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CONJUGATE POINTS ON EXTREMAL ROCKET PATHS*

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June 1968

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Abstract

Conjugate points on extremal finite thrust rocket trajectories in a central gravitational field are investigated using both analytical and numerical methods. When three state variables (two velocity components and the radius) are considered, it appears that conjugate points occur only after the circumferential velocity component has changed sign, irrespective of the choice of initial conditions, a result suggested by Darboux's theorem and symmetry considerations. Conjugate points are also found in problems featuring four state variables (two position and two velocity components), although no general qualitative rules governing them have been deduced. Numerical results are presented for launches from the moon.

Nomenclature

B	function defined by Eq. (8b)
C	function defined by Eq. (8c)
g	acceleration of gravity at r_o
g_e	32.174
I_{sp}	specific impulse
m	mass, $m_o\left[1 - (T/W_o)(t/I_{sp})\right]$
r	radius
T	thrust
t	time
u	radial velocity component
v	circumferential velocity component
W	weight, $g_e m$
x_i	state variables; u, v, r, θ
θ	central angle measured from an arbitrary reference line drawn through the center of attraction
λ_i	Lagrange multipliers
ψ	thrust direction angle measured from the local horizontal
$(\ )_f$	final value of a variable
$(\ )_o$	initial value of a variable
$(\dot{\ })$	time derivative of a variable

Introduction

Of the methods in use for optimizing rocket trajectories, many do not test whether or not the Jacobi necessary condition is satisfied or otherwise guarantee the minimizing character of the extremal obtained. Thus the question of possible occurrence of conjugate points on rocket extremals is of some interest, and in the present paper we investigate the simplified case of planar vacuum flight in an inverse square law gravity field under constant thrust. The results indicate that conjugate points do indeed occur and shed some light upon the circumstances incident to their appearance.

In the following we briefly review the first order necessary conditions and the conjugate point condition for the rocket trajectory problem. We then demonstrate that for Mayer problems the conjugate point test may be performed using a Lagrange multiplier and the determinant of a single minor of the usual test matrix. The situation for extremals in u, v, r-space, of interest for commonly occurring problems featuring central angle unspecified, is then examined. Vertical launch trajectories are shown to be extremals. The state variation defined by an extremal adjacent to this trajectory obeys a single second order differential equation which may exhibit an oscillatory solution whose zeros specify conjugate points. It is proven analytically that, for the case of nonvertical launch with zero initial velocity, the sign of the circumferential velocity component must change before a conjugate point occurs. It is conjectured that this is also true for trajectories with arbitrary initial velocity. A plausibility argument and numerical results are offered in substantiation. Numerical confirmation of the existence of conjugate points in u, v, r, θ-space is also presented.

The problems studied in Refs. 1, 2, and 3 exhibit features that we will find on extremal rocket paths also. These simple problems are instructive because their two-dimensional character permits exhaustive plotting. The problem of Ref. 1 exhibits a line of symmetry, the initial point is on this line, and conjugate points occur on extremals after they have crossed it. These features also characterize the problem of Ref. 2, and this paper investigates a starting point which is off the line of symmetry. In this case the conjugate points still do not occur until after the line of symmetry has been crossed. The extremal family of Ref. 3, whose members extend beyond envelope contacts which are also corners, is of interest since three exceptional members continue to minimize beyond the envelope contact points. The Jacobi necessary condition does not apply to these arcs in the absence of the smoothness properties assumed in the classical derivation of the condition.

Extremal Rocket Paths

The planar flight of a constant thrust nonthrottleable rocket in a central gravitational field is governed by the following equations.

$$\dot{u} = \frac{v^2}{r} - g\left(\frac{r_o}{r}\right)^2 + \frac{g_e (T/W_o)\sin\psi}{1 - (T/W_o)(t/I_{sp})} \tag{1a}$$

$$\dot{v} = -\frac{uv}{r} + \frac{g_e (T/W_o)\cos\psi}{1 - (T/W_o)(t/I_{sp})} \tag{1b}$$

$$\dot{r} = u \tag{1c}$$

$$\dot{\theta} = v/r \tag{1d}$$

The quantities u, v, r, θ are state variables and will sometimes be designated by the generic symbol x_i. The Lagrange multipliers λ_i associated with the state variables obey

$$\dot{\lambda}_u = (v/r)\lambda_v - \lambda_r \tag{2a}$$

$$\dot{\lambda}_v = (-2v\lambda_u + u\lambda_v - \lambda_\theta)/r \tag{2b}$$

$$\dot{\lambda}_r = (v^2\lambda_u - uv\lambda_v + v\lambda_\theta)/r^2 - (2gr_o^2/r^3)\lambda_u \tag{2c}$$

$$\dot{\lambda}_\theta = 0 \tag{2d}$$

In accordance with the Pontryagin maximum principle, the control variable ψ is obtained from

$$\tan \psi = \lambda_u / \lambda_v \tag{3}$$

with the quadrant determined uniquely by the signs of λ_u and λ_v.

The extremal family associated with the initial point $x_i(0)$ may be constructed by integrating the system of equations (1), (2), (3) with various sets of values for the $\lambda_i(0)$. When these extremals are stopped at a fixed final time their end points define a three-dimensional manifold which may be termed the isochronal hypersurface.

The Jacobi Necessary Condition

Consider the family of extremals that originate at a given point in state variable space. A point at which an extremal intersects one of its neighbors as it contacts an envelope is called a conjugate point. The Jacobi necessary condition[4] states that if a normal nonsingular extremal without corners provides a relative minimum, then no conjugate point precedes its terminal point.

For purposes of locating conjugate points Eqs. (1), (2), and (3) are linearized:

$$\delta\dot{u} = \frac{2v}{r}\,\delta v - \left(\frac{v^2}{r^2} + g\,\frac{r_o^2}{r^3}\right)\delta r + \frac{g_e (T/W_o)\cos\psi}{1 - (T/W_o)(t/I_{sp})}\,\delta\psi \tag{4a}$$

$$\delta\dot{v} = -\frac{v}{r}\,\delta u - \frac{u}{r}\,\delta v + \frac{uv}{r^2}\,\delta r - \frac{g_e (T/W_o)\sin\psi}{1 - (T/W_o)(t/I_{sp})}\,\delta\psi \tag{4b}$$

$$\delta\dot{r} = \delta u \tag{4c}$$

$$\delta\dot{\theta} = r^{-1}\delta v - v r^{-2}\delta r \tag{4d}$$

$$\delta\dot{\lambda}_u = r^{-1}\lambda_v\delta v - vr^{-2}\lambda_v\delta r + vr^{-1}\delta\lambda_v - \delta\lambda_r \tag{5a}$$

$$\begin{aligned}\delta\dot{\lambda}_v &= (-2\lambda_u\delta v - 2v\delta\lambda_u + \lambda_v\delta u + u\delta\lambda_v - \delta\lambda_\theta)/r \\ &\quad + (2v\lambda_u - u\lambda_v + \lambda_\theta)r^{-2}\delta r\end{aligned} \tag{5b}$$

$$\begin{aligned}\delta\dot{\lambda}_r &= \left[(2v\lambda_u - u\lambda_v + \lambda_\theta)\delta v + v^2\delta\lambda_u - v\lambda_v\delta u - uv\delta\lambda_v + v\delta\lambda_\theta\right]/r^2 \\ &\quad - \left[2(v^2\lambda_u - uv\lambda_v + v\lambda_\theta)/r^3 - 6gr_o^2r^{-4}\lambda_u\right]\delta r - 2gr_o^2r^{-3}\delta\lambda_u\end{aligned} \tag{5c}$$

$$\delta\dot{\lambda}_\theta = 0 \tag{5d}$$

$$\delta\psi = (\lambda_v\delta\lambda_u - \lambda_u\delta\lambda_v)/(\lambda_u^2 + \lambda_v^2) \tag{6}$$

Four sets of initial conditions are assigned to this differential system. The $\delta x_i(0)$ of each set are all zero while the $\delta\lambda_i(0)$ of the j^{th} set are equated to Kronecker's δ_{ij}. A 4×4 matrix of $\delta x_i(t)$ rows is formed by integrating Eqs. (4), (5), and (6) with these initial conditions simultaneously with Eqs. (1), (2), and (3). The element in the j^{th} row and i^{th} column of this test matrix is thus equal to $\partial x_i(t)/\partial\lambda_j(0)$.

The four $\delta x_i(t)$ vectors lie in the hyperplane tangent to the isochronal defined in the previous section. Thus the $\delta x_i(t)$ vectors span at most three dimensions and the rank of the 4×4 test matrix is at most three. These properties may be derived analytically by means of the first integral $\sum_{i=1}^{4} \delta x_i(t)\lambda_i(t) = \text{constant}$.

(The latter relation may be checked by deriving and using Eqs. (2), (3), and (4).) Since the δx_i vanish initially we have

$$\sum_{i=1}^{4} \delta x_i(t)\ \lambda_i(t) = 0 \tag{7}$$

so that the four $\delta x_i(t)$ vectors are linearly dependent.

If at a particular time neighboring extremals have identical values for all of the $x_i(t)$ except one (designated $x_j(t)$), then from Eq. (7) the values of $x_j(t)$ must also coincide — unless $\lambda_j(t)$ is zero. Therefore to test for conjugate points one first establishes whether or not it is possible to cover with extremals the projection on a three dimensional space of the neighborhood of each point $x_i(t)$, $t_o < t < t_f$ lying on the nominal extremal. If the three-space chosen excludes the j^{th} dimension, then one examines the determinant of the minor of any of the elements of the j^{th} column of the test matrix $\partial x_i(t)/\partial\lambda_j(0)$. Whenever the determinant is zero there is an adjacent extremal with three of its $x_i(t)$ identical to those of the nominal, and if $\lambda_j(t)$ is nonzero the fourth coordinate also coincides. Thus when both conditions hold the determinants of the fifteen other minors must also vanish and the test matrix degenerates from rank three to rank two. It is possible to generalize this statement to abnormal problems and assert that whenever the test matrix degenerates in rank from any value as time increases an intersecting neighboring extremal exists.

This procedure may be found in Ref. 5 in application to the normal case and is the extension to Mayer problems of the test used in Lagrange problems presented in publications such as Ref. 4. It applies to all fixed initial point problems regardless of whether the payoff conditions call for a fixed or variable final point.

Vertical Trajectories in u, v, r-Space

Since the central angle θ does not appear in many engineering problems it is of some interest to consider trajectories described by merely three state variables — u, v, and r. Therefore, throughout this section and the next Eq. (1d) will be ignored and λ_θ will be set to zero, which is possible because λ_θ is constant by virtue of Eq. (2d). The special case of purely vertical motion will be examined first.

Euler Condition

Suppose that the following initial conditions are assigned to the state variables and multipliers: $u = v = 0$, $r = r_o$, $\lambda_u > 0$, $\lambda_v = 0$, $\lambda_r < 0$. It can be seen from Eqs. (1), (2), and (3) that the resulting extremal path will be vertical and will be upward provided that the acceleration due to the thrust exceeds that due to gravity. The algebraic signs that the λ_i's possess at the initial time will persist throughout the trajectory however long it may be maintained. Vertical ascents may also be obtained when λ_r is initially positive, although λ_u may in this case eventually become negative and cause a reversal of the thrust direction. The arc will have a corner at the reversal point. The term vertical will be employed only for those vertical paths that do not have corners.

Thus a vertical trajectory is an extremal and by virtue of the freedom of choice of the multiplier initial values has, apart from scaling factor choice, a one-parameter family of multiplier vectors. Because there is a one-parameter family of multiplier vector directions and because the relation (7) implies orthogonality between the isochronal and the multiplier vector, a corner is exhibited by the isochronal at the point associated with the vertical path.

Jacobi Condition

Since the initial value of λ_r/λ_u associated with the vertical trajectory is not unique, a particular initial λ_r/λ_u will generate the same trajectory as $(\lambda_r + \delta\lambda_r)/\lambda_u$ or $\lambda_r/(\lambda_u + \delta\lambda_y)$. Consequently, the two sets of δx_i obtained from Eqs. (4) with $\delta\lambda_u(0)$ nonzero in one case and $\delta\lambda_r(0)$ nonzero in the other case vanish for all t. When $\delta\lambda_v(0) = 1$, $\delta u(t)$ and $\delta r(t)$ are identically zero, but not $\delta v(t)$. Thus in the 3 x 3 test matrix of the Jacobi condition there is but one nonzero element. A conjugate point occurs when the matrix degenerates from rank one to rank zero, i.e., when this element returns to its initial value of zero.

A single second order equation for $\delta v(t)$ may be obtained by inserting Eq. (6) into Eq. (4b), differentiating Eq. (4b) and using Eqs. (1), (2), (4), and (5).

$$\delta\ddot{v} + B\delta\dot{v} + C\delta v = 0 \tag{8a}$$

$$B \equiv - \frac{T/W_o}{I_{sp} - (T/W_o)t} - \frac{\lambda_r}{\lambda_u} \tag{8b}$$

$$C \equiv \left[\frac{3g_e I_{sp} - u}{I_{sp} - (T/W_o)t}\right]\left[\frac{T/W_o}{r}\right] - \frac{g r_o^2}{r^3} - \frac{2u^2}{r^2} - \frac{u\lambda_r}{r\lambda_u} \tag{8c}$$

We see that $\delta v(t)$ is a function of λ_r/λ_u and investigation shows that there is a family of neighboring extremals that intersect the nominal arc at widely varying times and define a family of conjugate points. On account of the abnormality associated with the nonuniqueness of the multipliers and because the vertical trajectory is also the locus of singular points (the corners at which thrust reversals occur), the smoothness assumptions of the classical Jacobi condition derivation are not met. That smoothness assumptions are essential for the necessity of the Jacobi condition and not merely a convenience in the derivation has been shown by Dreyfus from the dynamic programming viewpoint (Ref. 6); hence the condition is not applicable to the vertical trajectory.

Numerical Results

Conjugate points on vertical trajectories are determined by intersecting adjacent extremals that are nonvertical. Computation has confirmed that the latter have conjugate points neighboring to their intersection points with the vertical arc. Thus although the Jacobi condition is inapplicable to vertical trajectories, a study of these extremals gives a general idea of how soon conjugate points may be expected on closely related paths.

Launches from the surface of the moon will be investigated numerically. Equations (1) can be used since the moon has no atmosphere and if its rotation is neglected the initial values of u and v are zero. The initial r is 5,702,395 feet. The gravitational acceleration on the moon's surface is taken as $g = 5.324$ ft/sec^2.

The values of $m/m_o = 1 - (T/W_o)(t/I_{sp})$ at the earliest conjugate points on trajectories with various values of T/W_o are plotted in Fig. 1 for a particular chemical rocket $(I_{sp} = 300$ sec) and a particular nuclear rocket $(I_{sp} = 1000$ sec). The smallest T/W_o considered was 0.2, which was barely large enough to overcome the gravitational attraction. For each trajectory the first of the one parameter family of conjugate points was obtained with $\lambda_u(0) = \lambda_v(0) = 0$ and $\lambda_r(0) < 0$.

Nonvertical Trajectories in u, v, r-Space

We now direct attention to nonvertical trajectories governed by the first three members of Eqs. (1) and (2). At first the initial conditions will be: u arbitrary, v vanishing, and r positive.

Equations (1), (2), and (3) indicate that for any particular extremal with time history $u(t)$, $v(t)$, $r(t)$, $\lambda_u(t)$, $\lambda_v(t)$, and $\lambda_r(t)$ there is a mirror image with time history $u(t)$, $-v(t)$, $r(t)$, $\lambda_u(t)$, $-\lambda_v(t)$ and $\lambda_r(t)$. This is because Eqs. (1a), (1c), (2a), and (2c) are even in v and λ_v whereas Eqs. (1b) and (2b) are odd. The $v = 0$ plane is a plane of symmetry in u, v, r-space. An extremal and its mirror image will intersect with equal time values if $v(t)$ happens to return to zero.

Here we digress to recall Darboux's theorem.[7] This theorem states that every extremal path ceases to provide the absolute minimum time to its current end point at a point that precedes the conjugate point. (This point coincides with the conjugate point only in certain exceptional cases.) The point at which two nonneighboring extremals intersect with equal time values will be called the Darboux point. The preceding paragraph's argument shows that the trajectories of the present problem do not reach Darboux points until the $v(t)$ functions return to the $v = 0$ plane of symmetry. It follows that a conjugate point cannot occur until after $v(t)$ has changed sign.

Extremal paths along which $v(t)$ returned to zero were calculated numerically and it was verified that the conjugate points always occurred afterward.

Now consider an extremal family whose $v(0)$ is nonzero. The initial point has been shifted to one side of the $v = 0$ plane of symmetry. Will the Darboux points shift to the same side or to the opposite side? A careful computational study indicates that the shift is to the opposite side. When the initial point lies on the plane of symmetry, then two extremals that meet at their common Darboux point lie on opposite sides of the plane, and when the initial point is moved a little to one side, two such extremals still lie on opposite sides of the plane most of the time. Each crosses the plane once, rather than one twice and the other not at all. Thus it appears that conjugate points do not occur on any of the trajectories in u, v, r-space until after $v(t)$ has changed sign; however, in the absence of exhaustive calculations for all possible initial conditions and thrust/weight ratios, this must be regarded as a conjecture.

Trajectories in u, v, r, θ-Space

We restrict our investigation to the initial conditions $u = v = \theta = 0$, $r = r_o$. Equations (1), (2), and (3) indicate that a given extremal with time history $u(t)$, $v(t)$, $r(t)$, and $\theta(t)$ has a mirror image with $u(t)$, $-v(t)$, $r(t)$, and $-\theta(t)$. Intersections between mirror image trajectories were common in the three state variable problem of the previous section but here they are exceptional since $v(t)$ and $\theta(t)$ must return to zero simultaneously. This is unfortunate since (except when $v(t)$ and $\theta(t)$ vanish simultaneously) it appears not to be possible to formulate qualitative rules governing Darboux and conjugate points.

When $\lambda_u(0) > 0$, $\lambda_v(0) = 0$, $\lambda_r(0) < 0$, and $\lambda_\theta = 0$; Eqs. (1), (2), and (3) indicate that the resulting extremal is a vertically ascending trajectory. The 4×4 test matrix of δx_i vectors contains but four elements that are not identically zero. They are the $\delta v(t)$ and $\delta\theta(t)$ generated with $\delta\lambda_v(0)$ nonzero in one case and with $\delta\lambda_\theta(0)$ nonzero in the other. Thus the test matrix has at most rank two. When it degenerates in rank an adjacent extremal can be found that intersects the nominal at a conjugate point.

Conjugate points on vertical launch trajectories from the moon were found with the aid of a computer. As in the three-dimensional state space problem a one parameter family of conjugate points was exhibited along each path. The values of m/m_o at the earliest conjugate points were plotted in Fig. 2. The significant features to be observed are the extremely short range of the curve for the nuclear rocket ($I_{sp} = 1000$) and the nonexistence (for $T/W_o > 0.2$) of the curve for the chemical rocket ($I_{sp} = 300$).

Conclusions

A family of conjugate points has been found on extremal rocket paths in u, v, r-space. Numerical results indicate that they appear only after $v(t)$ — the horizontal velocity component — has changed sign. When $v(0) = 0$ this can be proved analytically from symmetry considerations and Darboux's theorem. Since $v(t)$

seldom changes its sign on powered flight trajectories of engineering interest, the result is reassuring. Computer searches for other families of conjugate points were conducted without success.

Conjugate points have also been found in u, v, r θ-space, although in the case of lunar trajectories they are apparently quite rare. Of course if families exhibit envelopes in this space they will also when Cartesian or other coordinates are used. No general qualitative rules governing these conjugate points have been discovered.

References

1. Bryson, A. E., "Optimal programming and control," Proceedings of the IBM Scientific Computing Symposium on Control Theory and Applications, October 19-21, 1964, p. 19f.
2. Edelbaum, T. N., "Optimum power-limited orbit transfer in strong gravity fields," AIAA J. 3, 921-925, 1965.
3. Moyer, H. G., "Minimum impulse coplanar circle-ellipse transfer," AIAA J. 3, 723-726, 1965.
4. Bliss, G. A., Lectures on the Calculus of Variations, University of Chicago Press, Chicago, Illinois, 1946.
5. Cicala, P., An Engineering Approach to the Calculus of Variations, Levrotto-Bella, Torino, Italy, 1957, pp. 57-58, 94.

6. Dreyfus, S. E., Dynamic Programming and the Calculus of Variations, Acadmic Press, New York, 1965, pp. 103-119, 141-144.

7. Bolza, O., Vorlesungen über Variationsrechnung (Teubner, Leipzig, 1909) as reprinted by Chelsea Publishing Company, New York, 1963, p. 438 ff.

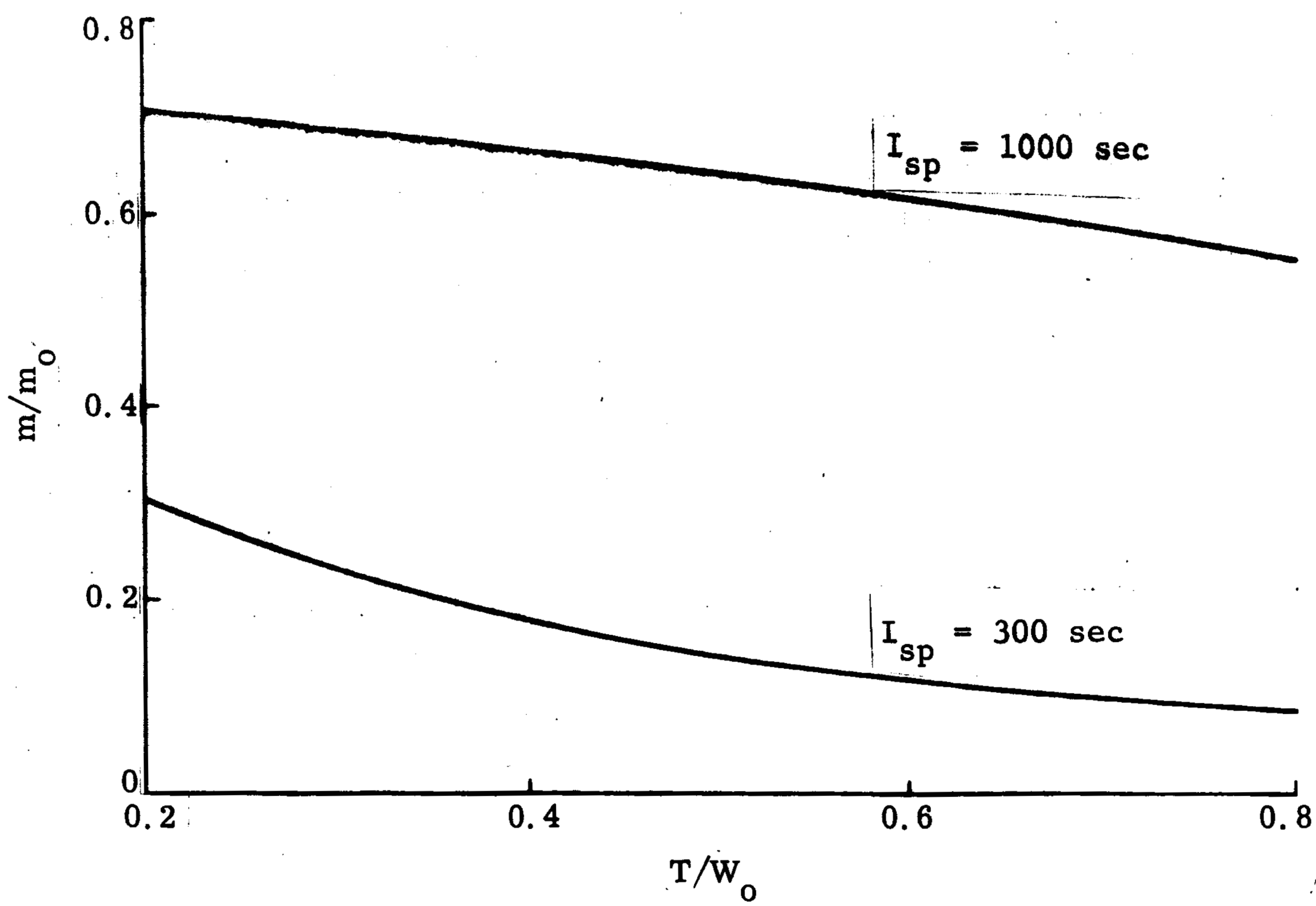

Fig. 1 Mass Ratio at First Conjugate Point for Vertical Launches from the Moon (u,v,r - Space)

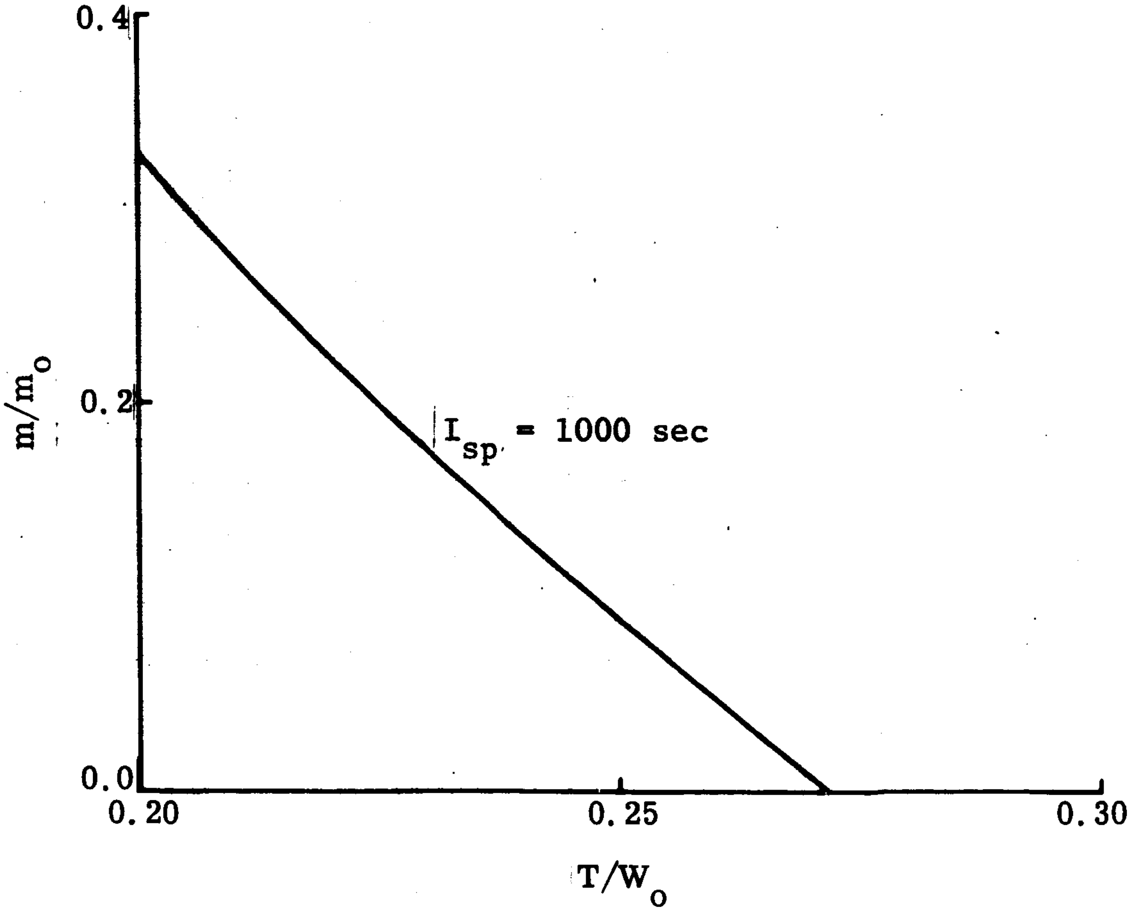

Fig. 2 Mass Ratio at First Conjugate Point for Vertical Launches from the Moon (u, v, r, θ - Space)