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THERMAL STRESSES IN A SOLID WEAKENED BY AN EXTERNAL CIRCULAR CRACK[1]

by

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Abstract

Linear thermoelastic problems are solved for the thermal stress and displacement fields in an elastic solid of infinite extent weakened by a plane of discontinuity or crack occupying the space outside of a circular region. The faces of the crack are heated by maintaining them at certain temperature and/or by some prescribed heat flux the distributions of which are such that their magnitudes diminish at infinity. Special emphasis is given to the case when the circular region surrounded by the external crack is insulated from heat flow. The solution to this thermal stress problem may be combined with that of applying appropriate tractions to the crack faces, thus providing the necessary ingredients for extending the Dugdale hypothesis to thermally-stressed bodies containing cracks. More specifically, the results of the analysis permit an estimate of the plastic zone size and the plastic energy dissipation for an external circular crack. Information of this kind contributes to the understanding of the mechanics of fracture initiation in ductile materials.

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Introduction

Previous efforts on steady-state thermoelastic problems have been focused mainly on problems dealing with bars, plates and cylinders. A complete account of these developments is clearly beyond the scope of this article. On the other hand, systematic study of the effect of plane cracks on thermal stresses set up in an elastic solid has a quite recent history started in the past few years.

Beginning with the work of Olesiak and Sneddon [1][4], the method of dual integral equations in the Hankel transforms was used to determine the distribution of temperature and stress in a solid containing a penny-shaped crack across whose surfaces there is a prescribed flux of heat. By having the same thermal conditions on the upper and lower faces of the crack, the problem was reduced to one of specifying certain mixed boundary conditions on a semi-infinite solid. The case of heat supplied antisymmetrically with respect to the crack plane was treated by Florence and Goodier [2]. Using potential function theory, Kassir and Sih [3] presented explicit solutions to a class of three-dimensional thermal stress problems with an elliptical crack whose faces are thermally disturbed by both symmetric and antisymmetric temperatures and/or temperature gradients. Their results include those in [1,2] as limiting cases. Further, Kassir and Sih [3] showed that for any small region around the outer boundary of an elliptically-shaped crack the thermal stresses and displacements correspond to a situation which is locally one of plane strain as derived earlier by Sih [4] using the equations of two-dimensional thermoelasticity.

This investigation presents an analysis of the steady-state axisymmetric thermoelastic problem concerning two semi-infinite solids joined over a

[4]Numbers in brackets designate References at end of paper.

circular region. The unconnected portion of the solids may be regarded as an external penny-shaped crack. Thermal boundary conditions are standard in that the temperature or heat flux must be known at the surfaces of the crack in such a way that the temperature distribution in the solid is determined uniquely. With this temperature distribution known, introduction of a thermoelastic potential reduces the problem to one in axisymmetric isothermal elasticity with body forces.

For definiteness sake, the circular region connecting the two semi-infinite solids is assumed to be insulated[5] from heat flow, while the crack surface is heated by temperature $T(r)$ that may vary as a function of the radial distance r from the center of the circular region of unit radius. Two special cases are considered in detail. In the first case, $T(r)$ is a constant prescribed over an annulus region surrounding the circle $r = 1$. In the second case, it is assumed that the function $T(r)$ varies according to r^{-n}, where $n > 1$. The problem in which the crack surface is heated by some flux of heat may be solved in the same fashion.

Another objective of this work is to calculate the stress-intensity factors [5] the critical values of which control the onset of crack propagation in brittle materials. For ductile materials, the Dugdale hypothesis [6] may be adopted by assuming that the plastic zone developed at the crack border can be approximated by a very thin layer in the form of a ring. An estimate of the plastic energy dissipation of the crack can also be obtained from the results presented in this paper.

Axisymmetric Equations of Thermoelasticity

Let an external penny-shaped crack be situated in the plane $z = 0$

[5] No additional difficulties are encountered if heat is allowed to flow through the circular region. Alterations in the thermal boundary conditions are discussed in the Appendix.

and be opened out by the application of heat to its surfaces such that the deformation is symmetrical about the z-axis. Referring to cylindrical coordinates (r,θ,z), the stress components are independent of the angle θ, and all derivatives with respect to θ vanish. The components of the displacement vector $\underset{\sim}{u}$ for axially symmetrical deformation are (u,o,w), and the non-vanishing components of the stress tensor $\underset{\sim}{\sigma}$ will be denoted by σ_r, σ_θ, σ_z and τ_{rz}.

If the heat flux vector does not depend on the components of strain, then the displacement equations of equilibrium become

$$2(1-\nu)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+(1-2\nu)\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial r\,\partial z}=2(1+\nu)\alpha\frac{\partial T}{\partial r},$$

$$(1-2\nu)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+2(1-\nu)\frac{\partial^2 w}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)=2(1+\nu)\alpha\frac{\partial T}{\partial z}, \qquad (1)$$

and can be solved independently from the equation of steady-state heat conduction

$$\nabla^2 T(r,z)=0 \; . \qquad (2)$$

Here, T is the temperature increase referred to some reference state and ∇^2 stands for

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2} \; .$$

In eqs. (1), α is the coefficient of linear expansion and ν is Poisson's ratio of the material.

When both the mechanical and thermal properties of the solid are assumed to be isotropic and homogeneous, the stress components may be obtained from the displacement components by means of the Duhamel-Neumann law, which in dyadic notation takes the form

$$\underset{\sim}{\sigma}=\mu\left\{\nabla\underset{\sim}{u}+\underset{\sim}{u}\nabla+\frac{2}{1-2\nu}\left[\nu\,\nabla\cdot\underset{\sim}{u}-(1+\nu)\alpha T\right]\underset{\sim}{I}\right\}, \qquad (3)$$

in which μ is the shear modulus, $\underset{\sim}{I}$ the unit dyad and ∇ the usual del operator.

Kassir and Sih [3][6] have shown that the solution of eqs. (1) may be represented in terms of certain harmonic functions for problems involving surfaces of discontinuities or plane cracks. Suppose that the displacements and stresses induced by T are of the symmetric pattern, then

$$\begin{aligned} &u,\ \sigma_r,\ \sigma_\theta,\ \sigma_z\ ;\ \text{even in } z \\ &w,\ \tau_{rz}\ ;\ \text{odd in } z \end{aligned} \tag{4}$$

Adopting eqs. (12) in [3] to the axisymmetric problem under consideration, the displacements u and w can be expressed in terms of two harmonic functions f(r,z) and Ω(r,z):

$$\begin{aligned} u &= (1-2\nu)\frac{\partial f}{\partial r} + \int_z^\infty \frac{\partial \Omega}{\partial r}\,dz + z\frac{\partial F}{\partial r}, \\ w &= -2(1-\nu)\frac{\partial f}{\partial z} + z\frac{\partial F}{\partial z}, \end{aligned} \tag{5}$$

where

$$F = \Omega + \frac{\partial f}{\partial z},$$

and

$$\nabla^2 f(r,z) = 0, \qquad \nabla^2 \Omega(r,z) = 0.$$

The thermoelastic potential Ω(r,z) can be determined from the temperature field as

$$\frac{\partial \Omega}{\partial z} = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha\, T(r,z), \tag{6}$$

and can be associated with the Boussinesq logarithmic potential for a disk whose boundary conforms to that of a crack. At infinity, although the potential Ω(r,z) is permitted to become unbounded, the regularity condition of the displacement vector requires Ω(r,z) to have bounded derivatives of all orders with respect to r and z. The limits of integration appearing in the first of eqs. (5) were chosen to ensure the boundness of u as $z \to \infty$.

[6] The harmonic-function representation in [3] was developed originally for solving non-axially symmetric problems of plane cracks.

Now, substituting eqs. (5) into (3) yield the following expressions for the stresses:

$$\frac{\sigma_r}{2\mu} = (1-2\nu)\frac{\partial^2 f}{\partial r^2} - 2\nu\frac{\partial^2 f}{\partial z^2} + \int_z^\infty \frac{\partial^2 \Omega}{\partial r^2}dz - 2\frac{\partial \Omega}{\partial z} + z\frac{\partial^2 F}{\partial r^2},$$

$$\frac{\sigma_\theta}{2\mu} = (1-\nu)\frac{1}{r}\frac{\partial f}{\partial r} - 2\nu\frac{\partial^2 f}{\partial z^2} + \frac{1}{r}\int_z^\infty \frac{\partial \Omega}{\partial r}dz - 2\frac{\partial \Omega}{\partial z} + z\frac{1}{r}\frac{\partial F}{\partial r},$$

$$\frac{\sigma_z}{2\mu} = -\frac{\partial F}{\partial z} + z\frac{\partial^2 F}{\partial z^2}, \tag{7}$$

$$\frac{\tau_{rz}}{2\mu} = z\frac{\partial^2 F}{\partial r \partial z}, \quad \tau_{r\theta} = \tau_{\theta z} = 0.$$

Considerations of the eveness and oddness of the displacements and stresses as stated in eqs. (4) together with the prescribed thermal conditions on the crack surfaces reduce the crack problem to one of an elastic half-space with mixed boundary conditions on the plane $z = 0$. In view of symmetry, the plane $z = 0$ must be free from the shearing stress τ_{rz} and $w(r,o)$ must vanish inside the circular region $r \leq 1$. Without loss in generality, the crack surfaces may be assumed to be free from mechanical loads, i.e., $\sigma_z = 0$ for $r \geq 1$ and $z = 0^{\pm}$. The case when the external penny-shaped crack is subjected to surface tractions has already been treated by Lowengrub and Sneddon [7][7], and will not be repeated here. Thus, the requisite thermal and elastic boundary conditions on the plane $z = 0$ are taken to be

$$\frac{\partial T}{\partial z} = 0, \quad 0 \leq r < 1,$$
$$T = T(r), \quad r > 1, \tag{8}$$

and

$$w = 0, \quad 0 \leq r < 1,$$
$$\sigma_z = 0, \quad r > 1, \tag{9}$$
$$\tau_{rz} = 0, \quad 0 \leq r < \infty.$$

[7]In what follows, their solution [7] will be added onto that obtained in this paper for computing the size of the plastic zone at the crack boundary.

It should be mentioned that the antisymmetric problem in which

$$\begin{aligned} u, \sigma_r, \sigma_\theta, \sigma_z &\; ; \text{ odd in } z \\ w, \tau_{rz} &\; ; \text{ even in } z \end{aligned} \tag{10}$$

may also be formulated by following the procedure of Kassir and Sih [3]. Hence, the two problems, one symmetric and the other antisymmetric, may be superimposed to yield the solution to problems of the infinite solid with any thermal conditions specified on the external penny-shaped crack.

Steady-State Temperature Distribution

For a semi-infinite solid $z \geq 0$ that is free from disturbance at infinity, the appropriate solution of eq. (2) is [1]

$$T(r,z) = \int_0^\infty A(s)\, e^{-sz} J_0(rs)\, ds , \quad z \geq 0 . \tag{11}$$

In eq. (11), J_o is the zero order Bessel function of the first kind and A(s) is a function of the parameter s to be determined from the thermal boundary conditions in eqs. (8) with $T(r) = T_o\, g(r)$, where T_o is a constant. The function g(r) is to be bounded at infinity and the integral

$$\int_1^\infty g(r)\, dr ,$$

is to be absolutely convergent.

With the help of eq. (11), the conditions in eqs. (8) lead to the dual integral equations

$$\begin{aligned} \int_0^\infty s A(s)\, J_0(rs)\, ds &= 0 , \quad 0 \leq r < 1 \\ \int_0^\infty A(s)\, J_0(rs)\, ds &= T_0\, g(r) , \quad r > 1 \end{aligned} \tag{12}$$

which determine the only unknown A(s). The solution of these equations has been given by many previous authors[8] and can be found in the open literature:

[8] See for example [8].

$$A(s) = -T_o\left(\frac{2s}{\pi}\right)^{\frac{1}{2}}\int_1^\infty t^{\frac{1}{2}}\,J_{\frac{1}{2}}(st)\,dt\left[\frac{d}{dt}\int_t^\infty \frac{r\,g(r)\,dr}{\sqrt{r^2-t^2}}\right] . \tag{13}$$

Making use of the relation

$$J_{\frac{1}{2}}(st) = \left(\frac{2}{\pi s t}\right)^{\frac{1}{2}} \sin(st) ,$$

and integrating eq. (13) by parts render

$$A(s) = \frac{2}{\pi}T_o\left\{\sin(s)\int_1^\infty \frac{r\,g(r)\,dr}{\sqrt{r^2-1}} + s\int_1^\infty \cos(st)\left[\int_t^\infty \frac{r\,g(r)\,dr}{\sqrt{r^2-t^2}}\right]dt\right\} .$$

For convenience, introduce the function

$$\phi(t) = \frac{2}{\pi}T_o\int_t^\infty \frac{r\,g(r)\,dr}{\sqrt{r^2-t^2}} , \tag{14}$$

so that, after a little manipulation, A(s) becomes

$$A(s) = -\int_1^\infty \sin(st)\,\phi'(t)\,dt , \tag{15}$$

where $\phi'(t) = d\phi/dt$. Eq. (15) may now be inserted into eq. (11) to give the temperature distribution

$$T(r,z) = -\int_0^\infty e^{-sz}J_o(rs)\,ds\left[\int_1^\infty \sin(st)\,\phi'(t)\,dt\right] , \quad z \geq 0 . \tag{16}$$

For the purpose of setting up the mechanical boundary conditions in the subsequent work, it suffices to compute the temperature on the plane z = 0. Hence, after a permissible reversal of the order of the integrations with respect to s and t, and upon using the identity

$$\int_0^\infty \sin(st)\,J_o(rs)\,ds = \begin{cases} (t^2-r^2)^{-\frac{1}{2}} , & r < t < \infty \\ 0 , & 0 \leq t < r \end{cases}$$

it is found that

$$T(r,0) = -\int_1^\infty \frac{\phi'(t)\,dt}{\sqrt{t^2-r^2}} , \quad 0 \leq r < 1 \tag{17}$$

and

$$T(r,o) = -\int_r^{\infty} \frac{\phi'(t)\,dt}{\sqrt{t^2-r^2}} \quad , \quad r > 1 \tag{18}$$

in which $\phi'(t)$ can be calculated from eq. (14) once $g(r)$ is given. Two examples of interest will be cited.

(1) Consider the problem of heating up the faces of an external circular crack over a ring whose inner and outer radii are unity and a, respectively. In this case, $g(r)$ takes the form

$$g(r) = H(a-r) = \begin{cases} 1, & a > r \\ 0, & a < r \end{cases} \quad , \quad r > 1 \tag{19}$$

where $H(r)$ represents the Heaviside step function. A straightforward calculation gives

$$\phi(t) = \frac{2}{\pi} T_o \sqrt{a^2-t^2} \; H(a-t) \quad , \quad t < a$$

and hence $T(r,o)$ may be found from eqs. (17). The result is

$$T(r,o) = \frac{2}{\pi} T_o \int_1^a \frac{t\,dt}{\sqrt{(a^2-t^2)(t^2-r^2)}} = \frac{2}{\pi} T_o \sin^{-1}\left(\frac{a^2-1}{a^2-r^2}\right)^{\frac{1}{2}}, \quad 0 \leq r < 1 \tag{20}$$

and the condition $T(r,o) = T_o$ for $r > 1$ can be verified from eq. (18).

(2) If the temperature variation on the crack faces is such that

$$g(r) = r^{-n} \quad , \quad n > 1 \; ; \; r > 1 \tag{21}$$

then eq. (14) yields

$$\phi(t) = \frac{T_o}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}-\frac{1}{2})}{\Gamma(\frac{n}{2})} t^{1-n} \quad , \quad n > 1$$

where $\Gamma(n)$ is the Gamma function. Putting $\phi(t)$ into eq. (17) and carrying out the integration, $T(r,o)$ is obtained:

$$T(r,o) = \frac{T_o (n-1)\,\Gamma(\frac{n}{2}-\frac{1}{2})}{2\sqrt{\pi}\;\Gamma(\frac{n}{2})} r^{-n} \; B_{r^2}\left(\frac{n}{2},\frac{1}{2}\right), \quad 0 \leq r < 1 \; ; \; n > 1 \tag{22}$$

Note that $B_x(m,n)$ is the incomplete Beta function defined by

$$B_x(m,n) = \int_0^x y^{m-1} (1-y)^{n-1} dy \quad , \quad Re[m] > 0 \ ; \ Re[n] > 0 \ .$$

The complete Beta function B(m,n) may be related to the Gamma functions as

$$B(m,n) = \int_0^1 y^{m-1} (1-y)^{n-1} dy = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \ .$$

Alternatively, the temperature variation in eq. (22) may also be expressed in terms of the Gauss hypergeometric function ${}_2F_1(a,b;c;x)$ through the known identity

$$B_x(m,n) = \frac{1}{m} x^m {}_2F_1(m, 1-n \ ; \ 1+m ; x) \ ,$$

and hence eq. (22) may also be expressed as

$$T(r,o) = \frac{T_o}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)} {}_2F_1(\frac{n}{2}, \frac{1}{2} ; \frac{n}{2}+1 ; r^2) \ , \quad 0 \leq r < 1 \ ; \ n > 1 \tag{23}$$

When n = 2,4,etc., the hypergeometric function in eq. (23) reduces to elementary functions:

(a) <u>n = 2.</u>

$$T(r,o) = \frac{T_o}{r^2} \left[1 - (1-r^2)^{\frac{1}{2}} \right] \ , \quad 0 \leq r < 1$$

(b) <u>n = 4.</u>

$$T(r,o) = \frac{T_o}{r^4} \left[1 - (1+\frac{r^2}{2})(1-r^2)^{\frac{1}{2}} \right] , \quad 0 \leq r < 1$$

Similar expressions of T(r,o) for other values of n may also be deduced, but they will not be considered here. For r > 1, the prescribed temperature distribution of $T(r,o) = T_o r^{-n}$ can be easily recovered by putting $\phi(t)$ into eq. (18).

Temperature distributions corresponding to other types of thermal boundary conditions are worked out in the Appendix.

<u>Thermal Stresses and Displacements</u>

It is seen from eqs. (5) and (7) that the evaluation of the displacements and stresses does not warrant an explicit expression of the thermoelastic potential $\Omega(r,z)$ with respect to r and z.

First of all, eqs. (6) and (11) may be combined to eliminate $T(r,z)$:

$$\frac{\partial\Omega}{\partial z} = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha\int_0^\infty A(s)\, e^{-sz} J_0(rs)\, ds \ , \quad z \geq 0 \ , \tag{24}$$

from which

$$\frac{\partial\Omega}{\partial r} = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha\int_0^\infty A(s)\, e^{-sz} J_1(rs)\, ds \ , \quad z \geq 0 \ , \tag{25}$$

is obtained. The arbitrary function of integration may be set to zero, since $\partial\Omega/\partial r$ must vanish in the limit as $z \to \infty$. Eq. (25) may be integrated with respect to z from o to ∞ giving

$$\int_z^\infty \frac{\partial\Omega}{\partial r} dz = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha\int_0^\infty \frac{1}{s} A(s)\, e^{-sz} J_1(rs)\, ds \ , \quad z \geq 0 \ . \tag{26}$$

Having determined the temperature field $T(r,z)$ or $A(s)$ for various prescribed thermal conditions, it is clear that the quantities

$$\frac{\partial\Omega}{\partial z} \ , \quad \frac{\partial\Omega}{\partial r} \ , \quad \int_z^\infty \frac{\partial\Omega}{\partial r} dz \ , \quad \text{etc.}$$

appearing in eqs. (5) and (7) can be calculated in a straightforward manner.

It is now more pertinent to find the unknown harmonic function $\partial f/\partial z$ from the mechanical boundary conditions in eqs. (9). A quick glance at eqs. (7) reveals that on the plane $z = 0$ τ_{rz} vanishes automatically and the remaining two conditions in eqs. (9) require that

$$\begin{aligned} \frac{\partial f}{\partial z} &= 0 \ , && 0 \leq r < 1 \\ \frac{\partial^2 f}{\partial z^2} &= -\frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha\, T(r,0) \ , && r > 1 \end{aligned} \tag{27}$$

where

$$\frac{\partial f}{\partial z} \ , \quad \frac{\partial^2 f}{\partial z^2} \to 0 \ , \quad \text{as } z \to \infty \ .$$

Taking into account the axisymmetric nature of the thermal loading, $\partial f/\partial z$ may be represented by the Hankel integral

$$\frac{\partial f}{\partial z} = \int_0^\infty \frac{1}{s} B(s)\, e^{-sz} J_0(rs)\, ds \, , \quad z \geq 0 \quad . \tag{28}$$

By virtue of eqs. (27), the function B(s) has to be found from the pair of simultaneous equations

$$\begin{aligned} \int_0^\infty \frac{1}{s} B(s)\; J_0(rs)\, ds &= 0 \, , \qquad 0 \leq r < 1 \\ \int_0^\infty B(s)\, J_0(rs)\, ds &= \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right) \alpha\, T(r) \, , \qquad r > 1 \end{aligned} \tag{29}$$

in which T(r) represents the axisymmetric temperature variation prescribed on the plane z = 0. Lowengrub and Sneddon [8] and others have shown that the satisfaction of eqs. (29) can be achieved by expressing B(s) in terms of the function

$$\psi(t) = \left(\frac{1+\nu}{1-\nu}\right) \frac{\alpha}{\pi} \int_t^\infty \frac{\eta\, T(\eta)\, d\eta}{\sqrt{\eta^2 - t^2}} \, , \tag{30}$$

through an integral of the form

$$B(s) = s \int_1^\infty \psi(t) \cos(st)\, dt \, , \tag{31}$$

With a knowledge of B(s), the problem of determining the displacements and stresses in the elastic solid is reduced to quadrature.

For the purpose of finding the displacements on the crack surfaces, eq. (31) is inserted into eq. (28):

$$\frac{\partial f}{\partial z} = \int_0^\infty \left[\int_1^\infty \psi(t) \cos(st)\, dt \right] e^{-sz} J_0(rs)\, ds \, , \quad z \geq 0 \tag{32}$$

Upon differentiation and integration with respect to the variables r and z, respectively, $\partial f/\partial r$ is derived:

$$\frac{\partial f}{\partial r} = \int_0^{\infty}\left[\int_1^{\infty}\psi(t)\cos(st)\,dt\right] e^{-sz} J_1(rs)\,ds, \quad z \geq 0 \,. \tag{33}$$

The condition $\partial f/\partial r \to 0$ as $z \to \infty$ has been enforced by neglecting the arbitrary function of r, which arises from the process of integration. Setting $z = 0$ and applying the identities

$$\int_0^{\infty}\cos(st)\,J_0(rs)\,ds = \begin{cases}(r^2-t^2)^{-\frac{1}{2}}, & r > t\\ 0, & r < t\end{cases}$$

and

$$\int_0^{\infty}\cos(st)\,J_1(rs)\,ds = \begin{cases}r^{-1}, & r > t\\ r^{-1}\left[1 - t(t^2-r^2)^{-\frac{1}{2}}\right], & r < t\end{cases}$$

eqs. (32) and (33) may be written as

$$\frac{\partial f}{\partial z} = \begin{cases}\displaystyle\int_1^{r}\frac{\psi(t)\,dt}{\sqrt{r^2-t^2}}, & r > 1\\ 0, & 0 \leq r < 1\end{cases} \tag{34}$$

and

$$\frac{\partial f}{\partial r} = \begin{cases}\displaystyle\frac{1}{r}\int_1^{r}\psi(t)\,dt, & r > 1\\ \displaystyle\frac{1}{r}\int_1^{\infty}\left[1 - t(t^2-r^2)^{-\frac{1}{2}}\right]\psi(t)\,dt, & 0 \leq r < 1\end{cases} \tag{35}$$

Hence, eqs. (26), (34) and (35) may be substituted into eqs. (5) and the resulting expressions for the displacements on the crack plane are

$$u(r,0) = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha\int_0^{\infty}\frac{1}{s}A(s)\,J_1(rs)\,ds + (1-2\nu)\frac{1}{r}\int_1^{r}\psi(t)\,dt, \quad r > 1$$

$$w(r,0) = -2(1-\nu)\int_1^{r}\frac{\psi(t)\,dt}{\sqrt{r^2-t^2}}, \quad r > 1 \tag{36}$$

Similarly, the displacements u and w for points inside the circular region of unit radius can be found:

$$u(r,0) = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha\int_0^\infty \frac{1}{s}A(s)\,J_1(rs)\,ds + (1-2\nu)\frac{1}{r}\int_0^\infty \left[1-t(t^2-r^2)^{-\frac{1}{2}}\right]\psi(t)\,dt\,, \quad 0\le r<1$$

$$w(r,0) = 0\,, \qquad 0 \le r < 1 \tag{37}$$

The functions A(s) and $\psi(t)$ in the above expressions are defined by eqs. (15) and (30), respectively.

Of particular interest is the stress component σ_z from which the crack-border stress-intensity factor formula may be extracted. This factor has been known to control the instability behavior of cracks in the theory of brittle fracture [5]. To this end, eqs. (6) and (32) are substituted into the third of eqs. (7) and hence for z = 0 σ_z becomes

$$\sigma_z(r,0) = 2\mu\left\{\int_0^\infty \left[s\int_1^\infty \psi(t)\cos(st)\,dt\right] J_0(rs)\,ds - \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha T(r)\right\}.$$

Therefore, it is not difficult to show that

$$\sigma_z(r,0) = -2\mu\left[\frac{\psi(r)}{\sqrt{1-r^2}} + \int_1^\infty \frac{\psi'(t)\,dt}{\sqrt{t^2-r^2}} + \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha T(r)\right], \quad 0\le r<1 \tag{38}$$

and $\sigma_z(r,o) = 0$ for $r > 1$. Notice that only the leading term in eq. (38) contributes to the singular behavior of σ_z, while the other two expressions are bounded as $r \to 1$. Thus, by letting $\epsilon = 1 - r$ and $\epsilon \to o$, σ_z becomes

$$\sigma_z(r,0) = -\frac{2\mu\,\psi(1)}{\sqrt{2\epsilon}} + \cdots$$

where terms of order higher than $\epsilon^{-1/2}$ have been dropped. The coefficient of $1/\sqrt{2\epsilon}$, say k_1, is the crack-border stress-intensity factor for the opening mode of crack extension, i.e.,

$$k_1 = -2\mu\,\psi(1) = -\left(\frac{1+\nu}{1-\nu}\right)\frac{2\mu\alpha}{\pi}\int_1^\infty \frac{\eta\,T(\eta)\,d\eta}{\sqrt{\eta^2-1}}\,. \tag{39}$$

By the same procedure, the other stress components may also be expressed in terms of T(r), the prescribed temperature distribution on the crack.

External Crack Around Insulated Circular Region

To fix ideas, the displacements and stresses on the plane z = 0 corresponding to the temperature distribution discussed earlier may be expressed explicitly in terms of $T(r) = T_o\, g(r)$. Appropriate elimination of A(s) and $\psi(t)$ in eqs. (36) and (37) gives the displacement field

$$u(r,0) = \left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha T_o}{\pi r}\begin{cases}(1-\sqrt{1-r^2})\displaystyle\int_1^\infty \frac{\eta g(\eta)d\eta}{\sqrt{\eta^2-1}} + 2(1-\nu)\int_1^\infty \left[1-\frac{t}{\sqrt{t^2-r^2}}\right]\left[\int_t^\infty \frac{\eta g(\eta)d\eta}{\sqrt{\eta^2-t^2}}\right]dt, & 0 \le r < 1 \\ \displaystyle\int_1^\infty \frac{\eta g(\eta)d\eta}{\sqrt{\eta^2-1}} + 2(1-\nu)\int_1^r \left[\int_t^\infty \frac{\eta g(\eta)d\eta}{\sqrt{\eta^2-t^2}}\right]dt, & r > 1\end{cases} \tag{40}$$

and

$$W(r,0) = -\frac{2(1+\nu)\alpha T_o}{\pi}\begin{cases}0, & 0 \le r < 1 \\ \displaystyle\int_1^r \left[\int_1^\infty \frac{\eta g(\eta)d\eta}{\sqrt{\eta^2-t^2}}\right]\frac{dt}{\sqrt{r^2-t^2}}, & r > 1\end{cases} \tag{41}$$

Eq. (30) may be combined with eq. (38) to put the normal stress component in the form

$$\sigma_z(r,0) = -\frac{E\alpha T_o}{(1-\nu)\pi}(1-r^2)^{-\frac{1}{2}}\begin{cases}\displaystyle\int_1^\infty \frac{\eta g(\eta)d\eta}{\sqrt{\eta^2-1}}, & 0 \le r < 1 \\ 0, & r > 1\end{cases} \tag{42}$$

where E is Young's modulus of elasticity. In deriving eq. (42), it is interesting to note that the two non-singular terms in eq. (38) cancelled each other.

Let g(r) in eqs. (40) through (42) be given by eqs. (19) and (21). The calculation of u, w and σ_z involves a considerable amount of detailed work which will be omitted. The final results are:

(1) Step function.

In this case, the displacement component in the radial direction is

$$u(r,0)=\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha T_0}{\pi r}\begin{cases}\nu\sqrt{a^2-1}\,\left(1-\sqrt{1-r^2}\right)+(1-\nu)a^2\left[\sin^{-1}\left(\frac{a^2-1}{a^2}\right)^{\frac{1}{2}}-\left(1-\frac{r^2}{a^2}\right)\sin^{-1}\left(\frac{a^2-1}{a^2-r^2}\right)^{\frac{1}{2}}\right], & 0\le r<1\\ \nu\sqrt{a^2-1}+(1-\nu)a^2\left[\sin^{-1}\left(\frac{r}{a}\right)-\sin^{-1}\left(\frac{1}{a}\right)+\frac{r}{a}\sqrt{1-\frac{r^2}{a^2}}\;\right], & 1<r<a\\ \nu\sqrt{a^2-1}+(1-\nu)\,a^2\sin^{-1}\left(\frac{a^2-1}{a^2}\right)^{\frac{1}{2}}, & r>a\end{cases} \tag{43}$$

and the normal displacement component is given by

$$W(r,0)=-\frac{2(1+\nu)\alpha r T_0}{\pi}\begin{cases}0, & 0\le r<1\\ \frac{a}{r}\left[E\left(\frac{r}{a},\frac{\pi}{2}\right)-E\left(\frac{r}{a},\alpha_1\right)\right], & 1<r<a\\ E\left(\frac{a}{r},\frac{\pi}{2}\right)-E\left(\frac{a}{r},\alpha_2\right)-\left(1-\frac{a^2}{r^2}\right)\left[K\left(\frac{a}{r},\frac{\pi}{2}\right)-K\left(\frac{a}{r},\alpha_2\right)\right], & r>a\end{cases} \tag{44}$$

in which $E(\frac{r}{a},\alpha_1)$ and $K(\frac{a}{r},\alpha_2)$ are the incomplete elliptic integrals of the second and first kind, respectively, where

$$\alpha_1=\sin^{-1}\left(\frac{1}{r}\right),\quad 0<\alpha_1<\frac{\pi}{2}\qquad \text{and}\qquad \alpha_2=\sin^{-1}\left(\frac{1}{a}\right),\quad 0<\alpha_2<\frac{\pi}{2}$$

When α_1 or $\alpha_2 \to \pi/2$, E and K become the complete elliptic integrals. The normal stress component is

$$\sigma_z(r,0)=-\frac{E\alpha T_0}{(1-\nu)\pi}(1-r^2)^{-\frac{1}{2}}\begin{cases}\sqrt{a^2-1}, & 0\le r<1\\ 0, & r>1\end{cases} \tag{45}$$

and it follows that

$$k_1=-\frac{E\alpha T_0}{(1-\nu)\pi}\sqrt{a^2-1}\,. \tag{46}$$

(2) Radial decay.

If the temperature on the crack varies in accordance with eq. (21), then for $n > 2$

$$u(r,0) = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha T_0\,\Gamma(\frac{n}{2}-\frac{1}{2})}{\sqrt{\pi}\,r\,\Gamma(\frac{n}{2})}\begin{cases} 1-\sqrt{1-r^2}+(1-\nu)\left[\frac{2}{n-2} - r^{2-n}B_{r^2}(\frac{n}{2}-1,\frac{1}{2})\right], & 0\le r<1 \\ 1+\frac{2(1-\nu)}{n-2}\,(1-r^{2-n}), & r>1 \end{cases} \tag{47}$$

and for n > 1

$$W(r,0) = -\frac{(1+\nu)\alpha T_0\,\Gamma(\frac{n}{2}-\frac{1}{2})}{2\sqrt{\pi}\,\Gamma(\frac{n}{2})}\begin{cases} 0, & 0\le r<1 \\ r^{1-n}\,B_{1-r^{-2}}(\frac{1}{2},1-\frac{n}{2}), & r>1 \end{cases} \tag{48}$$

As before, the incomplete Beta function $B_x(m,n)$ in eqs. (47) and (48) may be related to the hypergeometric function as follows:

$$u(r,0) = \frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha T_0\,\Gamma(\frac{n}{2}-\frac{1}{2})}{\sqrt{\pi}\,r\,\Gamma(\frac{n}{2})}\left\{1-\sqrt{1-r^2}+\frac{2(1-\nu)}{n-2}\left[1-{}_2F_1(\tfrac{n}{2}-1,\tfrac{1}{2};\tfrac{n}{2};r^2)\right]\right\},\quad 0\le r<1;\ n>2 \tag{49}$$

and

$$W(r,0) = -\frac{(1+\nu)\alpha T_0\,\Gamma(\frac{n}{2}-\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\frac{n}{2})}\,\frac{\sqrt{r^2-1}}{r^n}\;{}_2F_1(\tfrac{1}{2},\tfrac{n}{2};\tfrac{3}{2};\tfrac{r^2-1}{r^2}),\quad r>1;\ n>1 \tag{50}$$

For $g(r) = r^{-n}$ and $n > 1$, eq. (42) reduces to

$$\sigma_z(r,0) = -\frac{E\alpha T_0}{(1-\nu)\pi}\,(1-r^2)^{-\frac{1}{2}}\begin{cases} \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\frac{n}{2}-\frac{1}{2})}{\Gamma(\frac{n}{2})}, & 0\le r<1 \\ 0, & r>1 \end{cases} \tag{51}$$

Therefore, the k_1-factor is obtained:

$$k_1 = -\frac{E\alpha T_0\,\Gamma(\frac{n}{2}-\frac{1}{2})}{2(1-\nu)\sqrt{\pi}\,\Gamma(\frac{n}{2})},\qquad n>1 \tag{52}$$

To recapitulate, the stress-intensity factors given in eqs. (46) and (52) can be associated with the forces which motivate and produce crack extension owing to thermal disturbances. The critical values of k_1 for a particular material can usually be measured experimentally. Moreover, if the material undergoes plastic yielding at the crack border, where the thermal stresses are exceedingly high, there will be a localized zone of plasticity surrounding the periphery of the crack. The size of this plastic zone for an external penny-shaped crack will be estimated in the next section.

Thermal Plastic Zone Size

An ideal elastic-plastic model for the plane extension problem of a straight crack in a thin sheet has been proposed by Dugdale [6]. This model will be adopted to estimate the extent of plastic yielding at the edge of an external circular crack. The material near the crack is assumed to flow after yielding at a constant tensile stress q_o and the plastic zone is confined to a thin layer of width ω around the uncracked portion of the plane $z = 0$. The parameter ω will be determined from the finiteness condition of σ_z at the leading edge of the plastic zone.

Mathematically, the solid may be assumed to deform elastically under the action of thermal loading together with a mechanical compressive stress, $-q_o$, distributed over the surface of a ring of inner radius $r = 1$ and outer radius $r = 1 + \omega$. For this problem, σ_z can be obtained by superimposing the solution of Lowengrub and Sneddon [7] onto that of eq. (42). The normal stress component for the combined thermal and mechanical problem is

$$\sigma_z(r,0) = \begin{cases} \dfrac{E}{\sqrt{1-r^2}}\left[-\dfrac{\alpha T_o}{(1-\nu)\pi}\displaystyle\int_1^\infty \dfrac{\eta\, q(\eta)\, d\eta}{\sqrt{\eta^2-t^2}} + \dfrac{\phi_1(1)}{1+\nu}\right] + \dfrac{E}{1+\nu}\displaystyle\int_1^\infty \dfrac{\phi_1'(t)\, dt}{\sqrt{t^2-r^2}}\,, & 0 \le r < 1 \\ -p(r)\,, & r > 1 \end{cases} \tag{53}$$

where

$$\phi_1(t) = \frac{1}{\pi\mu}\int_t^\infty \frac{\xi\, p(\xi)\, d\xi}{\sqrt{\xi^2-t^2}}\,,$$

and

$$p(r) = -q_o\, H(1+\omega-r)\,.$$

Since σ_z is to be bounded at $r = 1$, the singular terms in eq. (53) must be removed by taking

$$\alpha\mu T_o \int_1^\infty \frac{\eta\, g(\eta)\, d\eta}{\sqrt{\eta^2 - t^2}} = \left(\frac{1-\nu}{1+\nu}\right) \int_1^\infty \xi \frac{p(\xi)\, d\xi}{\sqrt{\xi^2 - t^2}} \quad .$$

Setting $2\mu(1+\nu) = E$ and performing the integration with respect to ξ lead to the equation

$$E\alpha T_o \int_1^\infty \frac{\eta\, g(\eta)\, d\eta}{\sqrt{\eta^2 - t^2}} = -2(1-\nu)\, q_o \sqrt{\omega(\omega+2)} \quad , \tag{54}$$

for evaluating the plastic zone size ω. For illustration, formulas for ω are worked out for the two previously mentioned examples.

(1) If $g(r) = H(1+\beta-r)$, then eq. (54) may be integrated and solved for ω:

$$\omega = -1 + \left[1 + \beta(\beta+2)\gamma^2\right]^{\frac{1}{2}} \quad . \tag{55}$$

The quantity

$$\gamma = -\frac{E\alpha T_o}{2(1-\nu)\, q_o} \quad ,$$

may be interpretated as the ratio of the applied thermal stress to the yield stress of the material q_o, and β in eq. (55) is the width of the region heated by the constant temperature T_o. A plot of ω versus γ for various values of β are shown in Fig. 1. The curves are similar in trend to that found by Dugdale [6] for the two-dimensional problem of an isothermal crack.

(2) For $g(r) = r^{-n}$ with $n > 1$, the plastic zone size is found to be

$$\omega = -1 + \left\{1 + \frac{\pi}{4}\left[\gamma\, \frac{\Gamma(\frac{n}{2}-\frac{1}{2})}{\Gamma(\frac{n}{2})}\right]^2\right\}^{\frac{1}{2}} \quad , \tag{56}$$

whose variations with γ for different values of n are plotted in Fig. 2. As to be expected, the size of the plastic zone increases as the temperature T_o is raised.

Concluding Remarks

The linear thermoelastic problem of an elastic solid containing an external penny-shaped crack has been formulated and solved. The temperature and/or heat flux can be applied either symmetrically or antisymmetrically with respect to the plane in which the crack occupies. The solution offers the possibility of a theory of brittle fracture for crack propagation caused by heating. This can be verified by experimentally measuring the critical values of the stress-intensity factors as proposed earlier.

The obtained displacement field also permits an evaluation of the plastic energy dissipation for cracking induced by thermal stresses. This, however, will be treated separately in another paper.

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APPENDIX

Temperature fields pertaining to thermal boundary conditions not covered in the text will be presented below.

Case A. Instead of applying temperature to the crack, heat flux may be specified on the flat surfaces r > 1 and $z = 0^{\pm}$. The distribution of temperature that satisfies the set of conditions

$$\frac{\partial T}{\partial z} = \begin{cases} 0 , & 0 \leq r < 1 \\ Q(r) , & r > 1 \end{cases} \tag{57}$$

is given by

$$T(r,z) = -\int_0^{\infty} \left[\int_1^{\infty} \eta\, Q(\eta)\, J(s\eta)\, d\eta \right] e^{-sz} J_0(rs)\, ds \ , \quad z \geq 0 \ . \tag{58}$$

Consider two special cases of Q(r):

(1) Suppose that $Q(r) = Q_o H(a-r)$, where Q_o is a constant. Then, it can be shown that for z = 0

$$T(r,0) = \frac{2Q_o}{\pi} \begin{cases} E(r) , & 0 < r < 1 \\ r\left[K(\frac{1}{r}) - (1-\frac{1}{r^2}) E(\frac{1}{r}) \right] , & 1 < r < \infty \end{cases} \tag{59}$$

and

$$T(r,0) = -\frac{2aQ_o}{\pi} \begin{cases} E(\frac{r}{a}) , & 0 < r < a \\ \frac{r}{a}\left[K(\frac{a}{r}) - (1-\frac{a^2}{r^2}) E(\frac{a}{r}) \right] , & a < r < \infty \end{cases} \tag{60}$$

The complete elliptical integrals of the first and second kind are denoted by K and E.

(2) In the case, when $Q(r) = Q_o r^{-n}$ with n > 1, the temperature field is

$$T(r,0) = \begin{cases} \frac{1}{1-n}\, {}_3F_2(\frac{1}{2}, \frac{1}{2}, \frac{n}{2}-\frac{1}{2}; 1, 1; r^2) , & 0 < r < 1 \ ; \ n > 1 \\ \frac{1}{(2-n)r}\left[{}_3F_2(\frac{1}{2}, \frac{1}{2}, 1-\frac{n}{2}; 1, 1; \frac{1}{r^2}) - r^{2-n}\, {}_3F_2(\frac{1}{2}, \frac{1}{2}, 1-\frac{n}{2}; 1, 2-\frac{n}{2}; 1) \right] , & r > 1 ; \ 1 < n < 2 \end{cases} \tag{61}$$

The second of eqs. (61) is valid only for values of n between i and 2 and hence it is somewhat limited in application.

Case B. Another possible case is when the uncracked region $o \leq r < 1$ and $z = 0$ is permitted to conduct heat such that

$$T = \begin{cases} 0, & 0 \leq r < 1 \\ T_0 h(r), & r > 1 \end{cases} \tag{62}$$

and thus

$$T(r,z) = T_0 \int_0^\infty \left[\int_1^\infty \eta \, h(\eta) \, J_0(s\eta) \, d\eta \right] s \, e^{-sz} J_0(rs) \, ds \, , \quad z \geq 0 \, . \tag{63}$$

(1) For $h(r) = H(a-r)$, eq. (63) is expressible in terms of the Lipschitz-Hankel type of integrals

$$T(r,z) = T_0 \left[a \int_0^\infty e^{-sz} J_1(as) \, J_0(rs) \, ds - \int_0^\infty e^{-sz} J_1(s) \, J_0(rs) \, ds \right], \quad z \geq 0 . \tag{64}$$

These integrals have been evaluated and tabulated numerically in [9].

(2) If $h(r) = r^{-n}$ and $n > 1$, the temperature field becomes

$$T(r,z) = T_0 \int_0^\infty \left[n \, I_1^*(m,s) - J_1(s) \right] e^{-sz} J_0(rs) \, ds , \quad z \geq 0 \tag{65}$$

where

$$I_1^*(m,s) = \int_1^\infty \xi^{-m} J_1(s\xi) \, d\xi \, .$$

Titles of Figures

Fig. 1 . Widths of plastic zone for constant heating.

Fig. 2. Widths of plastic zone for temperature decaying radially.

Fig. 1. Widths of plastic zone for constant heating.

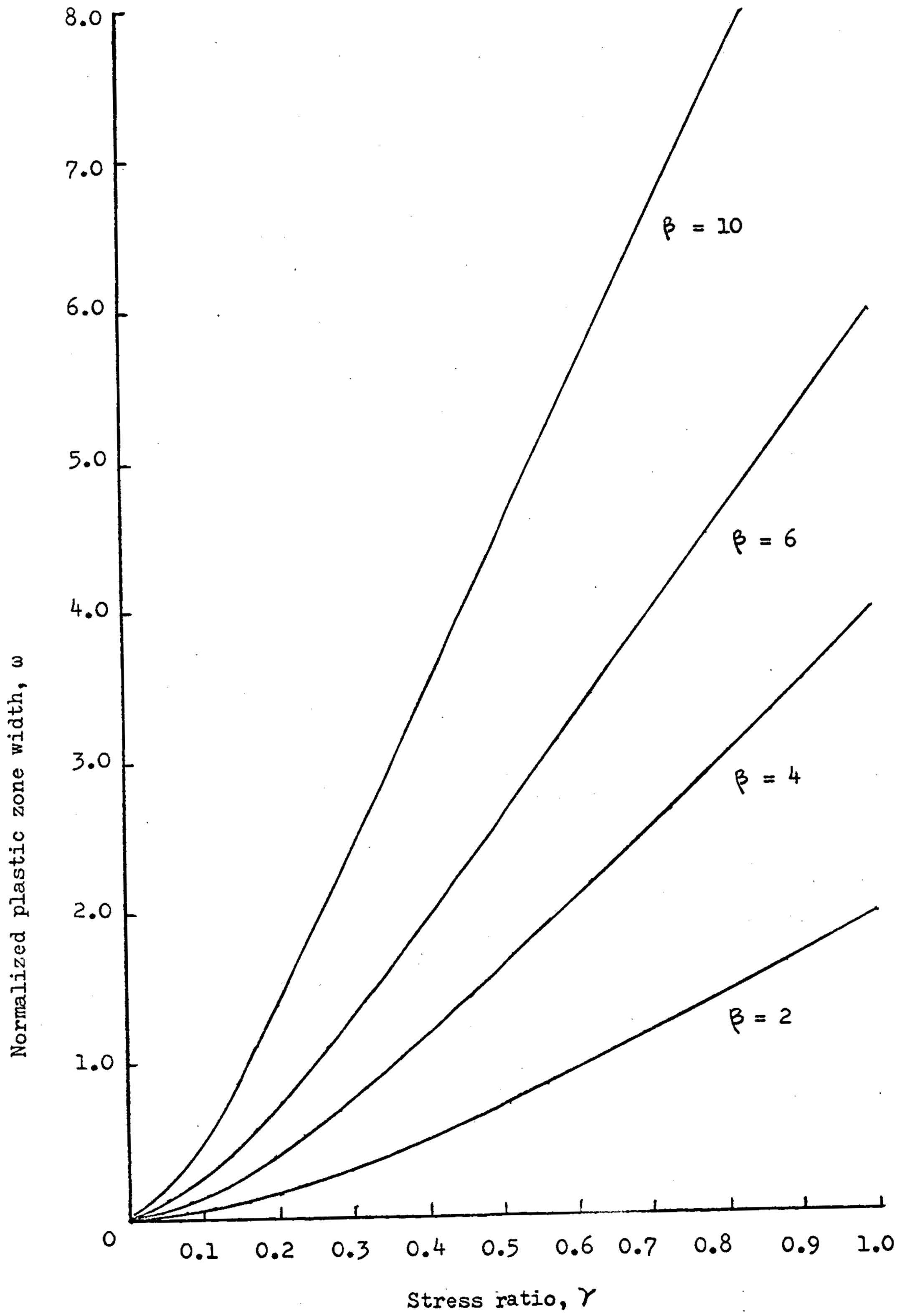

Fig. 2. Widths of plastic zone for temperature decaying radially.