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NETWORK-TOPOLOGICAL FORMULATION OF ANALYSES OF GEOMETRICALLY AND MATERIALLY NONLINEAR SPACE FRAMES

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ABSTRACT

The analysis of geometrically nonlinear and materially nonlinear structures is often regarded as outside the realm of applicability of linear graph theory and algebraic topology. That such is not the case is demonstrated in this paper.

This paper presents a network-topological formulation of the matrix force method for computer analysis of space frames. The automatic selection of redundant forces is accomplished by using linear graph theory, corresponding to the mesh method of network analysis. Nonlinear effects of large deformations and material nonlinearity are incorporated through use of interative procedures.

The equations of the formulation are given and examples are included to demonstrate the method.

LIST OF SYMBOLS

B_o = matrix relating internal forces to the applied loads.

B_x = matrix relating internal forces to the redundant forces.

e = number of members in a structure.

$\mathbf{f}_\alpha$ = primitive flexibility matrix of member α.

F, $\mathscr{F}$ = unassembled flexibility matrix and flexibility matrix of the entire structure, respectively.

m = number of basic meshes in a graph

$n, \bar{n}$ = number of nodes and number of nondatum nodes, respectively in a graph.

$\mathfrak{p}_\alpha$ = internal force vector of member α.

P = force vector applied at the nodes.

Q = incidence matrix of a linear graph.

r_α = arm matrix of member α.

u_α = displacement vector of member α.

U = displacement vector of the nondatum nodes of a structure.

X_α = redundant forces in statically indeterminate structure.

x_i = local coordinates of member α.

Z_i = global coordinates of structure.

λ^α = orthogonal transformation matrix which rotates local coordinates of member α into global coordinates.

INTRODUCTION

When forces are selected as unknowns, a significant part of the formulation of the analysis of complex structures involves purely static considerations. Force-displacement relations for various components of the system can be derived beforehand in the form of flexibility matrices, but the degree of redundancy, the relations between internal forces and applied loads, and the influence of redundant forces are established by successive applications of the familiar laws of static equilibrium. Though simple in concept, the process of computing static relations in complex structures can be extremely complicated. The action of a unit force may influence stress resultants throughout the system. Further, when electronic computation is required, it is often necessary to include the statics of the basic structure as input data. This means that the analyst if often forced to solve by-hand a substantial portion of the problem before it is brought to the computer.

Fortunately, the static relations between kinetic variables in analyses of complex space frames are directly related to the mode of connection of the structural members. The degree of redundancy, for example, can be related to the number of closed rings and the number of releases in the system; the influence of applied loads on various stress resultants is related to the paths through which the loading must be transmitted to the supports; etc. Collectively, these characteristics are said to depict the "connectivity" or the "topology" of the system, and their description is the principal objective of a branch of mathematics known as network topology. Briefly, network topology, or linear graph theory, is a mathematical discipline whose province it is to determine the relations of the characteristics of an entire set of variables (the system) to the characteristics of the individual members (the components of the system) along with their mode of connection. This branch of mathematics provides means to automatically generate static relations in complex structures.

The topological theorems derived by Euler and the network theory of Kirchhoff and Maxwell are the principal sources from which network topology has evolved. Applications to electrical networks can be found in the books by Reed [1], Reed and Seshu [2], and Kron [3] and in the papers of Reza [4], Doyle [5], Gould [6], Roth [7], and Okada [8], among others. That the theory is not limited to electrical systems is amply demonstrated in the works of Kron [9,10] and Koenig and Blackwell [11], wherein applications to hydraulic and mechanical systems are also presented.

The first application of topological theorems to the analysis of complex structural systems appears to have been presented by Kron [12]. In this paper, and in a subsequent paper [13], a "method of tearing" is employed which makes it possible to analyze structures in successive stages. More recently Lind [14] used a different topological approach to analyze pin-connected structures and Henderson and Bickley [15], Henderson [16], Morice [17] and DiMaggio [18] employed topological principles to study the determinacy of structures. The most thorough and rigorous accounts of the subject in relation to structural applications are found in the works of Samuelsson [19] and Roussopoulos [20]. The book by the latter author presents an independent and somewhat different approach than is found elsewhere.

Further applications to structural systems are given in the papers by Langefors [21], DiMaggio and Spillers [22], Spillers [23,24,25], Wu [26], and Fenves et al. [27,28,29].

In this paper, a modified version of the network-topological formulation of structural problems is used in conjunction with the matrix force method to develop a general technique for analyzing space frames. Basic definitions and topological theorems are reviewed along with their application to linear structural systems. It is then shown that the topological aspects of the formulation depict only the connectivity of the system and, in the present case, lead to purely static relations. Hence, certain types of nonlinear problems can be formulated with equal facility using basic topological theorems. The application of the method to the analysis of a class of geometrically and materially nonlinear structures is then discussed.

NETWORK TOPOLOGY AND GRAPH THEORY

A list of some of the basic definitions and equations of network topology is given as follows:

1. Abstract set. An abstract set, in the present sense, is a collection of a finite number of two types of objects: nodes, N_1, $N_2,\ldots,N_n$ and branches $b_{12}(N_1,N_2)$, $b_{23}(N_2,N_3),\ldots,$ $b_{n-1,n}(N_{n-1},N_n)$. The subset $N(N_1,N_2\ldots,N_n)$ is called the node or vertex set and each pairing $b_{ij} = (N_i,N_j)$ defines a branch.

2. Topological graph. A topological graph, or system graph, is a geometrical representation of an abstract set, as defined above. A topological graph of a given abstract set is constructed by assigning to each node in the set a point in three-dimensional space and to each branch in the set a line segment or curve in three-dimensional space.

A topological graph is shown in Fig. 1a.

3. Oriented branch. An oriented branch of a topological graph is an oriented line segment together with two endpoints. The endpoints are called nodes of the

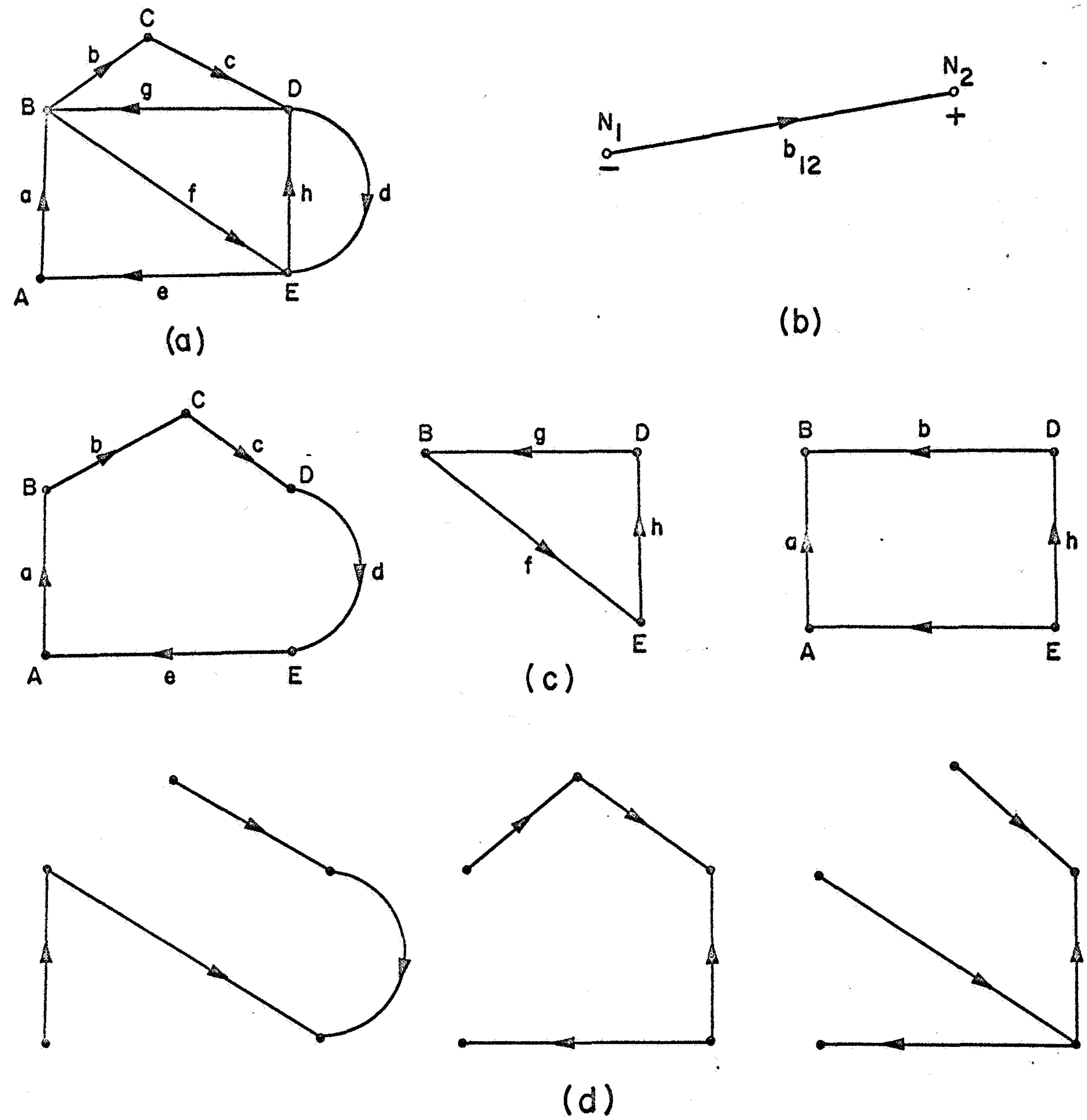


FIG. I TOPOLOGICAL GRAPHS

branch.

An oriented branch b_{12} with end nodes N_1, N_2 is shown in Fig. 1b. The nodes N_1 and N_2 are said to be incident with branch b_{12} and vice-versa. The orientation of the branch is indicated by the arrow. N_1 and N_2 are called the initial and final nodes, respectively, of branch b_{12}. Branch b_{12} is said to be positively incident on node N_1 and negatively incident on node N_2.

4. Subgraph. A subgraph is a graphical representation of any subset of branches of an abstract set.

5. Path. A path is a subgraph of a topological graph containing a sequence of branches $b_{01}, b_{12}, \ldots, b_{n,n+1}$ such that each pair of successive branches has a common endpoint. For example, branches a,b,c,d form a path in the topological graph in Fig. 1a.

6. Connected graph. If at least one path exists between any two distinct nodes of a graph, the graph is called a connected graph.

7. Complement of a subgraph. The complement of subgraph S of a graph G is the subgraph remaining in G when the elements of S are removed.

8. Separate part. A separate part is a connected subgraph that contains no nodes in common with its complement.

9. Mesh. A mesh or circuit is a closed path which is such that every node on the path is incident to two and only two branches. Examples of meshes are shown in Fig. 1c.

10. Tree. A tree is a connected subgraph containing no meshes and all nodes. Further, there can be one and only one branch incident to any pair of nodes in a tree. It follows that in a tree containing b branches and n nodes,

$$b = n - 1 \tag{1}$$

Examples of trees are indicated in Fig. 1d.

11. Links. Links (or chords) are the branches of the complement of a tree.

12. Basic mesh. A basic mesh is the unique mesh formed by adding a cord to a tree without introducing at least one new node. It is easily shown that if m is the number of basic meshes in a topological graph containing b branches and n nodes,

$$m = b - n + s \tag{2}$$

where s is the number of separate parts. Ordinarily s = 1.

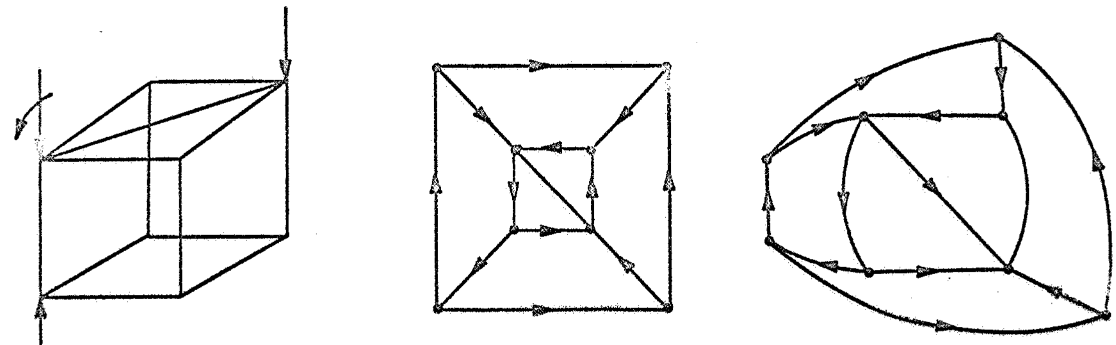
Other definitions pertaining to structural applications are given where they first appear in the text to follow.

TOPOLOGICAL GRAPHS OF STRUCTURAL SYSTEMS

The topological model for a frame structure is constructed by representing each member of the structure by a branch and each joint by a node in one-to-one correspondence. For example, topological graphs of the space frame in Fig. 2a are shown in Figs. 2b and 2c. The two graphs are isomorphic and, hence, topologically equivalent.

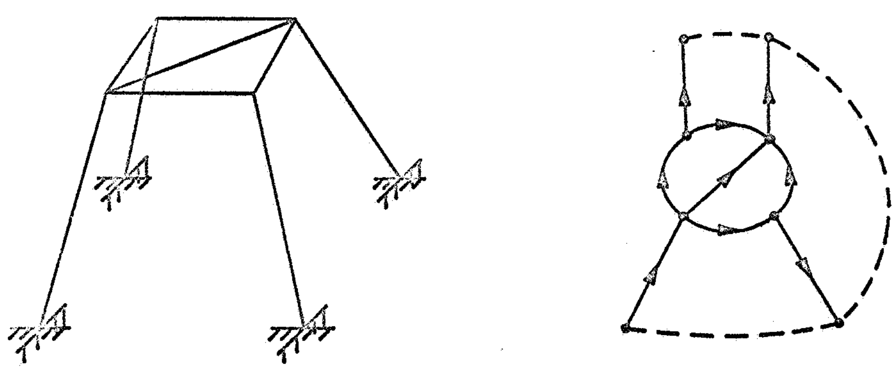
The topological graph for a supported structure, such as that shown in Fig. 3, is formed by constructing the subgraph representing the unsupported structure and then adding tree-connected branches to represent the foundation. These branches are called imaginary branches and are indicated by dashed lines in the figure. The remaining branches in the graph are called real branches. The nodes on the foundation which are incident with the imaginary branches are, in this paper, called datum nodes and the remaining nodes of the topological graph are called nondatum modes. If a topological graph contains no separate parts other than the graph itself (i.e., s = 1), it is easily shown that the total number m of basic meshes is given by

$$m = e - \overline{n} \tag{3}$$



(a) (b) (c)

FIG. 2 SPACE FRAME AND TWO ISOMORPHIC TOPOLOGICAL GRAPHS.



(a) (b)

FIG. 3 SUPPORTED SPACE FRAME AND ITS TOPOLOGICAL GRAPH.

where e is the number of real branches and $\bar{n}$ is the number of nondatumn nodes. Only the real branches are considered in the analysis of a structure.

In the present topological formulation, it is first assumed that rigid connections exist at the ends of all members. The order of statical indeterminacy is then 6m, where m is the number of links (basic meshes) in the graph of the structure. Releases that may exist in the structure are introduced after the topological formulation is completed.

A statically indeterminate structure can be divided into as many statically determinate substructures as desired, up to the number of datum nodes. Figure 4a indicates four statically determinate substructures of the structure shown in Fig. 3 and Fig. 4b shows one statically determinate substructure. This is accomplished by the selection of the m links.

The orientation of the branches in the graph of a structure is arbitrary. A convenient convention is to orient the tree branches so that they are negatively included in the node to datum paths. The orientation of the links is also arbitrary and the negative ends of the links will correspond to the cuts in the structure. The orientation of the tree branches and links for the structure of Figs. 4a and 4b are shown respectively in Figs. 4c and 4d. The heavy lines represent the tree branches and the light lines represent the links.

The convention used in this paper will be to number the tree branches first and the links last, and to number the non-datum nodes first and the datum nodes last.

STRUCTURAL MATRICES

Consider a complex space frame consisting of e members rigidly connected at n nodes, as is shown in Fig. 5a. The members of the space frame are assumed to be three-dimensional bar elements, each of which can transmit six stress resultants:

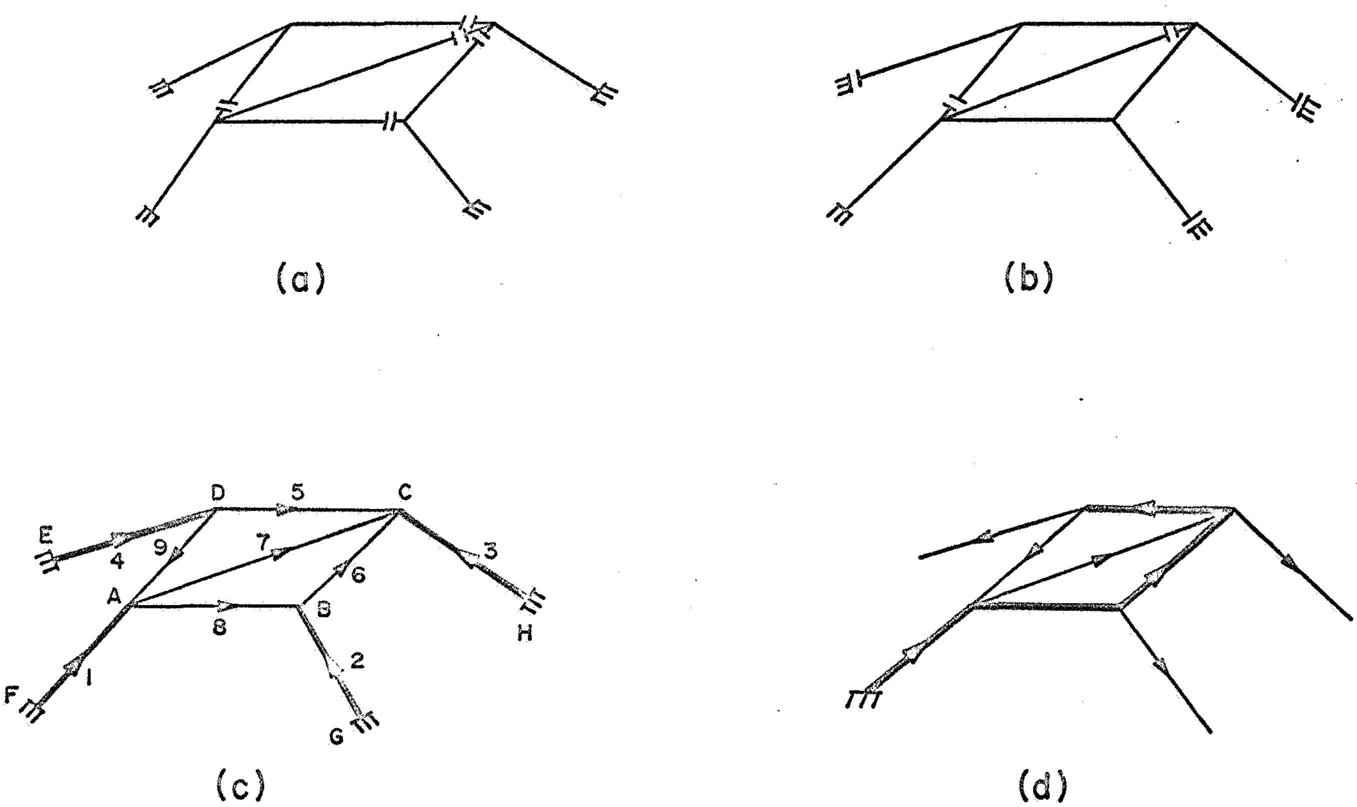


FIG. 4 LINKS AND TREE BRANCHES FOR STATICALLY INDETERMINATE STRUCTURE.

an axial force, two bending moments, a twisting moment, and two shearing forces parallel to the cross section. A typical bar element α between nodes M and N is shown in Fig. 5b. The bar is acted upon by forces and moments at each node which are shown referred to a local coordinate system associated with element α. These force systems are arranged in 6 x 1 vectors $p'_{M\alpha}$ and $p'_{N\alpha}$ called the element force vectors for element α:

$$p'_{M\alpha} = \{p_{Mi\alpha}\} \ , \quad p'_{N\alpha} = \{p_{Ni\alpha}\} \quad (i = 1,2,\ldots,6) \tag{4}$$

The first three entries in an element force vector represent the forces parallel to the respective local coordinates x_1, x_2, and x_3 whereas the remaining entries are moments about the respective axes. The prime ($'$) indicates that the components of these vectors are referred to the local coordinates of element α.

The vectors $p'_{M\alpha}$ and $p'_{N\alpha}$ are related by statics according to the formulas

$$p'_{M\alpha} = r'_{MN} p'_{N\alpha} \quad \text{and} \quad p'_{N\alpha} = r'_{NM} p'_{M\alpha} \tag{5}$$

where

$$r'_{MN} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -L & 0 & 1 & 0 \\ 0 & L & 0 & 0 & 0 & 1 \end{bmatrix} = r'^{-1}_{NM} \tag{6}$$

Here L is the length of the member.

Similarly, there corresponds to each of the force vectors $p'_{M\alpha}$ and $p'_{N\alpha}$ 6 x 1 displacement vectors $u'_{M\alpha}$ and $u'_{N\alpha}$ whose components are the displacements and

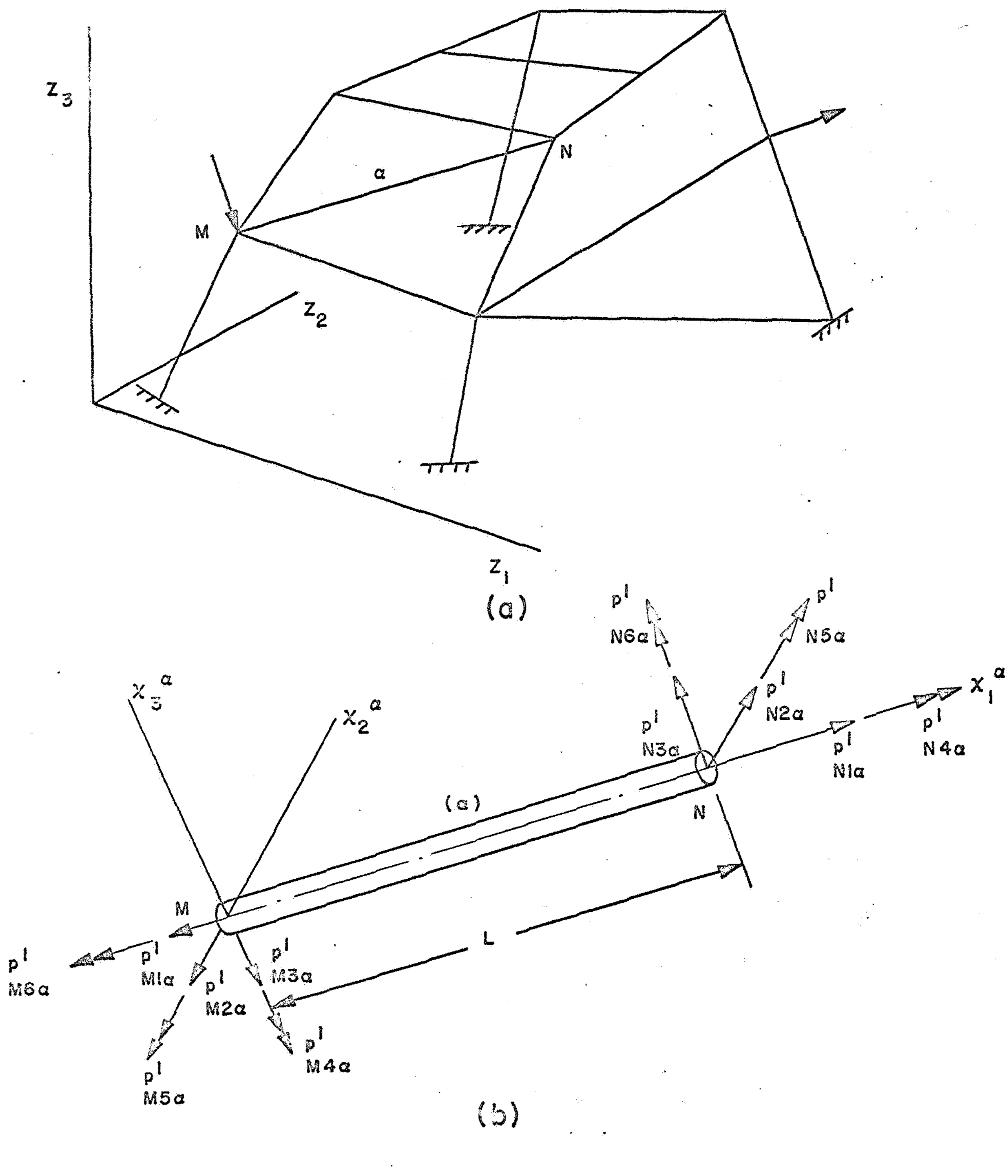


FIG. 5 A COMPLEX SPACE FRAME AND A TYPICAL BAR ELEMENT a

and rotations corresponding to the components of $p'_{M\alpha}$ and $p'_{N\alpha}$. That is, $u'_{Mi\alpha}$ $(i = 1,2,3)$ are the displacement components of node M in the directions of the local coordinates x_i^α and $u'_{Mi\alpha}(i = 4,5,6)$ are the rotations at M about the x_i^α axes. The displacement vectors $u'_{M\alpha}$ and $u'_{N\alpha}$ are related according to the formula

$$u'_{M\alpha} = r'^{T}_{NM} u'_{N\alpha} \tag{7}$$

where the T superscript means the transpose of the matrix.

It is necessary to consider only one of the two force and displacement vectors for the member α to characterize the behavior of that element, since Eqs. (6) and (7) establish dependencies between quantities associated with each end. In the following, force and displacement vectors associated with the node on which the element is negatively incident are taken as the characteristic vectors of the element. Thus, for an element α the characteristic force vector is $p'_\alpha = p'_{N\alpha}$ and the characteristic displacement vector is $u'_\alpha = u'_{N\alpha}$.

A 6 x 6 flexibility matrix for element α can be obtained which relates the element force and displacement vectors according to

$$u'_\alpha = f'_\alpha p'_\alpha \tag{8}$$

where, for a straight bar,

$$f'_\alpha = \begin{bmatrix} \frac{L}{AE} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{L^3}{3EI_3}\Psi_1 & 0 & 0 & 0 & \frac{L^2}{2EI_3}\Psi_2 \\ 0 & 0 & \frac{L^3}{3EI_2}\Psi_3 & 0 & -\frac{L^2}{2EI_2}\Psi_4 & 0 \\ 0 & 0 & 0 & \frac{L}{GJ}\Psi_5 & 0 & 0 \\ 0 & 0 & -\frac{L^2}{2EI_2}\Psi_4 & 0 & \frac{L}{EI_2}\Psi_6 & 0 \\ 0 & \frac{L^3}{2EI_3}\Psi_2 & 0 & 0 & 0 & \frac{L}{EI_3}\Psi_7 \end{bmatrix} \tag{9}$$

In this equation, I_1 and I_2 are the principal moments of inertia, J is the torsion constant, E is the elastic modulus, and G is the shear modulus. The functions Ψ_i are defined by the equations

$$\Psi_1 = \frac{3}{\lambda_1^3} (\lambda_1 - \tanh \lambda_1) \; ; \; \Psi_2 = \frac{2}{\lambda_1^2} (\operatorname{sech} \lambda_1 - 1) \; ; \; \Psi_3 = \frac{3}{\lambda_2^2} (\lambda_2 - \tanh \lambda_2).$$

$$\Psi_4 = \frac{2}{\lambda_2^2} (\operatorname{sech} \lambda_2 - 1) \; ; \; \Psi_5 = \frac{L^2 GJ}{E\Gamma\lambda_3^3} (\lambda_3 - 2 \coth \lambda_3 + 2 \operatorname{csch} \lambda_3) \tag{10a}$$

$$\Psi_6 = \frac{1}{\lambda_2} \tanh \lambda_2 \; ; \; \Psi_7 = \frac{1}{\lambda_1} \tanh \lambda_1$$

if the axial force is tensile and

$$\Psi_1 = \frac{3}{\lambda_1^2} (-\lambda_1 + \tan \lambda_1) \; ; \; \Psi_2 = \frac{2}{\lambda_1^2} (\sec \lambda_1 - 1) \; ; \; \Psi_3 = \frac{3}{\lambda_2^2} (-\lambda_2 + \tan \lambda_2)$$

$$\Psi_4 = \frac{2}{\lambda_2^2} (\sec \lambda_2 - 1) \; ; \; \Psi_5 = \frac{L^2 GJ}{E\Gamma\lambda_3^3} (\lambda_3 - 2 \cot \lambda_3 - 2 \csc \lambda_3) \tag{10b}$$

$$\Psi_6 = \frac{1}{\lambda_2} \tan \lambda_2 \; ; \; \Psi_7 = \frac{1}{\lambda_1} \tan \lambda_1$$

if the axial force is compressive. Here Γ is the warping constant, $\lambda_1^2 = p_1 L^2/EI_3$, $\lambda_2^2 = p_1 L^2/EI_2$, and $\lambda_3^2 = (GJ + r_s^2 p_1)\, L^2/E\Gamma$, r_s being the polar radius of gyration about the shear center.

To analyze a structure containing many elements, as in Fig. 3, it is desirable to relate all the quantities associated with the elements to the same coordinates. Hence, a global coordinate system Z_1, Z_2, Z_3 is established at an arbitrary point in the structure. An orthogonal transformation matrix λ^α exists for each element α which rotates the element's local coordinate system into the global coordinate system.

The element force and displacement vectors $\mathcal{P}'_\alpha$ and $\mathcal{U}'_\alpha$ are referred to the global system by the transformations

$$p_\alpha = T_\alpha p'_\alpha \text{ and } u_\alpha = T_\alpha u'_\alpha \tag{11}$$

where

$$T_\alpha = \begin{bmatrix} \lambda^\alpha & 0 \\ 0 & \lambda^\alpha \end{bmatrix} \tag{12}$$

The flexibility and arm matrices of the element α are expressed in the global system by the congruent transformations

$$f_\alpha = T_\alpha f'_\alpha T_\alpha^T \text{ and } r_\alpha = T_\alpha r'_\alpha T_\alpha^T \tag{13}$$

The complete set of force vectors corresponding to all the e elements of the structure are now arranged in a 6e x 1 vector p called the internal force vector of the system:

$$p = \{p_1, p_2, \ldots, p_\alpha, \ldots, p_e\} \tag{14}$$

Similarly, the displacement vectors corresponding to each force vector ($u_\alpha = u_{N\alpha}$) are arranged in a 6e x 1 vector u:

$$u = \{u_1, u_2, \ldots, u_\alpha, \ldots, u_e\} \tag{15}$$

Vectors u and p are related as follows:

$$u = F p \tag{16}$$

where F is the 6e x 6e unassembled flexibility matrix of the structure:

$$F = \begin{bmatrix} f_1 & 0 \ldots & 0 \ldots & 0 \\ 0 & f_2 \ldots & 0 \ldots & 0 \\ & \ddots & & \\ & & f_\alpha & \\ & & & \ddots \\ 0 & 0 & & f_e \end{bmatrix} \tag{17}$$

In addition to the internal forces, external forces, represented by 6 x 1 vectors P_M and P_N, act at nodes M and N. The (absolute) displacements and rotations of nodes M are likewise represented by 6 x 1 vectors U_M and U_N. Further, the complete set of external force vectors corresponding to all of the $\bar{n}$ nodes are arranged in $6\bar{n}$ x 1 vector

$$P = \{P_1, P_2, \ldots, P_{\bar{n}}\} \tag{18}$$

and the displacement vectors corresponding to each node are arranged in a $6\bar{n}$ x 1 vector

$$U = \{U_1, U_2, \ldots, U_{\bar{n}}\} \tag{19}$$

INCIDENCE MATRIX OF A STRUCTURE

The incidence relationships between the nodes and branches of the structure's graph are specified by an incidence matrix Q which is formulated after the links are specified and the nodes and branches are numbered. Using notation similar to DiMaggio and Spillers [22], the matrix Q is defined as containing the generic element $Q_{\alpha\bar{n}}$ where

$$Q_{\alpha\bar{n}} = \begin{cases} 0 & \text{if element } \alpha \text{ is not incident on node } \bar{n} \\ -r^T_{NM\alpha} & \text{if element } \alpha \text{ is positively incident on node } \bar{n} \\ I & \text{if element } \alpha \text{ is negatively incident on node } \bar{n} \end{cases} \tag{20}$$

The generic element $Q_{\alpha\bar{n}}$ is a 6 x 6 matrix for a space frame and the matrix Q is of order 6e x $6\bar{n}$.

The static relation between the internal forces and the applied loads is given by

$$P = Q^T p \tag{21}$$

THE MATRIX FORCE METHOD

The incidence matrix Q can be used to obtain the fundamental static matrices used in Argyris' force method [30]. To this end, first partition the incident matrix as follows:

$$Q = \begin{bmatrix} Q_T \\ Q_L \end{bmatrix} \tag{22}$$

where Q_T is the $6\bar{n} \times 6\bar{n}$ matrix corresponding to the tree branches and Q_L is the $6m \times 6\bar{n}$ matrix corresponding to the links. Equation (21) can thus be written

$$\begin{bmatrix} Q_T^T & Q_L^T \end{bmatrix} \begin{Bmatrix} \mathcal{P}_T \\ \mathcal{P}_L \end{Bmatrix} = P \tag{23}$$

The internal forces $\mathcal{P}_L$ are equivalent to the redundant forces X at the negative ends of the links.

Expanding Eq. (23) and solving for the internal forces in the tree branches gives

$$\mathcal{P}_T = Q_T^{T-1} P - Q_T^{T-1} Q_L^T X \tag{24}$$

Since $\mathcal{P}_L = X$, the internal force matrix can be written for the structure as

$$\mathcal{P} = B_o P + B_x X \tag{25}$$

where

$$B_o = \begin{bmatrix} Q_T^{T-1} \\ O \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} Q_T^{T-1} & Q_L^T \\ I & \end{bmatrix} \tag{26}$$

The redundants of Eq. (24) can be determined by imposing compatibility at the cuts in the links. The redundants in terms of the applied loads are given by

$$X = -(B_x^T F B_x)^{-1} B_x^T F B_o P \tag{27}$$

where F is the unassembled flexibility matrix defined in Eq. (17). The internal forces in the structure are then given by

$$p = B\,P \tag{28}$$

where

$$B = B_o - B_x(B_x^T F\; B_x)^{-1}(B_x^T F\; B_o) \tag{29}$$

The displacement vector U, corresponding to the nodes, is given by

$$U = B^T u = \mathcal{F}\,P \tag{30}$$

where

$$\mathcal{F} = B^T F\; B \tag{31}$$

is the flexibility of the entire structure.

As an example, consider the space frame in Fig. 3. The nodes and members are numbered as shown in Fig. 4c, according to the convention described earlier. Members 1,2,3, and 4 are tree branches and 5,6,7,8, and 9 are links. The transpose of the incidence matrix is written as

$$Q^T = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & -r_{AC} & -r_{AB} & I \\ 0 & I & 0 & 0 & 0 & -r_{BC} & 0 & I & 0 \\ 0 & 0 & I & 0 & I & I & I & 0 & 0 \\ 0 & 0 & 0 & I & -r_{DC} & 0 & 0 & 0 & -r_{DA} \end{bmatrix} \tag{32}$$

From Eq. (26), the B_o and B_x matrices are

$$B_o = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ & & O & \end{bmatrix} \qquad B_x = \begin{bmatrix} 0 & 0 & r_{AC} & r_{AB} & -I \\ 0 & r_{BC} & 0 & -I & 0 \\ -I & -I & -I & 0 & 0 \\ r_{DC} & 0 & 0 & 0 & r_{DA} \\ & & I & & \end{bmatrix} \tag{33}$$

These matrices are introduced into Eqs. (27) through (31) and the analysis is completed.

Structural Releases. The above formulation is based on the assumption that each member in the space frame transmits six stress resultants. In many structures, however, various types of structural releases (e.g., hinges, free ends, etc.) are present and the procedure discussed previously must be modified accordingly. This modified procedure also allows alternate redundants to be selected so that the conditioning of the matrix $B_x^T F B_x$ is improved for inversion.

Let X' denote the 6m x 1 vector of redundant forces, each component of which is referred to its appropriate local coordinate system. Assuming that r releases are present in the links of the system, a matrix D exists such that

$$X' = D \begin{bmatrix} X^* \\ 0 \end{bmatrix} \tag{34}$$

where D is a nonsingular matrix of ones and zeros and X^* is the (6m - r) x 1 vector of redundant forces in the released system.

The original redundant force vector X in global coordinates is related to X' by the formula

$$X = \Lambda X' \tag{35}$$

in which Λ is a 6m x 6m transformation matrix. Thus

$$X = A \; X' = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} X^* \\ 0 \end{Bmatrix} = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} X^* \tag{36}$$

where $A = \Lambda \; D$. Introducing Eq. (36) into Eq. (25), a new B_x matrix

$$B_x^* = B_x \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \tag{37}$$

is obtained which relates $\mathbf{p}$ to the redundants in the released system. The analysis proceeds as before except that $\mathbf{B}_x^*$ is used instead of $\mathbf{B}_x$.

If releases are present in the tree branches, both $\mathbf{B}_o$ and $\mathbf{B}_x$ must be modified though a procedure similar to that outlined above.

NONLINEAR ANALYSES

Once the topological formulation of a linear structural problem is completed, the structural matrices can be modified to account for nonlinear structural behavior. In general, there are two sources of nonlinearity in the structural problem: (1) geometric nonlinearity, which occurs when deformations are of such magnitude that their influence in equilibrium considerations cannot be ignored, (2) material nonlinearity, which occurs when the stress-strain relations of the structural materials are nonlinear. The remainder of this paper is devoted to the discussion of approximate procedures for handling both types of nonlinearity.

<u>Geometric Nonlinearity</u>. The effects of large deformations are accounted for by computing the static matrices from the geometry of the deformed rather than the undeformed structure. Consider, for example, the forces at node B of member AB shown in Fig. 6. The components of the internal force vector $\mathbf{p}_B$ are, by definition, the stress resultants developed parallel and normal to the bar's cross section at B. These are indicated by dashed lines in the figure. Clearly, these vectors do not act parallel to the local coordinate axes due to the rotations $\theta_{B1} = u_{B4}$, $\theta_{B2} = u_{B5}$, $\theta_{B3} = u_{B6}$ of end B. In the linear theory, however, no distinction is made between the deformed and the undeformed structure in applying statics. Hence, the linear theory assumes that these forces are parallel to the original local coordinate axes, as is indicated by the solid lines in the figure.

Let $\mathbf{p}_B^o$ denote the stress-resultant vector at B as given by the linear theory and let $\bar{\mathbf{p}}_B$ denote the force vector at B with components parallel to the local

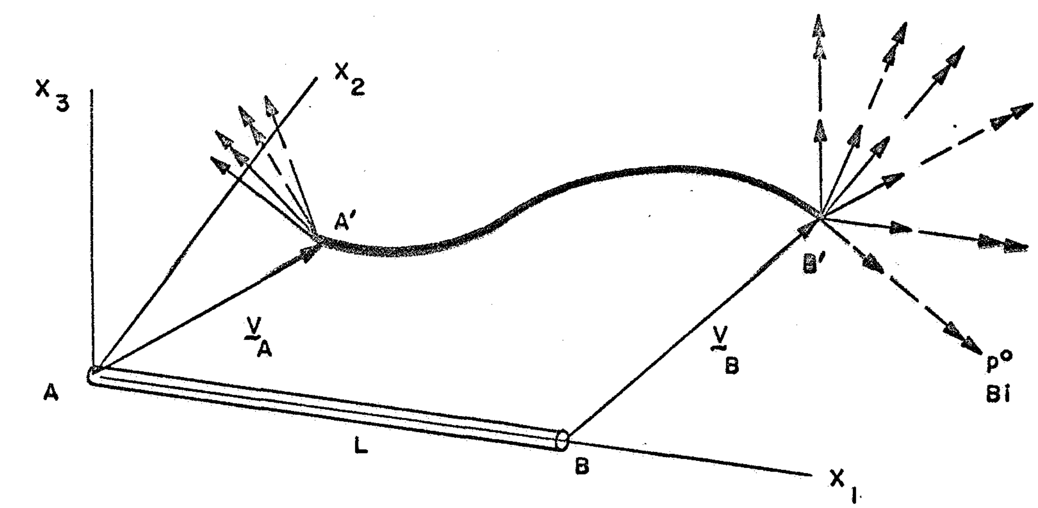


FIG. 6 NODE FORCES ON DEFORMED BAR

coordinates. If terms of quadratic and higher order in the rotations are neglected, it is easily shown that

$$\bar{p}_B = \begin{bmatrix} C_B & 0 \\ 0 & C_B \end{bmatrix} p_B^o = R_B p_B^o \tag{38}$$

where

$$C_B = \begin{bmatrix} 1 & -\theta_{B3} & \theta_{B2} \\ \theta_{B3} & 1 & -\theta_{B1} \\ -\theta_{B2} & \theta_{B1} & 1 \end{bmatrix} \tag{39}$$

In addition, the distance between A and B has changed due to deformation of the member so that instead of Eq. (6) the arm matrix is given by

$$\bar{r}_{AB} = \left[\begin{array}{ccc|c} & I & & 0 \\ \hline 0 & u_{A3}-u_{B3} & u_{B2}-u_{A2} & \\ u_{B3}-u_{A3} & 0 & -L+u_{B1}-u_{A1} & I \\ u_{A2}-u_{B2} & L-\;_{B1}+u_{A1} & 0 & \end{array}\right] \tag{40}$$

The procedure for including such nonlinear effects is outlined as follows:

(1) Perform a linear analysis of the structure using flexibility matrices for which the functions Ψ_i in Eq. (9) are unity (that is, neglect the influence of axial loads on the flexibilities).

(2) Using the results of step 1 and Eqs. (9) compute new flexibility matrices (that is, this time account for the influence of axial loads).

(3) Using Eqs. (38), (39), and (40) and the displacements and rotations obtained in step 1, compute modified force vectors, arm matrices, and node-incidence matrices.

(4) Introduce the results of steps 2 and 3 into Eqs. (27) through (32) and,

hence, analyze the structure using the modified matrices.

(5) Use the results of step 4 to compute new flexibility, node force, and incidence matrices and repeat the process until the solution converges to a desired degree.

Stability. A slight modification of the above procedure can be used to compute buckling loads. Briefly, the final flexibility matrix $\mathcal{F}$ is computed using the general element flexibilities given in Eq. (9). Each element flexibility is espressed in terms of $\lambda \mathbf{P}^*$, where $\mathbf{P}^*$ is a reference external load vector and λ is a load parameter such that $\lambda > 0$. The stability criterion is

$$\det(\mathbf{K}) = \mathbf{0} \tag{41}$$

where $\mathbf{K} = \mathcal{F}^{-1}$. By then assigning successive values to λ, the load $\mathbf{P}$ is applied in increments λ_i in such a way that for the j th increment, Eq. (41) is a function of $(\sum_i^j \lambda_i)\, \mathbf{P}^*$. The value $\bar{\lambda}$ of $\sum_i^j \lambda_i$ which satisfies Eq. (41) is the critical load parameter for the structure and $\bar{\lambda}\, \mathbf{P}^*$ is the vector of critical loads.

Material Nonlinearity. The static and kinematic conditions to be satisfied in a structural system depend upon the connectivity of the system but they are independent of the material properties of the system. Hence, if a structure is constructed of nonlinear materials, its topological graph leads to the same incidence matrices and static matrices $\mathbf{B}_o$ and $\mathbf{B}_x$ that are obtained for a linear structure with the same topological graph. The nonlinearity enters the problem only through the force-displacement relations, since, for a nonlinear material, the flexibilities are nonlinear functions of the node forces.

In the analysis of elasto-plastic and certain nonlinearly elastic structures, the force-displacement relations for a typical bar element can be written in the form

$$\mathbf{u} = \mathbf{f} \;\; \mathbf{p} + \mathbf{g}\,(p_{Ni}) \tag{42}$$

where $\mathbf{f}$ is the flexibility matrix given by the linear theory [Eq. (9)] and $\mathbf{g}\,(p_{Ni})$ is a 6 x 1 matrix whose elements are nonlinear functions of the generalized node forces p_{Ni}. A formula for the matrix $\mathbf{g}$ for a material exhibiting a Ramberg-Osgood type stress-strain law was given by Oden [31] and is not discussed here. After the appropriate transformations, a final system of nonlinear equations in the redundants is obtained of the form

$$B_x^T F\, B_o\;\; P + B_x^T F\, B_x X \;+ B_x^T\; G^{(P)} + B_x^T\;\; G^{(X)} = 0 \tag{43}$$

where $G\,(P)$ and $G\,(X)$ contain nonlinear functions of the applied forces and the redundants. Once these equations are solved, the internal forces are determined using Eq. (25), element displacements are calculated using Eq. (42), and node displacements are computed using Eq. (30).

Of the variety of methods available to solve systems of equations of the form in Eq. (43), one of the most expedient is a variation of the Newton-Raphson method [32]. In this method, the system of nonlinear equations is first written in the form

$$H\,(X) = 0 \tag{44}$$

which is then expanded in a Taylor's series about X^{o}, the solution to the linearized problem. Taking only two terms of this expansion, one finds

$$H\,(X) = \;H\,(X_o) + J_o(\,X^{(1)} - X^{o}) \tag{45}$$

where $X^{(1)}$ is the corrected solution and $J_o = [\,\frac{\partial H_i(X^o)}{\partial X_j}\,]$ is the Jacobian matrix corresponding to X^{o}. Equations (45) are now linear in $X^{(1)}$. These are solved for the corrected solution vector $X^{(1)}$ and the process is repeated until Eq. (44) is satisfied to a desired degree of accuracy. A general recurrence formula is obtained by solving Eq. (45) for the corrected solution vector of the j + 1 th cycle:

$$X^{j+1} = X^{j} - J_{j}^{-1} H(X^{j}) \tag{46}$$

CONCLUSIONS

Network topology provides efficient means to establish the kinematic and kinetic conditions to be satisfied in a structural problem. From purely topological properties of a given structural system, the fundamental static matrices B_o and B_x of the matrix force method can be automatically generated. In the case of large deformations, these matrices become functions of the displacements and can be generated through an iterative analysis procedure. In materially nonlinear structures, the static matrices are unaffected and nonlinearity is introduced only through the force-displacement relations of the structure.

LIST OF FIGURES

BIBLIOGRAPHY

1. Reed, M. B., Foundation for Electric Network Theory, Prentice-Hall Co., New Jersey, 1961.

2. Seshu, S. and Reed, M. S., Linear Graphs and Networks, Addison-Wesley Co., Massachusetts, 1962.

3. Kron, G., Tensor Analysis of Networks, John Wiley and Sons, New York, 1939.

4. Reza, F. M., "Some Topological Considerations in Network Theory," Transactions of the IRE in Circuit Theory, March 1958, pp. 30-42.

5. Doyle, T. C., "Topological and Dynamical Invariant Theory of an Electrical Network," Journal of Mathematics and Physics, Vol. 34, 1955, pp. 81-94.

6. Gould, R. L., "Application of Graph Theory to the Synthesis of Contact Networks," Proceedings of the International Symposium on Switching Circuits, Harvard University, April 1957.

7. Roth, J. P., "An Application of Algebraic Topology to Numerical Analysis: On the Existence of a Solution to the Network Problem," Proceedings of the National Academy of Science, Vol. 41, 1955, pp. 518-521.

8. Okada, S., "Topology Applied to Switching Circuits," Proceedings of the Polytechnic Institute of Brooklyn Symposium on Information Networks, Vol. 3, New York, 1954, pp. 267-290.

9. Kron, G., "A Method to Solve Very Large Physical Systems in Easy Stages," Proceedings, IRE, Vol. 42, 1954, pp. 680-686.

10. Kron, G., "A Set of Principles to Interconnect the Solutions of Physical Systems," Journal of Applied Physics, Vol. 24, 1953, pp. 965-980.

11. Koenig, H. E. and Blackwell, W. A., Electromechanical System Theory, McGraw-Hill Book Company, New York, 1961.

12. Kron, G., "Tensorial Analysis and Equivalent Circuits of Elastic Circuits," Journal of the Franklin Institute, Vol. 238, 1944, pp. 399-442.

13. Kron, G., "Solving Highly Complex Structures in Easy Stages," Journal of Applied Mechanics, ASME, Vol. 22, 1955, pp. 235-244.

14. Lind, N. C., "Analysis of Structures by System Theory," Journal of the Structures Division, ASCE, Vol. 88, ST-2, April 1962, pp. 1-22.

15. Henderson, J. C. de C. and Bickley, W. G., "Statical Indeterminancy of a Structure," Aircraft Engineering, Vol. 27, 1955, pp. 400-402.

16. Henderson, J. C. de C., "Topological Aspects of Structural Linear Analysis," Aircraft Engineering, Vol. 32, 1960, pp. 137-141.

17. Morice, P. B., Linear Structural Analysis, Thames and Hudson, London, 1959, pp. 48-59.

18. DiMaggio, F. L., "Statical Indeterminacy and Stability of Structures," Journal of the Structural Division, ASCE, Vol. 89, No. ST3, June 1963, pp. 63-75.

19. Samuelsson, A., "Linear Analysis of Frame Structures By Use of Algebraic Topology," Dissertation, Chalmer's University of Technology, Gothenburg, Sweden, 1962.

20. Roussopoulos, A. I., Theory of Elastic Complexes, Elsevier Publishing Co., Amsterdam, 1965.

21. Langefors, B., "Algebraic Topology for Elastic Network," TN-49, SAAB Aircraft Co., April 1961.

22. DiMaggio, F. L. and Spillers, W. R., "Network Analysis of Structures," Journal of the Engineering Mechanics Division, ASCE, Vol. 91, No. EM3, June 1965, pp. 169-188.

23. Spillers, W. R., "Network Analysis for the Truss Problem," Journal of the Engineering Mechanics Division, ASCE, Vol. 88, No. EM6, December 1962, pp. 33-40.

24. Spillers, W. R., "Applications of Topology in Structural Analysis," Journal of the Structural Division, ASCE, Vol. 89, No. ST4, August 1963, pp. 301-313.

25. Spillers, W. R., "Network Analogy for Linear Structures," Journal of the Engineering Mechanics Division, ASCE, Vol. 89, No. EM4, August 1963, pp. 21-29.

26. Wu, T. S., "Structural Analysis by System Theory," Developments in Theoretical and Applied Mechanics, Vol. 2, Proceedings of the Second Southeastern Conference on Theoretical and Applied Mechanics, Pergamon Press, London, 1965, pp. 605-628.

27. Fenves, S. J. and Branin, F. H., "Network-Topological Formulation of Structural Analysis," Journal of the Structural Division, ASCE, Vol. 89, No. ST4, August 1963, pp. 483-514.

28. Fenves, S. J., Logcher, R. D., and Mauch, S. P., Stress: A Reference Manual, The M.I.T. Press, Cambridge Mass., 1965.

29. Fenves, S. J., "Structural Analysis by Networks, Matrices, and Computers," Journal of the Structural Division, ASCE, Vol. 92, No. ST1, February 1966, pp. 199-221.

30. Argyris, J. H. and Kelsey, S., Energy Theorems and Structural Analysis, Butterworths, London, 1960.

31. Oden, J. T., "Analysis of Linear and Nonlinear Space Frames by the General

Conjugate Structure Analogy," Proceedings of the World Conference on Space Structures, London, 1966.

32. Schmidt, P. E., "Optimization, Constrained Optimization, and Nonlinear Programming of Complex Design Functions," Report M1-4-2-1, Phillips Petroleum Co., 1962.