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The Analysis of Structurally Orthotropic Shells

By Means of the Compliance Method

I. Introduction

The objective of this report is to state the six month progress on NASA Grant NGR 44-001-031, Supplement No. 2. The current grant is a renewal of NASA Grant NGR 44-001-031, Supplement No. 1, which in turn is a renewal of NGR 44-001-031. The current grant has four objectives

1. Continuation of experimental verification of compliance relations as applied to right circular cylindrical shells
2. A study and analysis of linear shell theory elastic constant instability conditions for symmetrically loaded shells of revolution
3. Generalization of linear shell theory elastic constant instability conditions in order that the condition be avoided
4. Non linear analysis for the determination of stable shell configurations when linear shell theory elastic constant instability is manifest.

At present, the greatest effort has been directed toward objective number two. However, all of the objectives have been acted upon in varying degrees of effort, though none have been brought to a state of completion. A detailed discussion of the results obtained for each of the listed objectives is given in the following portion of the report.

II. Discussion of Results

1. First Objective

A comparison of analytical and experimental results for a rib stiffened circular cylindrical panel has been given in the report on NGR-44-001-031, Supplement No. 1, December 22, 1967. Since that time, an error in the analytical results has been uncovered. The error is due to an incorrect statement of the boundary conditions. Hence the equations of the analysis are in the process of being reprogrammed with the expectation of better correlation between analytical and experimental data.

At the present time, experiments on a full cylindrical shell are in a conceptual stage. Until a good correlation is obtained between the analytical and experimental results for the stiffened panel, it is felt that full cylinder testing would be of no avail.

2. Second Objective

The problem of elastic constant instability had been first noted and reported in NGR 44-001-031, December 1, 1966. A development analagous to that given by Sanders for first order shell theory had been given for a displacement formulation of the plane anisotropic shell equations. In reducing the equations to the case of a circular cylindrical shell, a non-satisfaction of boundary conditions had been noted for certain combinations of elastic constants.

In order to more fully investigate this effect, a classical first order derivation for a symmetrically edge loaded, elastically plane anisotropic right circular cylindrical shell had been undertaken.

The analysis and its results are given in Appendix A.

As is noted in the derivation, eleven cases result in a non-satisfaction of the boundary conditions. However, though these cases are possible mathematically, a physical constraint exists which may limit the physically possible cases. This constraint is involved with the expression for the potential energy in that this quantity must be positive definite. Thus a number of inequality relations on the elastic constants will result which in turn may rule out from consideration some of the mathematically possible cases noted.

At the present time, a check is being completed on the strain energy function and the resulting relations for the elastic constants.

3. Third Objective

The general first order plane anisotropic shell equations have been developed. However an elastic constant stability analysis of these equations will not be attempted until the second objective is completed and the results indicate that mathematically and physically possible unstable states exist.

4. Fourth Objective

An engineering oriented non-linear analysis of shell equations is in the process of development. The term "engineering oriented" is utilized in that unlike general non-linear studies, the present study only considers first order non-linearties. That is, quadratic in displacements and/or their derivatives are retained but cubics and higher order products are neglected. In this way it is hoped that the equilibrium, compatibility and constitutive equations may be stated in terms of physically measurable quantities other than displacement components.

The derivations for the first quadratic form, the equilibrium and constitutive equation has been completed. However, considerable difficulty has been encountered in the development of the second quadratic form. The principal difficulty lies in defining the angle of rotation for an element. At the present time, the effort is being directed toward this definition.

Appendix A

Displacements and Elastic Constant Stability of a Plane Anisotropic Edge Loaded Semi-Infinite Cylinder

Nomenclature

a_{ij}	elastic constants
A, B	Lame' surface parameters
A_{ij}	elastic coefficients occurring in constitutive equations
e_{ij}	membrane strains
k_α, k_β	principal curvature of the undeformed reference surface
M_{ij}	bending and twisting moment stress resultant
R	radius of the shell reference surface
T_{ij}	membrane force stress resultant
u, v, w	displacement components in the directions α, β and γ
x, θ	principal coordinates in the longitudinal and circumferential direction of the cylinder
z	direction normal to shell surface
α, β	principal curvilinear coordinate curves
δ	shell thickness
$\kappa_\alpha, \kappa_\beta$	curvature change of the reference surface
τ	torsion of the reference surface

Introduction

An intrinsic plane anisotropic symmetry condition in the elastic constants is a relatively rare phenomena in shell structural materials. However with the advent of composites and laminates, materials exhibiting intrinsic orthotropic elastic constant symmetries are becoming increasingly important as primary structural materials in high strength-low weight applications.

Plane anisotropic elastic materials contain but one plane of elastic symmetry and hence one principal elastic axis. The number of independent elastic constants associated with such a material is thirteen. Orthotropic elastic materials contain three planes of elastic symmetry, three principal elastic axis and contain nine independent elastic constants. Relative to an orthogonal 1-2-3 coordinate system, the simplest mathematical form, or equivalentely, the canonical form of the plane anisotropic and orthotropic stress strain relations is obtained under the following conditions. In the first case, the principal elastic axis is coincident with one of the coordinate axis and in the second case when the three principal elastic axis constitute the coordinate axis. Assuming that the 3 axis is the principal elastic axis for the plane anisotropic material, the canonical form of the stress-strain equations for the two anisotropies is given as follows. For the plane anisotropic material

$$e_{11} = a_{1111}\sigma_{11} + a_{1112}\sigma_{12} + a_{1122}\sigma_{22} + a_{1133}\sigma_{33}$$
$$e_{12} = a_{1112}\sigma_{11} + a_{1212}\sigma_{12} + a_{1222}\sigma_{22} + a_{1233}\sigma_{33}$$
$$e_{13} = a_{1313}\sigma_{13} + a_{1323}\sigma_{23}$$
$$e_{22} = a_{1122}\sigma_{11} + a_{1222}\sigma_{12} + a_{2222}\sigma_{22} + a_{2233}\sigma_{33}$$
$$e_{23} = a_{1323}\sigma_{13} + a_{2323}\sigma_{23}$$
$$e_{33} = a_{1133}\sigma_{11} + a_{1233}\sigma_{12} + a_{2233}\sigma_{22} + a_{3333}\sigma_{33}$$

For the orthotropic material

$$e_{11} = a_{1111}\sigma_{11} + a_{1122}\sigma_{22} + a_{1133}\sigma_{33}$$

$$e_{12} = a_{1112}\sigma_{11} + a_{1222}\sigma_{22} + a_{1233}\sigma_{33}$$

$$e_{13} = a_{1313}\sigma_{13}$$

$$e_{22} = a_{1122}\sigma_{11} + a_{2222}\sigma_{22} + a_{2233}\sigma_{33}$$

$$e_{23} = a_{2323}\sigma_{23}$$

$$e_{33} = a_{1133}\sigma_{11} + a_{2233}\sigma_{22} + a_{3333}\sigma_{33}$$

Since the stresses may be considered independent variables, the mathematical characteristics of two sets of canonical stress-strain relations fundementally are different from each other. Whereas the canonical orthotropic formulation is identical mathematically to that which would be encountered in isotropic elasticity, the canonical plane anisotropic formulation contains coupling effects not found in the canonical orthotropic equations. In particular, the normal strains in the canonical plane anisotropic formulation are functions not only of normal stresses but also of shear stresses and visa-versa.

Thin shell theory is based on three independent formulations. The equilibrium equations, the compatibility equations and the constitutive equations. The constitutive equations relate the kinetics of the shell reference surface to its kinematics. As a consequence, this formulation is the only one which utilizes the stress-strain relations.

As mentioned, the canonical stress-strain relations for materials possessing orthotropic and plane anisotropic elastic properties fundamentally differ from each other in mathematical form. It would then be reasonable to expect that the corresponding constitutive shell equations for the two materials would also exhibit this difference. That is, the constitutive equations for plane anisotropic materials would be expected to contain reference surface shear force stress resultants and twisting moments in the expressions for the reference surface normal strains and curvature changes.

Given a shell geometry, loading and boundary conditions, the equilibrium and compatibility equations remain identical in form whatever the material properties may be. However the constitutive equations are a necessary part of the formulation. Hence if they differ in mathematical form from one material to another, the corresponding resultant shell equations will reflect this difference. From the discussion of the previous paragraph, it is obvious then that the resulting shell equations when developed from the canonical form of the plane anisotropic stress-strain relations will differ in mathematical form from those developed from either the isotropic or canonical orthotropic stress-strain relations.

Shell coordinization is based on the principal curvilinear curves of the undeformed reference surface and the normal direction to that surface. In the use of orthotropic materials such as composites or laminates in shell construction, the orientations of the principal elastic axis of the material very often are skewed relative to the shell coordinates. This condition is especially true if the shell is of the layered type in which case the skewing may vary from layer to layer. However the physical characteristics of the orthotropic layer or layers is such that when formed into a shell structure, one principal elastic axis remains normal to the shell reference surface.

Unlike isotropic elasticity, anisotropic elastic constants vary in magnitude with coordinate transformation. The transformation law for elastic constants from one coordinization to another is given as

$$a'_{ijkl} = l_{im}\, l_{jn}\, l_{kp}\, l_{lq}\, a_{mnpq}$$

The quantities l_{ij} are the direction cosines of the i^{th} new axis with respect to the j^{th} original axis. It is obvious then that given an orthotropic material, the canonical form previously given for the stress-strain relations is applicable only for a unique coordinization, namely one in which the coordinate axis coincide with the principal elastic axis. Also, it is equally obvious that given a general coordinization, the mathematical form of the orthotropic stress-strain relations becomes undistinguishable from that encountered in general anisotropy. The latter case, where no elastic symmetries are present, contains twenty one independent elastic constants whereas the former still contains only nine independent elastic constants.

From the discussion of the preceeding paragraph it is apparent that the canonical form of the orthotropic stress-strain relations cannot be utilized when dealing with the skewed orthotropic shell layers. Since the reference surface principal curvilinear curves and normal form the coordinate system, then the stress-strain relations of the material must be expressed relative to that coordinate system. Thus a coordinate transformation of the canonical orthotropic stress-strain equations must be effected.

That the principal elastic axis normal to the shell surface coincides with a coordinate axis allows a relatively simple transformation. The coordinate transformation will consist of a rotation about this elastic axis through an angle equal to the skew value. Thus the new coordinization for the elastic constants will be the same as that for the shell. The transformed stress-strain relations will be identical in mathematical form to the canonical plane anisotropic stress-strain equations previously given. However, these transformed orthotropic stress-strain relations still will contain only nine independent elastic constants.

The mathematical equivalence of the canonical form of the plane anisotropic and the transformed orthotropic stress-strain relations is extremely important. If the shell is single layered, then it follows directly that the constitutive equations will be identical mathematically to those obtained from the canonical form of the plane anisotropic shell equations. In the case of the multi-layered shell wherein each of the layers is orthotropic but the principal elastic axis are skewed different amounts in different layers, the problem is more complex. However even in this case it can be shown that the appropriate constitutive equations may be derived from the canonical form of the stress-strain equation of a hypothesized single layered plane anisotropic continuum.

The discussion of the preceeding paragraphs allows the following important conclusion to be drawn. The mathematical form of the constitutive equations derived from the canonical form of the plane anisotropic stress-strain relations is applicable to not only materials possessing intrinsic elastic plane anisotropy but also to materials possessing elastic orthotropy and whose principal elastic axis are skewed to the principal geometric directions. Composites, laminates and fiber windings form a relatively important class of structural shell materials. Since these materials in many applications fit the latter category, then the importance of the analysis of the plane anisotropic constitutive equations and hence the plane anisotropic shell equations is self evident.

At the present time, very few analytical solutions for plane anisotropic materials have appeared in the literature. Ambartsumian (1) discusses plane anisotropy and even formulates constitutive shell equations for plane anisotropic layered materials. However his shell equation solutions are applicable only to materials exhibiting the canonical form of the orthotropic stress-strain relations relative to the shell coordinate system. A perturbation analysis for solving plane anisotropic shells of revolution has been developed by Dong (2). Though the method is applicable for general plane anisotropies, the slow convergence of the perturbation series relegates the method to the treatment of mild plane anisotropies.

The difficulty in solving the plane anisotropic shell equations occurs because of the inherent coupling effect which exists in the shell constitutive equations between the membrane shearing and axial deformations and between the reference surface bending and twisting deformations. Unlike the constitutive equations of either isotropic or canonical orthotropic elasticity, this coupling effect prevents the separation of reference surface twisting and shearing from bending and stretching. Thus the constitutive equations and hence the resulting shell equations for elastic plane anisotropic materials fundamentally are different from their isotropic or canonically orthotropic counterparts. As a consequence, the well developed techniques and symmetry conditions associated with the solution of isotropic and canonically orthotropic shell problems cannot be extended to the analysis of plane anisotropic materials.

The paper is concerned with the first order linear analysis and solution of a semi-infinite right circular cylindrical thin elastic shell. The material properties are assumed to be canonically plane anisotropic relative to the shell coordinate system and the loading is assumed to consist of an axially symmetric edge bending moment and transverse shear.

The result of the analysis is a fourth order ordinary linear homogeneous differential equation defining the normal displacement of the shell reference surface. Hence the solution of the equation poses no difficulty in that it may be found by elementary mathematical techniques.

The problem solution is interesting because of its simplicity. Thus it illustrates several features peculiar to plane anisotropic shell analysis and which frequently are obscured by more complex shell problems. First, the derivation of the shell equations does illustrate that plane anisotropy and the resulting constitutive equations destroy axial symmetry even though the loading and shell geometry may exhibit this symmetry. Secondly, the fact that the solution is of a relatively simple form allows a qualitative and quantitative characterization of plane anisotropy as weak, mild and strong. As is shown the character of the solution and hence the mode shape of the normal displacement changes with each of the degrees of plane anisotropy. Thirdly, it illustrates a type of instability peculiar only to materials whose stress-strain equations are stated in plane anisotropic canonical form. Instability, as referred to in this paper, is defined as that condition wherein the normal displacement as determined from linear theory tends to infinity.

The instability uncovered in the analysis is solely a function of the elastic constants and is independent of the external loading. Hence in nature it differs from the elastic instability encountered in either isotropic or canonically orthotropic shell analysis. In order to differentiate the two instabilities, the former will be termed elastic constant instability.

Though the elastic constant instability is a function of all of the elastic constants, one particular set of elastic constants must be present if the instability condition is to occur. These constants are the coupling constants, that is, those constants which differentiate the canonical form of the plane anisotropic stress-strain relations from the canonical form of the orthotropic stress-strain relations. Hence as previously stated, the elastic constant instability is peculiar only to shells whose stress-strain equations are in canonical plane anisotropic mathematical form.

Because of the many possible numerical values that anisotropic elastic constants may possess, algebraic rather than numerical solutions are presented. In order to verify that the elastic constant stability is a possible physical condition, a single layered skewed orthotropic material is used to generate the form of the plane anisotropic stress-strain relations. The result is that instability always occurs at a skew angle of 45°. Further, the instability is independent of the magnitude of the elastic constants appearing in the cannonical form of the orthotropic stress-strain equations.

Problem Solution

A. General Right Circular Cylindrical Shell Equations

The reference surface of a right circular cylindrical shell of constant thickness $\mathcal{S}$ and mean line radius $\mathcal{R}$ will be chosen to be the mean surface. This surface is positioned midway between the upper and lower bounding surfaces of the cylindrical shell. Hence the reference surface also is of right circular cylindrical geometry and is of radius R.

The principal curvilinear coordinate curves of the undeformed reference surface are in the longitudinal and circumferential directions of the surface. These curves may be parameterized by the variables x and θ, where x is a linear measure in the longitudinal direction and θ is an angular measure in the circumferential direction. The coordinate z is measured normal to the undeformed surface and is assumed positive when acting in the convex direction to the surface. The positive directions of the x and θ coordinate curves are such that unit vectors tangent to these coordinate curves and directed toward increasing coordinate values together with a unit vector normal to the surface and acting in the positive z direction form a right handed orthonormal triad of vectors. The reference surface and its coordinization is shown in Fig. 1.

Relative to the coordinization described, the linear equilibrium equations may be stated in stress resultant form. Utilizing the positive convention for force and moment stress resultants as given by Novozhilov (3), the following equations result.

(1)

$$\frac{\partial T_{xx}}{\partial x} + \frac{1}{R}\frac{\partial T_{\theta x}}{\partial \theta} + q_x = 0$$

$$\frac{1}{R}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{x\theta}}{\partial x} + \frac{1}{R} T_{\theta z} + q_\theta = 0$$

$$\frac{\partial T_{xz}}{\partial x} + \frac{1}{R}\frac{\partial T_{\theta z}}{\partial \theta} - \frac{1}{R} T_{\theta\theta} + q_z = 0$$

$$\frac{1}{R}\frac{\partial M_{\theta\theta}}{\partial\theta} + \frac{\partial M_{x\theta}}{\partial x} - T_{\theta x} = 0$$

$$T_{x\theta} - T_{\theta x} - \frac{1}{R} M_{\theta x} = 0$$

Let u, v, w represent the reference surface displacement vector components. The components and respectively will be on the direction of the positive tangent vectors to the and coordinate curves. The component will be normal to the surface and positive in the positive direction. The reference surface deformations, that is, strains, curvature changes and torsion, are expressible in terms of the displacement components as the following.

$$e_{xx} = \frac{\partial u}{\partial x} \qquad K_x = -\frac{\partial^2 w}{\partial x^2} \tag{2}$$

$$e_{x\theta} = \frac{1}{R}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial x} \qquad \Upsilon = -\frac{1}{R}\frac{\partial^2 w}{\partial x\,\partial\theta} + \frac{1}{R}\frac{\partial v}{\partial x}$$

$$e_{\theta\theta} = \frac{1}{R}\frac{\partial v}{\partial\theta} + \frac{w}{R} \qquad K_\theta = -\frac{1}{R^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{R^2}\frac{\partial v}{\partial\theta}$$

The compatibility relations for the reference surface deformations are given as the following set of equations.

$$\frac{\partial K_\theta}{\partial x} - \frac{1}{R}\frac{\partial \Upsilon}{\partial\theta} = 0 \tag{3}$$

$$\frac{\partial K_x}{\partial\theta} - R\frac{\partial \Upsilon}{\partial x} + \frac{\partial e_{x\theta}}{\partial x} - \frac{1}{R}\frac{\partial e_{xx}}{\partial\theta} = 0$$

$$\frac{1}{R}K_x + \frac{\partial^2 e_{\theta\theta}}{\partial x^2} - \frac{1}{R}\frac{\partial^2 e_{x\theta}}{\partial x\,\partial\theta} + \frac{1}{R^2}\frac{\partial^2 e_{xx}}{\partial\theta^2} = 0$$

The relation between the stress resultants and the stress distribution through the shell thickness is stated in integral form. These relations are the following.

$$T_{xx} = \int_{-\delta/2}^{\delta/2} \sigma_{xx}\left(1+\frac{z}{R}\right)dz \qquad M_{xx} = \int_{-\delta/2}^{\delta/2} \sigma_{xx}\, z\left(1+\frac{z}{R}\right)dz$$

$$T_{x\theta} = \int_{-\delta/2}^{\delta/2} \sigma_{x\theta}\left(1+\frac{z}{R}\right)dz \qquad M_{x\theta} = \int_{-\delta/2}^{\delta/2} \sigma_{x\theta}\, z\left(1+\frac{z}{R}\right)dz$$

$$T_{\theta x} = \int_{-\delta/2}^{\delta/2} \sigma_{x\theta}\, dz \qquad M_{\theta x} = \int_{-\delta/2}^{\delta/2} \sigma_{x\theta}\, z\, dz$$

$$T_{\theta\theta} = \int_{-\delta/2}^{\delta/2} \sigma_{\theta\theta}\, dz \qquad M_{\theta\theta} = \int_{-\delta/2}^{\delta/2} \sigma_{\theta\theta}\, z\, dz$$

Up to this point, the relations stated are linearly exact. However the definition of the shell problem is not complete in that the constitutive equations have not been given. In order to state these relations, it must be possible to integrate the stress resultant stress equations which implies that the stress distribution through the shell thickness is known.

The Kirchhoff hypotheses define the displacement variation and hence also the strain variation through the shell thickness. However these variations are hypothesized rather than being fact and hence lead to analysis errors. For a cylindrical shell of the type discussed, the error at least is of order (δ/R).

The simplest form of the strain distribution through the shell thickness and yet which contains the maximum accuracy allowable from the Kirchhoff hypotheses is one which is a linear function of the coordinate z . Thus the strains $e_{xx}, e_{\theta\theta}, e_{x\theta}$, at points away from the reference surface may be expressed as functions of the reference surface deformations and the coordinate z . These relations are given as the following.

$$e_{xx}(z) = e_{xx} + z\, K_x$$

$$e_{\theta\theta}(z) = e_{\theta\theta} + z\, K_\theta$$

$$e_{x\theta}(z) = e_{x\theta} + 2z\, \Upsilon$$

If the strain variation in the shell is known, then the stress variation can be calculated from the appropriate stress-strain relations. For the problem being considered, the shell material is assumed to exhibit the canonical form of the plane anisotropic stress-strain relations relative to the coordinate axis chosen. The direction is assumed to be the direction of a principal elastic axis. Thus the stress-strain relations are the following.

$$e_{xx} = a_{1111}\sigma_{xx} + a_{1112}\sigma_{x\theta} + a_{1122}\sigma_{\theta\theta} + a_{1133}\sigma_{zz}$$
$$e_{x\theta} = a_{1112}\sigma_{xx} + a_{1212}\sigma_{x\theta} + a_{1222}\sigma_{\theta\theta} + a_{1233}\sigma_{zz}$$
$$e_{xz} = a_{1313}\sigma_{13} + a_{1323}\sigma_{\theta z}$$
$$e_{\theta\theta} = a_{1122}\sigma_{xx} + a_{1222}\sigma_{x\theta} + a_{2222}\sigma_{\theta\theta} + a_{2233}\sigma_{zz}$$
$$e_{\theta z} = a_{1323}\sigma_{xz} + a_{2323}\sigma_{\theta z}$$
$$e_{zz} = a_{1133}\sigma_{xx} + a_{1233}\sigma_{x\theta} + a_{2233}\sigma_{\theta\theta} + a_{3333}\sigma_{zz}$$

The four subscript system for the elastic constants is a necessary one if the stress-strain laws and the elastic constant transformation laws are to be presented in tensorial form. However structural shell applications where often analyses are based on contradictory hypotheses negate or at least minimize the use of tensorial representations. Hence for the sake of convenience in presentation, a double subscript notation is adopted for the elastic constants. Such in fact will be the situation in this paper. Thus when rewritten in a double subscript notation, the canonical form of the plane anisotropic stress-strain law is the following.

$$e_{xx} = a_{11}\sigma_{xx} + a_{12}\sigma_{x\theta} + a_{14}\sigma_{\theta\theta} + a_{16}\sigma_{zz}$$
$$e_{x\theta} = a_{12}\sigma_{xx} + a_{22}\sigma_{x\theta} + a_{24}\sigma_{\theta\theta} + a_{26}\sigma_{zz}$$

$$e_{xz} = a_{33}\,\sigma_{xz} + a_{35}\,\sigma_{\theta z}$$
$$e_{\theta\theta} = a_{14}\,\sigma_{xx} + a_{24}\,\sigma_{x\theta} + a_{44}\,\sigma_{\theta\theta} + a_{46}\,\sigma_{zz}$$
$$e_{\theta z} = a_{35}\,\sigma_{xz} + a_{55}\,\sigma_{\theta z}$$
$$e_{zz} = a_{16}\,\sigma_{xx} + a_{26}\,\sigma_{x\theta} + a_{46}\,\sigma_{\theta\theta} + a_{66}\,\sigma_{zz}$$

If the strain variation has been determined from the Kirchhoff hypotheses, then certain consistent simplifications and omissions must be introduced into stress-strain relations in order that the stress distribution be applicable to the analysis. To begin with, the Kirchhoff hypotheses neglect normal stresses and normal strains normal to the reference surface. Hence the stresses, σ_{zz}, and the strains, e_{zz}, must be set to zero. Setting these two quantities equal to zero at each point in the material leads to a simultaneous condition of curvilinear plane stress and plane strain. However Kozik (4) has shown that the stress resultant formulation of the linear thin elastic shell equations is insensitive within the Kirchhoff error bounds to this contradiction and further that for such an analysis the stress-strain equation for the normal strain e_{zz} should be suppressed.

Secondly, the Kirchhoff hypotheses neglect the effects of transverse shear strains. However this neglect is an implicit rather than explicit result of the hypotheses. Hence rather than setting the transverse shear strain expressions equal to zero, the stress-strain equations relating these quantities to the corresponding transverse shear stresses are suppressed. Thus the stress-strain relations appropriate to an analysis based on the Kirchhoff hypotheses are the following.

$$
\begin{aligned}
e_{xx} &= a_{11}\sigma_{xx} + a_{12}\sigma_{x\theta} + a_{14}\sigma_{\theta\theta} \\
e_{x\theta} &= a_{12}\sigma_{xx} + a_{22}\sigma_{x\theta} + a_{24}\sigma_{\theta\theta} \\
e_{\theta\theta} &= a_{14}\sigma_{xx} + a_{24}\sigma_{x\theta} + a_{44}\sigma_{\theta\theta}
\end{aligned}
\tag{4}
$$

As is noted, the number of independent elastic constants appearing in the pertinent stress-strain relations is six rather than the thirteen encountered in general plane anisotropy.

The stresses may now be solved in terms of the strains. The resulting relations are the following equations.

$$
\begin{aligned}
\sigma_{xx} &= A_{11} e_{xx} + A_{12} e_{x\theta} + A_{14} e_{\theta\theta} \\
\sigma_{x\theta} &= A_{12} e_{xx} + A_{22} e_{x\theta} + A_{24} e_{\theta\theta} \\
\sigma_{\theta\theta} &= A_{14} e_{xx} + A_{24} e_{x\theta} + A_{44} e_{\theta\theta}
\end{aligned}
\tag{5}
$$

In these expressions, the constants A_{ij} are defined in terms of the elastic constants. Thus

$$
\begin{aligned}
A_{11} &= (a_{22}a_{44} - a_{24}^2)/\Delta & A_{22} &= (a_{11}a_{44} - a_{14}^2)/\Delta \\
A_{12} &= (a_{14}a_{24} - a_{12}a_{44})/\Delta & A_{24} &= (a_{12}a_{14} - a_{11}a_{24})/\Delta \\
A_{14} &= (a_{12}a_{24} - a_{14}a_{22})/\Delta & A_{44} &= (a_{11}a_{22} - a_{12}^2)/\Delta
\end{aligned}
\tag{6}
$$

and

$$
\Delta = a_{11}a_{22}a_{44} + 2a_{12}a_{14}a_{24} - a_{12}^2 a_{44} - a_{11}a_{24}^2 - a_{14}^2 a_{22}
$$

Knowing the strain variation, the stress variation is determined and hence the stress resultant-stress equations integrated. The constitutive equations now can be derived. In stating these equations, again use is made of the Kirchhoff minimum error bound of δ/R . Thus the simplest formulation of the constitutive equations, and yet one which preserves the maximum possible accuracy, consists of the following relations.

$$
\begin{aligned}
T_{xx} &= (A_{11} e_{xx} + A_{12} e_{x\theta} + A_{14} e_{\theta\theta})\,\delta\,; \\
T_{x\theta} = T_{\theta x} &= (A_{12} e_{xx} + A_{22} e_{x\theta} + A_{24} e_{\theta\theta})\,\delta\,; \\
T_{\theta\theta} &= (A_{14} e_{xx} + A_{24} e_{x\theta} + A_{44} e_{\theta\theta})\,\delta\,; \\
M_{xx} &= (A_{11} K_x + 2 A_{12} \Upsilon + A_{14} K_\theta)\,\delta^3/12 \\
M_{x\theta} = M_{\theta x} &= (A_{12} K_x + 2 A_{22} \Upsilon + A_{24} K_\theta)\,\delta^3/12 \\
M_{\theta\theta} &= (A_{14} K_x + 2 A_{24} \Upsilon + A_{44} K_\theta)\,\delta^3/12
\end{aligned}
\tag{7}
$$

The boundary conditions also are effected by the Kirchhoff hypotheses. Analyses based on these assumptions can not account for all the possible stress resultants acting on a free boundary of a shell. Hence equivalent shear stress resultants must be defined on the boundary and further, the explicit statement of the twisting moment stress resultant be suppressed. The effective shear stress resultants acting on free edges coinciding with the coordinate curves are the following.

$x = \text{constant}$ $\qquad\qquad$ $\theta = \text{constant}$

$$T_{xz}(eff) = T_{xz} + \frac{1}{R}\frac{\partial M_{x\theta}}{\partial \theta}, \quad T_{\theta z}(eff) = T_{\theta z} + \frac{\partial M_{x\theta}}{\partial x} \tag{8}$$

$$T_{x\theta}(eff) = T_{x\theta} + \frac{1}{R} M_{x\theta}, \quad T_{\theta x}(eff) = T_{x\theta}$$

B. Equation Simplification

Assume that the cylindrical shell is infinitely long and is subjected to a uniform distribution of edge bending moment, M_0, and transverse shear, Q_0, as shown in Fig. 2. The external surface loading is assumed to be non-existent. Equations 1 - 8 given in the previous section and which define the pertinent cylindrical shell equations assume a simpler form.

In simplifying these equations note must be taken of the fact that the geometry and loading are axially symmetrical with respect to the axis of the cylinder. Hence it appears reasonable to assume that the shell dependent variables will be independent of the coordinate θ. However, further simplification is not possible.

The constitutive equations for the given shell contain coupling effects between the membrane axial force stress resultants and the membrane shear strains. Further, the coupling effect also exists between the membrane bending moment and the reference surface torsion. Thus unlike an isotropic or canonically orthotropic analysis, given an axial force or bending moment stress resultant, it follows that there will also be a membrane shear force stress resultant and a twisting moment stress resultant. Hence in eneral, it must also be assumed that there will be correspondingly a membrane shear strain and a reference surface torsion. It also follows that there will exist a tangential displacement, v, in the θ direction.

The simplified shell equations take the following form. Since the variables are only a function of X, then ordinary derivatives may be used.

Equilibrium

$$\frac{dT_{xx}}{dx} = 0 \tag{9}$$

$$\frac{dT_{x\theta}}{dx} + \frac{1}{R} T_{\theta z} = 0$$

$$\frac{dT_{xz}}{dx} - \frac{1}{R} T_{\theta\theta} = 0$$

$$\frac{dM_{x\theta}}{dx} - T_{\theta z} = 0$$

$$\frac{dM_{xx}}{dx} - T_{xz} = 0$$

Displacement-deformation

$$e_{xx} = \frac{du}{dx} \qquad K_x = -\frac{d^2 w}{dx^2} \tag{10}$$

$$e_{x\theta} = \frac{dv}{dx} \qquad \Upsilon = \frac{1}{R}\frac{dv}{dx}$$

$$e_{\theta\theta} = \frac{w}{R} \qquad K_\theta = 0$$

Compatibility

$$-R\frac{d\Upsilon}{dX} + \frac{de_{x\theta}}{dX} = 0 \tag{11}$$

$$\frac{1}{R} K_x + \frac{d^2 e_{\theta\theta}}{dX^2} = 0$$

Constitutive

$$T_{xx} = (A_{11} e_{xx} + A_{12} e_{x\theta} + A_{14} e_{\theta\theta})\delta \qquad (12)$$

$$T_{x\theta} = (A_{12} e_{xx} + A_{22} e_{x\theta} + A_{24} e_{\theta\theta})\delta$$

$$T_{\theta\theta} = (A_{14} e_{xx} + A_{24} e_{x\theta} + A_{44} e_{\theta\theta})\delta$$

$$M_{xx} = (A_{11} K_x + 2 A_{12} \tau)\, \delta^3/12$$

$$M_{x\theta} = (A_{12} K_x + 2 A_{22} \tau)\, \delta^3/12$$

$$M_{\theta\theta} = (A_{14} K_x + 2 A_{24} \tau)\, \delta^3/12$$

Boundary Conditions

$$T_{xx} = 0 \qquad (13)$$

$$M_{xx} = M_0 \quad ; \quad x = 0$$

$$T_{xz} = Q_0$$

$$T_{x\theta} + \frac{1}{R} M_{x\theta} = 0$$

$$u, v, w \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty$$

C. Resultant Differential Equation

Ultimately the resultant shell equations will be transformed into displacement form. The compatibility equations are assumed to be satisfied identically in displacement form and hence these equations need not be considered in the analysis.

The constitutive equations when expressed in stress resultant displacement form are as given in the following set of relations.

$$T_{xx} = \left(A_{11} \frac{du}{dx} + A_{12} \frac{dv}{dx} + \frac{A_{14}}{R} w\right)\delta$$

$$T_{x\theta} = \left(A_{12} \frac{du}{dx} + A_{22} \frac{dv}{dx} + \frac{A_{24}}{R} w\right)\delta$$

$$T_{\theta\theta} = \left(A_{14}\frac{du}{dx} + A_{24}\frac{dv}{dx} + \frac{A_{44}}{R}w\right)\delta$$

$$M_{xx} = -\left(A_{11}\frac{d^2w}{dx^2} - \frac{2A_{12}}{R}\frac{dv}{dx}\right)\delta^3/12$$

$$M_{x\theta} = -\left(A_{12}\frac{d^2w}{dx^2} - \frac{2A_{22}}{R}\frac{dv}{dx}\right)\delta^3/12$$

$$M_{\theta\theta} = -\left(A_{14}\frac{d^2w}{dx^2} - \frac{2A_{24}}{R}\frac{dv}{dx}\right)\delta^3/12$$

In order that the stress resultant equations be transformed to displacement form, the transverse shear stress resultants must be eliminated from the equations. Thus from the last two equilibrium equations,

$$T_{xz} = \frac{dM_{xx}}{dx} \qquad T_{\theta z} = \frac{dM_{x\theta}}{dx}$$

Substituting these relations into the second and third equilibrium equations, these two equations and the first equation become the following.

$$\frac{d}{dx}(T_{xx}) = 0$$

$$\frac{d}{dx}\left(T_{x\theta} + \frac{1}{R}M_{x\theta}\right) = 0$$

$$\frac{d^2M_{xx}}{dx^2} - \frac{1}{R}T_{\theta\theta} = 0$$

The first two equations can be integrated directly. Further since the integrated equations are applicable for any value of x, they are also applicable at the boundary. However the integrated equations are equal identically to two of the boundary conditions. Thus the above three equations may now be written as

$$T_{xx} = 0$$

$$T_{x\theta} + \frac{1}{R} M_{x\theta} = 0$$

$$\frac{d^2 M_{xx}}{dx^2} - \frac{1}{R} T_{\theta\theta} = 0$$

Note that two of the boundary conditions are identically satisfied.

$$T_{xz} = \frac{dM_{xx}}{dx} \quad ; \quad T_{\theta z} = \frac{dM_{x\theta}}{dx}$$

Substituting these relations into the second and third equilibrium equations, these two equations together with the first equilibrium equation yield the following.

$$\frac{d}{dx}(T_{xx}) = 0$$

$$\frac{d}{dx}\left(T_{x\theta} + \frac{1}{R} M_{x\theta}\right) = 0$$

$$\frac{d^2 M_{xx}}{dx^2} - \frac{1}{R} T_{\theta\theta} = 0$$

The first of these two equations can be integrated directly. Further, the constants of integration can be evaluated directly since the integrals of the first two equations are equal identically to two of the boundary conditions. Thus the preceeding three equilibrium equations may now be written as,

$$T_{xx} = 0$$

$$T_{x\theta} + \frac{1}{R} M_{x\theta} = 0$$

$$\frac{d^2 M_{xx}}{dx^2} - \frac{1}{R} T_{\theta\theta} = 0$$

Note that the first two equations guarantee identical satisfaction of two of the boundary conditions.

The stress resultant-displacement form of the constitutive equations when substituted into the three equilibrium equations transforms these latter equations into displacement form. Substituting and simplyfying by neglecting terms of order δ^2/R^2 in comparison to unity,

$$A_{11}\frac{du}{dx} + A_{12}\frac{dv}{dx} + \frac{A_{14}}{R} w = 0$$

$$A_{12}\frac{du}{dx} + A_{22}\frac{dv}{dx} + \frac{A_{24}}{R} w - A_{12}\frac{\delta^2}{12R}\frac{d^2w}{dx^2} = 0$$

$$-A_{11}\frac{\delta^2}{12}\frac{d^4w}{dx^4} - \frac{A_{44}}{R^2}w + \frac{A_{12}\delta^2}{6R}\frac{d^3v}{dx^3} - \frac{A_{24}}{R}\frac{dv}{dx} - \frac{A_{14}}{R}\frac{du}{dx} = 0$$

The first two of the equations may be solved for the derivatives du/dx and dv/dx as functions of w. Thus the result becomes

$$\frac{du}{dx} = -\frac{(A_{14}A_{22} - A_{12}A_{24})}{(A_{11}A_{22} - A_{12}^2)}\frac{w}{R} - \frac{A_{12}^2}{12(A_{11}A_{22} - A_{12}^2)}\frac{\delta^2}{R}\frac{d^2w}{dx^2} \tag{14}$$

$$\frac{dv}{dx} = -\frac{(A_{11}A_{24} - A_{12}A_{14})}{(A_{11}A_{22} - A_{12}^2)}\frac{w}{R} + \frac{A_{11}A_{12}}{(A_{11}A_{22} - A_{12}^2)}\frac{\delta^2}{12R}\frac{d^2w}{dx^2} \tag{15}$$

Equations (14) and (15) when substituted into the third equilibrium equation transform that equation into one containing only the variable w. Hence the third equilibrium equation, when simplified, is given as the following.

$$A_{11}\frac{\delta^2}{12}\frac{d^4w}{dx^4}+\frac{3A_{12}(A_{11}A_{24}-A_{12}A_{14})}{(A_{11}A_{22}-A_{12}^2)}\frac{\delta^2}{12R^2}\frac{d^2w}{dx^2}+ \tag{16}$$

$$\left[A_{44}-\frac{A_{24}(A_{11}A_{24}-A_{12}A_{14})}{(A_{11}A_{22}-A_{12}^2)}-\frac{A_{14}(A_{14}A_{22}-A_{12}A_{24})}{(A_{11}A_{22}-A_{12}^2)}\right]\frac{w}{R^2}=0$$

Equations (14), (15) and (16) define the displacement components of the shell. However in order to determine the tangential displacements, the normal displacement w must be known. Hence the solution of equation (16) ultimately defines the displacement solution of the shell equations.

The equation defining the normal displacement w, equation (16), is of relatively simple form. That is, it is a fourth order linear homogeneous differential equation with constant coefficients. In this respect the equation is similar in form to the equation given by Timoshenko (5) for an isotropic shell of similar geometry and loading. However unlike the isotropic analysis, the present equation contains a second derivative of the displacement w.

If the constitutive equations were reduced to their corresponding isotropic form, the coupling constants, A_{12} and A_{24}, would assume zero values. Equation (14) would simplify by the omission of the differential term, equation (15) would yield zero value for the displacement v, and equation (16) would simplify by the omission of the second derivative term. In particular, the equations would degenerate to the isotropic form given by Timoshenko. Thus the net result of plane anisotropy on the present problem is the inclusion of differential terms in the equation for the displacement u and w and the existence of the tangential displacement v.

In order to effect a solution for the displacement components, the boundary conditions must be stated in displacement form. However, since two of the boundary conditions are identically satisfied from the formulation of the displacement equations, only three boundary conditions are in existence. Further, one of the boundary conditions already is stated in displacement form. Hence only two boundary conditions need be transformed.

Substituting the constitutive stress resultant-displacement equations into the boundary conditions, the result becomes the following.

$$M_0 = \left(-A_{11}\frac{d^2w}{dx^2} + 2\frac{A_{12}}{R}\frac{dv}{dx}\right)\frac{\delta^3}{12}$$

$$(x = 0)$$

$$Q_0 = \left(-A_{11}\frac{d^3w}{dx^3} + 2\frac{A_{12}}{R}\frac{d^2v}{dx^2}\right)\frac{\delta^3}{12}$$

However these two equations can also be transformed into equations containing only the normal displacement w. Thus substituting equation (15) and its derivative into the transformed boundary equations, the final set of boundary conditions on the displacement components, u, v, w, become the following.

$$\frac{12 M_o}{\delta^3} = - A_{11} \frac{d^2 w}{dx^2} - \frac{2 A_{12} (A_{11} A_{24} - A_{12} A_{14})}{(A_{11} A_{22} - A_{12}^2)} \frac{w}{R^2}$$

(17)

$$\frac{12 Q_o}{\delta^3} = - A_{11} \frac{d^3 w}{dx^3} - \frac{2 A_{12} (A_{11} A_{24} - A_{12} A_{14})}{(A_{11} A_{22} - A_{12}^2)} \frac{w}{R^2} \qquad (x = 0)$$

$$u, v, w \rightarrow 0, \quad x \rightarrow \infty$$

D. Equation Solution

It has been shown in the previous section that the solution for reference surface shell displacement components is solely dependent on the solution for the normal displacement component w. Hence the nature of solution for this displacement component also will define the nature of the remaining displacement components. Thus this section will be concerned solely with the solution of equation (16) and its boundary conditions (17).

One feature of the study will be the determination of the effects of the elastic constants, a_{ij}, which appear in the stress-strain equations. The constants, A_{ij}, appearing in the differential equation and boundary conditions are the effective elastic constants given in the constitutive equations. However equation (6) defines the relation between the two sets of constants. Thus it is possible to transform equations (16) and (17) into a form containing the stress-strain elastic constants. The resulting transformed equations are the following.

$$\frac{d^4 w}{dx^4} - \frac{3 a_{22}}{a_{44}} \frac{(a_{14} a_{24} - a_{12} a_{44})}{(a_{22} a_{44} - a_{24}^2)} \frac{1}{R^2} \frac{d^2 w}{dx^2} + \frac{12 \Delta}{a_{44} (a_{22} a_{44} - a_{24}^2)} \frac{w}{\delta^2 R^2} = 0 \qquad (18)$$

$$\frac{12\Delta}{(a_{22}a_{44}-a_{24}^2)}\frac{M_o}{\delta^3} = -\frac{d^2w}{dx^2} + 2\frac{a_{24}}{a_{44}}\frac{(a_{14}a_{24}-a_{12}a_{44})}{(a_{22}a_{44}-a_{24}^2)}\frac{w}{R^2}_{(x=0)} \quad (19)$$

$$\frac{12\Delta}{(a_{22}a_{44}-a_{24}^2)}\frac{Q_o}{\delta^3} = -\frac{d^3w}{dx^3} + \frac{2a_{24}}{a_{44}}\frac{(a_{14}a_{24}-a_{12}a_{44})}{(a_{22}a_{44}-a_{24}^2)}\frac{1}{R^2}\frac{dw}{dx}_{(x=0)} \quad (20)$$

$$w \rightarrow 0 \ ; \ x \rightarrow \infty \quad (21)$$

In order to facilitate the manipulation of the equations, define the quantities b and C as follows.

$$b = \frac{3a_{22}}{a_{44}}\frac{(a_{14}a_{24}-a_{12}a_{44})}{(a_{22}a_{44}-a_{24}^2)} \quad (22)$$

$$C = \frac{12}{a_{44}}\frac{\Delta}{(a_{22}a_{44}-a_{24}^2)} \quad (23)$$

The differential equation and boundary conditions then become the following.

$$\frac{d^4w}{dx^4} - \frac{b}{R^2}\frac{d^2w}{dx^2} + \frac{c}{\delta^2R^2}w = 0 \quad (24)$$

$$a_{44}\frac{c}{\delta^3}M_o = -\frac{d^2w}{dx^2} + \frac{2b}{3}\frac{w}{R^2} \quad (x=0) \quad (25)$$

$$a_{44}\frac{c}{\delta^3}Q_o = -\frac{d^3w}{dx^3} + \frac{2b}{3}\frac{1}{R^2}\frac{dw}{dx} \quad (x=0) \quad (26)$$

$$w \rightarrow 0 \ ; \ x \rightarrow \infty \quad (27)$$

Consider now assuming a solution of the type

$$w = Be^{\lambda(x/R)} \quad (28)$$

where λ and B are constants to be determined. Substituting the assumed solution into the differential equation, the following characteristic equation for λ results.

$$\lambda^4 - b\lambda^2 + \left(\frac{R}{\delta}\right)^2 c = 0$$

There are four roots, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, to the characteristic equation. However since two of the roots are the negatives of the remaining two, it is more convenient to state the solution in the following form.

$$\lambda_{1,2}^2 = \frac{1}{2}\left\{b + \left[b^2 - 4\left(\frac{R}{\delta}\right)^2 c\right]^{1/2}\right\} \tag{29}$$

$$\lambda_{3,4}^2 = \frac{1}{2}\left\{b - \left[b^2 - 4\left(\frac{R}{\delta}\right)^2 c\right]^{1/2}\right\} \tag{30}$$

Inspection of the roots reveals that depending on the magnitude of the constants b and c, and on the geometrical properties R and δ, the value of the roots may be real, imaginary or complex. The values of λ determine the solution for the normal displacement w. Hence the nature of the roots and the conditions which determine a change in their nature are extremely important to the analysis.

If the shell material stress-strain equations were either isotropic or canonically orthotropic in form, the constant b would be zero identically. Hence an increasing magnitude of b indicates an increasing effect from plane anisotropy. Because of this fact, an arbitrary quantitative characterization of the degree of plane anisotropy may be given. Letting the terms weak, mild, strong and very strong characterize plane anisotropies, and noting that the constants b and c may be either positive or negative, the following definitions are set forth.

1. Weak anisotropy

$$c > 0; \quad |b| << 2(R/\delta)\sqrt{c}$$

$$c < 0; \quad |b| \ll 2\,(R/\delta)\sqrt{|c|}$$

2. Mild anisotropy

$$b > 0; \quad c > 0; \quad b^2 \ll 4\,(R/\delta)^2 c$$
$$b < 0; \quad c > 0; \quad b^2 \ll 4\,(R/\delta)^2 c$$
$$b > 0; \quad c < 0; \quad b^2 \ll 4\,(R/\delta)^2 |c|$$
$$b < 0; \quad c < 0; \quad b^2 \ll 4\,(R/\delta)^2 |c|$$

3. Strong anisotropy

$$b > 0; \quad c > 0; \quad b^2 - 4\,(R/\delta)^2 c > 0$$
$$b > 0; \quad c > 0; \quad b^2 - 4\,(R/\delta)^2 c < 0$$
$$b < 0; \quad c > 0; \quad b^2 - 4\,(R/\delta)^2 c < 0$$
$$b > 0; \quad c > 0; \quad b^2 = 4\,(R/\delta)^2 c$$
$$b < 0; \quad c > 0; \quad b^2 - 4\,(R/\delta)^2 c > 0$$
$$b > 0; \quad c < 0; \quad b^2 - 4\,(R/\delta)^2 |c| > 0$$
$$b < 0; \quad c < 0; \quad b^2 - 4\,(R/\delta)^2 |c| > 0$$
$$b < 0; \quad c > 0; \quad b^2 = 4\,(R/\delta)^2 c$$

4. Very strong anisotropy

$$b > 0; \quad c > 0; \quad b^2 \gg 4\,(R/\delta)^2 c$$
$$b > 0; \quad c < 0; \quad b^2 \gg 4\,(R/\delta)^2 |c|$$
$$b < 0; \quad c < 0; \quad b^2 \gg 4\,(R/\delta)^2 |c|$$
$$b < 0; \quad c > 0; \quad b^2 \gg 4\,(R/\delta)^2 c$$

A single inequality sign in the above equations indicate that the terms being compared are of the same order of magnitude. An inequality on the zero term defines the algebraic sign of the term. When a double inequality sign is shown, it means one term is at an order of magnitude larger than the other. Hence the smaller term may be neglected in any process of addition or subtraction.

The condition that $\omega \to 0$ as $x \to \infty$ implies that solutions which satisfy the boundary conditions and are finite must be those which contain the negative exponential. However of all the possible cases stated, not all corresponding solutions satisfy this condition. Hence the calculated displacements are either indeterminate or infinite in value.

Solutions of ω which do not contain the negative exponential occur in all four categories of plane anisotropy. These solutions correspond to the following conditions:

1. Weak anisotropy

$$c < 0; \quad |b| \ll 2(R/\delta) \cdot \sqrt{|c|}$$

2. Mild anisotropy

$$b > 0; \quad c < 0; \quad b^2 \ll 4(R/\delta)^2 |c|$$
$$b < 0; \quad c < 0; \quad b^2 \ll 4(R/\delta)^2 |c|$$

3. Strong anisotropy

$$b < 0; \quad c > 0; \quad b^2 - 4(R/\delta)^2 c > 0$$
$$b > 0; \quad c < 0; \quad b^2 - 4(R/\delta)^2 |c| > 0$$
$$b < 0; \quad c < 0; \quad b^2 - 4(R/\delta)^2 |c| > 0$$
$$b < 0; \quad c > 0; \quad b^2 = 4(R/\delta)^2 c$$

4. Very strong anisotropy

$$b>0;\ c>0;\ b^2 \gg 4(R/\delta)^2 c$$
$$b>0;\ c<0;\ b^2 \gg 4(R/\delta)^2 |c|$$
$$b<0;\ c<0;\ b^2 \gg 4(R/\delta)^2 |c|$$
$$b<0;\ c>0;\ b^2 \gg 4(R/\delta)^2 c$$

The corresponding solutions are given as the following. Letting $c' = c$, and $\beta^4 = (R/\delta)^2 c'$

1. Weak anisotropy

$$\omega = A_1 e^{-\beta\frac{x}{R}} + A_2 e^{\beta\frac{x}{R}} + A_3 \cos\beta\frac{x}{R} + A_4 \sin\beta\frac{x}{R}$$

2. Mild anisotropy

a. $b>0;\ c<0;\ b^2 \ll 4(R/\delta)^2 |c|$

$$\omega = A_1 e^{-\beta\left(1+\frac{b}{4\beta^2}\right)\frac{x}{R}} + A_2 e^{\beta\left(1+\frac{b}{4\beta^2}\right)\frac{x}{R}}$$

$$+ A_3 \cos\left[\beta\left(1-\frac{b}{4\beta^2}\right)\frac{x}{R}\right] + A_4 \sin\left[\beta\left(1-\frac{b}{4\beta^2}\right)\frac{x}{R}\right].$$

b. $b<0;\ c<0;\ b^2 \ll 4(R/\delta)^2 |c|$

$$\omega = A_1 e^{-\beta\left(1+\frac{b}{4\beta^2}\right)\frac{x}{R}} + A_2 e^{\beta\left(1+\frac{b}{4\beta^2}\right)\frac{x}{R}}$$

$$+ A_3 \cos\left[\beta\left(1-\frac{b}{4\beta^2}\right)\frac{x}{R}\right] + A_4 \sin\left[\beta\left(1-\frac{b}{4\beta^2}\right)\frac{x}{R}\right].$$

3. Strong anisotropy

a. $b < 0\,;\; c > 0\,;\; b^2 - 4(R/\delta)^2 c > 0$

$$
\begin{aligned}
w = {} & A_1 \cos\left\{\left[\frac{-b-(b^2+16\beta^4)^{1/2}}{2}\right]^{1/2}\left(\frac{x}{R}\right)\right\} \\
& + A_2 \sin\left\{\left[\frac{-b-(b^2+16\beta^4)^{1/2}}{2}\right]^{1/2}\left(\frac{x}{R}\right)\right\} \\
& + A_3 \cos\left\{\left[\frac{-b+(b^2+16\beta^4)^{1/2}}{2}\right]^{1/2}\left(\frac{x}{R}\right)\right\} \\
& + A_4 \sin\left\{\left[\frac{-b+(b^2+16\beta^4)^{1/2}}{2}\right]^{1/2}\left(\frac{x}{R}\right)\right\}
\end{aligned}
$$

b. $b > 0\,;\; c < 0\,;\; b^2 - 4(R/\delta)^2 |c| > 0$

Let $\beta^* = \left[\frac{1}{2}(R/\delta)\,|c|^{1/2}\right]^{1/2}$

$$
\begin{aligned}
w = {} & A_1 e^{-\left\{\frac{1}{2}\left[(b^2+16\beta^{*4})^{\frac{1}{2}}+b\right]\right\}^{1/2}\left(\frac{x}{R}\right)} \\
& + A_2 e^{\left\{\frac{1}{2}\left[(b^2+16\beta^{*4})^{\frac{1}{2}}+b\right]\right\}^{\frac{1}{2}}\left(\frac{x}{R}\right)} \\
& + A_3 \cos\left\{\frac{1}{2}\left[(b^2+16\beta^{*4})^{1/2}-b\right]\right\}^{1/2}\left(\frac{x}{R}\right) \\
& + A_4 \sin\left\{\frac{1}{2}\left[(b^2+16\beta^{*4})^{1/2}-b\right]\right\}^{1/2}\left(\frac{x}{R}\right).
\end{aligned}
$$

c. $b < 0\,;\; c < 0\,;\; b^2 - 4(R/\delta)^2 |c| > 0$

$$
\begin{aligned}
w = {} & A_1 e^{-\left\{\frac{1}{2}\left[(b^2+16\beta^{*4})^{1/2}+b\right]\right\}^{1/2}\left(\frac{x}{R}\right)} \\
& + A_2 e^{\left\{\frac{1}{2}\left[(b^2+16\beta^{*4})^{1/2}+b\right]\right\}^{1/2}\left(\frac{x}{R}\right)} \\
& + A_3 \cos\left\{\frac{1}{2}\left[(b^2+16\beta^{*4})^{\frac{1}{2}}-b\right]^{\frac{1}{2}}\right\}\left(\frac{x}{R}\right) \\
& + A_4 \sin\left\{\frac{1}{2}\left[(b^2+16\beta^{*4})^{\frac{1}{2}}-b\right]^{\frac{1}{2}}\right\}\left(\frac{x}{R}\right).
\end{aligned}
$$

d. $b < 0;\ c > 0;\ b^2 = 4\,(R/\delta)^2 c$

$$\omega = A_1 \cos\left[\left(-\frac{b}{2}\right)^{1/2}\left(\frac{x}{R}\right)\right] + A_2 \sin\left[\left(-\frac{b}{2}\right)^{1/2}\left(\frac{x}{R}\right)\right]$$
$$+ A_3\left(\frac{x}{R}\right)\cos\left[\left(-\frac{b}{2}\right)^{1/2}\left(\frac{x}{R}\right)\right] + A_4\left(\frac{x}{R}\right)\sin\left[\left(-\frac{b}{2}\right)^{1/2}\left(\frac{x}{R}\right)\right]$$

4. Very strong anisotropy

a. $b > 0;\ c > 0;\ b^2 \gg 4\,(R/\delta)^2 c$

$$\omega = A_1 + A_2\left(\frac{x}{R}\right) + A_3 e^{b\left(\frac{x}{R}\right)} + A_4 e^{-b\left(\frac{x}{R}\right)}$$

b. $b > 0;\ c < 0;\ b^2 \gg 4\,(R/\delta)^2 |c|$

$$\omega = A_1 + A_2\left(\frac{x}{R}\right) + A_3 e^{b\left(\frac{x}{R}\right)} + A_4 e^{-b\left(\frac{x}{R}\right)}$$

c. $b < 0;\ c < 0;\ b^2 \gg 4\,(R/\delta)^2 |c|$

$$\omega = A_1 + A_2\left(\frac{x}{R}\right) + A_3 \cos\left(-b\frac{x}{R}\right) + A_4 \sin\left(-b\frac{x}{R}\right)$$

d. $b < 0;\ c > 0;\ b^2 \gg 4\,(R/\delta)^2 c$

$$\omega = A_1 + A_2\left(\frac{x}{R}\right) + A_3 \cos\left(-b\frac{x}{R}\right) + A_4 \sin\left(-b\frac{x}{R}\right).$$

The instability conditions noted occurs for certain ratios of the plan anisotropic elastic constants. In order to determine whether the phe... ... is physically possible, an orthotropic material skewed at 45° to the geometrical shell axes can be used to generate the plane anisotropic elastic constants. Assume that the canonical form of the orthotropic stress strain relations is given as the following.

$$e_{x'x'} = a_{11}' \sigma_{x'x'} + a_{14}' \sigma_{y'y'}$$

$$e_{x'y'} = a_{22}' \sigma_{x'y'}$$

$$e_{y'y'} = a_{14} \sigma_{x'x'} + a_{44}' \sigma_{y'y'}$$

The x' and y' axes lie in plane tangent to shell surface and the axes makes an angle of 45° with the shell x axes. Using the transformation laws for the elastic constants, the stress strain relations relative to the shell coordinate axes take the plane anisotropic canonical form. However the relation of the elastic constants of the plane anisotropic stress-strain law to the canonical orthotropic elastic constants is given as the following.

$$a_{11} = 1/4\ (a_{11}' + a_{44}' + 2\,a_{14}' + a_{22}')$$

$$a_{12} = 1/2\ (a_{44}' - a_{11}')$$

$$a_{14} = 1/4\ (a_{11}' + a_{44}' + 2a_{14}' + a_{22}')$$

$$a_{22} = (a_{11}' + a_{44}' - 2\,a_{14}')$$

$$a_{24} = 1/2\ (a_{44}' - a_{11}')$$

$$a_{44} = 1/4\ (a_{11}' + a_{44}' + 2\,a_{14}' + a_{22}')$$

Hence it follows that $a_{11} = a_{14} = a_{44}$, and $a_{12} = a_{24}$

The quantity Δ is identically zero and hence

$$C = 0$$

The quantity b also calculates to be zero.

Thus the anisotropy can be characterized as strong. The resulting solution for this particular problem is the following.

$$w = A_1 + A_2 (X/R) + A_3 (X/R)^2 + A_4 (X/R)^3$$

Again in this instance the type of instability previously noted occurs. Hence it must be concluded that at least some of the instabilities noted are physically possible conditions.

Conclusions

In dealing with the plane anisotropic problem, it is noted that an instability phenomena is mathematically plausible. Further, this phenomena is independent of the magnitude of the edge loading but dependent only on the nature of the loading. A physical justification for this condition can be given for at least one instance, namely, the situation when a single orthotropic layer is skewed at 45^{o} to the shell axes. Whether all of the instabilities noted are physically plausible is open to question.

The term instability is dealt with loosely. In many of the cases given in the paper the term simply means that all of the boundary conditions cannot be satisfied, especially the asymptotic decay of the normal displacement. Thus the normal displacement becomes indeterminate in value. However, when the normal displacement increases without bound it is also indeterminate and hence the common terminology for the two phenomena.

Whether in fact the instability is physically plausible or not, one fact does exist. Namely, that for certain values of the coupling constants appearing in the canonical plane anisotropic stress-strain relations, a solution satisfying the boundary conditions is not possible. Further, such a situation does not occur in either isotropic or canonically orthotropic analysis.

The formulation of the shell equations is based on the linearly exact expressions for the equilibrium equations, the compatibility equations, the displacement deformation relations and the stress-strain relations. The introduction of the Kirchhoff hypotheses and the consequent constitutive equations are the sole source of error beyond that which is expected in linear analysis. However the Kirchhoff hypotheses yield reasonable results especially for relatively simple problems of which the problem under discussion is an example. Hence if the instability noted in this paper is physically not plausible, then one of two or both conclusions must be drawn. First, that non-linear analysis is not applicable to all plane anisotropic shell problems even though loadings are extremely small. Second, that the Kirchhoff hypotheses are not applicable to all plane anisotropic shell problems even though the shell may be extremely thin and of the single layered type.

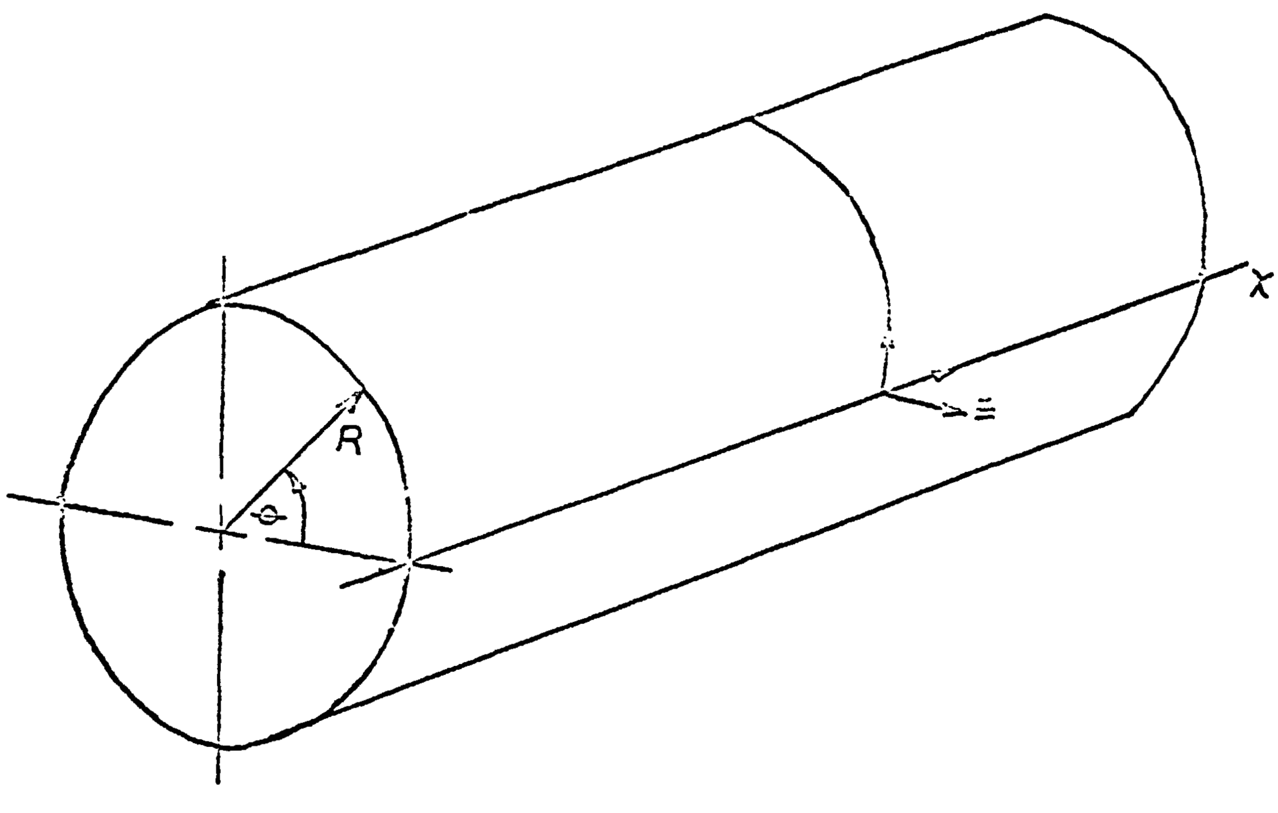

Figure 1. Middle surface coordinate system for right circular cylinder

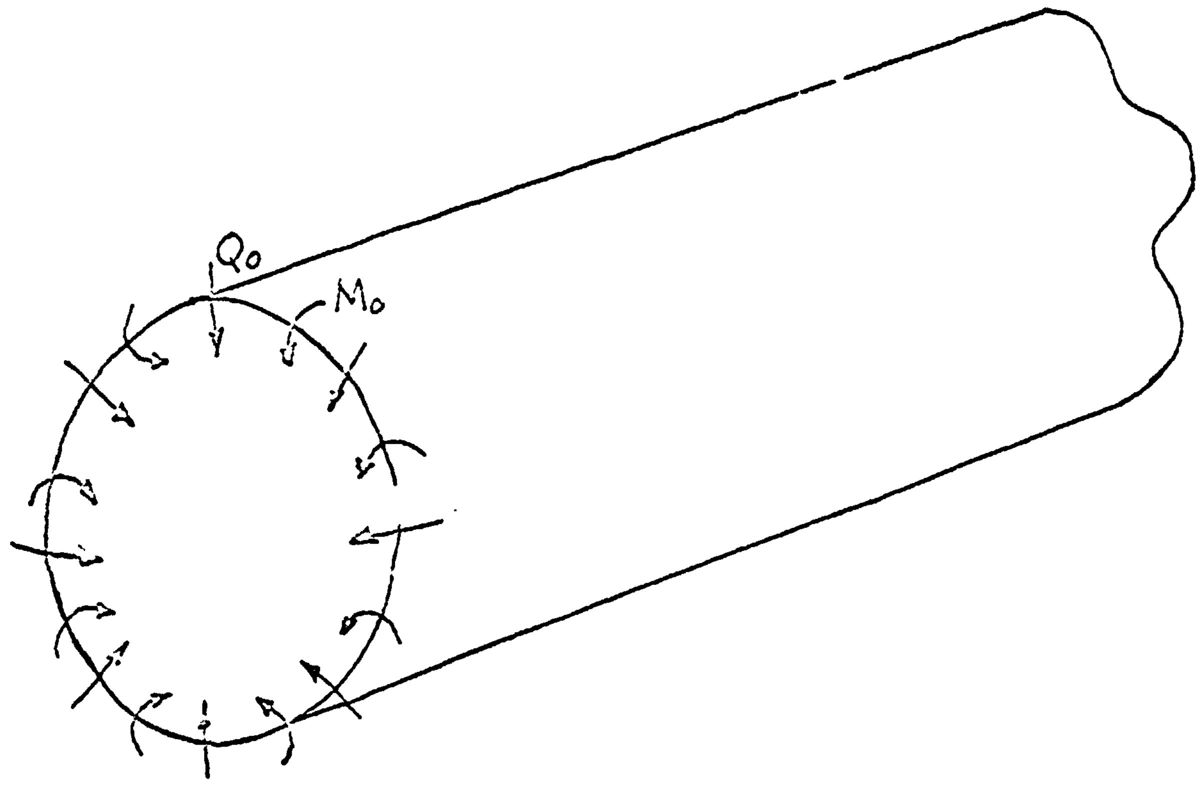

Figure 2. Semi-infinite right circular cylinder with uniformly applied edge loads