - H_{PX}\psi + O(\xi^2) \end{aligned} \tag{7}$$

where H_P and H_X are the gradients of the Hamiltonian evaluated at $p = P$, $x = X$ and

$$H_{XP} = \left[\frac{\partial^2 H}{\partial x_j \partial p_i}\right]_{\substack{x=X \\ p=P}} \qquad H_{PX} = \left[\frac{\partial^2 H}{\partial p_j \partial x_i}\right]_{\substack{x=X \\ p=P}} = H'_{XP}$$

$$H_{PP} = \left[\frac{\partial^2 H}{\partial p_j \partial p_i}\right]_{\substack{x=X \\ p=P}} \qquad H_{XX} = \left[\frac{\partial^2 H}{\partial x_j \partial x_i}\right]_{\substack{x=X \\ p=P}}$$

Since X and P are solutions to (1)-(4) it follows that

$$\dot{X} = H_P \qquad \dot{P} = -H_X$$

and hence, after dropping terms of $O(\xi^2)$, (7) becomes

$$\begin{aligned} \dot{\xi} &= H_{XP}\xi + H_{PP}\psi \\ \dot{\psi} &= -H_{XX}\xi - H_{PX}\psi \end{aligned} \tag{8}$$

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These differential equations are linear, and can be integrated to give

$$\xi(T) = \Phi_{11}(T, t)\,\xi(t) + \Phi_{12}(T, t)\,\psi(t)$$
$$\psi(T) = \Phi_{21}(T, t)\,\xi(t) + \Phi_{22}(T, t)\,\psi(t) \tag{9}$$

where $\Phi_{ij}(T, t)\,(i, j = 1, 2)$ are the $(n+1) \times (n+1)$ blocks of the $(2n+2) \times (2n+2)$ fundamental (transition) matrix of (8). Our objective is to find a relationship between the correction $\psi(t)$ to the adjoint vector $p(t)$ and the deviations $\xi(t)$ of the state from $X(t)$. To do this it is necessary to eliminate $\xi(T)$ and $\psi(T)$ from (9) by use of the boundary conditions for the original (exact) problem (1)-(4).

Consider a state variable x_i fixed at $t = T$. Then

$$\begin{aligned} x_i(T + dT) &= x_i(T) + \dot{x}_i(T)dT \\ &= X_i(T) + \xi_i(T) + \dot{X}_i(T)dT + \dot{\xi}_i(T)dT \end{aligned} \tag{10}$$

The last term is a second-order infinitesimal and can be dropped. If in the simplified problem the constraint is satisfied by X_i at time T, then in the exact problem the constraint must be satisifed at $T + dT$. Thus we must have $x_i(T + dT) = c_i = X_i(T)$ and hence (10) becomes:

$$\xi_i(T) = -\dot{X}_i dT \quad \text{for} \quad X_i(T) \text{ fixed} \tag{11}$$

Likewise, if $X_i(T)$ is free, the corresponding adjoint variable P_i is constrained to terminate at zero, and, reasoning as above, we conclude that:

$$\psi_i(T) = -\dot{P}_i dT \quad \text{for} \quad X_i(T) \text{ free} \tag{12}$$

Finally, we must have

$$\begin{aligned} dH &= \xi' \frac{\partial H}{\partial X} + \psi' \frac{\partial H}{\partial P} \\ &= -\dot{P}'\,\xi + \dot{X}'\,\psi = 0 \end{aligned} \tag{13}$$

Equations (10)-(13) give $n + 2$ relations. Since dT is an additional variable, there are just enough equations needed to solve (9) for $\psi(t)$ as a function of $\xi(t)$. It is readily established that upon elimination of $\psi(T)$ and $\xi(T)$, a linear relation between $\psi(t)$ and $\xi(t)$ is obtained:

$$\psi(t) = M(t)\,\xi(t) \tag{14}$$

Upon differentiation of (14) and substitution of the result into (8), there results

$$(dM/dt + MH_{XP} + H_{PX}M + MH_{PP}M + H_{XX})\xi = 0$$

If this relationship is to hold for any ξ, the matrix M must satisfy the matrix Riccati equation:

$$-dM/dt = MH_{XP} + H_{PX}M + MH_{PP}M + H_{XX} \tag{15}$$

It is evident that if M is a solution to (15) then M' is a solution to (15); thus the solution to (15) can be a symmetric matrix. In fact, when the adjoint vector p can be interpreted as the negative of the gradient of the optimum value of $x_0(T) = V$,[5] then

$$M = -\left[\frac{\partial^2 V}{\partial x_i \partial x_j}\right]_{x = X} \tag{16}$$

Because $\xi(t)$ is a change in $x(t)$ and $\psi(t)$ is the corresponding change in $p(t)$, it follows that

$$M = \left[\frac{\partial p_i}{\partial x_j}\right]_{x = X} \tag{17}$$

even when p cannot be interpreted as the negative gradient of V.

Constraints on the magnitude of the control variables, $|u_i| \leq U_i$ for example, will generally lead to a control law which is a discontinuous function of the adjoint variables, and hence, in a strict sense not all the partial derivatives required in (8) or (15) will exist. The discontinuous control variables can be treated by introduction of impulses (delta functions) which arise upon differentiation. These impulses can be handled by well-known formal methods. The examples below illustrate the validity of this approach. An alternative approach would be to approximate the amplitude constraints by constraints which do not lead to discontinuous controls.

Applications

Approximation by Simpler Process

In many problems it is possible to approximate the dynamic behavior of the process by a system of differential equations of considerably simpler form than those actually governing the process. Under favorable circumstances an analytic solution to the simplified problem can be found, but the use of the control law derived for the simplified process may not be entirely adequate for the exact dynamic model. If the neglected terms in the original dynamic model were accounted for approximately, however, it might be possible to improve performance to an acceptable level. The suboptimum control equations of the previous section provides a method of so doing.

The control u is generated as a function of the state x and the adjoint state p. Instead of using the exact (unknown) relation $p = p(x)$ between the adjoint state and the process state the adjoint state is approximated by

$$p = P + M\xi \tag{18}$$

where $P = P(X)$ is obtained as the analytic solution of the simplified problem and

$$M = M(X)$$

is the matrix M of the previous section, expressed as a function of the state X instead of time. (To eliminate time from M, it is necessary to express time in terms of the state variables along the optimum trajectory of the simplified system.)

The control system, using the optimum transformation from p to u, but the approximate relation (18) for the transformation from x to p has the configuration of Figure 1.

In many cases it will not be possible to obtain an explicit analytical expression for M(X). Numerical integration of the Riccati equation, using a function of the state rather than time as the independent variable, and analytical approximation of the result may prove feasible. An alternative procedure would be to use an asymptotic solution of (15), obtained by setting dM/dt to zero and solving the resulting algebraic system.

Linearization About Nominal Trajectory

The approximate control law of Figure 1 is suitable for processes which can be approximated by simpler ones for which an analytic expression for P(X) can be obtained and for which an exact or approximate expression for M(X) can be found. When this cannot be done, it may still be possible to employ the general technique, by numerically computing a solution P(t), X(t) to the exact problem (1)-(4). (Numerous methods for performing such computations have been given in the literature.) If the initial state x_0 is close to the nominal initial state X_0 and the disturbances are relatively small, then $\xi(t) = x(t) - X(t)$ will remain small and thus the adjoint state p(t) is well-approximated by

$$p(t) = P(t) + M(t)\ \xi(t) \tag{19}$$

To employ this technique X(t), P(t), and M(t) are stored in the controller (or generated by integration of (1)-(4) and (15) with nominal initial conditions X_0, P_0, M_0) and the control u(t) generated from p(t) and x(t) as shown in Figure 2.

This technique is very similar to the second-variation techniques of Kelley[2] and of Breakwell, Bryson and Speyer,[1] except that here the adjoint vector P(t) rather than the control u(t) is stored. The difference is relatively insignificant when u is a continuous function of p, but is of major significance when u is a bounded, discontinuous function of p. For example, if the optimum control law is of the form u = sgn (c'p) then there is no reasonable way to make a linear correction to the nominal control u, but $u = \text{sgn}\,[c'(P+M\xi)]$ is an entirely reasonable control law.

Linearization About P_0 and M_0

A third possible method of employing the suboptimum control technique is based on the interpretation of (17) as the Jacobian matrix of p with respect to x. As a consequence of this interpretation it follows that

$$\dot{p} = \left[\frac{\partial p_i}{\partial x_j}\right] \dot{x} = M\dot{x} \tag{20}$$

provided that the partial derivatives in the matrix M are evaluated at the true state x of the process. Thus the adjoint vector can be obtained by integration of (20);

$$p = p_0 + \int_{t_0}^{t} M\dot{x}\,dt \tag{21}$$

This relation leads to a control system with the configuration shown in Figure 3(a). It is noted that the derivatives of the state variables instead of the state variables themselves are the quantities fed back. Hence this technique is particularly applicable to problems in inertial guidance, where the principal sensors are accelerometers.

In the event that $\dot{x}$ cannot be sensed, an alternative configuration can be obtained by partial integration of (21):

$$p = p_0 + Mx - M_0x_0 - \int_{t_0}^{t} \dot{M}x\,dt \tag{22}$$

The right-hand side of (15) is used for $-\dot{M}$ in (22). The control system configuration corresponding to (22) is shown in Figure 3(b); it is seen that only the state x is required in the controller.

In either implementation the matrix M would be generated by real-time integration of (15) with the nominal initial condition M_0, and the nominal initial adjoint state P_0 would be used. Thus to achieve near-optimum performance, the actual initial state x_0 should be reasonably close to the nominal initial state X_0 for which M_0 and P_0 were computed. If the closed-loop system is asymptotically stable, however, the effects of using initially incorrect values of M_0 and p_0 will be only transient.

Examples

Approximation by Simpler Process

To illustrate the technique we consider a problem for which an exact solution is known, namely, to minimize $x_0(T) = T$ for the process

$$\begin{aligned} \dot{x}_0 &= 1 \\ \dot{x}_1 &= -x_1x_3+u \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= 0 \end{aligned} \tag{23}$$

subject to

$$x_1(T) = x_2(T) = 0 \text{ and } |u| \leq 1 \tag{24}$$

Since $\dot{x}_3 = 0$, $x_3 = a$ = constant and the problem is one of Bushaw's[6] and has the following solution for the minimum time

$$T = -U(x_1 + ax_2) + \frac{2 \log S}{a}, \qquad U = \pm 1$$

$$S = 1 + \left[(aUx_1 - 1)e^{aU(x_1 + ax_2)} + 1 \right]^{1/2} \qquad (25)$$

The adjoint variables as functions of the state can be solved for either by the formulas

$$p_i = -\frac{\partial T}{\partial x_i}$$

or by solving the exact two-point boundary-value problem. The results, which are rather unwieldy, are given in Reference 6. Upon performance of a second set of partial differentiations, the expressions for the elements m_{ij} of M can be found.[6]

For the simplified problem, we take $\xi = x_3 \equiv 0$, which reduces the process to a double-integrator. The Hamiltonian for this problem is

$$H = -1 + P_1 U + P_2 X_1 = 0$$

where $U = \text{sgn } P_1$. The canonical equations are

$$\begin{aligned} \dot{X}_0 &= 1 & \dot{P}_0 &= 0 \\ \dot{X}_1 &= \text{sgn } P_1 & \dot{P}_1 &= -P_2 \\ \dot{X}_2 &= X_1 & \dot{P}_2 &= 0 \end{aligned} \qquad (26)$$

and it is readily established that the adjoint variables as functions of the state (of the simplified problem) are

$$P_1 = U - \frac{X_1}{\left(\frac{1}{2} X_1^2 - U X_2\right)^{1/2}}$$

$$P_2 = \frac{U}{\left(\frac{1}{2} X_1^2 - U X_2\right)^{1/2}} \qquad (27)$$

The control U is negative above and positive below the "switch curve" given by

$$X_2 = -\frac{1}{2} U X_1^2 = \mp \frac{1}{2} X_1^2 \qquad (28)$$

For the complete problem the Hamiltonian is given by

$$H = p_0 + p_1(u - x_1 x_3) + p_2 x_1 \qquad (29)$$

and the maximum principle of Pontryagin gives the control as

$$u = \text{sgn } p_1$$

Hence, the suboptimum control law, according to (18) is

$$u = \text{sgn } (P_1 + m_{13} \xi) \qquad (30)$$

with $\xi = x_3 = a$, and $P_1(X_1, X_2)$ given by (27). It remains to compute m_{13}. The matrices appearing in (8) and the Riccati equation (15) are obtained from (29) with x_3 set to zero (after performance of the indicated partial differentiations) and are found to be:

$$H_{XP} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -X_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad H_{PX} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -X_1 & 0 & 0 \end{bmatrix} \qquad (31a)$$

$$H_{PP} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\delta(P_1) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad H_{XX} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_1 \\ 0 & 0 & 0 & 0 \\ 0 & -P_1 & 0 & 0 \end{bmatrix} \qquad (31b)$$

Note the appearance of the impulsive term $2\delta(P_1)$ which results in H_{PP} from differentiation of sgn P_1. The fundamental (transition) matrix Φ corresponding to (8) with $t_0 = 0$ can be written as the product of three matrices

$$\Phi(T, 0) = \Phi(T, t_s^+)\, \Phi(t_s^+, t_s^-)\, \Phi(t_s^-, 0) \qquad (32)$$

where $t_s = P_{10}/P_2$, is the switching time in the simplified problem. In the absence of the impulse $\Phi(t_s^+, t_s^-)$ would be the identity matrix. With the impulse present, however, we find that the upper right hand block $\Phi_{12}(t_s^+, t_s^-)$ of $\Phi(t_s^+, t_s^-)$ is given by

$$\Phi_{12}(t_s^+, t_s^-) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2U/P_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

upon use of the formula[7]

$$\delta(P_1(t)) = \frac{\delta(t - t_s)}{|\dot{P}_1(t_s)|} \qquad t_s^- < t_s < t_s^+$$

The calculation of the other elements of the fundamental matrix (32) is tedious but straightforward. After determining this matrix $\Phi(T, 0)$, the matrix M can be computed by applying the boundary conditions (10)-(13), which in this case are

$$\xi_1(T) = -\dot{X}_1(T)\, dT = U\, dT$$

$$\xi_2(T) = -\dot{X}_2(T)\, dT = 0$$

$$\psi_0(T) = -\dot{P}_0(T)\, dT = 0$$

$$\psi_3(T) = -\dot{P}_3(T)\, dT = 0$$

$$P_2(T)\ \xi_1(T) - U\psi_1(T) = 0$$

The solutions, after elimination of time in favor of $X_1 = X_1(t)$, are

$$\begin{aligned}
m_{00} &= m_{10} = m_{20} = m_{30} = 0 \\
m_{11} &= P_2^3 X_2 = U(\tfrac{1}{2} P_2^3 X_1^2 - P_2) \\
m_{12} &= -\tfrac{1}{2} P_2^3 X_1 \\
m_{13} &= \tfrac{1}{6} P_2^3 X_1^4 - P_2 X_1^2 + X_1 \\
m_{22} &= \tfrac{1}{2} UP_2^3 \\
m_{23} &= -\tfrac{1}{6} UP_2^3 X_1^3 \\
m_{33} &= U\left(\tfrac{1}{18} P_2^3 X_1^6 - \tfrac{1}{2} P_2 X_1^4 + \tfrac{2}{3} X_1^3 - \tfrac{1}{3} P_2^3\right)
\end{aligned} \qquad (33)$$

It can be verified that these

$$m_{ij} = \left[\frac{\partial^2 T}{\partial x_i \partial x_j}\right]_{x = X}$$

where T is given by (25), at all points where the partial derivatives exist, i.e. everywhere except on the switch curve itself. Thus, by the formal treatment of the impulse arising from differentiation of a discontinuous function we have succeeded in calculating the matrix of second partials of the minimum time for $x_3 = \xi = 0$ without first having found the expression for $T(X_1, X_2, \xi)$.

The suboptimum control law is given by (30), which upon use of (27) and (33) becomes

$$u = \text{sgn}\quad U - \frac{X_1}{\left(\frac{1}{2} X_1^2 - UX_2\right)^{\frac{1}{2}}} + \left(\tfrac{1}{6} P_2^3 X_1^4 - P_2 X_1^2 + X_1\right)\xi$$

The approximate switch curve has been computed numerically for $\xi = a = 0.3$ (not really very small) and is given by the curve labeled "approximate" in Figure 4. For purposes of comparison, the exact optimum switch curve

$$x_2 = -\frac{x_1}{a} + \frac{U}{a^2} \log(1 + aUx_1)$$

and the switch curve (28) for the simplified problem is also shown. It is evident that the use of the quasi-optimum control law will result in considerably better performance than the control law for the simplified problem.

<u>Linearization About Nominal Trajectory</u>

As the second example consider the process governed by the same equations as the simplified problem of the previous example. The nominal trajectory and adjoint variables are then given by

$$X_1(t) = \begin{cases} X_{10} + Ut & 0 \le t \le t_s \\ X_1(t_s) - Ut & t_s \le t \le T \end{cases}$$

$$X_2(t) = \begin{cases} X_{20} + X_{10}t + Ut^2/2 & 0 \le t \le t_s \\ X_2(t_s) + X_1(t_s)(t - t_s) - U(t - t_s)^2/2 & t_s \le t \le T \end{cases}$$

$$P_1(t) = P_{10} - P_{20}t$$

$$P_2(t) = P_{20}$$

where $t_s = P_{10}/P_{20}$

$U = \text{sgn } P_{10}$

and P_{10}, P_{20} are related to X_{10}, X_{20} by (27). The above expressions for X(t), P(t) are stored in the controller. The control law to be used is then

$$u = \text{sgn}\left(P_1(t) + m_{11}(t)\ \xi_1(t) + m_{12}(t)\ \xi_2(t)\right)$$

where $\xi_1(t) = x_1(t) - X_1(t)$, $\xi_2(t) = x_2(t) - X_2(t)$, $x_1(t)$ and $x_2(t)$ being the actual measured state variables. The gains m_{11} and m_{12} are obtained by numerical integration of the matrix Riccati equation (15) which in component form is

$$\begin{aligned}
\dot{m}_{11} &= -2m_{12} - 2m_{11}^2\ \delta(p_1) \\
\dot{m}_{12} &= -m_{22} - 2m_{11}m_{12}\ \delta(p_1) \\
\dot{m}_{22} &= -2m_{12}^2\ \delta(p_1)
\end{aligned} \qquad (34)$$

It is noted that the only dependence on the state vector or the adjoint vector is through $\delta(p_1)$. Hence during an interval for which $p_1(t)$ has constant sign the solutions to (34) are given by

$$m_{11}(t_2) = m_{11}(t_1) - 2m_{12}(t_1)\,\tau + m_{22}(t_1)\,\tau^2$$

$$m_{12}(t_2) = m_{12}(t_1) - m_{12}(t_1)\,\tau \qquad (35)$$

$$m_{22}(t_2) = m_{22}(t_1)$$

where

$$\tau = t_2 - t_1 \qquad t_2 \geq t_1.$$

The m_{ij} change discontinuously when p_1 goes through zero. Use of the approximation

$$|\dot{p}_1(t_s)|\delta(p_1(t)) = \begin{cases} 0 & t < t_s \\ 1/\epsilon & t_s \leq t \leq t_s + \epsilon \\ 0 & t > t_s + \epsilon \end{cases}$$

where $p_1(t_s) = 0$ results in the following expressions for the discontinuities in m_{ij}:

$$m_{11}^+ = m_{11}^- \,\theta$$

$$m_{12}^+ = m_{12}^- \,\theta$$

$$m_{22}^+ = m_{22}^- \quad 1 - \frac{2\theta}{|\dot{p}_1(t_s)|} \quad m_{12}^-$$

$$\text{where} \quad \theta = \left[1 + \frac{2m_{11}^-}{|\dot{p}_1(t_s)|}\right]^{-1}$$

The initial values of m_{ij} which are to be used with (34) or (35) are given by

$$m_{11}(0) = P_{20}^3 X_{20}$$

$$m_{12}(0) = -P_{20}^3 X_{10}/2$$

$$m_{22}(0) = UP_{20}^3/2$$

in accordance with (33) in the previous example.

To verify the performance of this technique this example was simulated on a digital computer for four deviations from the nominal initial conditions: $x_1(0) = 0$, $x_2(0) = 100$ as indicated in Table 1. The impulse function $\delta(p_1)$ in (34) was approximated by

$$\delta(p_1) = \begin{cases} 0 & \text{for } |p_1| > \epsilon \\ \frac{1}{2\epsilon} & \text{for } |p_1| \leq \epsilon \end{cases} \qquad (36)$$

using $\epsilon = 1.275 \times 10^{-3}$. The trajectories for each run and a portion of the exact switch curve are shown in Figure 5(A-D). The suboptimal control, the nominal control and the exact switch time t_s for the actual initial conditions are shown in Figure 6(A-D). It is believed that some of the overshoot of the switch curve and the resulting terminal error is due to the unavoidable time quantization which is introduced by digital computer simulation. Since u was changed and states observed only every 10^{-2} sec, in some cases p_1 passed through the band $|p_1| < \epsilon$ and the "impulse" of (36) was not introduced into (34). In other cases the "impulse" of (36) remained on even after $|p_1|$ exceeded ϵ. Even so, the performance (see Table 1) is acceptable: with no correction the input would have switched at the nominal switch time of 10 sec in each case, causing errors approximately ten times greater than those obtained using the suboptimal control law.

Summary and Conclusions

The relationship between a change ξ in the state vector x to a change ψ in the adjoint vector p has been used to obtain a class of suboptimum control techniques. The matrix M which transforms ξ into ψ can be obtained by solving either a linear, two-point boundary-value problem or a matrix Riccati equation. The suboptimum control laws take the form $u = \sigma(p,x)$ where $\sigma(p,x)$ is the function which maximizes the Hamiltonian and p is the approximate adjoint vector, generated by the equation

$$p = P + M\xi$$

where P is the adjoint vector for a neighboring optimization problem, or by

$$\dot{p} = M\dot{x}$$

Since the exact relation $u = \sigma(p,x)$ is used to generate the control, it is possible to deal with problems in which u is constrained. When there are bounds on the control variable which lead to a discontinuous transformation from x to p, the two-point boundary-value problem and the matrix Riccati equation will have impulsive coefficients. These impulses can be treated by appropriate formal methods.

The two examples given above demonstrate the effectiveness of this approach to suboptimum control. There are, however, several problems which require further investigation. In many cases the determination of the matrix M by the exact relationships may be impractical, and it would be desirable to be able to approximate M with a simpler matrix without seriously compromising the effectiveness of the technique. Use of the asymptotic solution to the matrix Riccati equation might be feasible in problems in which a nontrivial asymptotic solution exists. Another problem is that of evaluation of performance of the suboptimum control system: How small must the deviation ξ from the nominal state X be in order that the performance be acceptable? If stability is a relevant consideration (i.e. when the terminal time is infinite) and the optimum system is (asymptotically) stable, under what conditions will the suboptimum system be (asymptotically) stable? These, and related questions are currently under investigation.

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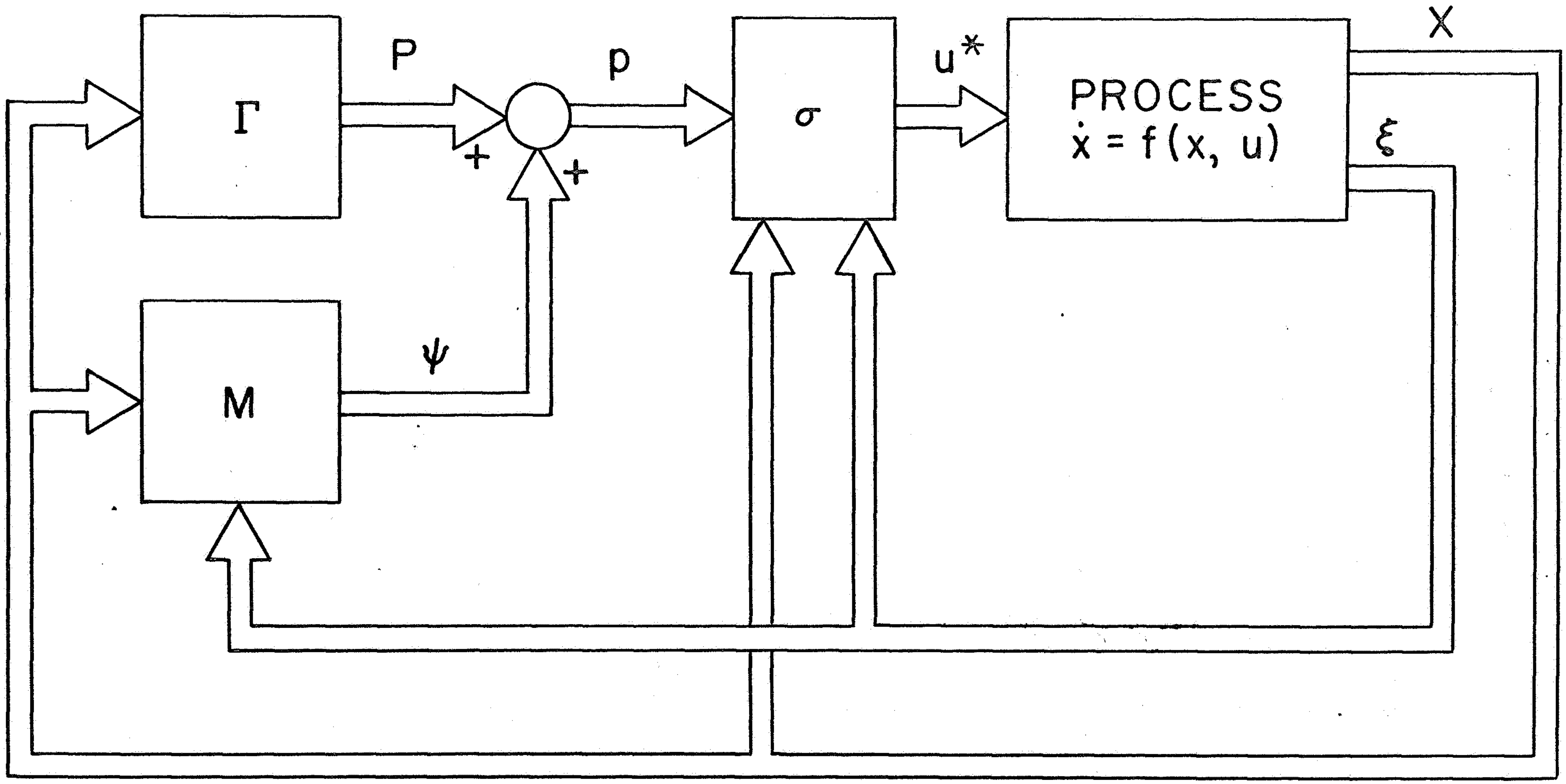
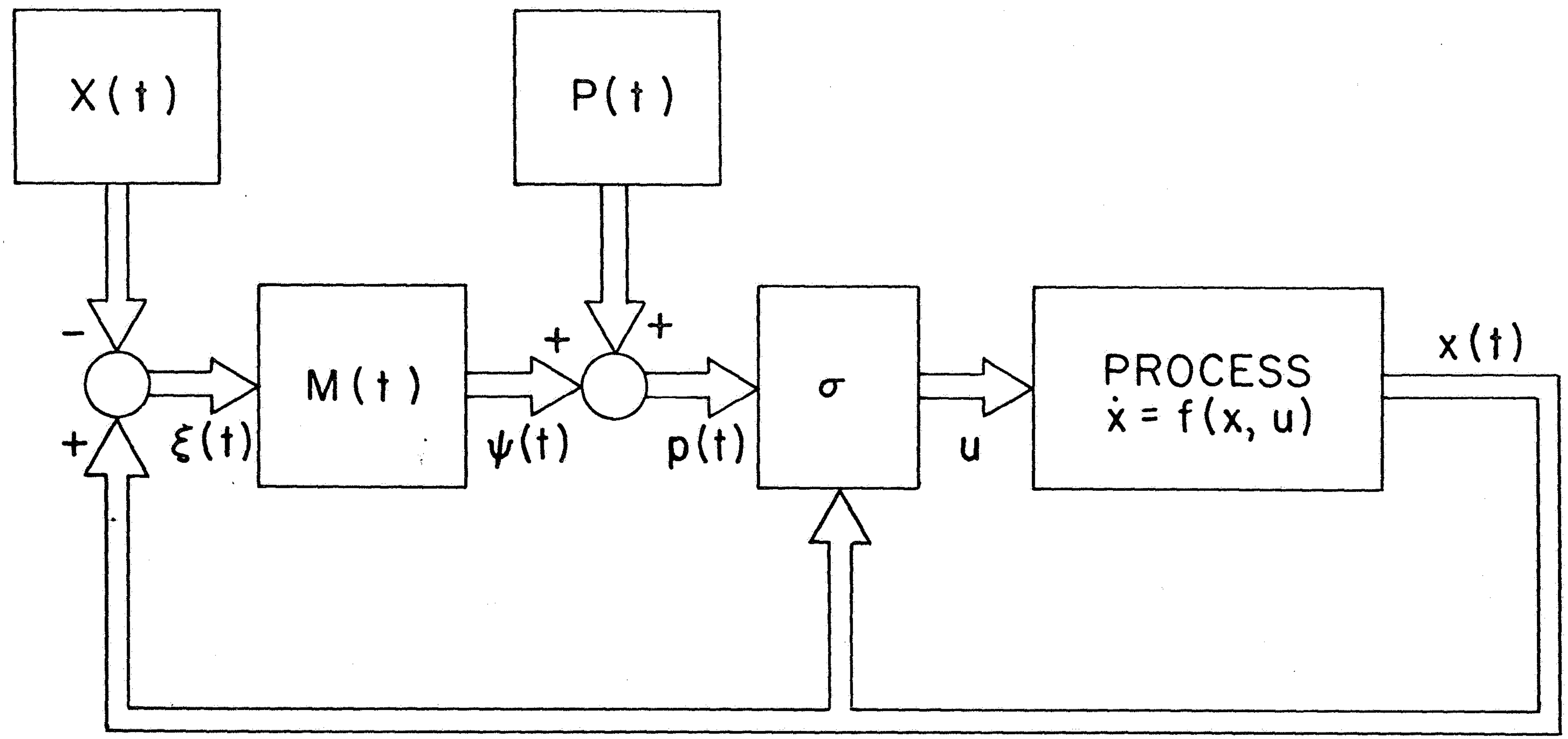
LIST OF CAPTIONS

TABLE 1

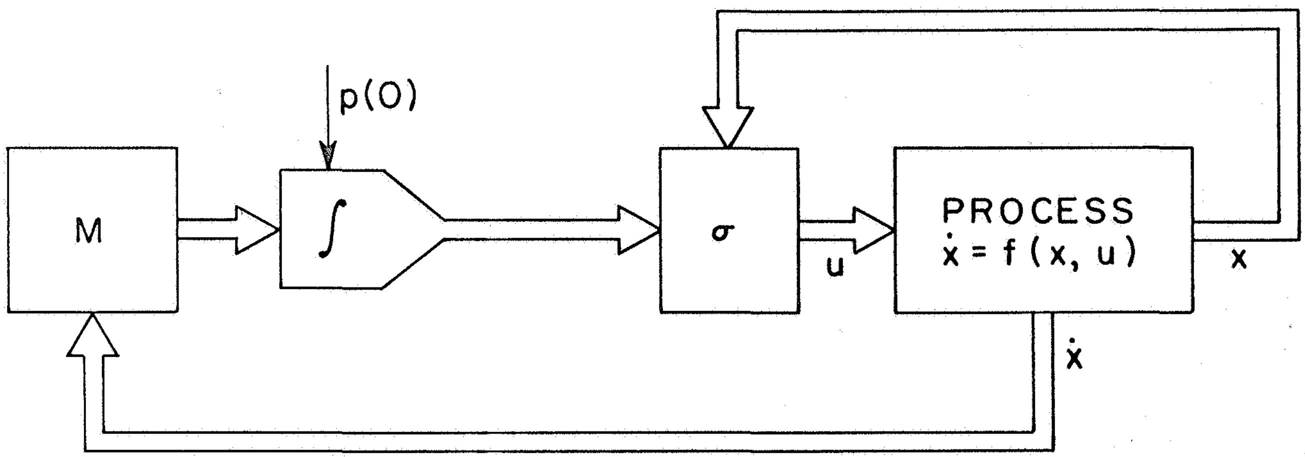
Summary of Performance of Suboptimum Controller

RUN	INITIAL STATE $X_1(0)$	INITIAL STATE $X_2(0)$	EXACT SWITCH TIME	SUBOPTIMUM CONTROL POSITION ERROR FOR $X_1(T) = 0$	SUBOPTIMUM CONTROL TERMINAL TIME, T
A	0	110	10.49	1.20	20.86
B	0	90	9.49	-.96	19.08
C	1	100	11.03	-1.73	21.23
D	-1	100	9.03	-1.27	19.19

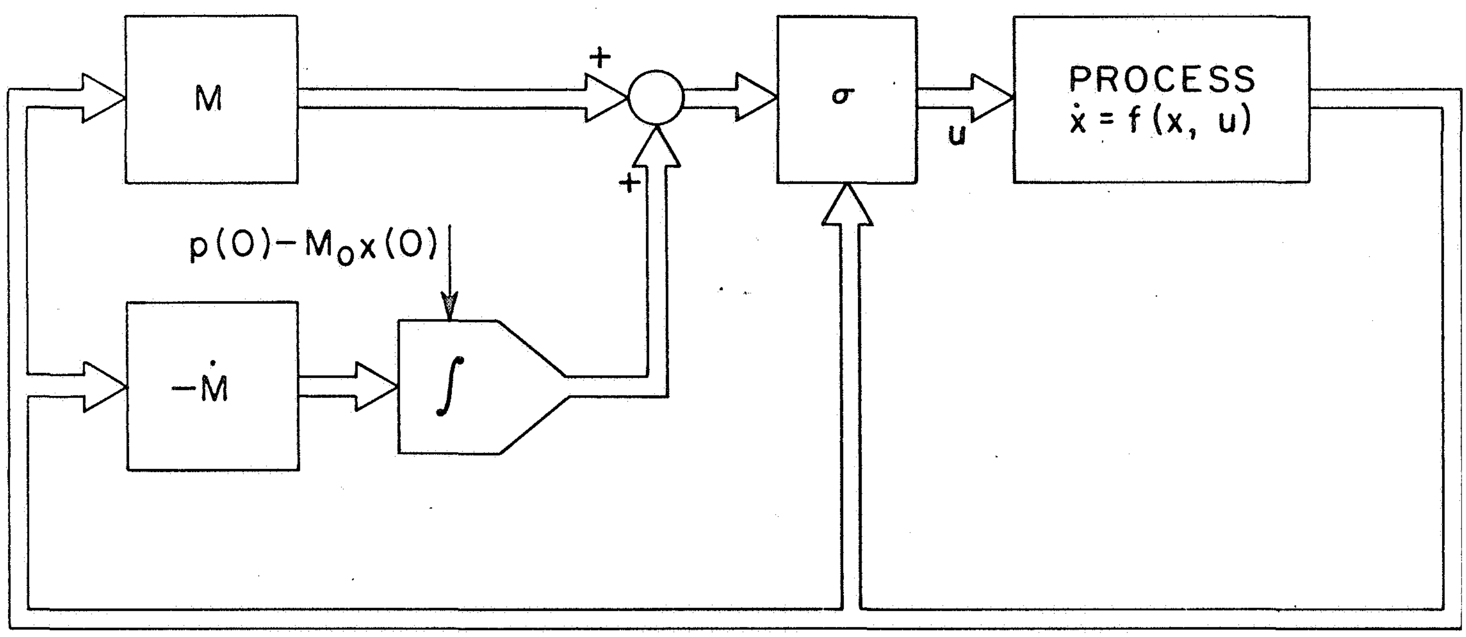

Γ
P
+
+
p
σ
u*
PROCESS
ẋ = f(x, u)
X
ξ
M
ψ

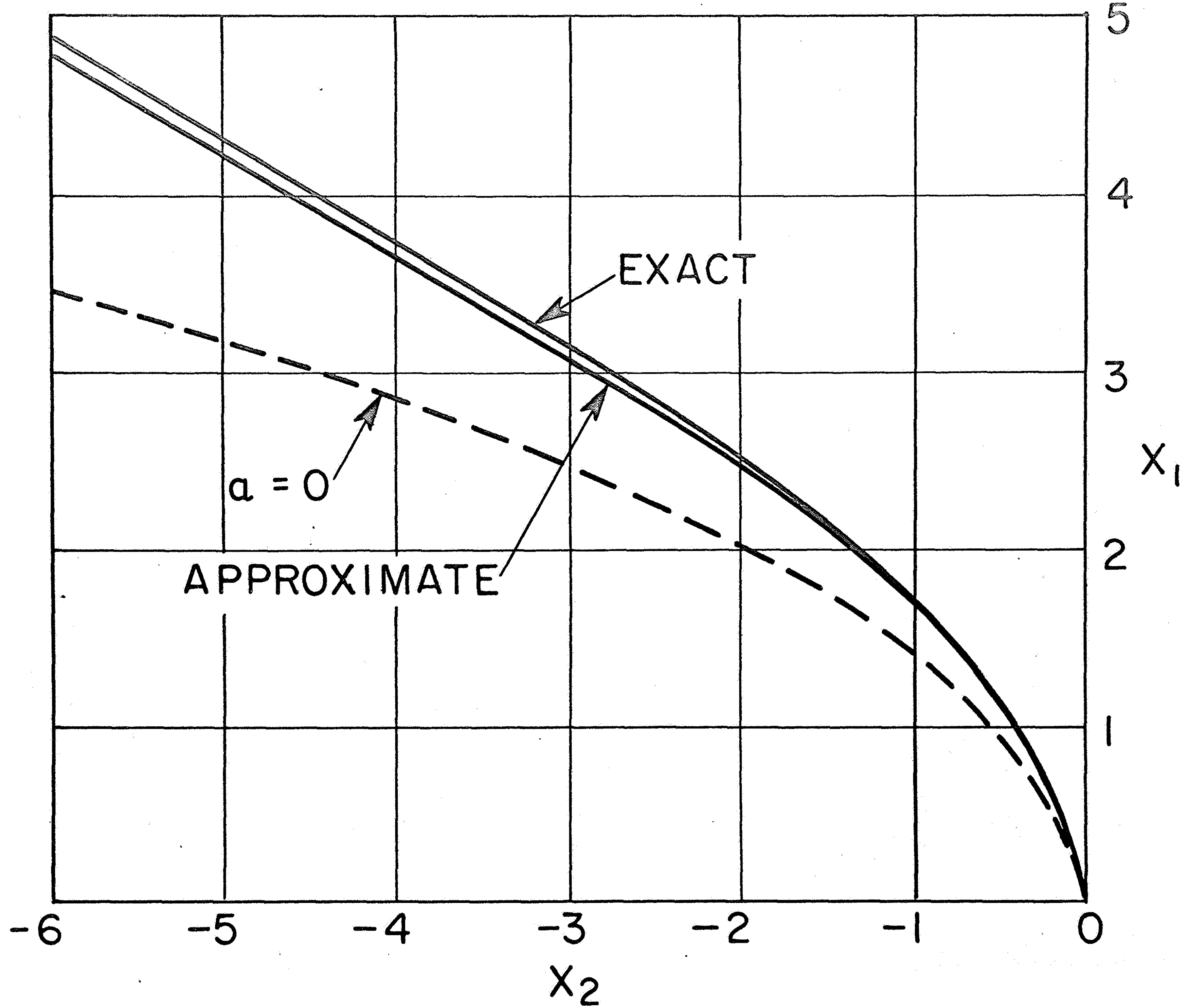

X(t)
P(t)
M(t)
σ
PROCESS
ẋ = f(x, u)
x(t)
ξ(t)
ψ(t)
p(t)
u
−
+
+
+

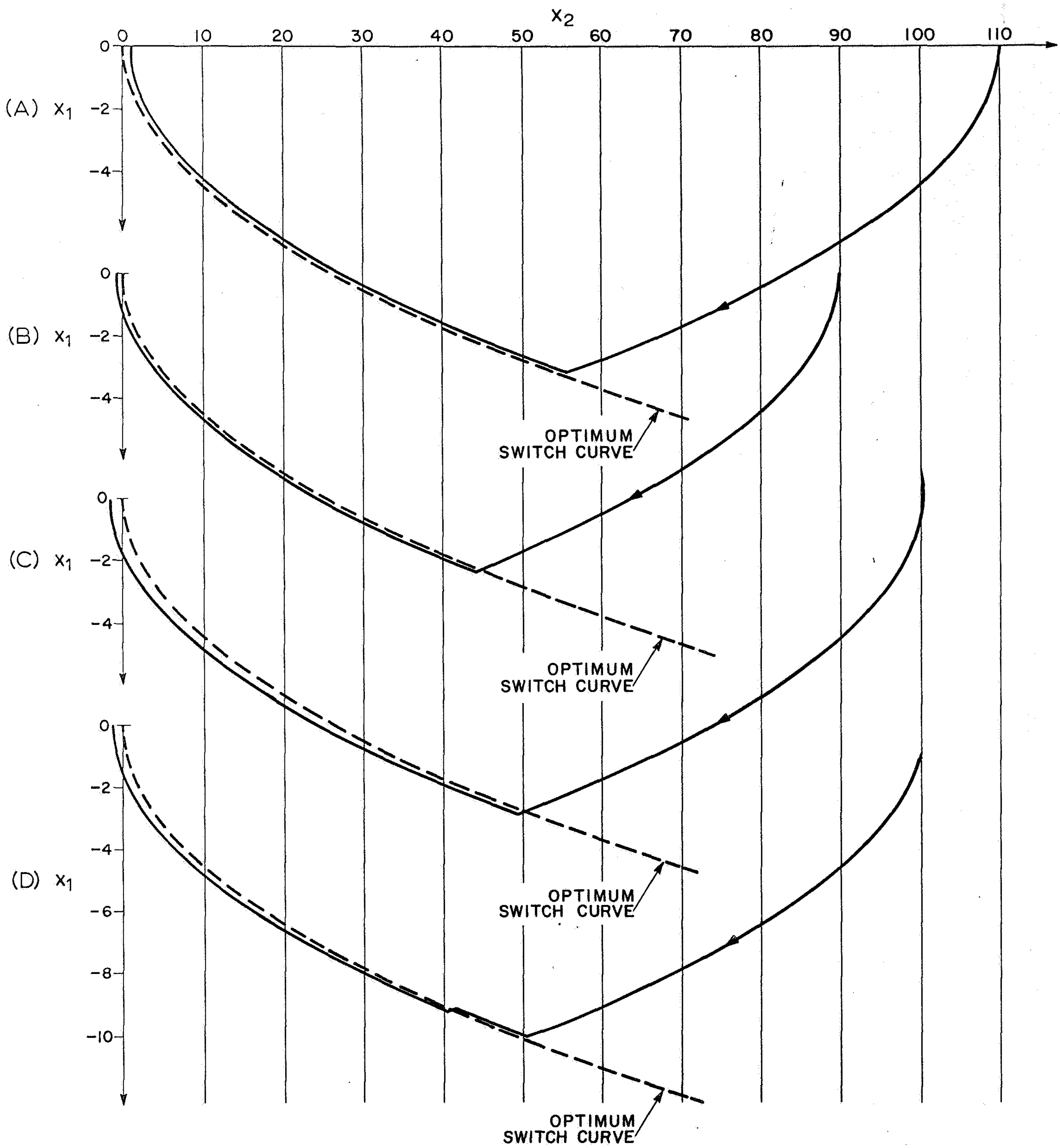
(A)



(B)




EXACT
APPROXIMATE
a = 0
5
4
3
2
1
0
-6
-5
-4
-3
-2
-1
X1
X2



x_2
0
10
20
30
40
50
60
70
80
90
100
110
(A) x_1
0
-2
-4
(B) x_1
0
-2
-4
OPTIMUM SWITCH CURVE
(C) x_1
0
-2
-4
OPTIMUM SWITCH CURVE
(D) x_1
0
-2
-4
-6
-8
-10
OPTIMUM SWITCH CURVE
OPTIMUM SWITCH CURVE

FIGURE 5

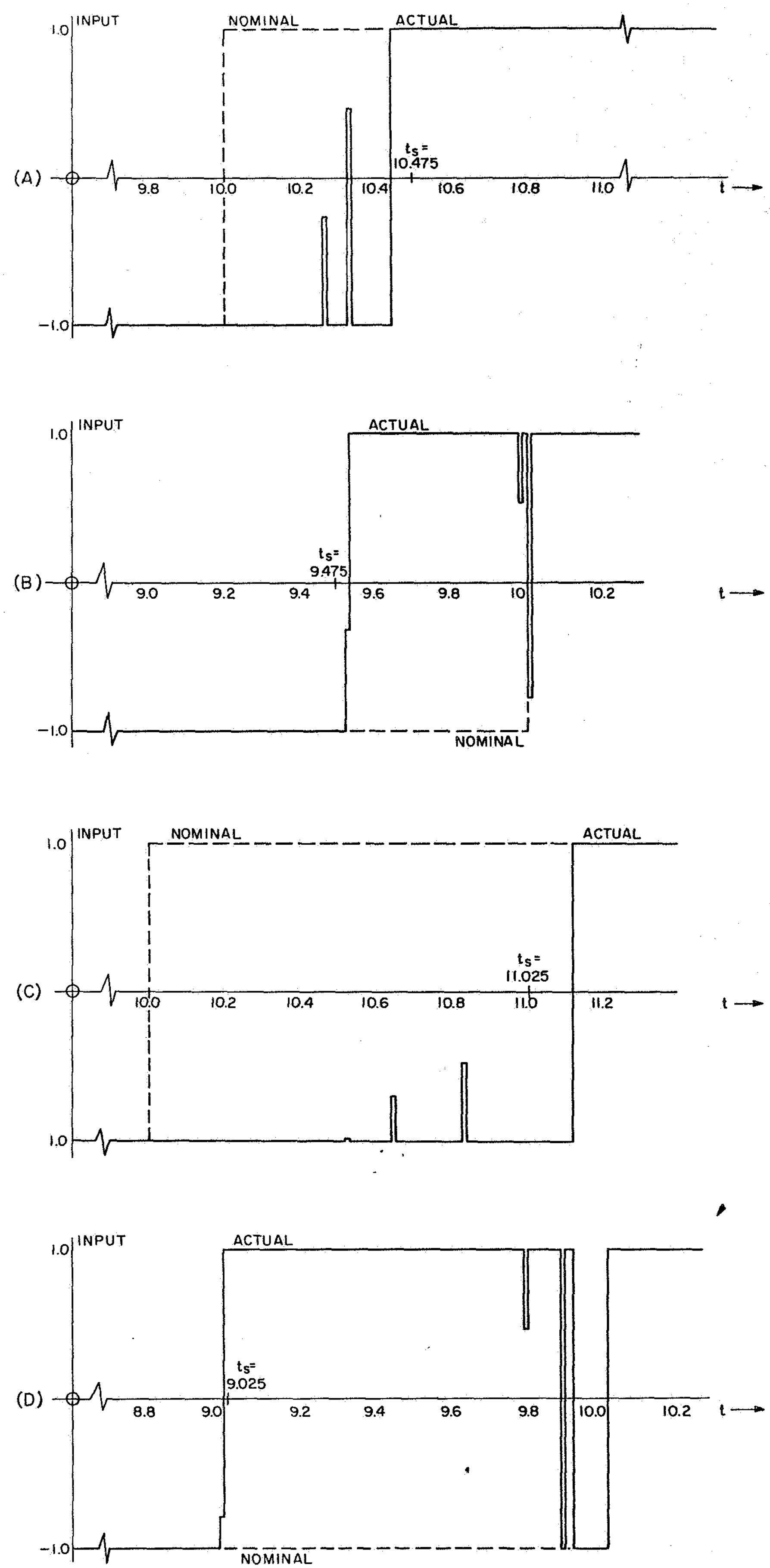

INPUT
1.0
NOMINAL
ACTUAL
(A)
t_s =
10.475
9.8
10.0
10.2
10.4
10.6
10.8
11.0
t
−1.0
INPUT
1.0
ACTUAL
(B)
t_s =
9.475
9.0
9.2
9.4
9.6
9.8
10
10.2
t
−1.0
NOMINAL
INPUT
1.0
NOMINAL
ACTUAL
(C)
t_s =
11.025
10.0
10.2
10.4
10.6
10.8
11.0
11.2
t
1.0
INPUT
1.0
ACTUAL
(D)
t_s =
9.025
8.8
9.0
9.2
9.4
9.6
9.8
10.0
10.2
t
−1.0
NOMINAL

FIGURE 6