ON THE STATISTICS OF CORRELATOR OUTPUTS

by

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Abstract. The characteristic function, probability density function, and probability distribution function are derived for the random variable z where

$$z = \sum_{j=1}^{2N} (s_j + n_j)(r_j + m_j).$$

The s_j and r_j are arbitrary numbers and the n_j and m_j are gaussian random variables. The probability Prob[z ≤ 0] is poltted for selected values of the pertinent parameters.

INTRODUCTION

Correlation detection is an optimum strategy for detecting signals in noise for various combinations of criteria of goodness and assumptions concerning the observed data [1-3]. Consider, for example, the important case where the criterion of goodness is taken to be minimum probability of error; then correlation detection is an optimum strategy provided only that the probability law governing the observed data satisfies certain symmetry conditions [4]. Furthermore, because they are easy to implement, correlation detectors are often used in situations where they are known to be suboptimal, but appear to be excellent approximations to the optimum detectors. [5-8].

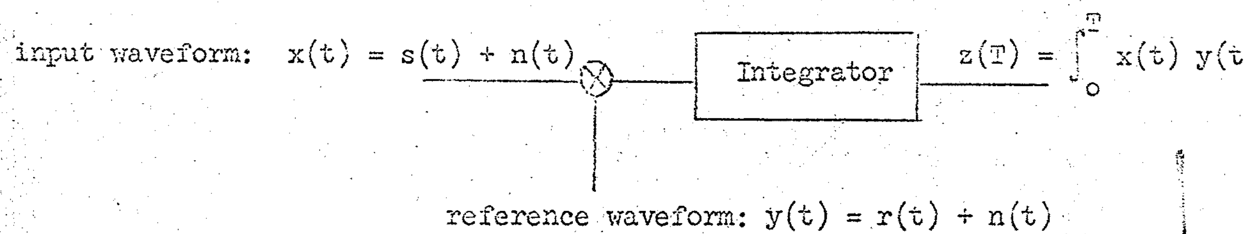

Fig. 1.

A simple correlator is shown in Fig. 1. The input waveform $x(t)$ consists of a deterministic signal $s(t)$ of T seconds duration plus a random perturbation $n(t)$ called noise. The reference waveform $y(t)$ also consists of a deterministic signal $r(t)$ of T seconds duration plus a random component $m(t)$ also called noise. Ordinarily, both $n(t)$ and $m(t)$ are assumed to be gaussian random processes. The output of the integrator $z(t)$ is sampled at $t = T$ and the statistic $z(T)$ used in the decision process. Hence, the average performance of the correlation detector depends on the statistics of the variate $z(T)$.

An attempt to determine the probability law governing $z(T)$ directly from

$$z(T) = \int_0^T [s(t) + n(t)]\,[r(t) + m(t)]\,dt$$

[redacted]

leads to severe mathematical difficulties. However, by employing an appropriate set of orthonormal basis functions $\{\psi_i(t)\}$, the required waveforms can be approximated by finite dimensional vectors, i.e.,

$$x(t) \doteq \sum_{j=1}^{2N} x_j \psi_j(t) \ , \qquad x_j = \int_0^T x(t)\ \psi_j(t)\ dt$$

$$y(t) \doteq \sum_{j=1}^{2N} y_j \psi_j(t) \ , \qquad y_j = \int_0^T y(t)\ \psi_j(t)\ dt$$

$$s(t) \doteq \sum_{j=1}^{2N} s_j \psi_j(t) \ , \qquad s_j = \int_0^T y(t)\ \psi_j(t)\ dt$$

$$r(t) \doteq \sum_{j=1}^{2N} r_j \psi_j(t) \ , \qquad r_j = \int_0^T r(t)\ \psi_j(t)\ dt$$

$$n(t) \doteq \sum_{j=1}^{2N} r_j \psi_j(t) \ , \qquad n_j = \int_0^T n(t)\ \psi_j(t)\ dt$$

$$m(t) \doteq \sum_{j=1}^{2N} m_j \psi_j(t) \ , \qquad m_j = \int_0^T m(t)\ \psi_j(t)\ dt$$

which, in turn, can be used to write

$$z(T) \doteq z = \int_0^T \sum_{j=1}^{2N} x_j \psi_j(t) \sum_{j=1}^{2N} y_j \psi_j(t)\ dt$$

$$= \sum_{j=1}^{2N} x_j y_j = \sum_{j=1}^{2N} (s_j + n_j)(r_j + m_j)$$

The problem of determining the probability law governing $z(T)$ and/or z has already been investigated for some special cases. For example, Kac and Siegert [9,10] considered the case $s(t) + n(t) = r(t) + m(t)$ (the square law detector). Green[11] derived an expression for the signal-to-noise ratio of the output of the correlation detector. Lampard [12] (See also Wishart and

Bartlett [13]) obtained the probability density function for the correlator output for $s(t) = r(t) = 0$ (noise only in both channels). Roe and White [14] obtained the probability density function for the correlator output for the special cases: $s(t) = r(t) = 0$ (noise only in both channels); $r(t) = 0$ (signal plus noise in the input channel, but noise only in the reference channel); $r(t) = s(t)$, $\overline{n^2(t)} = \overline{m^2(t)}$ (the same signal components in both channels with equal signal-to-noise ratios in both channels). Finally, Cooper [15] computed the probability density function for the correlator output for the following cases: $s(t) + n(t) = r(t) + m(t)$, $s(t) = \cos \omega t$ (quadratic detector); $s(t) = \cos \omega t$, $r(t) = 0$ (sine wave plus noise in one channel, noise only in other channel); $s(t) = r(t) = \cos \omega t$, $\overline{n^2(t)} = \overline{m^2(t)}$ (identical sine wave signals and equal signal-to-noise ratios in both channels).

No results for the case of unequal signal-to-noise ratios in the input and reference channels have been reported in the published literature. The statistics of $z(T)$ and/or z for this case are important since, in most practical situations, it is possible to obtain a reference waveform of considerably higher signal-to-noise ratio than the input waveform. Another important case, not previously considered, is that of non-identical signals in the two channels, i.e., $r(t) \neq s(t)$. This situation may arise due to imperfect time synchronization, e.g., $r(t) = s(t-\tau)$; or, when it is necessary to estimate $s(t)$ from previously received data and an unbiased estimate is not available.

In this paper the characteristic function, probability density function, and the probability distribution function for the random variable

$$z = \sum_{j=1}^{2N} (s_j + n_j)(r_j + n_j)$$

are computed for the general case, subject only to the following restrictions:

both signal vectors have finite energy, i.e., $\sum_{j=1}^{2N} s_j^2 < \infty$, $\sum_{j=1}^{2N} r_j^2 < \infty$; the random variables n_j, $j=1, \ldots, 2N$ form a set of 2N mutually independent gaussian random variables of identical variances, i.e., $\overline{(n_j - \overline{n_j})(n_k - \overline{n_k})} = \sigma_m^2 \delta_{jk}$; the random variables m_j form a set of 2N mutually independent gaussian random variables of identical variances, i.e., $\overline{(m_j - \overline{m_j})(m_k - \overline{m_k})} = \sigma_m^2 \delta_{jk}$. Note, however, that no loss of generality results from assuming that both the n_j and m_j have zero mean and unit variance. We shall first assume that the two noise vectors are sample functions from two independent random processes, i.e., $\overline{n_j m_k} = 0$. In a later section, the results are extended to the case of correlated noise vectors, i.e., $\overline{n_j m_k} = \rho\delta_{jk}$. Finally, graphical results are presented for selected values of the pertinent parameters.

PROBLEM STATEMENT

Let

$$x_j = s_j + n_j , \qquad j = 1, 2, . . ., 2N$$

$$y_j = r_j + m_j , \qquad j = 1, 2, . . ., 2N$$

where the s_j and r_j are known numbers and the n_j, m_j comprise a set of 4N mutually independent, zero mean, unit variance, gaussian random variables.

We also define

$$z_j = x_j \, y_j = (s_j + n_j)(r_j + m_j)$$

$$z = \sum_{j=1}^{2N} z_j = \sum_{j=1}^{2N} x_j \, y_j = \sum_{j=1}^{2N} (s_j + n_j)(r_j + m_j)$$

The procedure will be to compute the following, in the order listed:

$M_j(t)$ = characteristic function of z_j

$M(t)$ = characteristic function of z

$p(z)$ = probability density function of z

$P(B) = \int_{-\infty}^{B} p(z)\, dz$ = probability distribution function for z.

For ease of notation we also define the following symbols:

$$E_a = \frac{1}{2} \sum_{j=1}^{2N} (s_j^{\,2} + r_j^{\,2})$$ = average of the energies of s(t) and r(t).

$$E_c = \sum_{j=1}^{2N} s_j \, r_j$$ = cross energy of s(t) and r(t).

$$i = \sqrt{-1}$$

$K_k(z)$ = modified Bessel function of the second kind

$$\binom{j}{k} = \frac{j!}{k!(j - k)!}$$ = binomial coefficient

THE CHARACTERISTIC FUNCTION

We start by computing the characteristic function for z_j. $M_j(t)$ is the expectation of $\exp\left[i\ z_j\ t\right]$

$$M_j(t) = \int_{-\infty}^{+\infty} \exp[i\ z_j\ t]\ p(z_j)\ dt \tag{1}$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[i\ t\ x_j\ y_j]\ p(x_j,\ y_j)\ dx_j\ dy_j$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[i\ t\ x_j\ y_j]\ \frac{1}{2\pi} \exp[-(x_j-s_j)^2/2 - (y_j-r_j)^2/2]\ dx_j\ dy_j$$

The integrals can be evaluated in either order to obtain:

$$M_j(t) = (1 + t^2)^{-1/2} \exp\left[\frac{-\frac{1}{2}\ (s_j^2 + r_j^2) + i\ s_j\ r_j\ t}{1 + t^2}\right] \tag{2}$$

Since the z_j are statistically independent and identically distributed, the characteristic function for z is given by

$$M(t) = \prod_{j=1}^{2N} M_j(t)$$

$$= (1 + t^2)^{-N} \exp\left[\frac{-\frac{1}{2}\sum_{j=1}^{2N} (s_j^2 + r_j^2) + i \sum_{j=1}^{2N} r_j\ s_j\ t}{1 = t^2}\right]$$

$$= (1 + t^2)^{-N} \exp\left[\frac{-E_a\ t^2 + i\ E_c\ t}{1 + t^2}\right] \tag{3}$$

To facilitate later computations we prefer to express the characteristic function in the equivalent form:

$$M(t) = \exp[-E_a]\ (1 + t^2)^{-N} \exp\left[\frac{E_a + i\ E_c t}{1 + t^2}\right] \tag{4}$$

Eq. (4) points out that performance of the correlator does not depend on the waveshapes of the signal components in the two channels. All waveform pairs s(t), r(t) having the same average and cross energies perform equally well. It is also interesting that the correlator output does not depend on the two signal energies $\sum s_j^2$ and $\sum r_j^2$, but only on their average.

The probability density function for z is given by the inverse Fourier transform of M(t), i.e.,

$$p(z) = \frac{\exp[-E_a]}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp\left[\frac{E_a}{1+t^2}\right] \exp\left[\frac{iE_c t}{1+t^2}\right] \exp[-izt]}{(1 + t^2)^N} dt \qquad (5)$$

The integral appears somewhat formidable so we resort to a taylor series expansion of the first two exponential functions of the integrand:

$$p(z) = \frac{\exp[-E_a]}{2\pi} \int_{-\infty}^{+\infty} \sum_{k=0}^{\infty} \frac{\left[\frac{E_a}{1+t^2}\right]^k}{k!} \sum_{j=0}^{\infty} \frac{\left[\frac{iE_c t}{1+t^2}\right]^j}{j!} \frac{\exp[-izt]}{(1+t^2)^N} dt$$

$$= \frac{\exp[-E_a]}{2\pi} \sum_{k=0}^{\infty} \frac{(E_a)^k}{k!} \sum_{j=0}^{\infty} \frac{(-E_c)^j}{k!} \quad I(z) \qquad (6)$$

where I(z) is given by

$$I(z) = \int_{-\infty}^{+\infty} \frac{(-it)^j \exp[-izt]}{(1+t^2)^{N+j+k}} dt \qquad (7)$$

To evaluate the integral of Eq. (7) we first write

$$I(z) = \frac{d^j}{dz^j} \left\{ \int_{-\infty}^{+\infty} \frac{\exp[-izt]}{(1+t^2)^{N+j+k}} dt \right\}. \qquad (8)$$

The integral of Eq. (8) can be evaluated in terms of a modified Bessel function of the second kind, i.e., [16]

$$I(z) = \frac{d^j}{dz^j} \left\{ \frac{\sqrt{\pi}\, |z|^{j+k+N-1/2}\, K_{j+k+N-1/2}(|z|)}{2^{j+k+N-3/2}\, (j+k+N-1)!} \right\} \qquad (9)$$

I(z) can also be expressed in other forms that are more suitable for computational purposes. For example, we can use the finite series representation for the Bessel function [17]

$$K_{n+1/2}(z) = \sqrt{\frac{\pi}{2z}}\; e^{-z} \sum_{m=0}^{n} \frac{(n+m)!}{m!\,(n-m)!} \left(\frac{1}{2z}\right)^{m} \tag{10}$$

and obtain

$$I(z) = \frac{\pi}{2^{j+k+N-1}} \sum_{m=0}^{j+k+N-1} \binom{j+k+N-1+m}{m} \frac{\frac{d^j}{dz^j}\left\{|z|^{j+k+N-1-m} \exp[-|z|]\right\}}{(j+k+N-1-m)!\; 2^m} \tag{11}$$

Now applying Leibnitz's Rule [18]

$$\frac{d^j}{dz^j}\{\mu\, v\} = \sum_{\ell=0}^{j} \binom{j}{\ell} \frac{d^{j-\ell}}{dz^{j-\ell}}\{u\} \frac{d^j}{dz^j}\{v\} \tag{12}$$

we have, finally:

$$I(z) = \frac{\pi}{2^{j+k+N-1}} \sum_{m=0}^{j+k+N-1} \binom{j+k+N-1+m}{m} \left(\frac{1}{2}\right)^{m} \sum_{\ell=0}^{\min\{j,\,j+k+N-1-m\}} \binom{j}{\ell} (-1)^{\ell} \frac{|z|^{j+k+N-1-m-\ell} \exp[-|z|]}{(j+k+N-1-m-\ell)!} \cdot \begin{cases} 1, & z < 0 \\ (-1)^j, & z > 0 \end{cases} \tag{13}$$

THE PROBABILITY DISTRIBUTION FUNCTION

The probability distribution function for z is given by

$$P(B) = \int_{-\infty}^{B} p(z)\, dz$$

$$= \frac{\exp[-E_a]}{2\pi} \sum_{k=0}^{\infty} \frac{(E_a)^k}{k!} \sum_{j=0}^{\infty} \frac{(-E_c)^k}{k!} \quad J(B) \qquad (14)$$

where

$$J(B) = \int_{-\infty}^{B} I(z)\, dz = J(0) + \int_{0}^{B} I(z)\, dz \qquad (15)$$

In Appendix A we show that

$$J(0) = \begin{cases} \pi & , \quad j = 0 \\ \frac{\pi(-1)^{j-1}}{2^{j+2k+2N-1}} \sum_{n=0}^{j-1} \binom{j-1+2k+2N+n}{k+n+n} \binom{j-1}{n} \left(-\frac{1}{2}\right)^n, & j \neq 0 \end{cases} \qquad (16$$

and

$$\int_{-\infty}^{B} I(z)\, dz = \frac{\pi}{2^{j+k+N-1}} \sum_{m=0}^{j+k+N-1} \binom{j+k+N-1+m}{m} \left(\frac{1}{2}\right)^m \sum_{\ell=0}^{\min\{j, j+k+N-1-m\}} \binom{j}{\ell} (-1$$

$$\cdot \sum_{h=0}^{j+k+N-1-m-\ell-h} \frac{|B|^{j+k+N-1-m-\ell-h}}{(j+k+N-1-m-\ell-h)!} \exp[-|B|] \begin{cases} 1 & , \quad B < 0 \\ (-1)^j & , \quad B > 0 \end{cases} \qquad (17)$$

CORRELATED NOISES IN THE TWO CHANNELS

In this section we extend the analysis of the preceeding section to cover the case of correlated noises in the two channels, i.e., $\overline{n_j m_k} = \rho\delta_{jk}$. For this case, the characteristic function for z_j is still given by Eq. (1), but now with

$$p(x_j,y_j) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[\frac{x_j^2 - 2\rho x_j y_j + y_j^2}{2(1-\rho^2)}\right] \tag{22}$$

As before, the integrals are easily evaluated to obtain the characteristic function for z_j for the case of correlated noises:

$$M_j(t) = \frac{\exp\left[\dfrac{-\frac{1}{2}(r_j^2 + s_j^2 - 2\rho s_j r_j)\,t^2 + i\,r_j s_j\,t}{1 - 2i\rho t + (1-\rho^2)t^2}\right]}{1 - 2i\rho + (1-\rho^2)t^2} \tag{23}$$

Again assuming the z_j to be independent identically distributed random variables we can write immediately the characteristic function for z:

$$M(t) = \frac{\exp\left[\dfrac{-(E_a - \rho E_c)t^2 + i\,E_c t}{1 - 2i\rho t + (1-\rho^2)t^2}\right]}{(1 - 2i\rho t + (1-\rho^2)\,t^2)^N} \tag{24}$$

We next make a simple change of variable that transforms the characteristic function for correlated noises into the same form as the characteristic function for uncorrelated noises. Hence, the equations derived for the probability density function and probability distribution function derived for the case of uncorrelated noise processes are also valid for the case of correlated processes. Proceeding toward this end, we set

$$t = \frac{v + i\rho}{1 - \rho^2} \tag{25}$$

in Eq. (24) to obtain

$$M(v) = \frac{\exp\left[\dfrac{\dfrac{-E_a + \rho E_c}{1-\rho^2} v^2 + i\,\dfrac{(1+\rho^2)E_c - 2\rho E_a}{1-\rho^2} v + \dfrac{\rho^2 E_a - \rho E_c}{1-\rho^2}}{1 + v^2}\right]}{\left[\dfrac{1+v^2}{1-\rho^2}\right]^N} \tag{26}$$

Eq. (26) can also be written

$$M(v) = (1-\rho^2)^N \exp\left[\frac{\rho^2 E_a - \rho E_c}{1-\rho^2}\right] \frac{\exp\left[\dfrac{\dfrac{-(1+\rho^2)E_a + 2\rho E_c}{1-\rho^2} v^2 + i\,\dfrac{(1+\rho^2)E_c - 2\rho E_a)}{1-\rho^2} v}{1+v^2}\right]}{(1+v^2)^N}$$

which is identical in form to Eq. (3). From this point the procedure is the same as for the case of uncorrelated noises.

PROBABILITY OF ERROR

In most communication systems an unbiased detector is employed. For an unbiased correlation detector the probability of a decision error is simply the probability that the correlator output does not exceed zero, i.e.,

$$P_E = P(0) = \text{Prob}\left[\sum_{j=1}^{2N} (s_j + n_j)(r_j + m_j) < 0\right] \tag{18}$$

Combining Eqs. (14) and (16) we have

$$P_E = \frac{1}{2} - \frac{\exp[-E_a]}{2^{2N}} \sum_{k=0}^{\infty} \frac{(E_a/4)^k}{k!} \sum_{j=1}^{\infty} \frac{(E_c/2)^j}{j} \sum_{n=0}^{j-1} \binom{j-1+2k+2N+n}{k+N+n} \binom{j-1}{n} \left(-\frac{1}{2}\right)^n \quad (1$$

We now index the j summation from 0 rather than 1 and interchange the k summation with the j and n summations to obtain

$$P_E = \frac{1}{2} - \frac{\exp[-E_a]}{2^{2N}} \sum_{j=0}^{\infty} \frac{(E_c/2)^{j+1}}{(j+1)!} \sum_{n=0}^{j} \binom{j}{n} \left(-\frac{1}{2}\right)^n \sum_{k=0}^{\infty} \binom{j+2k+2N+n}{k+N+n} \frac{(E_a/4)^k}{k!} \tag{20}$$

One can also express the k summation as a hypergeometric function, i.e.,

$$P_E = \frac{1}{2} - \frac{1}{2^{2N}} \sum_{j=0}^{\infty} \frac{(E_c/2)^{j+1}}{(j+1)!} \sum_{n=0}^{j} \binom{j}{n} \binom{j+n+2N}{n+N} \left(-\frac{1}{2}\right)^n$$

$$\cdot\ {}_2F_2\left(N+1+\frac{n+j}{2}, N+\frac{1}{2}+\frac{n+j}{2}; 1+\frac{n+j}{2}, \frac{1}{2}+\frac{n+j}{2}; E_a\right) \exp\left[-E_a\right.$$

The probability of error P_E is easily evaluated for various values of N, E_c, and E_a on a digital computer. Fig. 2 illustrates the dependence of P_E on N, E_a, and E_c for selected values of these parameters. Next, consider the case of identical signal components, but different signal-to-noise ratios in the input and reference channels. For this case

$$z = \sum_{j=1}^{2N} (s_i + n_i)(\beta s_i + m_i) \tag{22}$$

If we let E represent the signal energy, i.e.,

$$E = \sum_{j=1}^{2N} s_j^{\,2} \tag{23}$$

then

$$E_a = \frac{E}{2}(1 + \beta^2) \;; \tag{24}$$

$$E_c = \beta E \tag{25}$$

In Fig. 3 P_E is plotted vs. E for selected values of β and N.

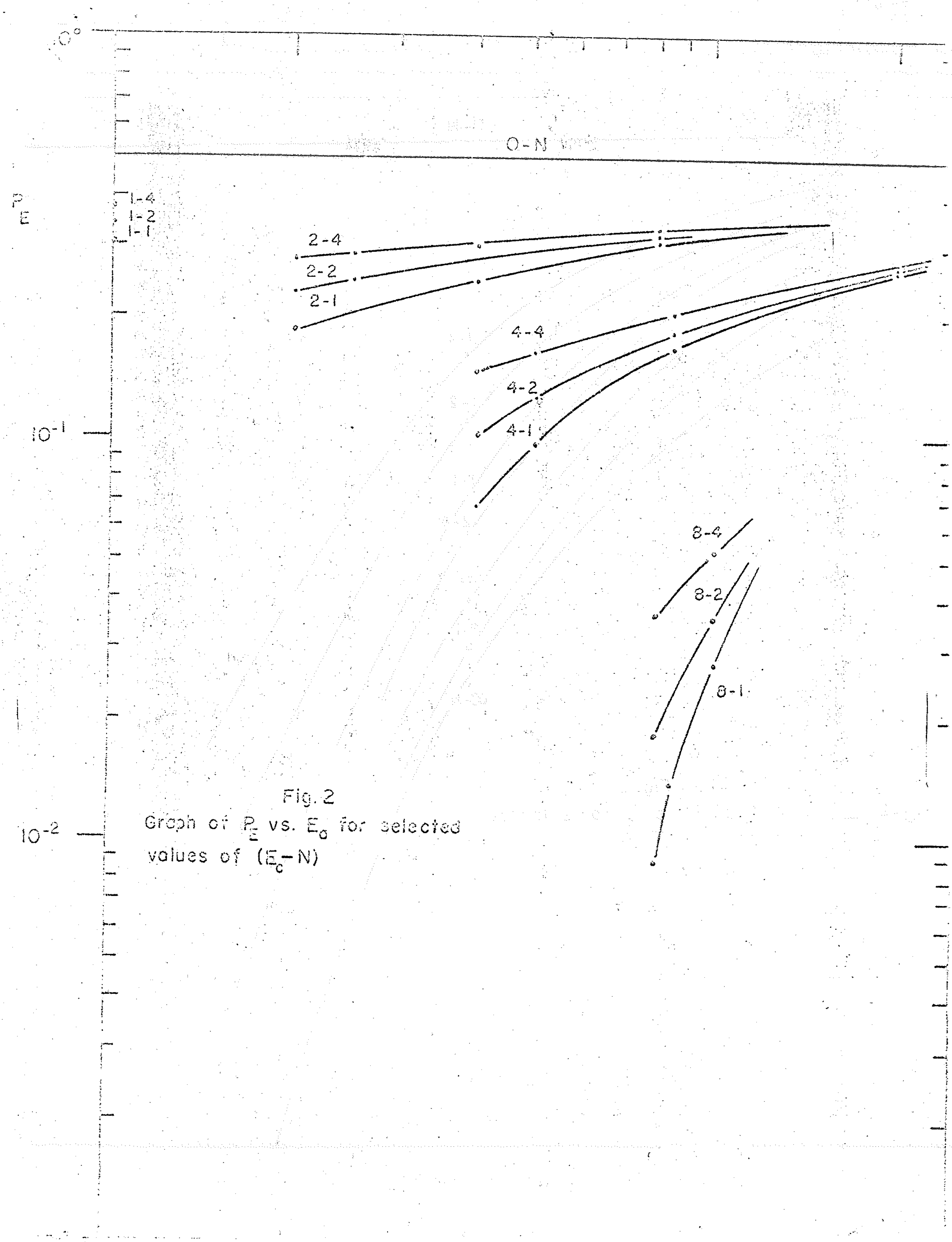

Fig. 2
Graph of P_E vs. E_a for selected values of $(E_c - N)$

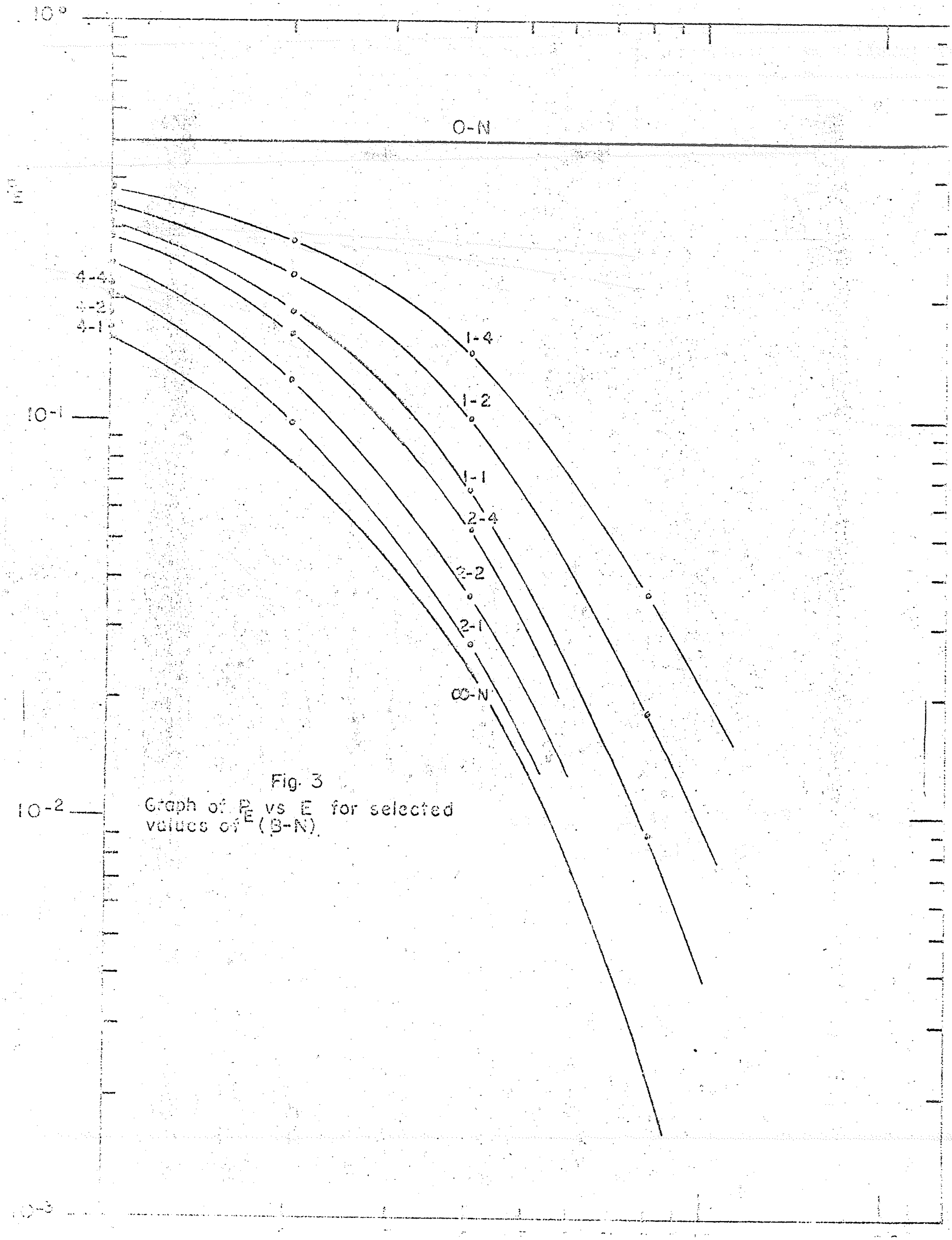

Fig. 3
Graph of P_E vs E for selected values of (B-N).

CONCLUSIONS

In the preceding sections the characteristic function, probability density function, and probability distribution function for the output of the correlation detector were computed and found to depend on:

- 2N - the demensionality of the vectors representing the input and reference waveforms;
- E_a - the average of the energies of the signal components of the input and reference waveforms;
- E_c - the inner product of the signal component vectors;
- ρ - the normalized crosscorrelation coefficient of the like coordinates of the noise components of the input and reference waveforms.

The dependence of the average performance of the correlation detector on the first three parameters is illustrated in Fig. 2. These curves are also valid for the case of correlated noise process in the two channels, but the three parameters E_a, E_c, and N indicated in the figures are now more complicated functions of the four parameters listed above. A more detailed investigation of the dependence of P_E on ρ was not investigated since, in most applications, it is reasonable to assume $\rho = 0$.

System performance for the special case of identical signal components, but unequal signal-to-noise ratios, in the two channels is illustrated in Fig. 3. Here the signal-to-noise ratio in one channel is E while the signal-to-noise ratio in the other channel is $\beta^2 E$.

Also uncovered during the course of this investigation was the identity

$$\sum_{n=0}^{m} \binom{n+m}{n} \left(\frac{1}{2}\right)^{n+m} = 1$$

which is proved in Appendix B and used in Appendix A. To the author's knowledge, this identity has not been noted previously.

ACKNOWLEDGMENT

The author is indebted to Mrs. Sharon Spelman for typing the manuscript and to Mr. M. Shapiro for writing the computer programs.

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APPENDIX A

In this appendix we compute $J(0) = \int_{-\infty}^{0} I(z)\,dz$ and $\int_{0}^{B} I(z)\,dz$ where $I(z)$ is given by

$$I(z) = \frac{\pi}{2^q} \sum_{m=0}^{q} \binom{q+m}{m} \left(\frac{1}{2}\right)^m \sum_{\ell=0}^{\min\{j,q-m\}} \binom{j}{\ell} (-1)^{\ell} \frac{|z|^{q-m-\ell} \exp[-|z|]}{(q-m-\ell)!} \begin{cases} 1, & z < 0 \\ (-1)^j, & z > 0 \end{cases} \tag{A-1}$$

$$q = j+k+N-1 \tag{A-2}$$

We shall first evaluate $J(0)$. We have

$$J(0) = \frac{\pi}{2^q} \sum_{m=0}^{q} \binom{q+m}{m} \left(\frac{1}{2}\right)^m \sum_{\ell=0}^{\min\{j,q-m\}} \binom{j}{\ell} \frac{(-1)^{\ell}}{(q-m-\ell)!} \int_{-\infty}^{0} (-z)^{q-m-\ell} \exp[z]\,dz \tag{A-3}$$

Now letting $y = -z$ we recognize the integral as Euler's integral, i.e.,

$$\int_{-\infty}^{0} (-z)^{q-m-\ell} \exp[z]\,dz = \int_{0}^{\infty} z^{q-m-\ell} \exp[z]\,dz = (q-m-\ell)! \tag{A-4}$$

so that we have

$$J(0) = \frac{\pi}{2^q} \sum_{m=0}^{q} \binom{q+m}{m} \left(\frac{1}{2}\right)^m \sum_{\ell=0}^{\min\{j,q-m\}} \binom{j}{\ell} (-1)^{\ell} \tag{A-5}$$

We next note that

$$\sum_{\ell=0}^{\min\{j,q-m\}} \binom{j}{\ell} (-1)^{\ell} = \sum_{\ell=0}^{\min\{j,j+k+N-1-m\}} \binom{j}{\ell} (-1)^{\ell} \tag{A-6}$$

$$= \begin{cases} 0, & j \neq 0,\ m \leq k+N-1 \\ (-1)^{j+k+N-1-m} \binom{j-1}{m-k-N}, & j \neq 0,\ m > k+N-1 \\ 1, & j = 0,\ \text{all } m \end{cases}$$

Using Eq. (A-6) in Eq. (A-5) we have

$$J(0) = \begin{cases} \frac{\pi}{2^q} \sum_{m=0}^{q} \binom{q+m}{m} \left(\frac{1}{2}\right)^m, & j = 0 \\ \frac{\pi}{2^q} \sum^{q} \binom{q+m}{\;} \binom{j-1}{\;} \left(\frac{1}{2}\right)^m (-1)^{q-m}, & j \neq 0 \end{cases} \tag{A-7}$$

Now using Eq. (B-1) in the $j = 0$ term, and setting $n = m-k-N$ in the $j \neq 0$ term we have, finally:

$$J(0) = \begin{cases} \pi , & j = 0 \\ \dfrac{\pi (-1)^{j-1}}{2^{q+k+N}} \displaystyle\sum_{n=0}^{j-1} \binom{q+k+N+n}{k+N+n} \binom{j-1}{n} \left(-\frac{1}{2}\right)^n , & j \neq 0 \end{cases} \tag{A-8}$$

The evaluation of $\int_0^B I(z)\, dz$ is straightforward. Using the relationship

$$\int_0^B z^{q-m-\ell} \exp[z]\, dz = e^B \sum_{h=0}^{q-m-\ell} \frac{(q-m-\ell)!}{(q-m-\ell-h)!} B^{q-m-\ell-h} (-1)^h \tag{A-9}$$

we find

$$\int_0^B I(z)\, dz = \frac{\pi}{2^q} \sum_{m=0}^{q} \binom{q+m}{m} \left(\frac{1}{2}\right)^m \sum_{\ell=0}^{\min\{j,q-m\}} \binom{j}{\ell} (-1)^\ell \tag{A-10}$$

$$\cdot \sum_{h=0}^{q-m-\ell} \frac{|B|^{q-m-\ell-h} e^{-|B|}}{(q-m-\ell-h)!} \begin{cases} 1 , & B < 0 \\ (-1)^j , & B > 0 \end{cases}$$

APPENDIX B

In this appendix we prove the identity

$$\sum_{n=0}^{m} \binom{n+m}{n} \left(\frac{1}{2}\right)^{n+m} = 1 , \qquad m = 0, 1, \ldots \tag{B-1}$$

The proof is by induction. Let

$$S_m = \sum_{n=0}^{m} \binom{n+m}{n} \left(\frac{1}{2}\right)^{n+m} \tag{B-2}$$

Then [19]

$$\begin{aligned} S_{m+1} &= \sum_{n=0}^{m+1} \binom{n+m+1}{n} \left(\frac{1}{2}\right)^{n+m+1} \\ &= \sum_{n=0}^{m+1} \left\{\binom{n+m}{n} + \binom{n+m}{n-1}\right\} \left(\frac{1}{2}\right)^{n+m+1} \\ &= \sum_{n=0}^{m+1} \binom{n+m}{n} \left(\frac{1}{2}\right)^{n+m+1} + \sum_{n=1}^{m+1} \binom{n+m}{n-1} \left(\frac{1}{2}\right)^{n+m+1} \end{aligned} \tag{B-3}$$

For the first term on the right hand side of Eq. (B-3) we write

$$\sum_{n=0}^{m+1} \binom{n+m}{n} \left(\frac{1}{2}\right)^{n+m+1} = \frac{1}{2} S_m + \binom{2m+1}{m+1} \left(\frac{1}{2}\right)^{2m+2} \tag{B-4}$$

For the second term on the right hand side of Eq. (B-4) we write

$$\begin{aligned} \sum_{n=1}^{m+1} \binom{n+m}{n-1} \left(\frac{1}{2}\right)^{n+m+1} &= \sum_{n=0}^{m} \binom{n+m+1}{n} \left(\frac{1}{2}\right)^{n+m+2} \\ &= \frac{1}{2} S_{m+1} - \binom{2m+2}{m+1} \left(\frac{1}{2}\right)^{2m+3} \end{aligned} \tag{B-5}$$

Combining Eqs. (B-3,4,5) we find

$$S_{m+1} = S_m \tag{B-6}$$

But from Eq. (B-2) we observe

$$S_0 = 1 \tag{B-7}$$

Hence, Eq. (B-1) is true.