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ON THE STRUCTURE OF THE BBGKY HIERARCHY FOR A BOLTZMANN GAS

R. Goldman
Institute for Fluid Dynamics and Applied Mathematics
University of Maryland, College Park, Maryland

and

E. A. Frieman
Department of Astrophysical Sciences
Princeton University
Princeton, New Jersey

ABSTRACT

We treat the evolution in time of a spatially uniform Boltzmann gas with no initial correlations. For the case of cut-off potentials and arbitrary initial velocity distribution functions, on using the expansion parameter nr_o^3, with n = particle density and r_o = range of binary potential, we show for terms of the order $(nr_o^3)^2$ and lower that the hierarchy is formally self-closing even with the inclusion of many body effects; *i.e.*, at a given order in (nr_o^3), with the exception of contributions linear in the single particle distribution function of the same order, the binary correlation function which determines the kinetic behavior of the single particle distribution function only depends on functions which themselves fully determinable within a prescribed iteration procedure. The actual convergence of the various orders of the formal expansion is discussed for initially aribtrary velocity distributions and for linearizations around Maxwellian velocity distributions.

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<u>Basic equation</u> In terms of the correlation functions:

$$g_s(\tilde{x}_1,\ldots\tilde{x}_s,\ v_1\ldots v_s,\tilde{t}),\quad s = 1,\ 2,\ \ldots\ ,$$

the BBGKY hierarchy in the Boltzmann approximation[1] takes the form:

$$\frac{\partial g_s}{\partial \tilde{t}} + H_s g_s - \sum_{j=1}^{s}\sum_{i<j}\sum_{\alpha=1}^{s-1}\theta_{ij}\, g_\alpha(i,\ldots)\, g_{s-\alpha}(j,\ldots) = \sum_{i=1}^{s}\int \theta_{i,s+1}\left(\sum_{\alpha=1}^{s} g_\alpha(i,\ldots)\ g_{s+1-\alpha}(s+1,\ldots) + g_{s+1}\right)d\Omega_{s+1}\ . \qquad (1)$$

The equations are in dimensionless units with distances in units of r_0, the range of the binary interaction and velocities in units of v_{av} the root mean square particle velocity; therefore times are in units of r_0/v_{av}, the time of a binary interaction.

Equations (1) have been obtained from the general hierarchy equations by choosing $\langle\phi\rangle/mv_{av}^2 \sim 1$, with $\langle\phi\rangle$ the characteristic strength of the potential, and $nr_o^3 = \varepsilon$, $\varepsilon \ll 1$. The g_s are defined in a recursive manner from the reduced distribution functions, f_s, by the relations:

$$f_1(1) = g_1(1)$$

$$f_2(1,2) = f_1(1)f_1(2) + g_2(1,2)$$

$$f_3(1,2,3) = f_1(1)f_1(2)f_1(3) + \sum_{p} f_1(1)g_2(2,3) + g_3(1,2,3)\ .$$

<u>Expansion procedure</u> Our expansion procedure is a version of the many time scale procedure[2] amended to allow for additional many space scale variations. We assume

$$g_s = \sum_{m=0}^{\infty} \varepsilon^m\ g_s^m(t,\varepsilon t,x_1,\varepsilon x_1,\ldots x_s,\varepsilon x_s)$$

[redacted]

with t and x defined by: $t = \tilde{t}$, $x = \tilde{x}$. Thus we obtain

$$\frac{\partial}{\partial \tilde{t}} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \varepsilon t} \quad ; \quad \frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \varepsilon x}$$

and we can write

$$H_s = H_s^0 + \varepsilon H_s^1$$

with

$$H_s^0 = \sum_{i=1}^{s} v_i \cdot \frac{\partial}{\partial x_i} - \sum_{i=1}^{s} \sum_{j=1}^{i-1} \frac{\partial \phi}{\partial x_i} (x_{ji}) \cdot \left(\frac{\partial}{\partial v_i} - \frac{\partial}{\partial v_j} \right)$$

and

$$H_s^1 = \sum_{i=1}^{s} v_i \cdot \frac{\partial}{\partial \varepsilon x_i} \quad ; \quad \frac{\partial \phi}{\partial \varepsilon x_i} \equiv 0 \quad .$$

Neglect of the $\varepsilon^n t$ and $\varepsilon^n x_i$ scales $(n \geq 2)$ for spatially homogeneous distribution functions has been justified in earlier work[3].

<u>Mutilated hierarchy</u> On equating the coefficient of each power of ε to zero in the expansion of (1) we obtain the basic hierarchy:

$$\begin{aligned} &\frac{\partial g_s^m}{\partial t} + H_s^0 g_s^m + \frac{\partial}{\partial \varepsilon t} g_s^{m-1} + H_s^1 g_s^{m-1} \\ &- \sum_{n=0}^{m} \sum_{j=1}^{s} \sum_{i<j} \sum_{\alpha=1}^{s-1} \theta_{ij}\, g_\alpha^n(i,\ldots)\, g_{s-\alpha}^{m-n}(j,\ldots) \\ &= \sum_{i=1}^{s} \int \theta_{i,s+1} \left(\sum_{n=0}^{m-1} \sum_{\alpha=1}^{s} g_\alpha^n(i,\ldots)\, g_{s+1-\alpha}^{m-1-n}(s+1,\ldots) \right. \\ &\left. + g_{s+1}^{m-1} \right) d\Omega_{s+1} \quad . \end{aligned} \qquad (1')$$

For $m = 0$ we use the above. For $m = 1$ we promote terms (to be specified later) from the $m = 2$ equation, which itself next receives contributions from the $m = 3$ equation. Therefore in place of the basic hierarchy for $m \geq 1$ we can write a mutilated hierarchy of the form

$$\frac{\partial g_s^m}{\partial t} + H_s^0 g_s^m + \frac{\partial}{\partial \varepsilon t} g_s^{m-1} - \sum_{n=0}^{m} \sum_{j=1}^{s} \sum_{i<j} \sum_{\alpha=1}^{s-1} \theta_{ij}\, g_\alpha^n(i,\ldots)\, g_{s-\alpha}^{m-n}(j,\ldots)$$

$$- \sum_{i=1}^{s} \int_{|x_{i,s+1}|<r_o} \theta_{i,s+1} \sum_{n=0}^{m-1} \sum_{\alpha=1}^{s} g_\alpha^n(i,\ldots)\, g_{s+1-\alpha}^{m-1-n}(s+1,\ldots) + g_{s+1}^{m-1}\, d\Omega_{s+1}$$

$$- \Delta(s,m) + \varepsilon\Delta(s,m+1). \qquad (1'')$$

For $m = 0$ or $m = 1$, $\Delta(s,m) = 0$. For $m \geq 2$ we choose:

$$\Delta(s,m) = -\frac{\partial}{\partial \varepsilon t} g_s^{m-1} - H_s^1 g_s^{m-1} + \Delta'(s,m)$$

with $\Delta'(s,m)$ still to be specified.

<u>Determination of g_s^0</u> On using equation (1") for $m = 0$ for a time τ of order unity ($\tau \approx$ collision duration) we have for $|x_{i,s+1}| \leq r_o$

$$g_{s+1}^0(t) = e^{-H_{s+1}^0 \tau} \left(\sum_{\alpha=1}^{s} g_\alpha^0(i,\ldots,t-\tau)\, g_{s+1-\alpha}^0(s+1,t-\tau) + g_{s+1}^0(t-\tau) \right)$$

$$- \sum_{\alpha=1}^{s} g_\alpha^0(i,\ldots,t)\, g_{s+1-\alpha}^0(s+1,\ldots,t) + \delta_{10}(i,s+1;1,\ldots,s+1;t) \qquad (2)$$

and

$$\theta_{i,s+1} = -H_{s+1}^0 + H_s^0 + v_{s+1} \cdot \frac{\partial}{\partial x_{s+1}} + \delta_{11}. \qquad (3)$$

If and only if i and $s+1$ interact with each other from $t-\tau$ to t, then δ_{10} and δ_{11} are zero. There are no secular contributions involving terms in δ_{10} and δ_{11}, since these are only non-zero for times of order unity.

By means of (3) and (2), the right hand side of (1") for $m = 1$ can be put in the form

$$R(s,0) \equiv \left(H_s^0 \sum_{i=1}^{s} \int_{|x_{i,s+1}|<r_o} e^{-H_{s+1}^0 \tau} - \sum_{i=1}^{s} \int_{|x_{i,s+1}|<r_o} e^{-H_{s+1}^0 \tau} H_{s+1}^0 \right.$$

$$+ \sum_{i=1}^{s} \int_{|x_{i,s+1}|<r_o} (v_{s+1} - v_i) \cdot \frac{\partial}{\partial x_{s+1}} e^{-H_{s+1}^0 \tau} + \sum_{i=1}^{s} \int_{|x_{i,s+1}|<r_o} \left. \delta_{11} e^{-H_{s+1}^0 \tau} \right)$$

$$\times \left(I_1(i,s+1,1,\ldots s,t) + e^{H_{s+1}^0 \tau} \delta_{10} \right) d\Omega_{s+1} \qquad (4)$$

with

$$I_1(i,s+1,1\ldots s,t) \equiv \left[\sum_{\alpha=1}^{s} g_\alpha^0(i,\ldots,t-\tau)\, g_{s+1-\alpha}^0(s+1,t-\tau) + g_{s+1}^0(t-\tau) \right].$$

The terms linear in H_s^0 and H_{s+1}^0 (with the addition of terms of order ε) contribute to a total derivative with respect to time of quantity of order unity. The terms linear in $v_{s+1} - v_i$ ($\equiv v_{s+1,i}$) have a contribution which can be put in the form

$$A(s,0) + B(s,0) + C(s,0) + \delta_{12}$$

with

$$A(s,0) = - \sum_{i=1}^{s} \int_{|x_{i,s+1}|=r_o,\, x_{i,s+1} \| -v_{i,s+1}} d_3 v_{s+1}\, d\sigma_{s+1}\, |v_{s+1,i}|\, g_1^0(s+1)\, g_s^0(t)$$

$$B(s,0) = \sum_{i=1}^{s} \int_{|x_{i,s+1}|=r_o,\, x_{i,s+1} \| v_{i,s+1}} d_3 v_{s+1}\, d\sigma_{s+1}\, |v_{s+1,i}|\, e^{-H_{s+1}^0 \tau}\, g_1^0(s+1)\, g_s^0$$

$$+ \sum_{i=1}^{s} \int_{|x_{i,s+1}|=r_o} d_3 v_{s+1}\, d\sigma_{s+1}\, d_{s+1}\, |v_{s+1,i}|\, e^{-H_{s+1}^0 \tau}\, g_1^0(i)\, g_s^0(s+1)$$

$$C(s,0) = \sum_{i=1}^{s} \int_{|x_{i,s+1}|=r_o} d_3 v_{s+1}\, d\sigma_{s+1}\; v_{s+1,i} \quad e^{-H^0_{s+1}\tau}$$
$$\times \left(\sum_{\alpha=2}^{s-1} g^0_\alpha(i,\ldots)\, g^0_{s+1-\alpha}(s+1,\ldots) + g^0_{s+1} \right)$$

with $d\sigma_{s+1}$ denoting a surface element. As long as all of the s particles in the collection $(1, 2,\ldots,s)$ are further apart than distance of order unity, $\delta_{12} = 0$. It has been shown earlier[3] for g^1_s to be bounded that one must take:

$$\left(\frac{\partial}{\partial \varepsilon t} + H^1_s\right) g^0_s = A(s,0)$$

This completes the definition of g^0_s.

Determination of g^1_s in general Within the $m = 2$ equations of (1") for $|x_{i,s+1}| < r_o$ we have as in the case of the $m = 1$ equations:

$$g^1_{s+1}(t) = e^{-H^0_{s+1}\tau} \left[\sum_{n=0}^{1} \sum_{=1}^{s} g^n_\alpha(i,\ldots)\, g^{1-n}_{s+1-\alpha}(s+1,\ldots) + g^1_{s+1}(t-\tau) \right]$$
$$- \sum_{n=0}^{1} \sum_{\alpha=1}^{s} g^n_\alpha(i,\ldots t)\, g^{1-n}_{s+1-\alpha}(s+1,\ldots)$$
$$+ \delta_{20}(i,s+1;\, 1,\ldots,s+1,\, t)\ . \tag{5}$$

For the particles $(1,\ldots,s)$ far apart we assume that δ_{20} is small compared to the terms linear in $e^{-H^0_{s+1}\tau}$. This has been verified in some detail for $s = 1$, 2 and 3, where one finds that δ_{20} is characteristically of order t^{-1} or x^{-1} compared to the terms linear in $e^{-H^0_{s+1}\tau}$. (Here and afterward x is taken to be the minimum interparticle separation of the s particles.)

The terms, excluding $\Delta(s,m) + \varepsilon\Delta(s,m+1)$, on the right hand side of (1") for $m = 2$ can therefore be written:

$$\Big(H_s^0 \sum_{i=1}^{s} \int_{|x_{i,s+1}|<r_o} e^{-H_{s+1}^0\tau} - \sum_{i=1}^{s}\int_{|x_{i,s+1}|<r_o} e^{-H_{s+1}^0\tau} H_{s+1}^0$$

$$+ \sum_{i=1}^{s}\int_{|x_{i,s+1}|<r_o} (v_{s+1}-v_i)\cdot\frac{\partial}{\partial x_{s+1}} e^{-H_{s+1}^0\tau} + \sum_{i=1}^{s}\int_{|x_{i,s+1}|<r_o} \delta_{11}\, e^{-H_{s+1}^0\tau}\Big)$$

$$\times\ (I_2(i,s+1,1,\ldots,s,t) + e^{H_{s+1}^0\tau}\delta_{20})\, d\Omega_{s+1} \tag{6}$$

with

$$I_2(i,s+1;1,\ldots,s,t) \equiv \left[\sum_{n=0}^{1}\sum_{\alpha=1}^{s} g_\alpha^n(i,\ldots)\, g_{s+1-\alpha}^{1-n}(s+1,\ldots) + g_{s+1}^1(t-\tau)\right] \tag{7}$$

By coupling the arguments following (4) with considerations which put the integral terms in Eq. (1") for $m = 1$ of order t^{-1} or x^{-1} compared with g_s^1, one finds that the dominant terms in (6) are linear in

$$(v_{s+1} - v_i)\cdot\frac{\partial}{\partial x_{s+1}} e^{-H_{s+1}^0\tau} I_2 . \tag{8}$$

The contribution from (8) within (6) may be put in the form

$$A(s,1) + B(s,1) + C(s,1) + \delta_{22}$$

with

$$A(s,1) = - \sum_{i=1}^{s} \int_{|x_{i,s+1}|=r_o, x_{i,s+1} \| -v_{i,s+1}} d_3 v_{s+1} \; d\sigma_{s+1} \; |v_{s+1,i}| \; g_1^0(s+1) \; g_s^1(t)$$

$$B(s,1) = \sum_{i=1}^{s} \int_{|x_{i,s+1}|=r_o, x_{i,s+1} \| v_{i,s+1}} d_3 v_{s+1} \; d\sigma_{s+1} \; |v_{s+1,i}| \; e^{-H_{s+1}^0 \tau}$$

$$\times \; (g_1^0(s+1) \; g_s^1(i,t) + g_s^1(s+1,\ldots t) \; g_1^0(i))$$

$$- \sum_{i=1}^{s} \int_{|x_{i,s+1}|=r_o, x_{i,s+1} \| v_{i,s+1}} d_3 v_{s+1} \; d\sigma_{s+1} \; |v_{s+1,i}| \; e^{-H_{s+1}^0 \tau} \; g_1^0(i) \; g_s^1(s+1,t)$$

$$C(s,1) = \sum_{i=1}^{s} \int_{|x_{i,s+1}|<r_o} d\Omega_{s+1} \quad \mathbb{V}_{s+1,i} \cdot \frac{\partial}{\partial x_{s+1}} \qquad e^{-H_{s+1}^0 \tau}$$

$$\times \left[\sum_{n=0}^{1} \sum_{\alpha=2}^{s-1} g_\alpha^n(i,\ldots) \; g_{s+1-\alpha}^{1-n}(s+1,\ldots,t-\tau) + g_1^1(i,t-\tau) \; g_s^0 \right.$$

$$\left. + \; g_s^0(i,\ldots,t-\tau) \; g_1^1(s+1,t-\tau) + g_{s+1}^1(s+1,t-\tau) \right] .$$

The term in δ_{22} is ~~of order~~ t^{-1} or x^{-1} compared to A + B. The term linear in $g_s^0\, g_1^1(s+1)$ evaluated at $x_{i,s+1} \| -v_{i,s+1}$ yields a higher order correction to g_s^0 which we will neglect.

We are free to "promote" the $A(s,1)$ term to the lower order equation with $m = 1$; since it leads to exponential damping and does not result in an essentially more complicated equation to solve, we do so. Therefore we have for (1") with $m \geq 2$, $s \geq 2$:

$$\Delta'(s,2) = -A(s,1) + \Delta''(s,2)$$

The term in Δ'' contains terms which if they were not "promoted" would lead to divergent behavior in g_s^2. Such terms can be estimated by usage of the values obtained for g_s^1 on the omission of the term in $\varepsilon\Delta(s,2)$.

For $s = 2$, $B(s,1)$ is of order $|x_{12}|^{-1}$ for $|x_{12}| \gg 1$. Consequently the contribution to g_2^2 through (1") is logarithmic. Therefore we write

$$g_2^1 = (g_2^1)_1 + (g_2^1)_2$$

and "promote" the terms within $B(2,1)$ linear in $(g_2^1)_1$. For $s = 3$ the contribution to g_3^2 is finite (at least as concerns x, v arguments which ultimately contribute to g_2 for $|x_{12}| < r_o$). Hence we do not "promote" $B(3,1)$. For $s \geq 4$ we assume that the conclusion for $B(s,1)$ is the same as for $B(3,1)$.

Hence the solution for g_s^1, $s \geq 3$ is a matter of simple iteration (subject to a knowledge of g_1^1 and g_2^1).

<u>Solution for g_2^1 in particular</u> For g_2^1 we have from (1"):

$$\left[\frac{\partial}{\partial t} g_2^1 + H_2^0 g_2^1 + \varepsilon \frac{\partial}{\partial \varepsilon t} g_2^1 + \varepsilon H_2^1 g_2^1 + \varepsilon \sum_{i=1}^{2} \int_{|x_{i,s}|=r_0, x_{i3}||-v_{i3}} d_3 v_3 \, d\sigma_3 |v_{3i}| g_1^0(3)\right] g_2^1$$

$$- \theta_{12}(g_1^0 g_1^1 + g_1^1 g_1^0)$$

$$= \varepsilon \sum_{i=1}^{2} \int_{|x_{i3}|=r_0, x_{i3}||v_{i3}} d_3 v_3 \, d\sigma_3 \, |v_{3i}| \quad e^{-H_3^0 \tau}$$

$$\times \quad (g_1^0(3) \; (g_2^1)_1 \; (i,\ldots,t) + (g_2^1)_1(3,\ldots,t) \; g_1^0(i))$$

$$- \varepsilon \sum_{i=1}^{2} \int_{|x_{i3}|=r_0, x_{i3}||-v_{i3}} d_3 v_3 \, d\sigma_3 \, |v_{3,i}| \quad e^{-H_3^0 \tau} \; g_1^0(i) \; (g_2^1)_1 \; (3,t)$$

$$+ B(2,0) + [R(2,0) - A(2,0) - B(2,0)] \quad . \tag{9}$$

We write

$$
\begin{aligned}
g_2^1 &= (g_2^1)_\alpha + (g_2^1)_\beta + (g_2^1)_\gamma \\
(g_2^1)_1 &= (g_2^1)_{\alpha|1} + (g_2^1)_{\beta|1} \\
(g_2^1)_2 &= (g_2^1)_{\alpha|2} + (g_2^1)_\gamma \\
(g_2^1)_\alpha &= (g_2^1)_{\alpha|1} + (g_2^1)_{\alpha|2} \\
(g_2^1)_\beta &= (g_2^1)_{\beta|1}
\end{aligned}
\tag{10}
$$

with:

$$
\left[\frac{\partial}{\partial t} + H_2^0 + \varepsilon \frac{\partial}{\partial \varepsilon t} + \varepsilon\ H_2^1 + \varepsilon \sum_{i=1}^{2} \int_{|x_{i,3}|=r_o, x_{i3}||-v_{i3}} d_3\ v_3\ d\sigma_3\ |v_{3,i}|\ g_1^0(3)\right] (g_2^1)_{\alpha|2}
$$

$$
- \theta_r (g_1^0\ g_1^1 + g_1^1\ g_1^0) = [R(2,0) - A(2,0) - B(2,0)]
\tag{11}
$$

$$
\left[\frac{\partial}{\partial t} + H_2^0 + \varepsilon \frac{\partial}{\partial \varepsilon t} + \varepsilon\ H_2^1 + \varepsilon \sum_{i=1}^{2} \int_{|x_{i,3}|=r_o, x_{i3}||-v_{i3}} d_3\ v_3\ d\sigma_3\ |v_{3,i}|\ g_1^0(3)\right] (g_2^1)_{\alpha|1}
$$

$$
= B(2,0)
\tag{12}
$$

$$\left[\frac{\partial}{\partial t} + H_2^0 + \varepsilon \frac{\partial}{\partial \varepsilon t} + \varepsilon H_2^1 + \varepsilon \sum_{i=1}^{2} \int_{|x_{i,3}|=r_0, x_{i3}||-v_{i3}} d_3 v_3 \, d\sigma_3 \, |v_{3,i}| \, g_1^0(3)\right] (g_2^1)_{\beta|1}$$

$$= \varepsilon \sum_{i=1}^{2} \int_{|x_{i3}|=r_0, x_{i3}||v_{i3}} d_3 v_3 \, d\sigma_3 \, |v_{3,i}| \, e^{-H_3^0 \tau} \left(g_1^0(3) \, (g_2^1)_{\alpha|1} + (g_2^1)_{\alpha|1} \, g_1^0(i)\right)$$

$$- \varepsilon \sum_{i=1}^{2} \int_{|x_{i3}|=r_0, x_{i3}||-v_{i3}} d_3 v_3 \, d\sigma_3 \, |v_{3,i}| \, e^{-H_3^0 \tau} \, g_1^0(i) \, (g_2^1)_{\alpha|1} \, (3,t) \quad . \tag{13}$$

$$\left[\frac{\partial}{\partial t} + H_2^0 + \varepsilon \frac{\partial}{\partial \varepsilon t} + \varepsilon H_2^1 + \varepsilon \sum_{i=1}^{2} \int_{|x_{i,3}|=r_0, x_{i3}||-v_{i3}} d_3 v_3 \, d\sigma_3 \, |v_{3,i}| \, g_1^0(3)\right] (g_2^1)_\gamma$$

$$= \varepsilon \sum_{i=1}^{2} \int_{|x_{i3}|=r_0, x_{i3}||v_{i3}} d_3 v_3 \, d\sigma_3 \, |v_{3,i}| \, e^{-H_3^0 \tau} \, g_1^0(3) \, (g_2^1)_{\beta|1} \tag{14}$$

$$+ (g_2^1)_{\beta|1} \, g_1^0(i) - \varepsilon \sum_{i=1}^{2} \int_{|x_{i3}|=r_0, x_{i3}||-v_{i3}} d_3 v_3 \, d\sigma_3 \, |v_{3,i}| \, e^{-H_3^0 \tau} g_1^0(i) \, (g_2^1)_{\beta|1} \, (3t) \; .$$

The term $[R(2,0) - A(2,0) - B(2,0)]$ falls off with increasing $|x_{12}|$ more rapidly than $|x_{12}|^{-2}$. Consequently $(g_2^1)_{\alpha|2}$ does not contribute to secular behavior in (g_2^2). The term $B(2,0)$ varies with increasing $|x_{12}|$ as $|x_{12}|^{-2}$. Consequently the right hand side of (13) is of the order $|x_{12}|^{-1}$ and the right hand side of (14) is of order $\varepsilon \, \ln(\varepsilon|x_{12}|)$ for $|x_{12}| \lesssim \varepsilon^{-1}$, from which $(g_2^1)_\gamma$ is of order ε for $|x_{12}| \lesssim \varepsilon^{-1}$.

For $\varepsilon x >> 1$ one has that both $(g_2^1)_{\beta|1}$ and $(g_2^1)_\gamma$ are of the form

$\varepsilon e^{-\alpha\varepsilon x}$ with α positive, of order unity, and bounded from below.

Determination of g_s^2 and g_2^2 in particular We now turn to the determination of g_s^2. The right hand side of (1") for $m = 3$ can be written, analogously to (6) and (7):

$$\Bigg[H_s^0 \sum_{i=1}^{s} \int_{|x_{i,s+1}|< r_o} e^{-H_{s+1}^0 \tau} - \sum_{i=1}^{s} \int_{|x_{i,s+1}} e^{-H_{s+1}^0 \tau} H_{s+1}^0$$

$$+ \sum_{i=1}^{s} \int_{|x_{i,s+1}|<r_o} (v_{s+1} - v_i) \cdot \frac{\partial}{\partial x_{s+1}} e^{-H_{s+1}^0 \tau} + \sum_{i=1}^{s} \int_{|x_{i,s+1}|<r_o} \delta_{11}\, e^{-H_{s+1}^0 \tau} \Bigg]$$

$$\times \left(I_3(i,s+1,1,\ldots,s,t) + e^{H_{s+1}^0 \tau}\, \delta_{30}\right) d\Omega_{s+1} \tag{15}$$

with

$$I_3(i,s+1;1,\ldots,s,t) \equiv \Bigg[\sum_{n=0}^{2} \sum_{\alpha=1}^{s} g_\alpha^n(i,\ldots)\, g_{s+1-\alpha}^{2-n}(s+1,\ldots) + g_{s+1}^2(t-\tau) \Bigg] \quad . \tag{16}$$

One may first calculate the various $g_s^2(t - \tau)$ without the contributions $\Delta'(s,3)$ and then determine the resulting corrections from the $\Delta'(s,3)$. For $s = 3$, at least as concerns g_3^2 which contributes directly to g_2 for $|x_{12}| < r_o$, the term in $v_{3i} \cdot \frac{\partial}{\partial x_3} e^{-H_3^0 \tau} g_3^2(t-\tau)$ does not contribute to g_2 in order ε^2 or lower. We assume that the iteration with $\Delta'(s,3)$ does not decrease the order of g_3^2, and we take the size of this term as representative of the size of other terms involving $g_3^2(t-\tau)$ within (15). Also we take the contribution from $s = 3$ to be a bound on the contributions from $s \geq 4$, so that these too can be neglected in finding g_2 to order ε^2 for $|x_{12}| < r_o$.

For $\Delta'(2,3)$ we note that due to the presence of $(g_2^1)_\gamma$ one expects that one may write

$$g_2^2 = g_{2|1}^2 + g_{2|2}^2$$

with $g_{2|1}^2$ of order unity on a length scale of order the mean free path and a time scale of order the mean free time. Correspondingly we have

$$\Delta'(2,3) = \sum_{i=1}^{2} \int_{|x_{i,3}|<r_b} d\Omega_3 \; v_{3i} \cdot \frac{\partial}{\partial x_3} \; e^{-H_2^0(i,3)\tau} \left(g_1^0(i)\, g_{2|1}^2 + g_1^0(3)\, g_{2|1}^2\right)$$

and

$$\frac{\partial}{\partial t} g_{2|1}^2 + H_2^0\, g_{2|1}^2 + \varepsilon \frac{\partial}{\partial \varepsilon t} g_{2|1}^2 + \varepsilon H_2^1\, g_{2|1}^2 - \theta_{12}\left(g_{1|0}^2(1)\, g_1^0 + g_{1|1}(2) g_1^0\right)$$

$$- \varepsilon \sum_{i=1}^{2} \int_{|x_{i,3}|<r_b} d\Omega_3 \; v_{3i} \cdot \frac{\partial}{\partial x_3} \; e^{-H_2^0(i,3)\tau} \left(g_1^0(i)\, g_{2|1}^2 + g_1^0(3)\, g_{2|1}^2\right)$$

$$= \sum_{i=1}^{2} \int_{|x_{i3}|=r_b,\, x_{i3} \| v_{i3}} d_3 v_3 \, d\sigma_3 \, |v_{3,i}| \; e^{-H_3^0 \tau} \left(g_1^0(3)\, (g_2^1(i,\ldots,t))\right)_\gamma$$

$$+ g_1^0(i) \;\; (g_2^1(3,\ldots,t))_\gamma) - \sum_{i=1}^{2} \int_{|x_{i3}|=r_b,\; x_{i3} \| -v_{i3}} d_3 v_3 \, d\sigma_3$$

$$|v_{3,i}| \; e^{-H_3^0 \tau} \; g_1^0(i) \; (g_2^1(3,\ldots,t))_\gamma \qquad (17)$$

For $g^2_{2|2}$ we note that the equation is of the form:

$$\left\{\frac{\partial}{\partial t} + H^0_2 + \varepsilon \frac{\partial}{\partial \varepsilon t} + \varepsilon H^1_2 + \varepsilon \sum_{i=1}^{2} \int_{|x_{i,3}| = r_o, x_{i3}||-v_{i3}} d_3v_3 d\sigma_3 |v_{3,i}| g^0_1(3) \right\} g^2_{2|2}$$

$$- \theta_{12} \left(g^2_{1|2}(1) g^0_1 + g^2_{1|2}(2) g^0_1 + g^1_1 g^1_1\right) = S(1, 2, t, \varepsilon t) \quad . \tag{18}$$

S decreases sufficiently rapidly with increasing $|x_{12}|$ that to order unity the solution for $g^2_{2|2}$, subject to a knowledge of g^1_1 and $g^2_{1|2}$, is obtained without iteration.

Discussion of Eq. (17) and comments on the general structure

Equation (17) treats the many body effects whose existence has been previously noted by other authors[4,5,6,7]. The behavior in Eq. (17) is not entirely unexpected since the bineary correlation function as distances of order the mean free path in general is of order ε^2. One expects that parts of the binary correlation function in order higher than ε^2 will have equations whose homogeneous parts are similar in form to the left hand side of (17). Since the correlation function for n bodies at distances of order the mean free path from each other is in general of order $\varepsilon^{2(n-1)}$ and higher, part of the n body correlation function will satisfy an integro-differential equation whose homogeneous terms are acted on by the operator

$$\left(\frac{\partial}{\partial t} + H^0_n + \varepsilon \frac{\partial}{\partial \varepsilon t} + \varepsilon H^1_n - \varepsilon \sum_{i=1}^{n} \int_{|x_{i,n+1}|<r_o} d\Omega_{n+1} v_{n+1,i} \cdot \frac{\partial}{\partial x_{n+1}} e^{-H^0_n(i,n+1)\tau} \right.$$

$$\left. \times \left(1 + \varepsilon(i,n+1)\right) g^0_1(n+1)\right)$$

with $\varepsilon(i,n+1)$ exchanging the labels i and $n+1$.

<u>Form of single particle distribution function</u>

The single particle distribution function g_1 is of the form

$$g_1 = g_1^0 + \varepsilon g_{10}^1 + \varepsilon^2 \ell n\, \varepsilon\, g_{11}^1 + \varepsilon^2 g_1^2$$

with g_1^0, g_{10}^1, g_{11}^1 and g_1^2 of order unity. Its behavior for all times is given by the equations:

$$\frac{\partial g_1^0}{\partial t} = 0 \tag{19}$$

$$\frac{\partial g_1^0}{\partial \varepsilon t} = \lim_{t\to\infty} \int \theta_{12} g_2^0 \, d\Omega_2 \tag{20}$$

$$\frac{\partial g_{10}^1}{\partial t} = (1-\lim_{t\to\infty}) \int \theta_{12}\, g_2^0 \, d\Omega_2 \tag{21}$$

$$\frac{\partial g_{10}^1}{\partial \varepsilon t} = \lim_{t\to\infty} \int \theta_{12}(1-P_1-P_2)\ (g_2^1)_\alpha \, d\Omega_2 \tag{22}$$

$$\frac{\partial g_{11}^1}{\partial t} = (1-\lim_{t\to\infty}) \int \theta_{12}\ P_3 (g_2^1)_\alpha\ S(1/\varepsilon - t)\ d\Omega_2 \tag{23}$$

$$\frac{\partial g_{11}^1}{\partial \varepsilon t} = \lim_{t\to\infty} \int \theta_{12}(P_1 (g_2^1)_\alpha + (1-P_4)(g_2^1)_\beta) d\Omega_2 \tag{24}$$

$$\begin{aligned} \frac{\partial g_1^2}{\partial t} + \varepsilon \frac{\partial g_1^2}{\partial \varepsilon t} = {} & (1-\lim_{t\to\infty}) \int \theta_{12}(1-P_1-P_2-P_3)(g_2^1)_\alpha \, d\Omega_2 \\ & + (1-\lim_{t\to\infty}) \int \theta_{12} P_3 (g_2^1)_\alpha\ S(t-1/\varepsilon)\ d\Omega_2 \\ & + (1-\lim_{t\to\infty}) \int \theta_{12}(P_1 (g_2^1)_\alpha + (1-P_4)\ (g_2^1)_\beta)\ d\Omega_2 \\ & + \int \theta_{12}(P_2 (g_2^1)_\alpha + P_4 (g_2^1)_\beta + (g_2^1)_\gamma + (g_2^1) \ + \varepsilon g_2^2) d\Omega_2 \end{aligned} \tag{25}$$

Here

$$S(x) = 1, \ x > 0$$
$$S(x) = 0, \ x < 0$$

and the operators P_i denote the existence of contributions from the functions following them.

Equation (20) is the Boltzmann equation. Equations (22) and (24) are linearized spatially homogeneous Boltzmann equations with known source terms. From Eq. (25) the behavior due to $g^2_{2|1}$ is given by

$$\frac{\partial g^2_{1|1}}{\partial t} = 0 \tag{26}$$

$$\frac{\partial g^2_{1|1}}{\partial \varepsilon t} = \int_{|x_{12}| = r_o} v_{21} \cdot d\sigma_2 dv_2 \, e^{-H^0_2 \tau} (g^2_{1|1}(1) g^0_1(2) + g^2_{1|1}(2) g^0_1(1) + g^2_{1|1}(2) g^0_1(1) + g^2_{2|1}) \tag{27}$$

Within (27), the term in $g^2_{2|1}$ is independent of $g^2_{1|1}$. The form of (27) follows from the fact that the inhomogeneous integral terms in (17) are of order ε.

<u>Description of the asymptotic time behavior of g_1 to order ε^2</u>

If the deviation of g^0_1 from its asymptotic value in time approaches zero exponentially or more rapidly, then g^1_{10}, g^1_{11} and $g^2_1 - g^2_{1|1}$ approach their asymptotic values exponentially. However, a normal mode[8,9] analysis of (17) and (27) reveals that $g^2_{1|1}$ may in principle approach its asymptotic value algebraically as $t^{-3/2}$ for $t >> 1$.

If the deviation of g_1^0 from its asymptotic value approaches zero less rapidly than exponentially, then in general one cannot even conclude that g_{10}^1 and g_1^2 approach limits.

However, provided g_{10}^1, g_{11}^1 and $g_{1|1}^2$ remain bounded one can, after a sufficiently long time, linearize (20) in the deviations of g_1^0 from its asymptotic Maxwellian velocity form. Then, for cut-off Maxwellian or "harder" cut-off potentials,[10] it appears that the deviation of g_1^0 from its asymptotic value decays exponentially in time and the corresponding conclusions as to g_{10}^1, g_{11}^1, $g_1^2 - g_{1|1}^2$ and $g_{1|1}^2$ follow.

Finally for the potentials just mentioned, if g_1^0 has a constant small deviation from the Maxwellian velocity distribution and one linearizes in this small deviation, the contribution from those modes which yield previous $t^{-3/2}$ contribution is finite.

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