Division of Engineering Mechanics

STANFORD UNIVERSITY

Technical Report No. 182

REORIENTATION OF THE HUMAN BODY BY MEANS OF ARM MOTIONS

by

B. C. Riddle and T. R. Kane

February 1968

ABSTRACT

This work deals with the determination of changes in orientation of the human body in a state of free fall (weightlessness), the changes resulting from specified motions of the arms relative to the remainder of the body. A computer program, written in Fortran IV for the IBM 360/67, is documented and examples in the use of the program are given.

ACKNOWLEDGMENT

Gratitude is expressed for financial assistance provided by the National Aeronautics and Space Administration, under Grant NGR-05-020-209.

TABLE OF CONTENTS

Section Page

I. INTRODUCTION

Within recent years, there has emerged considerable interest in the problem of maneuverability of the human body in a state of free fall (weightlessness). Studies have been made of devices such as hand held jet guns, jet shoes, and jet back packs for maneuvering the body in space. The present work deals with the reorientation of the body by means of relative motions of body parts during states of motion in which the angular momentum of the entire system relative to its mass center is equal to zero. More specifically, changes in orientation resulting from specified motions of the arms relative to the remainder of the body are considered. To this end, the human body is regarded as consisting of three rigid bodies: a main body, representing the head, neck, trunk, legs and feet; and two additional bodies representing the arms. (The phrase "free fall" refers to a state in which the system is subjected to no external forces other than gravitational forces which may be replaced with a single force applied at the system's mass center.)

The dynamical analysis of the problem is presented in Chapter II, and Chapter III is devoted to the documentation of a computer program that may be used to solve the equations developed in Chapter II.

II. THEORY

1.1 Consider a system, S, consisting of three rigid bodies B_0, B_1 and B_2, as shown in Fig. 1.1, where B_0 is composed of rigid bodies B_3, B_3', B_4, B_4' and B_5, as shown in Fig. 1.3. B_1 and B_2 are connected to B_0 at points A_1 and A_2, respectively, and are constrained to move relative to B_0 in a manner described in Sec. 1.2.

Several sets of unit vectors are needed in the sequel. Every vector shown in Figs. 1.1 - 1.3 belongs to one of a number of sets of three mutually perpendicular unit vectors, such as $\underline{\epsilon}_1$, $\underline{\epsilon}_2$, $\underline{\epsilon}_3$ and $\underline{n}_1^{(0)}$, $\underline{n}_2^{(0)}$, $\underline{n}_3^{(0)}$, etc. The vectors $\underline{n}_i^{(j)}$, $i = 1,2,3$, are parallel to principal axes of body B_j for the mass center B_j^* of B_j, $j = 1,2,3,4,5$. Unit vectors $\underline{n}_i^{(0)}$, $i = 1,2,3$ are parallel to principal axes of B_5 for B_5^*.

Body B_0 is assumed to be symmetric as regards geometrical and inertial properties, the plane of symmetry being parallel to the unit vectors $\underline{n}_1^{(0)}$ and $\underline{n}_3^{(0)}$. This is simply the usual assumption of left-right symmetry of the human body, which may be expected to be valid whenever the legs occupy symmetric positions relative to the trunk. (For further discussion of this assumption, see Sec. 1.5.)

The mass of B_j is denoted by m_j. The mass of B_1 and B_2 are equal to each other and are denoted by m_1. Similarly, the mass of B_3 and B_3', and the mass of B_4 and B_4', are equal to each other and are denoted by m_3 and m_4, respectively.

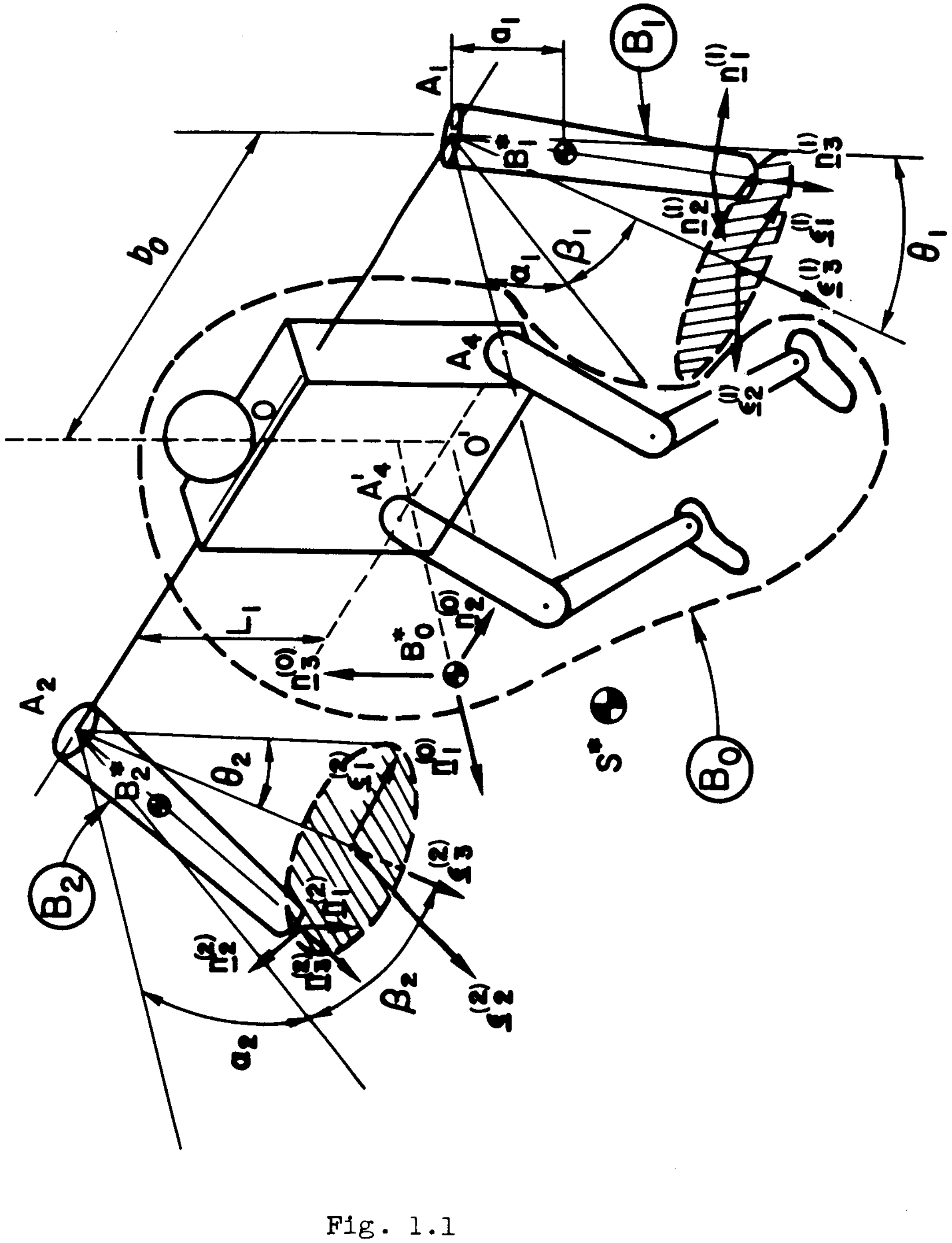

Fig. 1.1

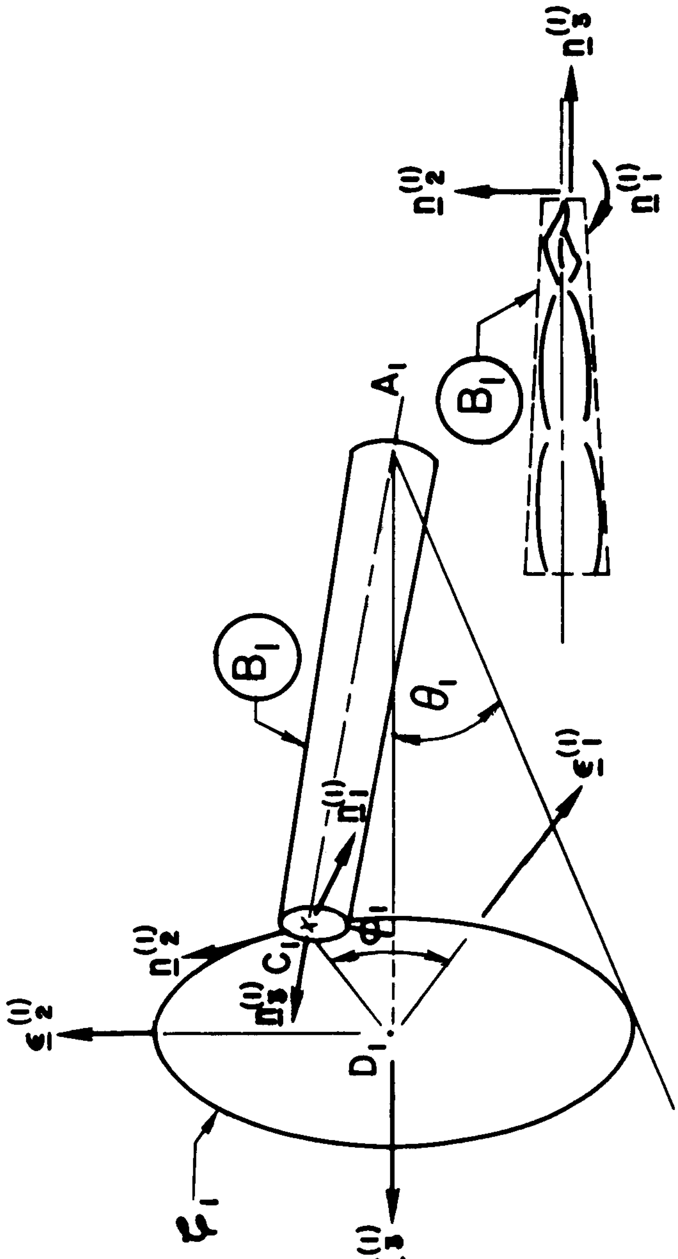

Fig. 1.2

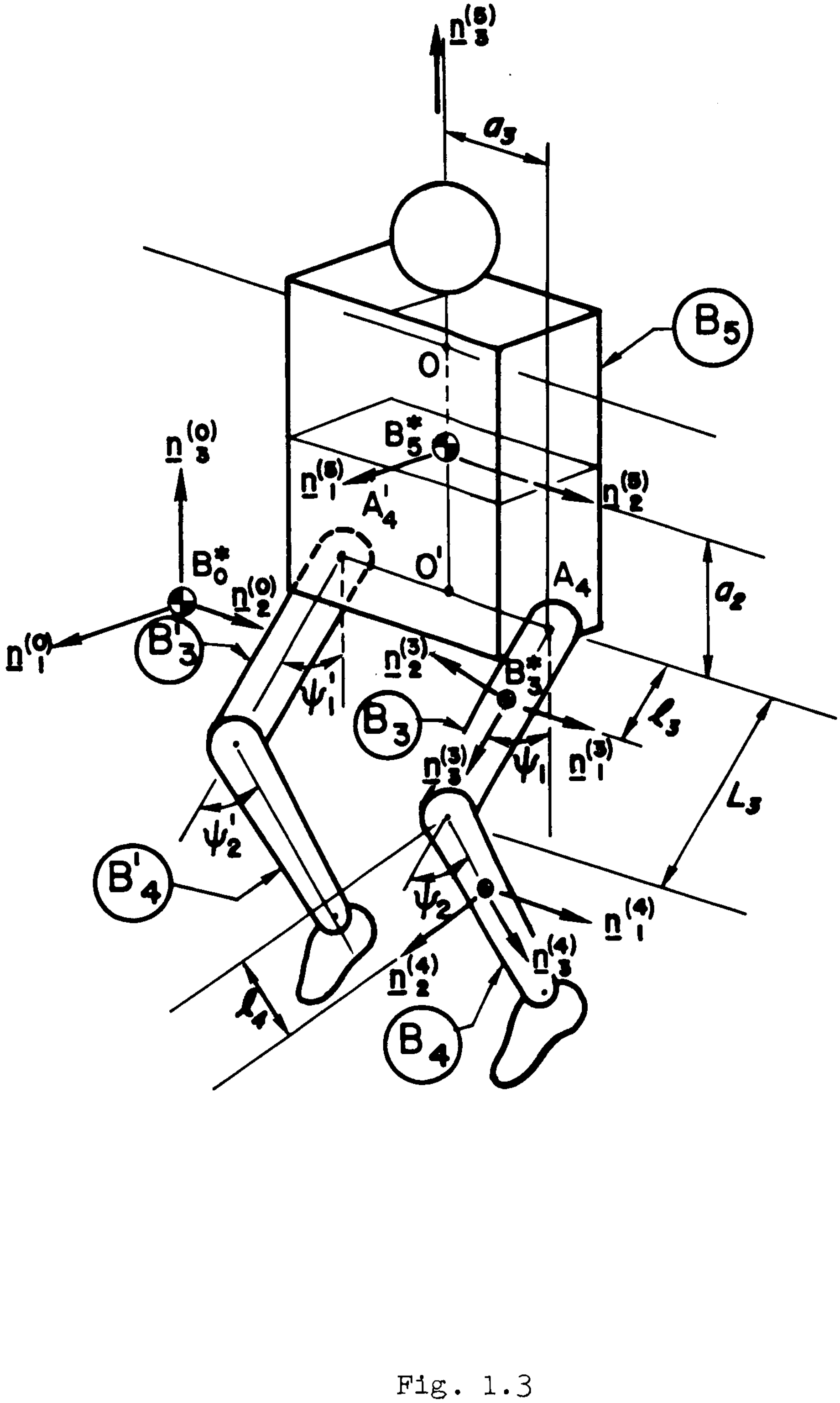
n(5)
3
a3
B5
O
B*5
n(0)
3
n(5)
1
A'4
n(5)
2
B*0
O'
A4
a2
n(0)
2
n(0)
1
B'3
n(3)
2
B*3
ψ'1
B3
l3
n(3)
3
ψ1
n(3)
1
L3
ψ'2
B'4
ψ2
n(4)
1
n(4)
3
n(4)
2
l4
B4

Fig. 1.3

The principal moments of inertia of body B_i for B_i^* are denoted by $I_j^{(i)}$, $j = 1,2,3$, the principal axis associated with $I_j^{(i)}$ being parallel to $\underline{n}_j^{(i)}$. Again, because of the symmetry characteristics of B_0, the principal moments of inertia of B_3' and B_4' are equal to corresponding principal moments of inertia of B_3 and B_4, respectively. Furthermore, it is assumed that B_1 and B_2 have equivalent inertial and geometrical properties, i.e., $I_j^{(2)} = I_j^{(1)}$, $j = 1,2,3$.

1.2 The prescribed motion of B_1 (B_2) relative to B_0 may be thought of as a "coning" motion in which the center line of $B_1(B_2)$ acts as a directrix of a cone, $\mathcal{C}_1$ ($\mathcal{C}_2$), with apex at A_1 (A_2), see Fig. 1.2. The variables α_1, β_1, θ_1, ϕ_1 (α_2, β_2, θ_2, ϕ_2), defined subsequently, are used in describing this "coning" motion.

Let $\mathcal{L}_i$ denote a line parallel to the vector $\underline{n}_i^{(0)}$, $i = 1,2,3$, and let $\mathcal{L}_i - \mathcal{L}_j$ denote a plane containing the lines $\mathcal{L}_i$ and $\mathcal{L}_j$, $i, j = 1,2,3$. Then, the axis of $\mathcal{C}_1$ is located with respect to B_0 by two angles α_1 and β_1, where α_1 is the angle between $\mathcal{L}_1$ and the projection of the axis of $\mathcal{C}_1$ onto the $\mathcal{L}_1 - \mathcal{L}_3$ plane, and β_1 is the angle between the axis and the projection of the axis onto the $\mathcal{L}_1 - \mathcal{L}_3$ plane.

A reference frame, E_1, fixed in $\mathcal{C}_1$, is defined by the unit vectors $\underline{\epsilon}_1^{(1)}$, $\underline{\epsilon}_2^{(1)}$, $\underline{\epsilon}_3^{(1)}$, where these vectors are fixed in the cone $\mathcal{C}_1$ and have directions such that $\underline{\epsilon}_1^{(1)} = \underline{n}_2^{(0)}$, $\underline{\epsilon}_2^{(1)} = \underline{n}_3^{(0)}$, and $\underline{\epsilon}_3^{(1)} = \underline{n}_1^{(0)}$ when $\alpha_1 = \beta_1 = 0$

(i.e., when the centerline of arm B_1 is forward parallel to $\underline{n}_1^{(0)}$). The centerline of B_1 is located with respect to E_1 by the angles θ_1 and ϕ_1, where θ_1 is the semi-vertex angle of $\mathcal{C}_1$, and ϕ_1 is a prescribed function of time indicating the position of the centerline of B_1 on $\mathcal{C}_1$ at time t, relative to the initial position at $t = 0$. Similarly, the "coning" motion of B_2 is described by the angles α_2, β_2, θ_2, ϕ_2.

B_1 and B_2, which represent the arms of the human body, are required to move relative to B_0 in a physiologically feasible manner. To express this requirement in analytical terms, let P_1 be the plane containing the points A_1, C_1, D_1 (Fig. 1.2); ${}^{P_1}\underline{\omega}^{B_1}$ the angular velocity of B_1 in a reference frame fixed in P_1; and ${}^{E_1}\underline{\omega}^{P_1}$ the angular velocity of P_1 in reference frame E_1. Appropriate motions of B_1 are then those that satisfy the constraint equation

$$ {}^{P_1}\underline{\omega}^{B_1} \cdot \underline{n}_3^{(1)} = - {}^{E_1}\underline{\omega}^{P_1} \cdot \underline{\epsilon}_3^{(1)} \qquad (1.1) $$

and, similarly, for B_2

$$ {}^{P_2}\underline{\omega}^{B_2} \cdot \underline{n}_3^{(2)} = - {}^{E_2}\underline{\omega}^{P_2} \cdot \underline{\epsilon}_3^{(2)} \qquad (1.2) $$

<u>1.3</u> The equations of motion are obtained by use of the angular momentum principle. Since it is assumed that the system is in a state of "free fall", as was discussed in the introduction, the sum of the moments of all contact and

gravitational forces about the system mass center is zero. Therefore, from the angular momentum principle, the angular momentum of the system with respect to the system mass center is constant in an inertial reference frame. In this report, this constant is taken to be zero.

The angular momentum, $\underline{A}^{S/S^*}$, of S relative to S^* in an inertial reference frame can be expressed as

$$\underline{A}^{S/S^*} = \underline{A}^{B_0/B_0^*} + \underline{A}^{B_1/B_1^*} + \underline{A}^{B_2/B_2^*} + \underline{A}^{B_0^*/S^*} + \underline{A}^{B_1^*/S^*} + \underline{A}^{B_2^*/S^*} \qquad (1.3)$$

where $\underline{A}^{B_i/B_i^*}$ denotes the angular momentum of B_i relative to B_i^* and $\underline{A}^{B_i^*/S^*}$ is the angular momentum relative to S^* of a particle of mass m_i located at B_i^*.

The analytical expressions for the elements of (1.3) will be meaningful only after further definition and relationships between existing quantities have been described. For example, to express $\underline{A}^{B_0/B_0^*}$ analytically, the inertia properties of B_0 must be known in terms of the inertia properties of B_3, B_3', B_4, B_4' and B_5. These properties as well as other necessary quantities are determined in sections that follow, and expressions for each term in (1.3) are then given.

<u>1.4</u> The following relationships between unit vectors are needed:

$$\left\{\underline{\epsilon}^{(1)}\right\} = \left[A_1^{(1)}\right] \left\{\underline{n}^{(0)}\right\} \tag{1.4}$$

where

$$\left\{\underline{\epsilon}^{(1)}\right\} \equiv \left\{\begin{matrix} \underline{\epsilon}_1^{(1)} \\ \underline{\epsilon}_2^{(1)} \\ \underline{\epsilon}_3^{(1)} \end{matrix}\right\} , \quad \left\{\underline{n}^{(0)}\right\} \equiv \left\{\begin{matrix} \underline{n}_1^{(0)} \\ \underline{n}_2^{(0)} \\ n_3^{(0)} \end{matrix}\right\}$$

and[†]

$$\left[A_1^{(1)}\right] \equiv \begin{bmatrix} -c_{\alpha_1} \; s_{\beta_1} & c_{\beta_1} & s_{\alpha_1} \; s_{\beta_1} \\ s_{\alpha_1} & 0 & c_{\alpha_1} \\ c_{\alpha_1} \; c_{\beta_1} & s_{\beta_1} & -s_{\alpha_1} \; c_{\beta_1} \end{bmatrix} \tag{1.5}$$

$$\left\{\underline{n}^{(1)}\right\} = \left[A_2^{(1)}\right] \left\{\underline{\epsilon}^{(1)}\right\} \tag{1.6}$$

where

$$\left\{n^{(1)}\right\} \equiv \left\{\begin{matrix} \underline{n}_1^{(1)} \\ \underline{n}_2^{(1)} \\ \underline{n}_3^{(1)} \end{matrix}\right\}$$

---

† s and c denote sine and cosine functions, i.e., $s_{\alpha_1} = \sin(\alpha_1)$, $c_{\beta_1} = \cos(\beta_1)$, etc.

and

$$\left[A_2^{(1)}\right] \equiv \begin{bmatrix} \left(c_{\theta_1} c_{\phi_1}^2 + s_{\phi_1}^2\right) & s_{\phi_1} c_{\phi_1} \left(c_{\theta_1} -1\right) & \left(-s_{\theta_1} c_{\phi_1}\right) \\ s_{\phi_1} c_{\phi_1} \left(c_{\theta_1} -1\right) & \left(c_{\theta_1} s_{\phi_1}^2 + c_{\phi_1}^2\right) & \left(-s_{\theta_1} s_{\phi_1}\right) \\ s_{\theta_1} c_{\phi_1} & s_{\theta_1} s_{\phi_1} & c_{\theta_1} \end{bmatrix} \quad (1.7)$$

Hence

$$\left\{\underline{n}^{(1)}\right\} \underset{\substack{(1.6)\\(1.4)}}{=} \left[A_2^{(1)}\right] \left[A_1^{(1)}\right] \left\{\underline{n}^{(0)}\right\} = T^{(1)} \left\{\underline{n}^{(0)}\right\} \quad (1.8)$$

where

$$\left[T^{(1)}\right] \equiv \left[A_2^{(1)}\right] \left[A_1^{(1)}\right] \quad (1.9)$$

so that, if $T_{ij}^{(1)}$ denotes the element in the i^{th} row and j^{th} column of $\left[T^{(1)}\right]$, then

$$T_{11}^{(1)} = s_{\alpha_1} s_{\phi_1} c_{\phi_1} \left(c_{\theta_1} -1\right) - c_{\alpha_1} s_{\beta_1} \left(c_{\theta_1} c_{\phi_1}^2 + s_{\phi_1}^2\right) - c_{\alpha_1} c_{\beta_1} s_{\theta_1} c_{\phi_1}$$

$$T_{12}^{(1)} = c_{\beta_1} \left(c_{\theta_1} c_{\phi_1}^2 + s_{\phi_1}^2\right) - s_{\beta_1} s_{\theta_1} c_{\phi_1}$$

$$T_{13}^{(1)} = c_{\alpha_1} s_{\phi_1} c_{\phi_1} \left(c_{\theta_1} -1\right) + s_{\alpha_1} s_{\beta_1} \left(c_{\theta_1} c_{\phi_1}^2 + s_{\phi_1}^2\right) + s_{\alpha_1} c_{\beta_1} s_{\theta_1} c_{\phi_1}$$

$$T_{21}^{(1)} = s_{\alpha_1}\left(c_{\theta_1} s_{\phi_1}^2 + c_{\phi_1}^2\right) - c_{\alpha_1} s_{\beta_1} s_{\phi_1} c_{\phi_1}\left(c_{\theta_1} - 1\right) - c_{\alpha_1} c_{\beta_1} s_{\theta_1} s_{\phi_1}$$

$$T_{22}^{(1)} = c_{\beta_1} s_{\phi_1} c_{\phi_1}\left(c_{\theta_1} - 1\right) - s_{\beta_1} s_{\theta_1} s_{\phi_1}$$

$$T_{23}^{(1)} = c_{\alpha_1}\left(c_{\theta_1} s_{\phi_1}^2 + c_{\phi_1}^2\right) + s_{\alpha_1} s_{\beta_1} s_{\phi_1} c_{\phi_1}\left(c_{\theta_1} - 1\right) + s_{\alpha_1} c_{\beta_1} s_{\theta_1} s_{\phi_1}$$

$$T_{31}^{(1)} = s_{\alpha_1} s_{\theta_1} s_{\phi_1} - c_{\alpha_1} s_{\beta_1} s_{\theta_1} c_{\phi_1} + c_{\alpha_1} c_{\beta_1} c_{\theta_1}$$

$$T_{32}^{(1)} = c_{\beta_1} s_{\theta_1} c_{\phi_1} + s_{\beta_1} c_{\theta_1}$$

$$T_{33}^{(1)} = c_{\alpha_1} s_{\theta_1} s_{\phi_1} + s_{\alpha_1} s_{\beta_1} s_{\theta_1} c_{\phi_1} - s_{\alpha_1} c_{\beta_1} c_{\theta_1} \qquad (1.10)$$

The only time dependent quantity in $\left[T^{(1)}\right]$ is the angle ϕ_1. Hence, the time derivatives of the elements of $\left[T^{(1)}\right]$ are

$$\dot{T}_{11}^{(1)} = \left[s_{\alpha_1}\left(c_{\theta_1} - 1\right)\left(c_{\phi_1}^2 - s_{\phi_1}^2\right) + 2\, c_{\alpha_1} s_{\beta_1} s_{\phi_1} c_{\phi_1}\left(c_{\theta_1} - 1\right) + c_{\alpha_1} c_{\beta_1} s_{\theta_1} s_{\phi_1}\right]\dot{\phi}_1$$

$$\dot{T}_{12}^{(1)} = \left[-2\, c_{\beta_1} s_{\phi_1} c_{\phi_1}\left(c_{\theta_1} - 1\right) + s_{\beta_1} s_{\theta_1} s_{\phi_1}\right]\dot{\phi}_1$$

$$\dot{T}_{13}^{(1)} = \left[c_{\alpha_1} \left(c_{\phi_1}^2 - s_{\phi_1}^2 \right) \left(c_{\theta_1} - 1 \right) - 2\, s_{\alpha_1} s_{\beta_1} s_{\phi_1} c_{\phi_1} \left(c_{\theta_1} - 1 \right) - s_{\alpha_1} c_{\beta_1} s_{\theta_1} s_{\phi_1} \right] \dot{\phi}_1$$

$$\dot{T}_{21}^{(1)} = \left[2\, s_{\alpha_1} s_{\phi_1} c_{\phi_1} \left(c_{\theta_1} - 1 \right) - c_{\alpha_1} s_{\beta_1} \left(c_{\phi_1}^2 - s_{\phi_1}^2 \right) \left(c_{\theta_1} - 1 \right) - c_{\alpha_1} c_{\beta_1} s_{\theta_1} c_{\phi_1} \right] \dot{\phi}_1$$

$$\dot{T}_{22}^{(1)} = \left[c_{\beta_1} \left(c_{\phi_1}^2 - s_{\phi_1}^2 \right) \left(c_{\theta_1} - 1 \right) - s_{\beta_1} s_{\theta_1} c_{\phi_1} \right] \dot{\phi}_1$$

$$\dot{T}_{23}^{(1)} = \left[2\, c_{\alpha_1} s_{\phi_1} c_{\phi_1} \left(c_{\theta_1} - 1 \right) + s_{\alpha_1} s_{\beta_1} \left(c_{\phi_1}^2 - s_{\phi_1}^2 \right) \left(c_{\theta_1} - 1 \right) + s_{\alpha_1} c_{\beta_1} s_{\theta_1} c_{\phi_1} \right] \dot{\phi}_1$$

$$\dot{T}_{31}^{(1)} = \left[s_{\alpha_1} s_{\theta_1} c_{\phi_1} + c_{\alpha_1} s_{\beta_1} s_{\theta_1} s_{\phi_1} \right] \dot{\phi}_1$$

$$\dot{T}_{32}^{(1)} = \left[- c_{\beta_1} s_{\theta_1} s_{\phi_1} \right] \dot{\phi}_1$$

$$\dot{T}_{33}^{(1)} = \left[c_{\alpha_1} s_{\theta_1} c_{\phi_1} - s_{\alpha_1} s_{\beta_1} s_{\theta_1} s_{\phi_1} \right] \dot{\phi}_1 \qquad (1.11)$$

All necessary transformation matrices relating the unit vectors fixed in B_2 $\underline{n}_i^{(2)}$, $i = 1,2,3$ and the unit vectors $\underline{n}_i^{(0)}$, $i = 1,2,3$ may be obtained from (1.4) - (1.11) by replacing the superscript (1) by (2) wherever it appears in the vectors and matrices, and by replacing the subscript 1 of the angles α_1, β_1, θ_1, ϕ_1 by a 2. For example, analogous to (1.5)

$$
\left[A_1^{(2)}\right] = \begin{bmatrix} -c_{\alpha_2}\, s_{\beta_2} & c_{\beta_2} & s_{\alpha_2}\, s_{\beta_2} \\ s_{\alpha_2} & 0 & c_{\alpha_2} \\ c_{\alpha_2}\, c_{\beta_1} & s_{\beta_2} & -s_{\alpha_2}\, c_{\beta_2} \end{bmatrix} \tag{1.12}
$$

1.5 In (1.3) the factors involving B_0 can be expressed analytically when the geometric and inertia properties of B_0 are known.

The assumption that B_0 is symmetric in all respects requires that B_3 and B_3' (B_4 and B_4') be identical rigid bodies, and that the positions of B_3 and B_3' (B_4 and B_4') relative to B_5 be constrained such that a line between any two corresponding points of B_3 and B_3' (B_4 and B_4') remains parallel to a line connecting the points A_4 and A_4' (i.e., the legs remain parallel for all configurations of B_0). Also, in this analysis B_3 and B_4 are required to remain in a plane parallel to the $\ell_1 - \ell_3$ plane (Sec. 1.2), and it is assumed that there exists a principal plane for B_3 and B_4 parallel to the $\ell_1 - \ell_3$ plane.

The principal moments of inertia of B_0 for B_0^* are obtained from second moments of B_0 relative to B_0^* for directions $\underline{n}_i^{(0)}$, $i = 1,2,3$, which are in turn obtained from the second moments of B_j relative to B_j^*, $j = 3,4,5$, for the direction $\underline{n}_i^{(0)}$, $i = 1,2,3$.

The second moment, $\underline{I}_i^{B_3/B_3^*}$, of B_3 relative to B_3^* for the direction $\underline{n}_i^{(0)}$, i = 1,2,3, is given by

$$\underline{I}_1^{B_3/B_3^*} = \left(c_{\psi_1}^2 \; I_2^{(3)} + s_{\psi_1}^2 \; I_3^{(3)}\right) \underline{n}_1^{(0)}$$

$$+ \; s_{\psi_1} \; c_{\psi_1} \left(I_2^{(3)} - I_3^{(3)}\right) \underline{n}_3^{(0)}$$

$$\underline{I}_2^{B_3/B_3^*} = I_1^{(3)} \; \underline{n}_2^{(0)}$$

$$\underline{I}_3^{B_3/B_3^*} = s_{\psi_1} \; c_{\psi_1} \left(I_2^{(3)} - I_3^{(3)}\right) \underline{n}_1^{(0)}$$

$$+ \left(s_{\psi_1}^2 \; I_2^{(3)} + c_{\psi_1}^2 \; I_3^{(3)}\right) \underline{n}_3^{(0)} \qquad (1.13)$$

Similarly,

$$\underline{I}_1^{B_4/B_4^*} = \left(c_{\psi_3}^2 \; I_2^{(4)} + s_{\psi_3}^2 \; I_3^{(4)}\right) \underline{n}_1^{(0)}$$

$$+ \; s_{\psi_3} \; c_{\psi_3} \left(I_2^{(4)} - I_3^{(4)}\right) \underline{n}_3^{(0)}$$

$$\underline{I}_2^{B_4/B_4^*} = I_1^{(4)} \; \underline{n}_2^{(0)}$$

$$\underline{I}_3^{B_4/B_4^*} = s_{\psi_3} \; c_{\psi_3} \left(I_2^{(4)} - I_3^{(4)}\right) \underline{n}_1^{(0)}$$

$$+ \left(s_{\psi_3}^2 \; I_2^{(4)} + c_{\psi_3}^2 \; I_3^{(4)}\right) \underline{n}_3^{(0)} \qquad (1.14)$$

where

$$\psi_3 = \psi_1 - \psi_2 \qquad (1.15)$$

Since B_3' and B_3 (B_4 and B_4') are identical,

$$\underline{I}_i^{B_3'/B_3'^*} = \underline{I}_i^{B_3/B_3^*}$$

$$\underline{I}_i^{B_4'/B_4'^*} = \underline{I}_i^{B_4/B_4^*} \quad , \qquad i = 1,2,3 \qquad (1.16)$$

Finally,

$$I_i^{B_5/B_5^*} = I_i^{(5)} \, \underline{n}_i^{(0)} \quad , \qquad i = 1,2,3 \qquad (1.17)$$

where, in (1.13) - (1.17), the quantities $I_i^{(j)}$, $i = 1,2,3$, $j = 3,4,5$, are principal moments of inertia defined in Sec. 1.1.

By the parallel axis theorem for second moments

$$\underline{I}_i^{B_j/B_0^*} = \underline{I}_i^{B_j/B_j^*} + \underline{I}_i^{B_j^*/B_0^*} \quad , \qquad i = 1,2,3, \; j = 3,4, \qquad (1.18)$$

where $\underline{I}_i^{B_j^*/B_0^*}$ is the second moment of the point B_j^* regarded as having a mass equal to the mass, m_j, of B_j, i.e.,

$$\underline{I}_i^{B_j^*/B_0^*} = m_j \, \underline{r}^{B_j^*/B_0^*} \times \left(\underline{n}_i^{(0)} \times \underline{r}^{B_j^*/B_0^*}\right), \quad i = 1,2,3 \qquad (1.19)$$

where

$$\underline{r}^{B_3^*/B_0^*} = \left[- \frac{2}{m_0} \left(m_3 \, \ell_3 \, s_{\psi_1} + m_4 \, \ell_4 \, s_{\psi_3} + m_4 \, L_3 \, s_{\psi_1}\right)\right]$$

$$+ \ell_3 \, s_{\psi_1} \Big] \underline{n}_1^{(0)} + [a_3] \, \underline{n}_2^{(0)}$$

$$+ \left[\frac{1}{m_0} \Big(- m_5 \, a_2 + 2 \, m_3 \, \ell_3 \, c_{\psi_1} + 2 \, m_4 \, \ell_4 \, c_{\psi_3} \right.$$

$$\left. + 2 \, m_4 \, L_3 \, c_{\psi_1} \Big) - \ell_3 \, c_{\psi_1} \right] \underline{n}_3^{(0)} \qquad (1.20)$$

$$\underline{r}^{B_4^*/B_0^*} = \Big[- \frac{2}{m_0} \Big(m_3 \, \ell_3 \, s_{\psi_1} + m_4 \, \ell_4 \, s_{\psi_3} + m_4 \, L_3 \, s_{\psi_1} \Big)$$

$$+ L_3 \, s_{\psi_1} + \ell_4 \, s_{\psi_3} \Big] \underline{n}_1^{(0)} + [a_3] \, \underline{n}_2^{(0)}$$

$$+ \left[\frac{1}{m_0} \Big(- m_5 \, a_2 + 2 \, m_3 \, \ell_3 \, c_{\psi_1} + 2 \, m_4 \, \ell_4 \, c_{\psi_3} \right.$$

$$\left. + 2 \, m_4 \, L_3 \, c_{\psi_1} \Big) - L_3 \, c_{\psi_1} - \ell_4 \, c_{\psi_3} \right] \underline{n}_3^{(0)}$$

$$(1.21)$$

and

$$m_0 = 2 \, m_3 + 2 \, m_4 + m_5 \qquad (1.22)$$

Again, because of equivalent inertial and geometrical properties and symmetrical locations of B_3 and B_3' and of B_4 and B_4',

$$\underline{I}_i^{B_3'/B_0^*} = \underline{I}_i^{B_3/B_0^*}$$

$$\underline{I}_i^{B_4'/B_0^*} = \underline{I}_i^{B_4/B_0^*} \quad , \qquad i = 1,2,3 \qquad (1.23)$$

Also,

$$\underline{I}_i^{B_5/B_0^*} = \underline{I}_i^{B_5/B_5^*} + \underline{I}_i^{B_5^*/B_0^*} \quad , \quad i = 1,2,3 \tag{1.24}$$

where

$$\underline{I}_i^{B_5^*/B_0^*} = m_5 \, \underline{r}^{B_5^*/B_0^*} \times \left(\underline{n}_i^{(0)} \times \underline{r}^{B_5^*/B_0^*}\right), \quad i = 1,2,3 \tag{1.25}$$

and

$$\underline{r}^{B_5^*/B_0^*} = \left[-\frac{2}{m_0}\left(m_3 \, \ell_3 \, s_{\psi_1} + m_4 \, \ell_4 \, s_{\psi_3} + m_4 \, L_3 \, s_{\psi_1}\right)\right] \underline{n}_1^{(0)}$$
$$+ \left[\frac{1}{m_0}\left(- m_5 \, a_2 + 2 \, m_3 \, \ell_3 \, c_{\psi_1} + 2 \, m_4 \, \ell_4 \, c_{\psi_3}\right.\right.$$
$$\left.\left. + 2 \, m_4 \, L_3 \, c_{\psi_1}\right) + a_2\right] \underline{n}_3^{(0)} \tag{1.26}$$

Finally, the second moment of B_0 relative to B_0^* for the direction $\underline{n}_i^{(0)}$ is

$$\underline{I}_i^{B_0/B_0^*} = 2 \, \underline{I}_i^{B_3/B_0^*} + 2 \, \underline{I}_i^{B_4/B_0^*} + \underline{I}_i^{B_5/B_0^*} \quad , \quad i = 1,2,3 \tag{1.27}$$

The principal moments of inertia of B_0 for B_0^*, $I_1^{(0)}$, $I_2^{(0)}$, $I_3^{(0)}$, are obtained from the three second $\underline{I}_i^{B_0/B_0^*}$, $i = 1,2,3$ as follows:

$$I_1^{(0)} = \frac{I_{11} + I_{33}}{2} + \left[\left(\frac{I_{11} - I_{33}}{2}\right)^2 + (I_{13})^2\right]^{1/2}$$

$$I_3^{(0)} = \frac{I_{11} + I_{33}}{2} - \left[\left(\frac{I_{11} - I_{33}}{2}\right)^2 + (I_{13})^2\right]^{1/2} \qquad (1.28)$$

where

$$I_{ij} \equiv \underline{I}_i^{B_0/B_0^*} \cdot \underline{n}_j^{(0)} \quad , \quad i, j = 1,2,3 \qquad (1.29)$$

As B_0 is symmetric with respect to the $\mathfrak{L}_1 - \mathfrak{L}_3$ plane, it follows that this plane is a principal plane; and $\underline{n}_2^{(0)}$ is parallel to principal axis of B_0 for B_0^*, the associated principal moment of inertia being given by

$$I_2^{(0)} = \underline{I}_2^{B_0/B_0^*} \cdot \underline{n}_2^{(0)} \qquad (1.30)$$

The angle, γ_1, between the axis of largest principal moment of inertia, $I_1^{(0)}$, and a line parallel to $\underline{n}_1^{(0)}$, is given by

$$\gamma_1 = \arctan\left(\frac{I_1^{(0)} - I_{11}}{I_{13}}\right) \qquad (1.31)$$

<u>1.6</u> One further topic must be discussed before analytical expressions for the terms of (1.3) are presented, namely the location of the system mass center, S^*, relative to a reference point, O, fixed at some point of S.

Let O be the point of intersection of the line between A_1 and A_2 and the $\mathfrak{L}_1 - \mathfrak{L}_3$ plane. Then O is fixed in B_5 and in the plane of symmetry of B_0, the $\mathfrak{L}_1 - \mathfrak{L}_3$ plane. The position vector of O relative to the mass center, S^*, of S is denoted by $\underline{r}^{O/S^*}$ and is given by

$$\underline{r}^{O/S^*} = \frac{1}{M}\left\{\left[m_O\ a_5 - m_1\ a_1\ \left(T_{31}^{(1)} + T_{31}^{(2)}\right)\right]\ \underline{n}_1^{(O)} + \left[- m_1\ a_1\ \left(T_{32}^{(1)} + T_{32}^{(2)}\right)\right]\ \underline{n}_2^{(O)} + \left[m_O\ (a_4 + L_1) - m_1\ a_1\ \left(T_{33}^{(1)} + T_{33}^{(2)}\right)\right]\ \underline{n}_3^{(O)}\right\} \qquad (1.32)$$

where

$$M = m_O + 2\ m_1$$

$$a_4 = \frac{1}{m_O}\left(- m_5\ a_2 + 2\ m_3\ \ell_3\ c_{\psi_1} + 2\ m_4\ \ell_4\ c_{\psi_3} + 2\ m_4\ L_3\ c_{\psi_1}\right) \qquad (1.33)$$

and

$$a_5 = \frac{-2}{m_O}\left(m_3\ \ell_3\ s_{\psi_1} + m_4\ \ell_4\ s_{\psi_3} + m_4\ L_3\ s_{\psi_1}\right)$$

If $\underline{r}^{O/S^*}$ is expressed as

$$\underline{r}^{O/S^*} \equiv r_1^{(O)}\ \underline{n}_1^{(O)} + r_2^{(O)}\ \underline{n}_2^{(O)} + r_3^{(O)}\ \underline{n}_3^{(O)} \qquad (1.34)$$

then it follows from (1.32) that

$$r_1^{(O)} = \frac{1}{M}\left[m_O\ a_5 - m_1\ a_1\ \left(T_{31}^{(1)} + T_{31}^{(2)}\right)\right]$$

$$r_2^{(O)} = \frac{1}{M}\left[- m_1\ a_1\ \left(T_{32}^{(1)} + T_{32}^{(2)}\right)\right]$$

$$r_3^{(0)} = \frac{1}{M}\left[m_0\ (a_4 + L_1) - m_1\ a_1\ \left(T_{33}^{(1)} + T_{33}^{(2)}\right)\right] \tag{1.35}$$

The time derivative of $\underline{r}^{O/S^*}$ in B_0 is given by

$$\frac{{}^{B_0}d}{dt}\left(\underline{r}^{O/S^*}\right) = \dot{r}_1^{(0)}\ \underline{n}_1^{(0)} + \dot{r}_2^{(0)}\ \underline{n}_2^{(0)} + \dot{r}_3^{(0)}\ \underline{n}_3^{(0)} \tag{1.36}$$

and from (1.35)

$$\dot{r}_1^{(0)} = -\frac{m_1\ a_1}{M}\left(\dot{T}_{31}^{(1)} + \dot{T}_{31}^{(2)}\right)$$

$$\dot{r}_2^{(0)} = -\frac{m_1\ a_1}{M}\left(\dot{T}_{32}^{(1)} + \dot{T}_{32}^{(2)}\right)$$

$$\dot{r}_3^{(0)} = -\frac{m_1\ a_1}{M}\left(\dot{T}_{33}^{(1)} + \dot{T}_{33}^{(2)}\right) \tag{1.37}$$

1.7 Finally, the terms comprising (1.3) can be clearly expressed analytically. $\underline{A}^{B_0/B_0^*}$ is given by

$$\begin{aligned}\underline{A}^{B_0/B_0^*} = &\Big[\left(I_1^{(0)}\ c_{\gamma_1}^2 + I_3^{(0)}\ s_{\gamma_1}^2\right)\Omega_1 + s_{\gamma_1}\ c_{\gamma_1} \\ &\left(I_1^{(0)} - I_3^{(0)}\right)\ \Omega_3\Big]\ \underline{n}_1^{(0)} + \left[I_2^{(0)}\ \Omega_2\right]\ \underline{n}_2^{(0)} \\ &+ \Big[s_{\gamma_1}\ c_{\gamma_1}\ \left(I_1^{(0)} - I_3^{(0)}\right)\ \Omega_1 + \Big(I_1^{(0)}\ s_{\gamma_1}^2 \\ &+ I_3^{(0)}\ c_{\gamma_1}^2\Big)\ \Omega_3\Big]\ \underline{n}_3^{(0)}\end{aligned} \tag{1.38}$$

where

$$\Omega_i = \underline{\omega}^{B_0} \cdot \underline{n}_i^{(0)} \quad , \quad i = 1,2,3$$

and $\underline{\omega}^{B_0}$ denotes the angular velocity of B_0 in an inertial reference frame.

Next,

$$\underline{A}^{B_1/B_1^*} = \left[A_{11}^{1/1^*} \dot{\phi}_1 + \sum_{i=2}^{4} A_{i1}^{1/1^*} \Omega_{i-1} \right] \underline{n}_1^{(0)} + \left[A_{12}^{1/1^*} \dot{\phi}_1 + \sum_{i=2}^{4} A_{i2}^{1/1^*} \Omega_{i-1} \right] \underline{n}_2^{(0)} + \left[A_{13}^{1/1^*} \dot{\phi}_1 + \sum_{i=2}^{4} A_{i3}^{1/1^*} \Omega_{i-1} \right] \underline{n}_3^{(0)} \qquad (1.39)$$

where

$$A_{ij}^{1/1^*} = \sum_{k=1}^{3} A_{ik}^{(1)} T_{kj}^{(1)} \quad , \quad i = 1,\dots,4, \; j = 1,2,3 \qquad (1.40)$$

and

$$A_{11}^{(1)} = - I_1^{(1)} s_{\theta_1} c_{\phi_1} \dot{\phi}_1 \quad , \quad A_{21}^{(1)} = I_1^{(1)} T_{11}^{(1)} \quad ,$$

$$A_{31}^{(1)} = I_1^{(1)} T_{12}^{(1)} \quad , \quad A_{41}^{(1)} = I_1^{(1)} T_{13}^{(1)} \quad ,$$

$$A_{12}^{(1)} = - I_2^{(1)} s_{\theta_1} s_{\phi_1} \dot{\phi}_1 \quad , \quad A_{22}^{(1)} = I_2^{(1)} T_{21}^{(1)} \quad ,$$

$$A_{32}^{(1)} = I_2^{(1)} T_{22}^{(1)} \quad , \quad A_{42}^{(1)} = I_2^{(1)} T_{23}^{(1)} \quad ,$$

$$A_{13}^{(1)} = I_3^{(1)} \left(c_{\theta_1} - 1\right) \dot{\phi}_1 \quad , \quad A_{23}^{(1)} = I_3^{(1)} T_{31}^{(1)} \quad ,$$

$$A_{33}^{(1)} = I_3^{(1)} T_{32}^{(1)} \quad , \quad A_{43}^{(1)} = I_3^{(1)} T_{33}^{(1)} \quad . \tag{1.41}$$

Similarly,

$$\begin{aligned} A^{B_2/B_2^*} = & \left[A_{11}^{2/2^*} \dot{\phi}_2 + \sum_{i=2}^{4} A_{i1}^{2/2^*} \Omega_{i-1}\right] \underline{n}_1^{(0)} \\ & + \left[A_{12}^{2/2^*} \dot{\phi}_2 + \sum_{i=2}^{4} A_{i2}^{2/2^*} \Omega_{i-1}\right] \underline{n}_2^{(0)} \\ & + \left[A_{13}^{2/2^*} \dot{\phi}_2 + \sum_{i=2}^{4} A_{i3}^{2/2^*} \Omega_{i-1}\right] \underline{n}_3^{(0)} \end{aligned} \tag{1.42}$$

where

$$A_{ij}^{2/2^*} = \sum_{k=1}^{3} A_{ik}^{(2)} T_{kj}^{(2)} \quad , \quad i = 1,\dots,4, \; j = 1,2,3. \tag{1.43}$$

and $A_{ij}^{(2)}$, $i = 1,\dots,4$, $j = 1,2,3$ are obtained from (1.41) by replacing the superscript (1) of $T_{ij}^{(1)}$ by (2), and the subscript 1 of θ_1 ϕ_1, $\dot{\phi}_1$ by a 2.

For example,

$$A_{21}^{(2)} = I_1^{(1)} T_{11}^{(2)} \; , \; A_{12}^{(2)} = - I_2^{(1)} s_{\theta_2} s_{\phi_2} \dot{\phi}_2, \text{ etc.} \tag{1.44}$$

The angular momentum, $\underline{A}^{B_0^*/S^*}$, relative to S^* of a particle of mass m_0 located at B_0^* is

$$\underline{A}^{B_0^*/S^*} = m_0\, \underline{r}^{B_0^*/S^*} \times \underline{v}^{B_0^*} \tag{1.45}$$

where

$$\underline{v}^{B_0^*} = \underline{\omega}^{B_0} \times \underline{r}^{B_0^*/S^*} + {}^{B_0}\frac{d}{dt}\left(\underline{r}^{B_0^*/S^*}\right) \tag{1.46}$$

and

$$\underline{r}^{B_0^*/S^*} = \left(r_1^{(0)} - a_5\right) \underline{n}_1^{(0)} + r_2^{(0)}\, n_2^{(0)}$$

$$+ \left(r_3^{(0)} - a_4 - L_1\right) \underline{n}_3^{(0)} \tag{1.47}$$

Since a_4, a_5, and L_1 are constants,

$${}^{B_0}\frac{d}{dt}\left(\underline{r}^{B_0^*/S^*}\right) = {}^{B_0}\frac{d}{dt}\left(\underline{r}^{0/S^*}\right)$$

$$= \dot{r}_1^{(0)}\, \underline{n}_1^{(0)} + \dot{r}_2\, \underline{n}_2^{(0)} + \dot{r}_3^{(0)}\, \underline{n}_3^{(0)} \tag{1.48}$$

where $\dot{r}_i^{(0)}$, $i = 1,2,3$ are given by (1.37).

Similarly,

$$\underline{A}^{B_1^*/S^*} = m_1\, \underline{r}^{B_1^*/S^*} \times \underline{v}^{B_1^*} \tag{1.49}$$

where

$$\underline{v}^{B_1^*} = \underline{\omega}^{B_0} \times \underline{r}^{B_1^*/S^*} + {}^{B_0}\frac{d}{dt}\left(\underline{r}^{B_1^*/S^*}\right) \tag{1.50}$$

$$\underline{r}^{B_1^*/S^*} = \left(r_1^{(0)} + a_1\ T_{31}^{(1)}\right)\ \underline{n}_1^{(0)} + \left(r_2^{(0)} + a_1\ T_{32}^{(1)} + b_0\right)\ \underline{n}_2^{(0)}$$

$$+ \left(r_3^{(0)} + a_1\ T_{33}^{(1)}\right)\ \underline{n}_3^{(0)} \qquad (1.51)$$

and

$$^{B_0}\frac{d}{dt}\left(\underline{r}^{B_1^*/S^*}\right) = \left(\dot{r}_1^{(0)} + a_1\ \dot{T}_{31}^{(1)}\right)\ \underline{n}_1^{(0)}$$

$$+ \left(\dot{r}_2^{(0)} + a_1\ \dot{T}_{32}^{(1)}\right)\ \underline{n}_2^{(0)}$$

$$+ \left(\dot{r}_3^{(0)} + a_1\ \dot{T}_{33}^{(1)}\right)\ \underline{n}_3^{(0)} \qquad (1.52)$$

Also,

$$\underline{A}^{B_2^*/S^*} = m_1\ \underline{r}^{B_2^*/S^*} \times \underline{v}^{B_2^*} \qquad (1.53)$$

where

$$\underline{v}^{B_2^*} = \underline{\omega}^{B_0} \times \underline{r}^{B_2^*/S^*} + {}^{B_0}\frac{d}{dt}\left(\underline{r}^{B_2^*/S^*}\right) \qquad (1.54)$$

$$\underline{r}^{B_2^*/S^*} = \left(r_1^{(0)} + a_1\ T_{31}^{(2)}\right)\ \underline{n}_1^{(0)}$$

$$+ \left(r_2^{(0)} + a_1\ T_{32}^{(2)} - b_0\right)\ \underline{n}_2^{(0)}$$

$$+ \left(r_3^{(0)} + a_1\ T_{33}^{(2)}\right)\ \underline{n}_3^{(0)} \qquad (1.55)$$

and

$$^{B_0}\frac{d}{dt}\left(\underline{r}^{B_2^*/S^*}\right) = \left(\dot{r}_1^{(0)} + a_1\ \dot{T}_{31}^{(2)}\right)\ \underline{n}_1^{(0)}$$

$$+ \left(\dot{r}_2^{(0)} + a_1 \dot{T}_{32}^{(2)}\right) \underline{n}_2^{(0)}$$

$$+ \left(\dot{r}_3^{(0)} + a_1 \dot{T}_{33}^{(2)}\right) \underline{n}_3^{(0)} \qquad (1.56)$$

Thus, all terms in (1.3) have been expressed analytically, with certain mathematical operations such as cross multiplication and series summation left unperformed. However, when all operations are performed, the terms of (1.3) may be grouped into the following form:

$$A^{S/S^*} = \left[A_{11}^{S/S^*} + A_{21}^{S/S^*}\,\Omega_1 + A_{31}^{S/S^*}\,\Omega_2 + A_{41}^{S/S^*}\,\Omega_3\right] \underline{n}_1^{(0)}$$

$$+ \left[A_{12}^{S/S^*} + A_{22}^{S/S^*}\,\Omega_1 + A_{32}^{S/S^*}\,\Omega_2 + A_{42}^{S/S^*}\,\Omega_3\right] \underline{n}_2^{(0)}$$

$$+ \left[A_{13}^{S/S^*} + A_{23}^{S/S^*}\,\Omega_1 + A_{33}^{S/S^*}\,\Omega_2 + A_{43}^{S/S^*}\,\Omega_3\right] \underline{n}_3^{(0)} \qquad (1.57)$$

As discussed in Sec. 1.3, the equations of motion are obtained by setting the coefficients of $\underline{n}_i^{(0)}$, $i = 1,2,3,$ in (1.57) equal to zero. This leads to three simultaneous nonhomogeneous linear algebraic equations in Ω_1, Ω_2, Ω_3, which may be solved simply by applying Cramer's Rule.

1.8 Recall that the objective of this work is to determine the orientation of the main body in terms of Euler angles. A step in this direction is to determine certain Euler parameters for the motion.

The angular velocity components $\Omega_1, \Omega_2, \Omega_3$, as found in Sec. 1.7, are used in Euler's kinematical equations for the determination of the Euler parameters ϵ_i, i = 1,2,3, and η of B_0 in an inertial reference frame:

$$\dot{\epsilon}_1 = \frac{1}{2}(\Omega_3 \epsilon_2 - \Omega_2 \epsilon_3 + \Omega_1 \eta)$$

$$\dot{\epsilon}_2 = \frac{1}{2}(-\Omega_3 \epsilon_1 + \Omega_1 \epsilon_3 + \Omega_2 \eta)$$

$$\dot{\epsilon}_3 = \frac{1}{2}(\Omega_2 \epsilon_1 - \Omega_1 \epsilon_2 + \Omega_3 \eta)$$

$$\dot{\eta} = -\frac{1}{2}(\Omega_1 \epsilon_1 + \Omega_2 \epsilon_2 + \Omega_3 \epsilon_3) \qquad (1.58)$$

If $\underline{N}_i$, i = 1,2,3 are three mutually perpendicular unit vectors, each parallel to an axis fixed in an inertial reference frame, then the Euler parameters may be used to express the nine direction cosines relating $\underline{n}_i^{(0)}$ and $\underline{N}_j$, i, j = 1,2,3, as shown in Table 1.1.

Table 1.1

	$\underline{N}_1$	$\underline{N}_2$	$\underline{N}_3$
$\underline{n}_1^{(0)}$	$1 - \epsilon_2^2 - \epsilon_3^2$	$\epsilon_1 \epsilon_2 + \epsilon_3 \eta$	$\epsilon_1 \epsilon_3 - \epsilon_2 \eta$
$\underline{n}_2^{(0)}$	$\epsilon_1 \epsilon_2 - \epsilon_3 \eta$	$1 - \epsilon_1^2 - \epsilon_3^2$	$\epsilon_2 \epsilon_3 + \epsilon_1 \eta$
$\underline{n}_3^{(0)}$	$\epsilon_1 \epsilon_3 + \epsilon_2 \eta$	$\epsilon_2 \epsilon_3 - \epsilon_1 \eta$	$1 - \epsilon_1^2 - \epsilon_2^2$

These relationships can also be expressed in the form

$$\{\underline{n}^{(0)}\} = [E_{ij}] \{\underline{N}\} \tag{1.59}$$

where

$$\{\underline{n}^{(0)}\} = \begin{Bmatrix} \underline{n}_1^{(0)} \\ \underline{n}_2^{(0)} \\ \underline{n}_3^{(0)} \end{Bmatrix} , \quad \{\underline{N}\} = \begin{Bmatrix} \underline{N}_1 \\ \underline{N}_2 \\ \underline{N}_3 \end{Bmatrix}$$

and

$$[E_{ij}] \equiv \begin{bmatrix} (1 - \epsilon_2^2 - \epsilon_3^2) & (\epsilon_1 \epsilon_2 + \epsilon_3 \eta) & (\epsilon_1 \epsilon_3 - \epsilon_2 \eta) \\ (\epsilon_1 \epsilon_2 - \epsilon_3 \eta) & (1 - \epsilon_1^2 - \epsilon_3^2) & (\epsilon_2 \epsilon_3 + \epsilon_1 \eta) \\ (\epsilon_1 \epsilon_3 + \epsilon_2 \eta) & (\epsilon_2 \epsilon_3 - \epsilon_1 \eta) & (1 - \epsilon_1^2 - \epsilon_2^2) \end{bmatrix} \tag{1.60}$$

1.9 In terms of three axis Euler angles, θ_1, θ_2, θ_3, with rotations performed in the order θ_i, about an axis parallel to $\underline{n}_i^{(0)}$, $i = 3,2,1$, the direction cosines relating $\underline{n}_i^{(0)}$ and $\underline{N}_j$, $i, j = 1,2,3$, are given by Table 1.2.

Table 1.2

	$\underline{N}_1$	$\underline{N}_2$	$\underline{N}_3$
$\underline{n}_1^{(0)}$	$c_{\theta_2} c_{\theta_3}$	$c_{\theta_2} s_{\theta_3}$	$- s_{\theta_2}$
$\underline{n}_2^{(0)}$	$s_{\theta_1} s_{\theta_2} c_{\theta_3} - c_{\theta_1} s_{\theta_3}$	$s_{\theta_1} s_{\theta_2} s_{\theta_3} + c_{\theta_1} c_{\theta_3}$	$s_{\theta_1} c_{\theta_2}$
$\underline{n}_3^{(0)}$	$c_{\theta_1} s_{\theta_2} c_{\theta_3} + s_{\theta_1} s_{\theta_3}$	$c_{\theta_1} s_{\theta_2} s_{\theta_3} - s_{\theta_1} c_{\theta_3}$	$c_{\theta_1} c_{\theta_2}$

The order of rotation described above was chosen so that the change in orientation associated with θ_1, θ_2, θ_3 may be demonstrated with a mechanical model similar to a two axis gyroscope, in which the man plays the part of the rotor mounted on the inner gimbal.

<u>1.10</u> The time dependence of the Euler angles θ_i, i = 1,2,3, can be found by equating certain elements of Table 1.1, Table 1.2, and using the definition of E_{ij} given by (1.60). The elements in the first row, third column, show that

$$\theta_2 = \sin^{-1}(-E_{13}) \tag{1.61}$$

It is convenient to require that

$$-\pi/2 \le \theta_2 \le \pi/2 \tag{1.62}$$

Next, if a quantity θ_3' is defined as

$$\theta_3' = \sin^{-1}\left(\frac{E_{12}}{c_{\theta_2}}\right)$$

and is required to satisfy

$$-\pi/2 \le \theta_3' \le \pi/2 \tag{1.63}$$

then

$$\begin{aligned} &\theta_3 = \theta_3' && \text{if } E_{11} \ge 0 \\ \text{and} \quad &\theta_3 = \pi - \theta_3' && \text{if } E_{11} < 0 \end{aligned} \tag{1.64}$$

Finally, if θ_1' is defined as

$$\theta_1' = \sin^{-1}\left(\frac{E_{23}}{c_{\theta_2}}\right) \tag{1.65}$$

with the requirement that

$$-\pi/2 \le \theta_1' \le \pi/2 \tag{1.66}$$

then

$$\begin{aligned} \theta_1 &= \theta_1' && \text{if} \quad E_{33} \ge 0 \\ \theta_1 &= \pi - \theta_1' && \text{if} \quad E_{33} < 0 \end{aligned} \tag{1.67}$$

These three Euler angles, θ_1, θ_2, θ_3, then determine the orientation of the main body, B_0, in an inertial reference frame.

1.11 The angles ϕ_1 and ϕ_2, defined in Sec. 1.2, may be any functions of time compatible with physiologically possible motions of the arms. However, if it is assumed that initially $(t = 0)$ B_0 is at rest in an inertial frame, then it follows from the requirement $\underline{A}^{S/S^*} = 0$, that $\dot{\phi}_1(0) = \dot{\phi}_2(0) = 0$, i.e., the arms, B_1 and B_2, have no motion relative to B_0.

It may be desired to find the reorientation of B_0 when the arms have completed one cycle of motion, i.e., when B_1 and B_2 have swept once around their respective cones, $\mathcal{C}_1$ and $\mathcal{C}_2$. The only additional requirement on ϕ_1

and ϕ_2 is that for some time T, $\phi_1(T) = \phi_2(T) = 2\pi$ (the arms have returned to their original position).

Once the matrix $[E_{ij}]$, see (1.60), has been determined for one cycle, the Euler angles for n cycles can be found by using the matrix $[E_{ij}]^n$ and proceeding as in Sec. 1.10.

1.12 The preceding analysis is presented in a form such that the pertinent equations may be readily solved by means of a digital computer for the desired quantities, θ_1, θ_2, θ_3, associated with the motion. A program for this solution is documented in Chapter III. In essence this program performs the operations indicated in equations (1.10), (1.11), (1.32), (1.37), (1.38), (1.41), (1.40), (1.39), (1.42), (1.47), (1.45), (1.49), (1.51), (1.55), (1.53), (1.58), (1.60), (1.61), (1.64), (1.67).

III. COMPUTER PROGRAM

In this chapter the problem discussed in Chapter II is programmed in Fortran IV language for solution on the IBM 360/67 digital computer.

To use this program one need only know what data must be supplied and in what units these data are given. For this purpose the input and output data are described in detail.

To aid the reader who may wish to follow the program through step by step, a documentation of each subroutine is presented. Usually it is less difficult to follow mathematical language than the corresponding transliteration of this language into computer program language. To help the reader in following this transliteration, many of the equations in the program are referred to by their corresponding equation numbers in Chapter II.

A listing of the complete program including each subroutine is given in the example problem discussed in Sec. 2.13.

2.1 The purpose of the main program, FREFALI, is to determine the orientation of the main body, B_0, with respect to an inertial reference frame by means of three-axis Euler angles, for a maneuver in which B_1 and B_2 perform one cycle of the "coning" motion described in Chapter II.

In this program the inertia and geometric properties of B_0 are supplied as input data. Therefore, the properties must be found by hand calculations (for simple configurations) or by using a separate program in which the subroutine INRTBO is used to make these calculations. In Sec. 2.2 a program including INRTBO is discussed.

PROGRAM: FREFAL1

SUBROUTINES REQUIRED: ELMAIN, ANGMOM, CURLAB, SIMEQ, DFEQKM, DCOSEP, EULER3.

INPUT DATA:

<u>First Data Card</u> Contains five numbers according to the format E12.4 in the following order:

$$\alpha_1,\ \alpha_2,\ \theta_1,\ \theta_2,\ m_0$$

The angles, α_1, α_2, θ_1, θ_2, are given in degrees and m_0 is given in slugs.

<u>Second Data Card</u> Contains five numbers according to the format E12.4 in the following order:

$$m_1,\ a_1,\ b_0,\ L_1,\ \gamma_1$$

where m_1 is in slugs; a_1, b_0, and L_1 are given in feet; and γ_1 is in radians.

<u>Third Data Card</u> Contains two numbers according to the format E12.4 in the following order:

$$a_4,\ a_5$$

where a_4 and a_5 are given in feet.

Fourth Data Card Contains two numbers according to the format E12.4 in the order:

$$\beta_1, \beta_2$$

where β_1 and β_2 are given in degrees.

Fifth Data Card Contains two numbers according to the format E12.4 in the order:

$$I_1^{(0)}, I_1^{(1)}$$

where these numbers are given in slug-ft^2.

Sixth Data Card Contains the numbers

$$I_2^{(0)}, I_2^{(1)}$$

given in slug-ft^2.

Seventh Data Card Contains the numbers

$$I_3^{(0)}, I_3^{(1)}$$

given in slug-ft^2.

The functions ϕ_1, $\dot{\phi}_1$, ϕ_2 and $\dot{\phi}_2$ must be supplied in ELMAIN as program statements.

OUTPUT DATA:

The values of the three-axis Euler angles $\theta_1, \theta_2, \theta_3$ are the main output quantities. These angles are calculated in the last subroutine, EULER3, called in FREFAL1. The angles are given in radians and in degrees; the first number in each column being in radians and the second number in degrees. Note that the angles are printed out only when IP2 = 1 in the argument of subroutine EULER3.

INTERMEDIATE DATA:

Other data calculated by the subroutines may be printed by using appropriate print options contained in each subroutine.

2.2 The inertial and geometrical properties of B_O for other than simple configurations are found by using the subroutine INRTBO. These calculations may be made through a separate program in which the properties of the bodies comprising B_O are the input data and the properties of B_O as a whole are the output data, or another main program, say FREFAL2, may be written which includes INRTBO. For the reader who may wish to make this modification, INRTBO has been included in the documentation of the subroutines.

2.3 If it is desired to obtain the three-axis Euler angles associated with a reorientation resulting from a maneuver in which the arms complete more than one full cycle of motion, then the main program XROTT may be used. XROTT uses the data obtained from one cycle of the motion to find the reorientation for n cycles, where n is any positive integer greater than one.

PROGRAM: XROTT

SUBROUTINES REQUIRED: MTXMLT, EULER3

INPUT DATA:

The input data consist of the elements of two matrices, E and B, of direction cosines. E is the matrix of direction cosines associated with one cycle of the maneuver and is obtained from FREFAL1 as output from the subroutine DCOSEP. B is identical to E when the calculations are made beginning with two cycles, then three cycles, etc., up to n cycles. However, if the reorientation for n cycles is known, then the reorientation for m cycles, $m > n$, may be found without starting over at two cycles and calculating to m cycles. To do this the B matrix is the matrix of direction cosines for n cycles, E is unchanged; and M in the argument of MTXMLT is equal to m-n, see the documentation of MTXMLT given in Sec. 2.10.

Data Cards

There are six data cards, each containing three numbers according to the format E12.4 representing one row of the two input matrices. The first three cards contain the elements of the E matrix, the first card representing the first row of E, etc. The next three cards contain the rows of B in order.

OUTPUT DATA:

Again the three-axis Euler angles are the output data and are displayed as described previously in Sec. 2.1.

2.4 SUBROUTINE: ANGMOM

PURPOSE: To determine the components of the angular momentum vector, $\underline{A}^{S/S^*}$, of the system, S, with respect to the mass center S^*. This subroutine determines the components A_{ij} of

$$\underline{A}^{S/S^*} = [A_{11} + (A_{21})\,\Omega_1 + (A_{31})\,\Omega_2 + (A_{41})\,\Omega_3]\,\underline{n}_1^{(0)}$$

$$+ [A_{12} + (A_{22})\,\Omega_1 + (A_{32})\,\Omega_2 + (A_{42})\,\Omega_3]\,\underline{n}_1^{(0)}$$

$$+ [A_{13} + (A_{23})\,\Omega_1 + (A_{33})\,\Omega_2 + (A_{43})\,\Omega_3]\,\underline{n}_3^{(0)} \tag{2.1}$$

where

$$\Omega_i = \underline{\omega}^{B_0} \cdot \underline{n}_i^{(0)} \quad , \quad i = 1,2,3 \tag{2.2}$$

and $\underline{\omega}^{B_0}$ denotes the angular velocity of B_0 in an inertial reference frame. See Chapter II, Sec. 1.7 for further details.

SUBROUTINES REQUIRED: CURLAB

```
CALL ANGMOM (GAM1, ALP, THT, PH, DPH, M0, M1, B1, B4, B5,
             B0, L1, I, ASS$, IP1, IP2, BET)
```

PARAMETERS:

PARM	TYPE	DESCRIPTION
GAM1	Real variable	The angle between the axis of maximum inertia of body B_0 for B_0^* and the $\underline{n}_1^{(0)}$ axis. Note: GAM1 = 0 for legs completely extended in the attention position. If the legs are in a "tucked" position, GAM1 is calculated in INERTBO.
ALP	Real array (2×1)	The angular position of the arms, bodies B_1 and B_2, in the $\ell_1 - \ell_3$ plane.
THT	Real array (2×1)	The cone angles, θ_1 and θ_2, of the motions of bodies B_1 and B_2. respectively.
PH	Real array (2×1)	The angular positions of B_1 and B_2 relative to their initial positions, specified as explicit functions of time.
DPH	Real array (2×1)	The time derivative of PH.
MO	Real variable	The mass of body B_0 which includes the head, neck, trunk and legs.
M1	Real variable	The mass of bodies B_1 and B_2.
B1	Real variable	The distance between B_1^* or B_2^* and the respective points of attachment A_1 or A_2.
B4	Real variable	The measure number of the position vector of O' with respect to B_0^* along the ℓ_3 axis. (B4 is negative as shown in Fig. 1.1.)

PARM	TYPE	DESCRIPTION
B5	Real variable	The measure number of the position vector of O' with respect to B_0^* along the ℓ_1 axis. (B5 is negative as shown in Fig. 1.1.)
BO	Real variable	The distance between the ℓ_3 axis and the pivot point A_1 or A_2.
L1	Real variable	The distance between points O and O'.
I	Real array (3×2)	The principal moment of inertia of bodies B_0 and B_1 (B_2 same as B_1) for their respective mass centers. [$I(1,1) = I_1^{B_0/B_0^*}$, $I(2,1) = I_2^{B_0/B_0^*}$ etc.]
ASS$	Real array (4×3)	The angular momentum components determined by this program.
IP1	Integer variable	Print option, IP1 = 1 => print input data, otherwise no print.
IP2	Integer variable	IP2 = 1 => print output data, otherwise no print.
BET	Real array (2×1)	Additional angles required to locate the cone axes of B_1 and B_2.

Note: Angles are in radians, masses in slugs, lengths in feet, moments of inertia in slug-feet^2.

2.5 SUBROUTINE: CURL

PURPOSE: To find the components of the cross product of two vectors $\underline{A}$ and $\underline{B}$ where

$$\underline{A} = a_1 \underline{n}_1 + a_1 \underline{n}_2 + a_3 \underline{n}_3$$

$$\underline{B} = b_1 \underline{n}_1 + b_2 \underline{n}_2 + b_3 \underline{n}_3 \qquad (2.3)$$

and

$$\underline{C} = \underline{A} \times \underline{B} = (a_2 b_3 - a_3 b_2) \underline{n}_1 + (a_3 b_1 - a_1 b_3) \underline{n}_2$$

$$+ (a_1 b_2 - a_2 b_1) \underline{n}_3 \qquad (2.4)$$

CALL CURL (A, B, C, IP1, IP2)

PARAMETERS:

PARM	TYPE	DESCRIPTION
A	Real array (3×1)	The two vectors involved in the operation $\underline{A} \times \underline{B}$.
B	Real array (3×1)	
C	Real array (3×1)	The resulting vector, $\underline{C} = \underline{A} \times \underline{B}$.
IP1	Integer variable	Print options; IP1 = 1 => print input data; IP2 = 1 => print output data, otherwise no print.
IP2	Integer variable	

2.6 SUBROUTINE: CURLAB

PURPOSE: To find the components of the cross product of two vectors $\underline{A}$ and $\underline{B}$ where

$$\underline{A} = (A_{11} + A_{21} + A_{31} + A_{41}) \underline{n}_1 + (A_{12} + A_{22} + A_{32} + A_{42}) \underline{n}_2$$

$$+ (A_{13} + A_{23} + A_{33} + A_{43}) \underline{n}_3 \qquad (2.5)$$

$$\underline{B} = (B_1)\ \underline{n}_1 + (B_2)\ \underline{n}_2 + (B_3)\ \underline{n}_3 \qquad (2.6)$$

If

$$\underline{C} = \underline{A} \times \underline{B} \qquad (2.7)$$

then

$$\underline{C} = (C_{11} + C_{21} + C_{31} + C_{41})\ \ln_1 + (C_{12} + \dots + C_{42})\ \ln_2$$

$$+ C_{13} + \dots + C_{43})\ \ln_3 \qquad (2.8)$$

where

$$C_{i1} = A_{i2}\ B_3 - A_{i3}\ B_2$$

$$C_{i2} = A_{i3}\ B_1 - A_{i1}\ B_3 \qquad (2.9)$$

$$C_{i3} = A_{i1}\ B_2 - A_{i2}\ B_1$$

SUBROUTINES REQUIRED: None

CALL CURLAB (A, B, C, IP1, IP2)

PARAMETERS:

PARM	TYPE	DESCRIPTION
A	Real array (4×3)	The vector expressed in (2.5).
B	Real array (3×1)	The vector expressed in (2.6).
C	Real array (4×3)	Result of $\underline{A} \times \underline{B}$ with elements given by (2.8) and (2.9).
IP1, IP2	Integer variable	Print options; IP1 = 1 => print input; IP2 = 1 => print output, otherwise no print.

2.7 SUBROUTINE: DCOSEP

PURPOSE: To calculate the nine direction cosines associated with the Euler parameters relating two sets of mutually perpendicular unit vectors, $\underline{n}_i$ and N_j, i, j = 1,2,3, from the following table:

Table 2.1

	$\underline{N}_1$	$\underline{N}_2$	$\underline{N}_3$
$\underline{n}_1$	$1 - \epsilon_2^2 - \epsilon_3^2$	$\epsilon_1 \epsilon_2 + \epsilon_3 \eta$	$\epsilon_1 \epsilon_3 - \epsilon_2 \eta$
$\underline{n}_2$	$\epsilon_1 \epsilon_2 - \epsilon_3 \eta$	$1 - \epsilon_1^2 - \epsilon_3^2$	$\epsilon_2 \epsilon_3 + \epsilon_1 \eta$
$\underline{n}_3$	$\epsilon_1 \epsilon_3 + \epsilon_2 \eta$	$\epsilon_2 \epsilon_3 - \epsilon_1 \eta$	$1 - \epsilon_1^2 - \epsilon_2^2$

Reference: Table 1.1, Sec. 2.8 of Chapter II.

SUBROUTINES REQUIRED: None

CALL DCOSEP (EPS, E, IP1, IP2)

PARAMETERS:

PARM	TYPE	DESCRIPTION
EPS	Real array (4×1)	The input values of the Euler parameters EPS(1) = ϵ_1, etc.
E	Real array (3×3)	The resulting matrix of direction cosines as given by Table 2.1.
IP1, IP2	Integer variable	IP1 = 1 => print input data; IP2 = 1 => print output data, otherwise no print.

2.8 SUBROUTINE: ELMAIN

PURPOSE:

1. To initialize the input data from FREFAL1 for use in the subroutines ANGMOM and SIMEQ which are called in ELMAIN.
2. To supply expressions for the quantities ϕ_1, $\dot{\phi}_1$, ϕ_2, $\dot{\phi}_2$, which are given by program statements in ELMAIN.
3. To provide Euler's kinematical equations through an ENTRY point (ENTRY ELRKEQ) when ELRKEQ is called in subroutine DFEQKM.

SUBROUTINES REQUIRED: ANGMOM, SIMEQ.

(These have been listed as required in FREFAL1 but are actually called in ELMAIN.)

CALL ELMAIN (GAM1, ALP, THT, BET, MO, M1, A1, A4, A5, BO, L1, I)

PARAMETERS: All parameters in the argument of ELMAIN are input quantities to FREFAL1 and are described later in the particular subroutines in which they are used, i.e., none of these quantities are used in expressions in ELMAIN.

2.9 SUBROUTINE: EULER3

PURPOSE: To find the three axis Euler angles θ_1, θ_2, θ_3 from the given direction cosine matrix when the rotations are performed in the order θ_3, θ_2, θ_1 about respective body fixed areas. See Sec. 1.9, Chapter II for the theory involved.

SUBROUTINES REQUIRED: None

CALL EULER3 (E, T, IP1, IP2)

PARAMETERS:

PARM	TYPE	DESCRIPTION
E	Real array (3×3)	The given matrix of direction cosines E(I, J) = E_{ij}.
T	Real array (3×1)	The resulting three axis Euler angles. [T(1) = θ_1, etc.]
IP1	Integer variable	IP1 = 1 => print input data, otherwise no print.
IP2	Integer variable	IP2 = 1 => print output data, otherwise no print.

Note: Angles are given in radians.

2.10 SUBROUTINE: INRTBO

PURPOSE: To determine the inertia properties and locate the principal axes of inertia of body B_0 for the mass center B_0^*. The theory is given in Chapter II, Sec. 1.5.

SUBROUTINES REQUIRED: CURL

CALL INRTBO (INRT, MASS, BETA, LL3, CL3, LL4, A2, A3, I, GAM1, A4, A5, IP1, IP2)

PARAMETERS:

PARM	TYPE	DESCRIPTION
INRT	Real array (3×3)	The inertia properties of the individual bodies for their mass centers stored by columns in the order: col. 1, B_3; col. 2, B_4; col. 3, B_5.
MASS	Real array (3×1) or (3)	The mass of the individual bodies; MASS(1) = m_3, MASS(2) = m_4, MASS(3) = m_5.
BETA	Real array (2)	The angles β_1 and β_2 described in Chapter II, Sec. 1.2. BET(1) = β_1, BET(2) = β_2.
LL3	Real variable	LL3 = ℓ_3 (see Fig. 1.3).
CL3	Real variable	CL3 = L_3 (see Fig. 1.3).
LL4	Real variable	LL4 = ℓ_4 (see Fig. 1.3).
A2	Real variable	$A_2 = a_2$ (see Fig. 1.3).
A3	Real variable	$A_3 = a_3$ (see Fig. 1.3).
I	Real array (3, 3)	Products and principal moments of inertia of B for B_0^*. I(I,J) = I_{ij}, see (1.29). Note: $I_{12} = I_{21} = I_{23} = I_{32} = 0$.
GAM1	Real variable	Locates the principal axis associated with $I_{11} = I_1$ with respect to the $\mathfrak{L}_1$ axis (the principal axis of B_5 for B_5^* associated with $I_1^{B_5/B_5^*}$).

PARM	TYPE	DESCRIPTION
A4	Real variable	$A_4 = a_4 = \underline{r}^{O'/B_O^*} \cdot \underline{n}_3^{(0)}$ = location of point O' relative to mass center B_O^* parallel to 3 axis. See (1.33).
A5	Real variable	$A5 = a_5 = \underline{r}^{O'/B_O^*} \cdot \underline{n}_1^{(0)}$ = location of point O' relative to B_O^* parallel to the ℓ_1 axis. See (1.33).
IP1	Integer	IP1 = 1 => print input data; IP1 $\neq$ 1 => no print out.
IP2	Integer	IP2 = 1 => print output data.

2.11 SUBROUTINE: MTXMLT

PURPOSE: To determine the matrix C resulting from the multiplication of two square matrices A and B of order 3, or from raising a square matrix of order 3 to the m^{th} power.

$$[C] = [A]\,[B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

SUBROUTINES REQUIRED: None

CALL MTXMLT (A, B, C, M, IPR1, IPR2, IPR3)

PARAMETERS:

PARM	TYPE	DESCRIPTION
A	Real array (3×3)	One of the input matrices.
B	Real array (3×3)	The other input matrix.
C	Real array (3×3)	The result of $A \times B$ or $(A)^m$.
M	Integer variable	$M \leq 2$ => mult. $A \times B$; $M = N > 2$ => raise A to n^{th} power.
IPR1	Integer variable	IPR1 = 1 => print input data, otherwise no print.
IPR2	Integer variable	IPR2 = 1 => print output data, otherwise no print.
IPR3	Integer variable	IPR3 = 1 => print intermediate data, otherwise no print.

2.12 SUBROUTINE: SIMEQ

PURPOSE: To solve the following system of three simultaneous linear algebraic equations in x_1, x_2, x_3.

$$\begin{aligned} a_{21} x_1 + a_{31} x_2 + a_{41} x_3 &= - a_{11} \\ a_{22} x_1 + a_{32} x_2 + a_{42} x_3 &= - a_{12} \\ a_{23} x_1 + a_{33} x_2 + a_{43} x_3 &= - a_{13} \end{aligned} \tag{2.11}$$

SUBROUTINES REQUIRED: None

CALL SIMEQ (A, X, IP1, IP2)

PARAMETERS:

PARM	TYPE	DESCRIPTION
A	Real array (4×3)	The matrix of coefficients (see the theory for complete description).
X	Real array (3×1)	The three unknown variables which are determined by this subroutine.
IP1	Integer variable	IP1 = 1 => print input data, otherwise no print.
IP2	Integer variable	IP2 = 1 => print output data, otherwise no print.

2.13 To illustrate the use of FREFAL1 and XROTT, let $\alpha_1 = \alpha_2 = 90^o$, $\theta_1 = \theta_2 = 30^o$, $\beta_1 = 45^o$, $\beta_2 = -45^o$, $\phi_1 = \phi_2 = \pi + \pi \sin(\pi t - \pi/2)$, $m_0 = 4.458$ slugs, $m_1 = 0.288$ slugs, $a_1 = 0.903$ feet, $a_4 = -0.06047$ feet, $a_5 = 0$, $b_0 = 0.665$ feet, $L_1 = 1.542$ feet, $I_1^{(0)} = 8.336$ slug-feet^2, $I_2^{(0)} = 8.150$ slug-feet^2, $I_2^{(1)} = 0.1320$ slug-feet^2, $I_3^{(1)} = 0.0020$ slug-feet^2. Calculate the reorientation of B_0 for one cycle of arm motion. Also, using XROTT, calculate the reorientation for 10 cycles.

```
$WATFOR
      REAL I(3,2),ALP(2),THT(2),BET(2),       Y(4), MO,M1,L1
      EXTERNAL ELRKEQ
      WRITE(6,6)
      READ(5,1) ALPH1D,ALPH2D,THET1D,THET2D,MO
      READ(5,1) M1,A1,B0,L1,GAM1
      READ(5,2) A4,A5
      READ(5,2) (BET(J),J = 1,2)
      WRITE(6,10) GAM1,ALPH1D,ALPH2D
      WRITE(6,7) THET1D,THET2D
      WRITE(6,11) MO,M1,A1
      WRITE(6,12)A4,A5,B0
      WRITE(6,13) L1
      WRITE(6,16) (BET(J),J = 1,2)
      READ(5,2) ((I(K,J),J=1,2),K=1,3)
      WRITE(6,14) (I(1,J),J=1,2)
      WRITE(6,15) (I(2,J),J=1,2)
      WRITE(6,14) (I(3,J),J=1,2)
      X = 0.0
      DO 30 J=1,3
   30 Y(J) = 0.0
      Y(4) = SQRT(2.0)
      DRC = 1.0/57.296
      ALPH1R = DRC*ALPH1D
      ALPH2R = DRC*ALPH2D
      THET1R = DRC*THET1D
      THET2R = DRC*THET2D
      ALP(1) = ALPH1R
      ALP(2) = ALPH2R
      THT(1) = THET1R
      THT(2)= THET2R
      BET(1) = DRC*BET(1)
      BET(2) = DRC*BET(2)
      CALL ELMAIN(GAM1,ALP,THT,BET,    MO,M1,A1,A4,A5,B0,L1,I)
      WRITE(6,202)
  100 CALL DFEQKM(4,X,0.05,Y,ELRKEQ,1.E-5,1.E-4,&200,.TRUE.)
      IF(X.LT.0.99) GO TO 400
      WRITE(6,450)
  400 CONTINUE
      WRITE(6,9) X
      EPMAG=Y(1)*Y(1)+Y(2)*Y(2)+Y(3)*Y(3)+Y(4)*Y(4)
      DELT=ABS(EPMAG-2.0)
      IF(DELT.LT.0.001) GO TO 150
      X=1.0
      WRITE(6,32) (Y(J),J=1,4)
      WRITE(6,32) DELT
      GO TO 300
  150 CONTINUE
      REAL E(3,3)
      REAL EPS(4)
      DO 20 K=1,4
   20 EPS(K) = Y(K)
      CALL DCOSEP(EPS,E,0,1)
      REAL T(3)
      CALL EULER3(E,T,0,1)
      IF(X.LT.0.99)  GO TO 100
      GO TO 300
```

```
200 WRITE(6,98) X
  1 FORMAT(5E12.4)
  2 FORMAT(2E12.4)
  6 FORMAT('1',58X,' INPUT DATA')
  7 FORMAT('0',28X,'THET1D =',E12.4,2X,'THET2D =',E12.4)
  9 FORMAT('0',49X,'TAU =',F7.2)
 10 FORMAT('0',28X,'GAM1 =',E12.4,2X,'ALPH1D =',E12.4,2X,'ALPH2D =',
   1                                                            E12.4)
 11 FORMAT('0',28X,'M0 =',E12.4,4X,'M1 =',E12.4,4X,'A1 =',E12.4)
 12 FORMAT('0',28X,'A4 =',E12.4,4X,'A5 =',E12.4,4X,'B0 =',E12.4)
 13 FORMAT('0',28X,'CL1 =',E12.4)
 14 FORMAT('0',30X,3E16.4)
 15 FORMAT('0',28X,'I =',3E15.4)
 16 FORMAT('0',28X,'BET1 =',E12.4,2X,'BET2 =',E12.4)
 32 FORMAT('0',15X,'Y1 =',E12.4,2X,'Y2=',E12.4,2X,'Y3=',E12.4,2X,
   1                 'Y4=',E12.4)
 33 FORMAT('0',15X,'DELT=',E12.4)
 98 FORMAT('SOLUTION ENDED AT',F10.7)
202 FORMAT('1',49X,'RESULTS')
450 FORMAT('1')
300 CONTINUE
    RETURN
    END
```

```
      SUBROUTINE ELMAIN(GAM1,ALP,THT,BET,    M0,M1,A1,A4,A5,B0,L1,I)
      REAL ALP(2),THT(2),PH(2),DPH(2),I(3,2),M0,M1,L1,BET(2)
      P2 = 0.0
      RETURN
      ENTRY     ELRKEQ(X,Y,DY)
      REAL X,Y(4),DY(4)
      REAL OMEG(3)
      REAL ASS$(4,3)
      REAL LM(3)
      IP2 = 0
      PI = 3.14159
      P = PI/2
      ARG = PI*X - P
      PH(1) =  PI + PI*SIN(ARG)
      PH(2) =   PH(1)
      DPH(1) = PI*PI*COS(ARG)
      DPH(2) =   DPH(1)
      IF(X.NE.P2) GO TO 10
      IP2 = 1
      P2 = 0.05 + P2
   10 CONTINUE
      CALL ANGMOM(GAM1,ALP,THT,PH,DPH,M0,M1,A1,A4,A5,B0,L1,I,ASS$,0,IP2,
     1                                                            BET)
      CALL SIMEQ(ASS$,OMEG,0,IP2)
      DO 30 K = 1,3
   30 LM(K) = ASS$(1,K)       + ASS$(2,K) *OMEG(1)+ASS$(3,K)*OMEG(2)+
     1                    ASS$(4,K)*OMEG(3)
      DELT =LM(1)*LM(1)+LM(2)*LM(2)+LM(3)*LM(3)
      IF(DELT.LT.0.0001) GO TO 50
      DO 60 J=1,4
   60 DY(J) = 0.0
      WRITE(6,70)
      WRITE(6,31) (LM(K),K=1,3)
      GO TO 900
C     EULER'S KINEMATICAL EQUATIONS FROM (1.58)
   50 DY(1) = (0.5)*(Y(4)*OMEG(1) + Y(2)*OMEG(3) - Y(3)*OMEG(2))
      DY(2) = (0.5)*(Y(4)*OMEG(2) + Y(3)*OMEG(1) - Y(1)*OMEG(3))
      DY(3) = (0.5)*(Y(4)*OMEG(3) + Y(1)*OMEG(2) - Y(2)*OMEG(1))
      DY(4) = -(0.5)*(Y(1)*OMEG(1) + Y(2)*OMEG(2) + Y(3)*OMEG(3))
   31 FORMAT('0',30X,'LM1 =',E12.4,2X,'LM2 =',E12.4,2X,'LM3 =',E12.4)
   70 FORMAT('0',20X,'ANG  MOM  NOT EQ ZERO')
  900 RETURN
      END
C
```

```
      SUBROUTINE ANGMOM(GAM1,ALP,THT,PH,DPH,M0,M1,B1,B4,B5,B0,L1,I,ASS$,
     1                                                   IP1,IP2,BET)
C
      REAL ALP(2),THT(2),PH(2),DPH(2),I(3,2),ASS$(4,3),ROS$(3),DROS$(3),
     1     A00$(4,3),A1(4,3),A11$(4,3),A2(4,3),A22$(4,3),R0$S$(3),
     2     OMEGB0(4,3),OR(4,3),IV(4,3),A0$S$(4,3),R1$S$(3),T1(3,3),
     3     T2(3,3),DT1(3,3),DT2(3,3),A1$S$(4,3),R2$S$(3),A2$S$(4,3)
      REAL M0,M1,L1,M,MF,BET(2)
      ALP1 = ALP(1)
      ALP2 = ALP(2)
      THT1 = THT(1)
      THT2 = THT(2)
      P1 = PH(1)
      P2 = PH(2)
      DP1 = DPH(1)
      DP2 = DPH(2)
      SA1 = SIN(ALP1)
      SA2 = SIN(ALP2)
      CA1 = COS(ALP1)
      CA2 = COS(ALP2)
      ST1 = SIN(THT1)
      ST2 = SIN(THT2)
      CT1 = COS(THT1)
      CT2 = COS(THT2)
      SP1 = SIN(P1)
      SP2 = SIN(P2)
      CP1 = COS(P1)
      CP2 = COS(P2)
      SP12 = SP1*SP1
      SP22 = SP2*SP2
      CP12 = CP1*CP1
      CP22 = CP2*CP2
      SP1CP1 = SP1*CP1
      SP2CP2 = SP2*CP2
      SB1 = SIN(BET(1))
      SB2 = SIN(BET(2))
      CB1 = COS(BET(1))
      CB2 = COS(BET(2))
      C1  = CT1 - 1.0
      C2  = CT2 - 1.0
C     FROM EQ. (1.10)
      T1(1,1) = SA1*SP1CP1*C1 - CA1*SB1*(CT1*CP12 + SP12) - CA1*CB1*ST1*
     1                                                               CP1
      T1(1,2) = CB1*(CT1*CP12 + SP12) - SB1*ST1*CP1
      T1(1,3) = CA1*SP1CP1*C1 + SA1*SB1*(CT1*CP12 + SP12) + SA1*CB1*ST1*
     1                                                               CP1
      T1(2,1) = SA1*(CT1*SP12 + CP12) - CA1*SB1*SP1CP1*C1 - CA1*CB1*ST1*
     1                                                               SP1
      T1(2,2) = CB1*SP1CP1*C1 - SB1*ST1*SP1
      T1(2,3) = CA1*(CT1*SP12 + CP12) + SA1*SB1*SP1CP1*C1 + SA1*CB1*ST1*
     1                                                               SP1
      T1(3,1) = SA1*ST1*SP1 - CA1*SB1*ST1*CP1 + CA1*CB1*CT1
      T1(3,2) = CB1*ST1*CP1 + SB1*CT1
      T1(3,3) = CA1*ST1*SP1 + SA1*SB1*ST1*CP1 - SA1*CB1*CT1
C
      T2(1,1) = SA2*SP2CP2*C2 - CA2*SB2*(CT2*CP22 + SP22) - CA2*CB2*ST2*
     1                                                               CP2
```

```
      T2(1,2) = CB2*(CT2*CP22 + SP22) - SB2*ST2*CP2
      T2(1,3) = CA2*SP2CP2*C2 + SA2*SB2*(CT2*CP22 + SP22) + SA2*CB2*ST2*
     1                                                                CP2
      T2(2,1) = SA2*(CT2*SP22 + CP22) - CA2*SB2*SP2CP2*C2 - CA2*CB2*ST2*
     1                                                                SP2
      T2(2,2) = CB2*SP2CP2*C2 - SB2*ST2*SP2
      T2(2,3) = CA2*(CT2*SP22 + CP22) + SA2*SB2*SP2CP2*C2 + SA2*CB2*ST2*
     1                                                                SP2
      T2(3,1) = SA2*ST2*SP2 - CA2*SB2*ST2*CP2 + CA2*CB2*CT2
      T2(3,2) = CB2*ST2*CP2 + SB2*CT2
      T2(3,3) = CA2*ST2*SP2 + SA2*SB2*ST2*CP2 - SA2*CB2*CT2
C
      F1 = CP12 - SP12
      F2 = CP22 - SP22
C     FROM EQ. (1.11)
      DT1(1,1) = (SA1*C1*F1 + 2.*CA1*SB1*SP1CP1*C1 + CA1*CB1*ST1*SP1)*
     1                                                              DP1
      DT1(1,2) = (-2.*CB1*SP1CP1 *C1 + SB1*ST1*SP1)*DP1
      DT1(1,3) = (CA1*F1*C1 - 2.*SA1*SB1*SP1CP1*C1 - SA1*CB1*ST1*SP1)*
     1                                                              DP1
      DT1(2,1) = (2.*SA1*SP1CP1*C1 - CA1*SB1*F1*C1 - CA1*CB1*ST1*CP1)*
     1                                                              DP1
      DT1(2,2) = (CB1*F1*C1 - SB1*ST1*CP1)*DP1
      DT1(2,3) = (2.*CA1*SP1CP1 *C1 + SA1*SB1*F1*C1 + SA1*CB1*ST1*CP1)*
     1                                                             DP1
      DT1(3,1) = (SA1*ST1*CP1 + CA1*SB1*ST1*SP1)*DP1
      DT1(3,2) = (-CB1*ST1*SP1)*DP1
      DT1(3,3) = (CA1*ST1*CP1 - SA1*SB1*ST1*SP1)*DP1
C
      DT2(1,1) = (SA2*C2*F2 + 2.*CA2*SB2*SP2CP2*C2 + CA2*CB2*ST2*SP2)*
     1                                                            DP2
      DT2(1,2) = (-2.*CB2*SP2CP2 *C2 + SB2*ST2*SP2)*DP2
      DT2(1,3) = (CA2*F2*C2 - 2.*SA2*SB2*SP2CP2*C2 - SA2*CB2*ST2*SP2)*
     1                                                            DP2
      DT2(2,1) = (2.*SA2*SP2CP2*C2 - CA2*SB2*F2*C2 - CA2*CB2*ST2*CP2)*
     1                                                            DP2
      DT2(2,2) = (CB2*F2*C2 - SB2*ST2*CP2)*DP2
      DT2(2,3) = (2.*CA2*SP2CP2 *C2 + SA2*SB2*F2*C2 + SA2*CB2*ST2*CP2)*
     1                                                             DP2
      DT2(3,1) = (SA2*ST2*CP2 + CA2*SB2*ST2*SP2)*DP2
      DT2(3,2) = (-CB2*ST2*SP2)*DP2
      DT2(3,3) = (CA2*ST2*CP2 - SA2*SB2*ST2*SP2)*DP2
C
      M = MO + 2.0*M1
      MF = 1.0/M
C     FROM EQ. (1.32)
      ROS$(1) = MF*(MO*B5 -M1*B1*(T1(3,1) + T2(3,1)))
      ROS$(2) = MF*(-M1*B1*(T1(3,2) + T2(3,2)))
      ROS$(3) = MF*(MO*(B4 + L1) - M1*B1*(T1(3,3) + T2(3,3)))
C     FROM EQ.(1.37)
      DROS$(1) = -MF*M1*B1*(DT1(3,1) + DT2(3,1))
      DROS$(2) = -MF*M1*B1*(DT1(3,2) + DT2(3,2))
      DROS$(3) = -MF*M1*B1*(DT1(3,3) + DT2(3,3))
C
      CG1 = COS(GAM1)
      CG12 = CG1*CG1
      SG1 = SIN(GAM1)
```

```
      SG12 = SG1*SG1
      SG1CG1 = SG1*CG1
C     FROM EQ. (1.38)
      A00$(1,1) = 0.0
      A00$(2,1) = I(1,1)*CG12 + I(3,1)*SG12
      A00$(3,1) = 0.0
      A00$(4,1) = SG1CG1*(I(1,1) - I(3,1))
      A00$(1,2) = 0.0
      A00$(2,2) = 0.0
      A00$(3,2) = I(2,1)
      A00$(4,2) = 0.0
      A00$(1,3) = 0.0
      A00$(2,3) = SG1CG1*(I(1,1) - I(3,1))
      A00$(3,3) = 0.0
      A00$(4,3) = I(1,1)*SG12 + I(3,1)*CG12
C     FROM EQ. (1.41)
      A1(1,1) = -I(1,2)*ST1*CP1*DP1
      DO 5 J = 2,4
    5 A1(J,1) = I(1,2)*T1(1,J-1)
      A1(1,2) = -I(2,2)*ST1*SP1*DP1
      DO 6 J = 2,4
    6 A1(J,2) = I(2,2)*T1(2,J-1)
      A1(1,3) = I(3,2)*(CT1 - 1.0)*DP1
      DO 7 J = 2,4
    7 A1(J,3) = I(3,2)*T1(3,J-1)
C     FROM EQ. (1.39)
      DO 107 J=1,3
  107 A11$(1,J) = ( A1(1,1)*T1(1,J)+A1(1,2)*T1(2,J)+A1(1,3)*T1(3,J) )
      DO 9 K = 1,3
      DO 8 J = 2,4
    8 A11$(J,K) = A1(J,1)*T1(1,K) + A1(J,2)*T1(2,K) + A1(J,3)*T1(3,K)
    9 CONTINUE
C
      A2(1,1) = -I(1,2)*ST2*CP2*DP2
      DO 10 J = 2,4
   10 A2(J,1) = I(1,2)*T2(1,J-1)
      A2(1,2) = -I(2,2)*ST2*SP2*DP2
      DO 11 J = 2,4
   11 A2(J,2) = I(2,2)*T2(2,J-1)
      A2(1,3) = I(3,2)*(CT2 - 1.0)*DP2
      DO 12 J = 2,4
   12 A2(J,3) = I(3,2)*T2(3,J-1)
C     FROM EQ. (1.42)
      DO 112 J=1,3
  112 A22$(1,J)=( A2(1,1)*T2(1,J)+A2(1,2)*T2(2,J)+A2(1,3)*T2(3,J) )
      DO 14 K = 1,3
      DO 13 J = 2,4
   13 A22$(J,K) = A2(J,1)*T2(1,K) + A2(J,2)*T2(2,K) + A2(J,3)*T2(3,K)
   14 CONTINUE
C     FROM EQ. (1.47)
      RO$S$(1) = ROS$(1) - B5
      RO$S$(2) = ROS$(2)
      RO$S$(3) = ROS$(3) - B4 - L1
C
      OMEGBO(1,1) = 0.0
      OMEGBO(2,1) = 1.0
      OMEGBO(3,1) = 0.0
```

```
      OMEGBO(4,1) = 0.0
      OMEGBO(1,2) = 0.0
      OMEGBO(2,2) = 0.0
      OMEGBO(3,2) = 1.0
      OMEGBO(4,2) = 0.0
      DO 16 J = 1,3
   16 OMEGBO(J,3) = 0.0
      OMFGBO(4,3) = 1.0
C
      CALL CURLAB(OMEGBO,RO$S$,OR,0,0)
      OR(1,1) = OR(1,1) + DROS$(1)
      OR(1,2) = OR(1,2) + DROS$(2)
      OR(1,3) = OR(1,3) + DROS$(3)
      CALL CURLAB(OR,RO$S$,IV,0,0)
C     FROM EQ. (1.45)
      DO 18 K = 1,4
      DO 17 J = 1,3
   17 AO$S$(K,J) = -MO*IV(K,J)
   18 CONTINUE
C     FROM EQ. (1.51)
      R1$S$(1) = ROS$(1) + B1*T1(3,1)
      R1$S$(2) = ROS$(2) + B1*T1(3,2) + BO
      R1$S$(3) = ROS$(3) + B1*T1(3,3)
C
      CALL CURLAB(OMEGBO,R1$S$,OR,0,0)
      OR(1,1) = OR(1,1) + DROS$(1) + B1*DT1(3,1)
      OR(1,2) = OR(1,2) + DROS$(2) + B1*DT1(3,2)
      OR(1,3) = OR(1,3) + DROS$(3) + B1*DT1(3,3)
      CALL CURLAB(OR,R1$S$,IV,0,0)
C     FROM EQ. (1.49)
      DO 20 K = 1,4
      DO 19 J = 1,3
   19 A1$S$(K,J) = -M1*IV(K,J)
   20 CONTINUE
C     FROM EQ. (1.55)
      R2$S$(1) = ROS$(1) + B1*T2(3,1)
      R2$S$(2) = ROS$(2) + B1*T2(3,2) - BO
      R2$S$(3) = ROS$(3) + B1*T2(3,3)
C
      CALL CURLAB(OMEGBO,R2$S$,OR,0,0)
      OR(1,1) = OR(1,1) + DROS$(1) + B1*DT2(3,1)
      OR(1,2) = OR(1,2) + DROS$(2) + B1*DT2(3,2)
      OR(1,3) = OR(1,3) + DROS$(3) + B1*DT2(3,3)
      CALL CURLAB(OR,R2$S$,IV,0,0)
C     FROM EQ. (1.53)
      DO 22 K = 1,4
      DO 21 J = 1,3
   21 A2$S$(K,J) = -M1*IV(K,J)
   22 CONTINUE
C     FROM EQ. (1.3)
      DO 24 K = 1,4
      DO 23 J = 1,3
   23 ASS$(K,J) = AOO$(K,J) + A11$(K,J) + A22$(K,J) + AO$S$(K,J) +
     1             A1$S$(K,J) + A2$S$(K,J)
   24 CONTINUE
      IF(IP1.NE.1) GO TO 449
      WRITE(6,1)
```

```
      WRITE(6,102)GAM1,(ALP(J),J=1,2)
      WRITE(6,103) (THT(J),J=1,2)
      WRITE(6,104) (PH(J),J=1,2)
      WRITE(6,105) (DPH(J),J=1,2)
      WRITE(6,3)      M0,M1,B1
      WRITE(6,4)     B4,B5,B0
      WRITE(6,25)    L1
      WRITE(6,29)(I(1,J),J=1,2)
      WRITE(6,30) (I(2,J),J=1,2)
      WRITE(6,29)(I(3,J),J=1,2)
  449 IF(IP2.NE.1) GO TO 900
      WRITE(6,450)
      WRITE(6,42) (ASS$(K,1),K=1,4)
      WRITE(6,41)  (ASS$(K,2),K=1,4)
      WRITE(6,42) (ASS$(K,3),K=1,4)
      WRITE(6,200) (ROS$(J), J=1,3)
      WRITE(6,201) (DROS$(J), J=1,3)
      WRITE(6,202) (A11$(1,J),J=1,3)
      WRITE(6,203) (A22$(1,J),J=1,3)
      WRITE(6,204) (RO$S$(J),J=1,3)
      WRITE(6,205) (A0$S$(1,J),J=1,3)
      WRITE(6,206) (A1$S$(1,J),J=1,3)
      WRITE(6,207) (A2$S$(1,J),J=1,3)
    1 FORMAT(49X,'INPUT DATA FOR ANGMOM')
    3 FORMAT('0',28X,'M0 =' ,E12.4,2X,'M1 =',E12.4,2X,'A1 =',E12.4)
    4 FORMAT('0',28X,'A4 =',E12.4,2X,'A5 =',E12.4,2X,'B0 =',E12.4)
   25 FORMAT('0',28X,'CL1=',E12.4)
   29 FORMAT('0',30X,3E16.4)
   30 FORMAT('0',27X,'I =',3E16.4)
   41 FORMAT('0',22X,'ANGMOM =',4E16.4)
   42 FORMAT('0',30X,4E16.4)
  102 FORMAT('0',28X,'GAM1 =',E12.4,2X,'ALPHA1 =',E12.4,2X,'ALPHA2 =',
     1                                                        E12.4)
  103 FORMAT('0',28X,'THETA1 =',E12.4,2X,'THETA2 =',E12.4)
  104 FORMAT('0',28X,'PHI1 =',E12.4,2X,'PHI2 =',E12.4)
  105 FORMAT('0',28X,'DPHI1 =',E12.4,2X,'DPHI2 =',E12.4)
  200 FORMAT('0',5X,'ROS$ ',3E16.4)
  201 FORMAT('0',5X,'DROS$',3E16.4)
  202 FORMAT('0',5X,'A11$ ',3E16.4)
  203 FORMAT('0',5X,'A22$ ',3E16.4)
  204 FORMAT('0',5X,'RO$S$',3E16.4)
  205 FORMAT('0',5X,'A0$S$',3E16.4)
  206 FORMAT('0',5X,'A1$S$',3E16.4)
  207 FORMAT('0',5X,'A2$S$',3E16.4)
  450 FORMAT('0',40X,'OUTPUT DATA FROM ANGMOM')
  900 CONTINUE
      RETURN
       END
C
```

```
      SUBROUTINE CURLAB(A,B,C,IP1,IP2)
C
      DIMENSION A(4,3),B(3),C(4,3)
      IF(IP1.NE.1) GO TO 10
      WRITE(6,15)
      WRITE(6,12) (A(I,1),I=1,4)
      WRITE(6,13) (A(I,2),I=1,4)
      WRITE(6,12) (A(I,3),I=1,4)
      WRITE(6,14) (B(I),I=1,3)
   10 DO 1 I =1,4
C     FROM EQ. (2.9)
    1 C(I,1) = B(3)*A(I,2) - B(2)*A(I,3)
      DO 2 J = 1,4
    2 C(J,2) = B(1)*A(J,3) - B(3)*A(J,1)
      DO 3 K = 1,4
    3 C(K,3) = B(2)*A(K,1) - B(1)*A(K,2)
      IF(IP2.NE.1) GO TO 900
      WRITE(6,16)
      WRITE(6,12) (C(I,1),I=1,4)
      WRITE(6,17) (C(I,2),I=1,4)
      WRITE(6,12) (C(I,3),I=1,4)
   12 FORMAT('0',30X,4E16.4)
   13 FORMAT('0',27X,'A =',4E16.4)
   14 FORMAT('0',27X,'B(1) =',E12.4,2X,'B(2) =',E12.4,2X,'B(3) =',E12.4)
   15 FORMAT('0',49X,'INPUT DATA FOR CURLAB')
   16 FORMAT('0',47X,'OUTPUT DATA FROM CURLAB')
   17 FORMAT('0',27X,'C =',4E16.4)
  900 RETURN
      END
```

```
      SUBROUTINE DCOSEP(EPS,E,IP1,IP2)
      REAL EPS(4),E(3,3)
      IF(IP1.NE.1) GO TO 20
      WRITE(6,1)
      WRITE(6,2) (EPS(I),I=1,2)
      WRITE(6,3) (EPS(J),J=3,4)
   20 EP12 = EPS(1)*EPS(1)
      EP22 = EPS(2)*EPS(2)
      EP32 = EPS(3)*EPS(3)
      EP1EP2 = EPS(1)*EPS(2)
      EP1EP3 = EPS(1)*EPS(3)
      EP1EP4 = EPS(1)*EPS(4)
      EP2EP3 = EPS(2)*EPS(3)
      EP2EP4 = EPS(2)*EPS(4)
      EP3EP4 = EPS(3)*EPS(4)
C     FROM TABLE 2.1
      E(1,1) = 1 - EP22 - EP32
      E(1,2) = EP1EP2 + EP3EP4
      E(1,3) = EP1EP3 - EP2EP4
      E(2,1) = EP1EP2 -EP3EP4
      E(2,2) = 1 - EP12 - EP32
      E(2,3) = EP2EP3 + EP1EP4
      E(3,1) = EP1EP3 + EP2EP4
      E(3,2) = EP2EP3 - EP1EP4
      E(3,3) = 1 - EP12 - EP22
      IF(IP2.NE.1) GO TO 900
      WRITE(6,4)
      WRITE(6,5) (E(1,J),J=1,3)
      WRITE(6,6) (E(2,J),J=1,3)
      WRITE(6,5) (E(3,J),J=1,3)
    1 FORMAT('0',49X,'INPUT DATA FOR DCOSEP ')
    2 FORMAT('0',27X,'EPS1 =',E14.4,6X,'EPS2 =',E14.4)
    3 FORMAT('0',27X,'EPS3 =',E14.4,6X,'ETA  =',E14.4)
    4 FORMAT('0',45X,'OUTPUT DATA FROM DCOSEP ')
    5 FORMAT('0',30X,3E16.4)
    6 FORMAT('0',27X,'E =',3E16.4)
  900 RETURN
      END
```

```
      SUBROUTINE EULER3(E,T,IP1,IP2)
C
      REAL E(3,3),T(3),TD(3)
C
      IF(IP1.NE.1) GO TO 10
      WRITE(6,5)
      WRITE(6,2) (E(1,J),J=1,3)
      WRITE(6,3) (E(2,J),J=1,3)
      WRITE(6,2) (E(3,J),J=1,3)
   10 PI = 3.14159
      P  = PI/2
C     FROM EQ. (1.61)
      T(2) = ARSIN(-E(1,3))
      ABT2 = ABS(T(2))
      DELT = P - ABT2
      IF (DELT.GT.0.001) GO TO 15
      IF (IP2.NE.1) GO TO 900
      WRITE(6,5)
      WRITE(6,1)
      GO TO 900
   15 CT2 = COS(T(2))
      ARG = E(1,2)/CT2
      TEST = ABS(ARG)
      IF(TEST.LT.1.0) GO TO 19
      IF(TEST.GT.1.001) GO TO 890
      IF(ARG.GT.0.0) GO TO 18
      T(3) = -P
      GO TO 21
   18 T(3) = P
      GO TO 21
   19 IF (E(1,1).LT.0) GO TO 20
      T(3) = ARSIN(E(1,2)/CT2)
      GO TO 25
   20 T(3) = PI - ARSIN(E(1,2)/CT2)
   21 ARG = E(2,3)/CT2
      TEST = ABS(ARG)
      IF(TEST.LT.1.0) GO TO 25
      IF(TEST.GT.1.001) GO TO 895
      IF(ARG.GT.0.0) GO TO 27
      T(1) = -P
      GO TO 40
   27 T(1) = P
      GO TO 40
   25 IF (E(3,3).LT. 0) GO TO 30
C     FROM EQ. (1.65)
      T(1) = ARSIN(E(2,3)/CT2)
      GO TO 40
   30 T(1) = PI - ARSIN(E(1,2)/CT2)
C
   40 IF (IP2.NE.1) GO TO 900
      DO 35 J = 1,3
   35 TD(J) = 57.296*T(J)
      WRITE(6,6)
      WRITE(6,4)
      WRITE(6,45) (T(I),I=1,3)
      WRITE(6,45) (TD(J),J = 1,3)
      GO TO 900
```

```
890 CONTINUE
    WRITE(6,891)
    WRITE(6,892) ARG
    GO TO 21
895 CONTINUE
    WRITE(6,891)
    WRITE(6,896) ARG
  1 FORMAT('0',2X,'T1 = T3 = 0.0   T2 = PI/2 ' )
  2 FORMAT('0',30X,3E16.4)
  3 FORMAT('0',27X,'E =',3E16.4)
  4 FORMAT('0',38X,'THET1',11X,'THET2',11X,'THET3')
  5 FORMAT('0',49X,'INPUT DATA FOR EULER3')
  6 FORMAT('0',45X,'OUTPUT DATA FROM EULER3')
 45 FORMAT('0',30X,3E16.4)
891 FORMAT('0',15X,'ARG. OF ASIN IMPROPER')
892 FORMAT('0',15X,'E(1,2)/CT2 =',E12.4)
896 FORMAT('0',15X,'E(2,3)/CT2 =',E12.4)
900 RETURN
    END
```

```
      SUBROUTINE SIMEQ (A,X,IP1,IP2)
C
      DIMENSION A(4,3),X(3),B(1,3)
      IF(IP1.NE.1) GO TO 10
      WRITE(6,1)
      WRITE(6,2) (A(I,1),I=1,4)
      WRITE(6,3) (A(I,2),I=1,4)
      WRITE(6,2) (A(I,3),I=1,4)
   10 D = A(2,1)*(A(3,2)*A(4,3) - A(4,2)*A(3,3)) + A(3,1)*(A(4,2)*A(2,3)
     1    -A(2,2)*A(4,3)) + A(4,1)*(A(2,2)*A(3,3) - A(3,2)*A(2,3))
      DA = ABS(D)
      IF (DA.LT.0.00001) GO TO 99
      DI = 1/D
      B(1,1) = -A(1,1)
      B(1,2) = -A(1,2)
      B(1,3) = -A(1,3)
   11 X(1) = DI*( B(1,1)*(A(3,2)*A(4,3) - A(4,2)*A(3,3)) + A(3,1)*(
     1        A(4,2)*B(1,3) - B(1,2)*A(4,3)) + A(4,1)*(B(1,2)*A(3,3) -
     2        A(3,2)*B(1,3)) )
      X(2) = DI*( A(2,1)*(B(1,2)*A(4,3) - A(4,2)*B(1,3)) + B(1,1)*(
     1        A(4,2)*A(2,3) - A(2,2)*A(4,3)) + A(4,1)*(A(2,2)*B(1,3) -
     2        B(1,2)*A(2,3)) )
      X(3) = DI*( A(2,1)*(A(3,2)*B(1,3) - B(1,2)*A(3,3)) + A(3,1)*(
     1        B(1,2)*A(2,3) - A(2,2)*B(1,3)) + B(1,1)*(A(2,2)*A(3,3) -
     2        A(3,2)*A(2,3)) )
      IF(IP2.NE.1) GO TO 100
      WRITE(6,5)
      WRITE(6,6) (X(I),I=1,3)
      GO TO 100
   99 CONTINUE
      DO 200 J=1,3
  200  X(J) = 0.0
      WRITE(6,95)
    1 FORMAT('0',45X,'INPUT DATA FOR SIMEQ')
    2 FORMAT('0',30X,4E16.4)
    3 FORMAT('0',27X,'A =',4E16.4)
    5 FORMAT('0',45X, 'OUTPUT DATA FROM SIMEQ')
    6 FORMAT('0',30X,'X1 =',E12.4,2X,'X2 =',E12.4,2X,'X3 =',E12.4)
   95  FORMAT('0',10X,'COEFFICIENT DETERMINANT IS SINGULAR')
  100 RETURN
      END
```

INPUT DATA

GAM1 = 0.0000E 00 ALPH1D = 0.9000E 02 ALPH2D = 0.9000E 02

THET1D = 0.3000E 02 THET2D = 0.3000E 02

M0 = 0.4458E 01 M1 = 0.2880E 00 A1 = 0.9030E 00

A4 = -0.6047E-01 A5 = 0.0000E 00 B0 = 0.6650E 00

CL1 = 0.1542E 01

BET1 = 0.4500E 02 BET2 = -0.4500E 02

	0.8336E 01	0.1325E 00
I =	0.8150E 01	0.1320E 00
	0.3900E 00	0.2000E-02

TAU = 1.00

OUTPUT DATA FROM DCOSEP

	0.8733E 00	0.4870E 00	0.9346E-02
E =	-0.4870E 00	0.8734E 00	-0.2430E-02
	-0.9346E-02	-0.2429E-02	0.1000E 01

OUTPUT DATA FROM EULER3

THET1	THET2	THET3
-0.2430E-02	-0.9346E-02	0.5087E 00
-0.1392E 00	-0.5355E 00	0.2915E 02

```
$WATFOR
      REAL F(3,3),C(3,3),T(3),B(3,3)
      WRITE(6,2)
      READ(5,1) ((E(I,J),J=1,3),I=1,3)
      WRITE(6,3) (E(1,J),J=1,3)
      WRITE(6,4) (E(2,J), J=1,3)
      WRITE(6,3) (E(3,J),J=1,3)
      READ(5,1)((B(I,J),J=1,3),I=1,3)
      WRITE(6,3) (B(1,J),J=1,3)
      WRITE(6,12) (B(2,J),J=1,3)
      WRITE(6,3) (B(3,J),J=1,3)
      M = 2
    7 CONTINUE
      WRITE(6,11) M
    8 CALL MTXMLT(E,B,C,  2,0,1,0)
      CALL EULER3(C,T,0,1)
      M = M + 1
      DO 10 I = 1,3
      DO 9 J = 1,3
    9 B(I,J) = C(I,J)
   10 CONTINUE
      IF(M.LE.10 ) GO TO 7
    1 FORMAT(3E12.4)
    2 FORMAT('1',58X,'INPUT DATA')
    3 FORMAT('0',30X,3E16.4)
    4 FORMAT('0',27X,'E =',3E16.4)
   11 FORMAT('0',53X,'M =',I3)
   12 FORMAT('0',27X,'B =',3E16.4)
      RETURN
      END
C
```

```
      SUBROUTINE MTXMLT (A,B,C,  M,IPR1,IPR2,IPR3)
      REAL      A(3,3),B(3,3),C(3,3)
C
      IF(IPR1.NE.1) GO TO 19
      WRITE(6,18)
      WRITE(6,1)
      WRITE(6,920) (A(1,J), J=1,3)
      WRITE(6,2)   (A(2,J), J=1,3)
      WRITE(6,920) (A(3,J), J=1,3)
      WRITE(6,920) (B(1,J), J=1,3)
      WRITE(6,3)   (B(2,J), J=1,3)
      WRITE(6,920) (B(3,J), J=1,3)
   19 L = 1
   20 CONTINUE
      L = L + 1
   21 DO 50 I = 1,3
      DO 40 J = 1,3
      C(I,J) = 0.0
      DO 30 N = 1,3
   30 C(I,J) = A(I,N)*B(N,J) + C(I,J)
   40 CONTINUE
   50 CONTINUE
      IF(IPR3.NE.1) GO TO 59
      WRITE(6,52)
      WRITE(6,920)  (C(1,J),J=1,3)
      WRITE(6,51 ) (C(2,J),J=1,3)
      WRITE(6,920)  (C(3,J),J=1,3)
   59 IF (M.LE.L) GO TO 900
   60 L = L + 1
      DO 90 I = 1,3
      DO 80 J = 1,3
      B(I,J) = C(I,J)
   80 CONTINUE
   90 CONTINUE
      GO TO 21
  900 IF(IPR2.NE.1) GO TO 950
      WRITE(6,910)
      WRITE(6,920) (C(1,J),J=1,3)
      WRITE(6,921) (C(2,J),J=1,3)
      WRITE(6,920) (C(3,J),J=1,3)
    1 FORMAT(50X,'INPUT DATA FOR MTXMLT')
    2 FORMAT('0',27X,'A =',3E16.4)
    3 FORMAT('0',27X,'B =',3E16.4)
   18 FORMAT('1')
   51 FORMAT('0',27X,'AB=',3E16.4)
   52  FORMAT('0',49X,'INTERMEDIATE DATA ')
  910 FORMAT ('0',49X,'OUTPUT FROM MTXMLT')
  920 FORMAT ('0',30X,3E16.4)
  921 FORMAT ('0',27X,'C =',3E16.4)
  950  RETURN
       END
```

```
      SUBROUTINE EULER3(E,T,IP1,IP2)
C
      REAL E(3,3),T(3),TD(3)
C
      IF(IP1.NE.1) GO TO 10
      WRITE(6,5)
      WRITE(6,2) (E(1,J),J=1,3)
      WRITE(6,3) (E(2,J),J=1,3)
      WRITE(6,2) (E(3,J),J=1,3)
   10 PI = 3.14159
      P  = PI/2
C     FROM EQ. (1.61)
      T(2) = ARSIN(-E(1,3))
      ABT2 = ABS(T(2))
      DELT = P - ABT2
      IF (DELT.GT.0.001) GO TO 15
      IF (IP2.NE.1) GO TO 900
      WRITE(6,5)
      WRITE(6,1)
      GO TO 900
   15 CT2 = COS(T(2))
      ARG = E(1,2)/CT2
      TEST = ABS(ARG)
      IF(TEST.LT.1.0) GO TO 19
      IF(TEST.GT.1.001) GO TO 890
      IF(ARG.GT.0.0) GO TO 18
      T(3) = -P
      GO TO 21
   18 T(3) = P
      GO TO 21
   19 IF (E(1,1).LT.0) GO TO 20
      T(3) = ARSIN(E(1,2)/CT2)
      GO TO 25
   20 T(3) = PI - ARSIN(E(1,2)/CT2)
   21 ARG = E(2,3)/CT2
      TEST = ABS(ARG)
      IF(TEST.LT.1.0) GO TO 25
      IF(TEST.GT.1.001) GO TO 895
      IF(ARG.GT.0.0) GO TO 27
      T(1) = -P
      GO TO 40
   27 T(1) = P
      GO TO 40
   25 IF (E(3,3).LT. 0) GO TO 30
C     FROM EQ. (1.65)
      T(1) = ARSIN(E(2,3)/CT2)
      GO TO 40
   30 T(1) = PI - ARSIN(E(1,2)/CT2)
C
   40 IF (IP2.NE.1) GO TO 900
      DO 35 J = 1,3
   35 TD(J) = 57.296*T(J)
      WRITE(6,6)
      WRITE(6,4)
      WRITE(6,45) (T(I),I=1,3)
      WRITE(6,45) (TD(J),J = 1,3)
      GO TO 900
```

```
890 CONTINUE
    WRITE(6,891)
    WRITE(6,892) ARG
    GO TO 21
895 CONTINUE
    WRITE(6,891)
    WRITE(6,896) ARG
  1 FORMAT('0',2X,'T1 = T3 = 0.0  T2 = PI/2 ' )
  2 FORMAT('0',30X,3E16.4)
  3 FORMAT('0',27X,'E =',3E16.4)
  4 FORMAT('0',38X,'THET1',11X,'THET2',11X,'THET3')
  5 FORMAT('0',49X,'INPUT DATA FOR EULER3')
  6 FORMAT('0',45X,'OUTPUT DATA FROM EULER3')
 45 FORMAT('0',30X,3E16.4)
891 FORMAT('0',15X,'ARG. OF ASIN IMPROPER')
892 FORMAT('0',15X,'E(1,2)/CT2 =',E12.4)
896 FORMAT('0',15X,'E(2,3)/CT2 =',E12.4)
900 RETURN
    END
```

```
                          M = 10

                      OUTPUT FROM MTXMLT

          0.3665E 00     -0.9300E 00     -0.1785E-01

    C =   0.9300E 00      0.3666E 00     -0.1216E-01

          0.1785E-01     -0.1217E-01      0.1000E 01

                   OUTPUT DATA FROM EULER3

            THET1           THET2           THET3

         -0.1217E-01      0.1785E-01     -0.1195E 01

         -0.6971E 00      0.1023E 01     -0.6846E 02
```