SPACE RESEARCH COORDINATION CENTER

N68-28844 (ACCESSION NUMBER)
43 (PAGES)
CR-95485 (NASA CR OR TMX OR AD NUMBER)
(THRU)
1 (CODE)
32 (CATEGORY)
FACILITY FORM 602

STABILITY CRITERIA FOR COMPLETELY SYMMETRICAL DISCRETE ELASTIC SYSTEMS

BY

S. J. BRITVEC

DEPARTMENT OF MECHANICAL ENGINEERING

GPO PRICE $

CFSTI PRICE(S) $

Hard copy (HC)

Microfiche (MF)

ff 653 July 65

SRCC REPORT NO. 79

UNIVERSITY OF PITTSBURGH

PITTSBURGH, PENNSYLVANIA

9 JULY 1968

The Space Research Coordination Center, established in May, 1963, has the following functions: (1) it administers predoctoral and postdoctoral fellowships in space-related science and engineering programs; (2) it makes available, on application and after review, allocations to assist new faculty members in the Division of the Natural Sciences and the School of Engineering to initiate research programs or to permit established faculty members to do preliminary work on research ideas of a novel character; (3) in the Division of the Natural Sciences it makes an annual allocation of funds to the interdisciplinary Laboratory for Atmospheric and Space Sciences; (4) in the School of Engineering it makes a similar allocation of funds to the Department of Metallurgical and Materials Engineering and to the program in Engineering Systems Management of the Department of Industrial Engineering; and (5) in concert with the University's Knowledge Availability Systems Center, it seeks to assist in the orderly transfer of new space-generated knowledge in industrial application. The Center also issues periodic reports of space-oriented research and a comprehensive annual report.

The Center is supported by an Institutional Grant (NsG-416) from the National Aeronautics and Space Administration, strongly supplemented by grants from the A. W. Mellon Educational and Charitable Trust, the Maurice Falk Medical Fund, the Richard King Mellon Foundation and the Sarah Mellon Scaife Foundation. Much of the work described in SRCC reports is financed by other grants, made to individual faculty members.

STABILITY CRITERIA FOR COMPLETELY SYMMETRICAL DISCRETE ELASTIC SYSTEMS

by

S. J. BRITVEC

UNIVERSITY OF PITTSBURGH

SCHOOL OF ENGINEERING

DEPARTMENT OF MECHANICAL ENGINEERING

PITTSBURGH, PENNSYLVANIA

JANUARY, 1968

SUMMARY

Completely symmetrical elastic systems are analysed in the vicinity of critical states which are realized simultaneously or nearly so by all the coordinates of deformation. It is shown that uncoupled and coupled modes of elastic deformation may occur in equilibrium and in motion. Stability of the systems is examined under conservative conditions and several criteria are obtained for the onset of motion from unstable equilibrium states when conditions of symmetry prevail within the system.

INTRODUCTION

Many two- and three-dimensional structural systems fall into the category of the so-called completely symmetrical discrete systems which consist of a finite number of elastic elements (the simplest type may be a prismatic rod). deformation of such systems may be described completely by a finite number of coordinates which represent rotations and displacements in space at some point or station on the elastic body. The generalized coordinates may therefore take positive and negative values. A system is said to be completely symmetrical if its total potential energy functional is independent of the sign in the variations of these coordinates. One such example are pin-jointed elastic systems under conservative loading conditions, e.g. slender trusses, space frameworks, reticulated surfaces and domes, to mention a few.

Systems of this kind possess several distinct properties which dominate their behaviour in equilibrium and in motion. Realization of unstable equilibrium states under conservative conditions and the capacity of a system for converting its total potential into kinetic energy from such states may be the most important properties from a pragmatic point of view. The first onset of elastic instability may occur in a critical state at no or very little prior deformation when branching of unstable equilibrium configurations becomes possible. Of particular interest are systems in which all or a dominant set of equally critical generalized coordinates are realized simultaneously or nearly so. Under perfect conditions this may occur in an all-critical state, but in the presence of distinct types of geometrical deviations or imperfections several critical states may accumulate in the proximity of one another. In either case several patterns of elastic deformation may result. Otherwise uncoupled modes of elastic deformation may thus be coupled in a simultaneous onset of elastic buckling. Branching equilibrium paths may be also stable, so that buckling is not always accompanied by unstable motion. Forms of coupled equilibrium paths have been studied recently by Chilver[1] and Supple.[2] These authors point out that highly non-linear equilibrium paths may exist in the proximity of two adjacent critical states.

The present paper discusses the forms of equilibrium paths in the vicinity of an all-critical state and the stability of completely symmetrical systems in the so-called coupled modes of elastic buckling. Geometrically imperfect conditions which may give rise to accumulation of several critical states and therefore to stable and unstable coupled buckling are examined on systems with two degrees of freedom. The paper also points out in a few systematic steps a general method by which stability or instability of coupled or any other equilibrium paths may be determined when energy losses in the process of buckling are negligible.

PERFECT SYSTEMS

1. REPRESENTATION OF DISCRETE SYSTEMS

The system consists of an elastic body which is composed of discrete elastic elements or stations connected in some way so that the aggregate constitutes a physical structure. The body is subjected to a conservative field of force, and it is conceived in such a way that interaction with this field occurs at discrete points within its structure. Usually these points are coincident with the joints between the constitutive elements but not necessarily so. The masses in the system are then concentrated at the points of application of the field of force. An example may be a reticulated surface which consists of light bars or a shell represented by discrete localized elastic stations connected together so that concentrated masses are localized at the points of connection.

Elastic deformations of a discrete system are defined by a finite number of quantities Q_i referred to as the generalized coordinates of deformation. This concept is not new, and it was in current use already in the last century (e.g. Lagrange). Generalized coordinates may represent rotations or displacements at localized points on the body. Thus the elastic energy is completely defined by the coordinates Q_i, $i=1,2,\ldots,n$.

We relate the generalized coordinates of the system to the field of force using the concept of conservation of energy. Denoting the total energy of the system by E we have between states I and II

$$\delta E = \delta U + \delta H + \delta T + \theta\ (\delta S_e + \delta S_i) \qquad (1)$$

Here

δU....change in elastic energy

δH....change in potential energy in the field of force

δT....change in kinetic energy

δS_e...entropy flow from the exterior

δS_i...entropy production in the interior

θ ....absolute temperature

Let

$$\delta V = \delta U + \delta H \qquad (2)$$

the change in the total potential energy of the system. Contemplating conservative systems with negligible entropy changes, (1) reduces to

$$\delta E = \delta V + \delta T = 0 \qquad (3)$$

If the change δT vanishes between two limiting states I and II then the system is either in a state of uniform motion or in a state of statical equilibrium. The necessary condition for equilibrium is therefore

$$\delta V_{I}^{II} = 0 \qquad (4)$$

i.e. that the total potential energy function has a stationary value.

The field of force is defined in localized regions at discrete points of the system. The amount of work necessary to displace a concentrated mass in this field by a small amount δs_k, k=1,2,3 between two adjacent points on a path is given by

$$\delta H = - F_k \, \delta s_k ; \text{ sum on } k \tag{5}$$

where F_k is defined as the local force of the field. The field is said to be locally conservative if this work is independent of the path taken, but it depends only on its terminal points. Substituting into (2) we obtain

$$\delta V = \delta U - F_k \cdot \delta s_k \tag{6}$$

The summation now extends over all the components at different points in the system. In applications potential fields are usually uniform so that the gradient vectors are parallel or constant locally. However, in different regions these vectors need not be parallel to one another.

Consider first the particular case when these gradients, and therefore the localized conservative forces, are proportional to a single parameter P. It is assumed that the displacements δs_k are a direct consequence of elastic deformation and therefore functions of the generalized coordinates Q_i. Then (6) may be written

$$\frac{\partial V}{\partial Q_i} = \frac{\partial U}{\partial Q_i} - F_k \frac{\partial s_k}{\partial Q_i} \tag{7}$$

$$i, j=1,2,\ldots,n$$

Since now F_k is also a function of P, the total potential energy function V takes the form

$$V = V\,(P, Q_i) \; ; \quad i=1,2,3,\ldots,n \tag{8}$$

In the continuation of this discussion we confine our attention to systems which are symmetrical with regard to the variations q_i in <u>all</u> the coordinates Q_i at a given state defined by $Q_i^{\,o}$ and P^o. To this end we write

$$V = V^o + v = V\,(P^o + p,\ Q_i^{\,o} + q_i) \tag{9}$$

The condition of complete symmetry requires that

$$V\,(P^o + p,\ Q_i^{\,o} + q_i) = V\,(P^o + p,\ Q_i^{\,o} - q_i) \qquad i = 1,2,\ldots,n \tag{10}$$

Equation (3) may be written

$$\delta T = -\,v \tag{3)bis}$$

where v is measured from some fundamental state. Then δT is positive only when v is negative. If v is positive in the entire adjacent physical neighbourhood of this state kinetic energy cannot be generated within the system. Consequently, unstable motion may only develop from a state of equilibrium at which the change in the total potential energy may become negative.

This concept may be used to analyse the instability and stability of elastic systems (Britvec,[3] 1960) if the variation v of the total potential energy function of the system is represented in a Taylor series by the increments q_i in the coordinates Q_i. Equilibrium paths are defined by condition (4) which is equivalent to

$$\partial v / \partial q_i = v_i = 0\ ;\ i=1,2,3,\ldots,n \tag{11}$$

To explore a general symmetrical system the energy function $V\,(P,Q_i)$ is written in series form at a fundamental equilibrium state $Q_i^{\,o}, P^o$. Thus

$$\begin{aligned} v = {} & V_i q_i + \frac{1}{2!} V_{ij} q_i q_j + \frac{1}{3!} V_{ijk} q_i q_j q_k + \frac{1}{4!} V_{ijkl} q_i q_j q_k q_l + \ldots \\ & + p \left[V_p + V_{pi}\, q_i + \frac{1}{2!} V_{pij}\, q_i\, q_j + \ldots \right. \\ & + \frac{1}{2!} p^2 \left[V_{pp} + V_{ppi}\, q_i + \frac{1}{2!} V_{ppij}\, q_i\, q_j + \ldots \right. \\ & + \ldots\ldots\ldots\ldots \end{aligned} \tag{12}$$

sum on i,j,k,l = 1,2,...,n

where $V_i = \partial V/\partial Q_i$, $V_{pi} = \frac{\partial^2 V}{\partial p \partial Q_i}$, etc. denote the partial derivatives with respect to the coordinates Q_i or q_i and p. Equilibrium conditions in the state $q_i = 0$ are satisfied if $v_i = 0$, i=1,2,...,n. If this state is realized at no prior deformation of the body, point $(P^o, Q_i{}^o)$ must lie on the P-axis. Consequently p is not necessarily zero when $q_i = 0$ and therefore

$$V_i = V_{pi} = V_{ppi} = \ldots\ldots\ldots\ldots = 0 \tag{13}$$

Similarly, if no changes in the potential energy occur without elastic deformation

$$V_p = V_{pp} = V_{ppp} = \ldots\ldots\ldots\ldots = 0 \tag{13a}$$

Then (12) reduces to

$$v = \frac{1}{2!} V_{ij} q_i q_j + \frac{1}{3!} V_{ijk} q_i q_j q_k + \frac{1}{4!} V_{ijkl} q_i q_j q_k q_l + \ldots$$
$$+ p \left[\frac{1}{2!} V_{pij} q_i q_j + \ldots \right. \tag{14}$$
$$+ \ldots\ldots\ldots\ldots$$

When the variations q_i are sufficiently small, the quadratic form

$$1/2! \, V_{ij} \, q_i \, q_j$$

determines the sign of v and therefore the stability of equilibrium in the fundamental state.

This form is expressed as a sum of squares by introducing in the fundamental state a set of eigenvectors b_j which define the axes of the local principal coordinate system u_j. In the case of a symmetrical system this step is not essential in the critical state only since the coefficients V_{ij} vanish on account of symmetry. However, in the subsequent stability analysis orthogonal principal systems play an important part, as it is convenient to retain Cartesian coordinates whenever possible. We recall briefly the essential steps in the development of the necessary machinery.

First, the eigenvalues λ_k of the local stability matrix $[V_{ij}]$ or $[v_{ij}]$ and the corresponding eigenvectors $\mathbf{b}_k$ are determined from

$$([V_{ij}] - \lambda_k I)\,\mathbf{b}_k = 0$$

and

$$\left|\,[V_{ij}] - \lambda_k I\,\right| = 0 \qquad k=1,2,\dots,n$$

For all eigenvectors this condition becomes

$$[V_{ij}]\,B - B\Lambda = 0$$

where the columns of B are the eigenvectors $\mathbf{b}_k$ and the elements of Λ, the eigenvalues λ_k. If $\mathbf{b}_k$ are orthogonal unit vectors in the directions of the local principal axes u_k, then $B^T B = I$, where I is the unit matrix. Hence

$$B^T\,[V_{ij}]\,B = \Lambda \qquad (15)$$

Putting $\mathbf{q} = B\mathbf{u}$, the dominant quadratic form in the expression for v becomes

$$\mathbf{q}^T\,[V_{ij}]\,\mathbf{q} = \mathbf{u}^T\Lambda\mathbf{u} = \lambda_i\,u_i^2 = V_{ii}\,u_i^2$$

sum on $i=1,2,\dots,n$

where now the coefficients V_{ii} represent partial derivatives of V with respect to the new coordinates u_i. At a new point on the fundamental (or another) path the quadratic form clearly resumes its non-diagonal form since the coefficients V_{ij} increase or decrease from zero. Therefore, if again local orthogonal principal axes are introduced at the new point, these axes are rotated in relation to the orthogonal axes at the previous point. It can be shown that only one quadratic form in the series may be diagonalized using a single set of orthogonal eigenvectors if no restrictions of commutation are to be imposed on the matrices in other forms.

The series (14) is now similar except that the q_i are replaced by the u_i and that the derivatives are evaluated with respect to the new coordinates. The coefficients λ_k, usually referred to as the local stability coefficients, are assumed to be initially positive on the P-axis.

Suppose now that all the stability coefficients λ_k vanish simultaneously in a critical state. The coordinates u_i are then said to be all-critical. The energy expression (14) takes a particularly simple form since all the derivatives of V with respect to the odd powers of u_i vanish on account of complete symmetry.

2. EQUILIBRIUM STATES

The first order equilibrium conditions for the completely symmetrical system are derived from (11) and (14). Putting $V_{ii} = 0$, $i=1,2,\ldots,n$,

$$v_\alpha = \frac{1}{2!} V_{\alpha\alpha ii} u_\alpha u_i^2 + \frac{1}{3!} V_{\alpha\alpha\alpha\alpha} u_\alpha^3 + p\left[V_{p\alpha\alpha} u_\alpha + \ldots\right] + \ldots = 0 \qquad (16)$$

The trivial solution $u_\alpha = 0$, $\alpha=1,2,\ldots,n$ implies that the P-axis is an equilibrium path. Other uncoupled solutions are possible, i.e. $u_\alpha \neq 0$, $u_i = 0$, $i=1,2,\ldots,n$, $i \neq \alpha$. These occur in the coordinate planes (p, u_α) and take the approximate form

$$\left(\frac{1}{2!} + \frac{1}{3!}\right) V_{\alpha\alpha\alpha\alpha} u_\alpha^3 + p \cdot V_{p\alpha\alpha} u_\alpha + \ldots = 0$$

The non-trivial solution is obtained on eliminating p from two equations $v_\alpha = 0$ and $v_\gamma = 0$, $\alpha=1,2,\ldots,n$, $\alpha \neq \gamma$ and putting $u_i^2 = x_i$ and $\gamma = 1$. Thus (n-1) linear equations are obtained in the form

$$\left(\frac{1}{3!} V_{p11} V_{\alpha\alpha\alpha\alpha} - \frac{1}{2!} V_{p\alpha\alpha} V_{11\alpha\alpha}\right) x_\alpha + \frac{1}{2!}\left(V_{p11} V_{\alpha\alpha ii} - V_{p\alpha\alpha} V_{11ii}\right) x_i = \left(\frac{1}{3!} V_{p\alpha\alpha} V_{1111} - \frac{1}{2!} V_{p11} V_{\alpha\alpha 11}\right) x_1 \qquad (17)$$

sum on $i=2,3,\ldots,n$; $i \neq \alpha$

where x_1 is now a reference coordinate. The solution takes the form

$$x_k = C_{k1}\, x_1 \;; \quad k = 2,3,\ldots,n \qquad (18)$$

C_{k1} are known constants. The path parameter is then given by

$$p = -\frac{1}{V_{p11}} \left(\frac{1}{3!} V_{1111} + \frac{1}{2!} V_{11ii}\, C_{i1}\right) x_1 \qquad (19)$$

sum on $i=1,2,\ldots,n$

with $V_{pii} < 0$.

The orthogonal projection of the coupled path in the (p, x_1) plane is a line and in the (p, u_1) plane a parabola, Figures 1 and 2. It can be shown that the rising paths may now be stable or unstable, whereas the falling paths are always unstable. In a given mode of buckling all the coordinates of deformation are coupled to the same order of magnitude. In many systems the coefficients V_{iijj} may assume several sets of values in the all-critical state; therefore several modes of buckling, involving the generalized coordinates in different ways, may be possible. A particular class of such systems is described in references 4 and 5.

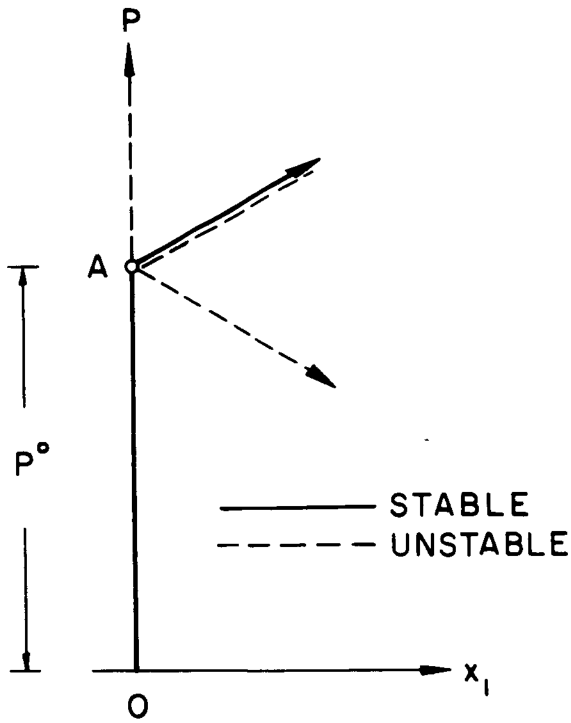

FIGURE 1, $\Delta P = 0$

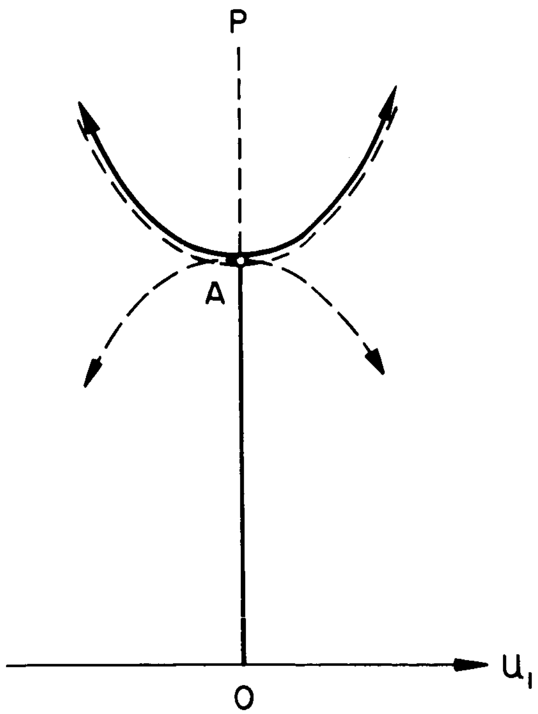

FIGURE 2, $\Delta P = 0$

4. SEVERAL LOAD PARAMETERS

Sometimes it is of interest to explore the system when the field of force at different points of application depends on several independent parameters P_l. Consider, therefore, the case when the number of load parameters corresponds to the number of degrees of freedom expressed by the coordinates u_i, so that any state of deformation defined by u_i may be also a state of equilibrium under a corresponding set of parameters P_l. This correspondence is considered in the behaviour of the system near the all-critical state. According to (6) and (7) we have for equilibrium

$$\frac{\partial V}{\partial u_\alpha} = \frac{\partial U}{\partial u_\alpha} - F_k\ (P_l)\ \frac{s_k}{\partial u_\alpha} = 0 \qquad (21)$$

$$\alpha = 1, 2, \ldots, n$$

where now differentiation is with respect to the coordinate u_α. Using the concise notation this may be written

$$V_\alpha = U_\alpha - F_k\ (P_l)\ (s_k)_\alpha = 0 \qquad (21a)$$

$$l, \alpha = 1, 2, \ldots, n$$

$$\text{sum on } k = 1, 2, \ldots, n$$

If the conservative forces F_k are functions of the parameters P_l, $l=1,2,\ldots,n$, the n equations (21) contain n independent quantities and they may be, therefore, differentiated partially with respect to a coordinate u_β. This also means that, given a set of coordinates u_α, $\alpha=1,2,\ldots,n$, which define the deformations of the elastic body in a state adjacent to the fundamental state considered, n forces F_k (or parameters P_l) can be found from these equations so that equilibrium of the system in the adjacent state is possible. (It is assumed here that the relationships between F_k and P_l are such that P_l are the real roots of Eq. 21 and also that no singularities are associated with the adjacent state, so that all the partial derivatives evaluated in this state are finite.) Differentiating now Eq. (21) with respect to u_β we obtain that on an equilibrium path,

$$\frac{\partial^2 U}{\partial u_\alpha \partial u_\beta} - F_k\ (P_l)\ \frac{\partial^2 s_k}{\partial u_\alpha \partial u_\beta} - \frac{\partial F_k}{\partial P_l}\frac{\partial P_l}{\partial u_\beta}\frac{\partial s_k}{\partial u_\alpha} = 0$$

or

$$U_{\alpha\beta} - F_k\ (P_l)\ (s_k)_{\alpha\beta} - \frac{\partial F_k}{\partial P_l}\ (P_l)_\beta\ (s_k)_\alpha = 0 \qquad (22)$$

Therefore $v_{\alpha\beta}$ may be expressed along an equilibrium path by

$$v_{\alpha\beta} = \frac{\partial F_k}{\partial P_l}\ (P_l)_\beta\ (s_k)_\alpha \qquad (23)$$

The adjacent states of equilibrium are now realized by means of slight changes in the parameters P_l. Since, in the all-critical state, $v_{\alpha\beta} = 0$, it follows that in that state $(s_k)_\alpha = 0$, as the other quantities are generally non-zero. By (21) then, also $U_\alpha = 0$ at $u_\alpha = 0$, i.e. the rates of the corresponding displacements of the points of force and the rate of the internal energy with respect to the coordinates of deformation vanish in the all-critical state.

Equilibrium condition (21) may be written

$$\left(\frac{\partial U}{\partial x_\alpha} - F_k \frac{\partial s_k}{\partial x_\alpha}\right) \frac{\partial x_\alpha}{\partial u_\alpha} = 0 \qquad (24)$$

Therefore, if u_α is generally non-zero,

$$\frac{\partial V}{\partial x_\alpha} = \frac{\partial U}{\partial x_\alpha} - F_k \frac{\partial s_k}{\partial x_\alpha} = 0 \qquad (25)$$

Further differentiation in states of equilibrium yields,

$$\frac{\partial^2 U}{\partial x_\alpha \partial x_\beta} - F_k \frac{\partial^2 s_k}{\partial x_\alpha \partial x_\beta} - \frac{\partial F_k}{\partial x_\beta} \frac{\partial s_k}{\partial x_\alpha} = 0 \qquad (26)$$

Then on a path of equilibrium states

$$\frac{\partial^2 V}{\partial x_\alpha \partial x_\beta} = \frac{\partial F_k}{\partial x_\beta} \frac{\partial s_k}{\partial x_\alpha} \qquad (27)$$

Substituting into (20) we get,

$$v = \frac{1}{2!} \frac{\partial F_k}{\partial x_\beta} \frac{\partial s_k}{\partial x_\alpha} x_\beta x_\alpha + \ldots \qquad (28)$$

If the parameters P_l are identified with the forces F_k, the last result may be written

$$v = \frac{1}{2!} \delta F_k \, \delta s_k + \ldots \qquad \text{sum on } k=1,2,\ldots,n$$

Hence we deduce

Theorem (i): The all-critical equilibrium state of a completely symmetrical system is unstable if in an adjacent equilibrium state, defined by the corresponding displacements δs_k,

$$v = \frac{1}{2!} \delta F_k \cdot \delta s_k$$

is negative, where the system may be maintained in equilibrium by the forces $F_k + \delta F_k$.

Theorem (ii): If in all adjacent states of the all-critical state

$$v = \frac{1}{2!} \delta F_k \cdot \delta s_k$$

is found to be positive, equilibrium in that state is stable.

(It is implied that equilibrium of the system may be realized by the parameter changes δF_k in the entire physical space u_i, $i=1,2,3,\ldots,n$.) Neutral equilibrium may occur to the accuracy of the analysis if $v = 0$ in the adjacent states. Special forms of the last two theorems were obtained in reference 5.

GEOMETRICALLY IMPERFECT SYSTEMS

COUPLED BUCKLING

An all-critical state may not always be realized under geometrically imperfect conditions in an otherwise completely symmetrical system. Presence of distinct types of imperfections may be responsible for slight differences in the critical stability coefficients λ_i. These imperfections need not impair the conservative nature of the system. Adjacent critical states, corresponding to different generalized coordinates, may, thus, accumulate on the fundamental equilibrium path. In this way other equilibrium states may be affected or generated so that several modes are coupled in a simultaneous onset of stable or unstable buckling. Similar situations may occur under perfect conditions in other elastic systems which are not discussed at present. To demonstrate the different equilibrium and stability properties of slightly imperfect symmetrical systems prone to coupled buckling, we consider the particular case of a system with two degrees of freedom, when the variable local stability coefficients λ_1 and λ_2 vanish at two adjacent points on the fundamental path.

COUPLED EQUILIBRIUM PATHS

Forms of equilibrium paths of this type have been analysed by Chilver[1] and Supple[2], and the reader is referred to their work concerning the equilibrium analysis which is only referred to when essential differences in the methods occur. It is an advantage to retain in the subsequent analysis the local rotating Cartesian coordinates, since otherwise analytical operations must be adapted to coordinate axes with varying obliquity.

Let the dominant quadratic form in the expression (14) for v be given by

$$(q_1\ q_2)\begin{pmatrix} V'_{11} & V'_{12} \\ V'_{21} & V'_{22} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

where now the primed coefficients denote partial differentiation with respect to q_1 and q_2. Suppose at the first critical point on the fundamental path the stability coefficient λ_1 referred to the local principal orthogonal axes u_i vanishes. Then λ_1 is the root of the equation

$$\begin{vmatrix} V'_{11} - \lambda_j & V'_{12} \\ V'_{21} & V'_{22} - \lambda_j \end{vmatrix} = 0 \qquad (29)$$

i.e.

$$\lambda_1 = \tfrac{1}{2}(V'_{11}+V'_{22}) - \sqrt{\tfrac{1}{4}(V'_{11}+V'_{22})^2 - V'_{11}\ V'_{22} + V_{12}{}^2} = V_{11} = 0 \qquad (30a)$$

This occurs when $-V'_{11}\ V'_{22} + V'_{12}{}^2 = 0$. The second stability coefficient is

$$\lambda_2 = \tfrac{1}{2}(V'_{11}+V'_{22}) + \sqrt{\tfrac{1}{4}(V'_{11}+V'_{22})^2 - V'_{11}\,V'_{22} + V'^{\,2}_{12}} = V'_{11}+V'_{22} =$$

$$V_{22} > 0 \qquad (30b)$$

At this point λ_2 is supposed to be positive. Unprimed coefficients now denote partial differentiation with respect to the new orthogonal coordinates u_i. The dominant quadratic form becomes in the new coordinate system

$$\lambda_1 u_1^2 + \lambda_2 u_2^2 = (u_1 \quad u_2) \begin{pmatrix} V_{11} & 0 \\ 0 & V_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and the expression for v, similarly to (14),

$$v = \frac{1}{2!} V_{ii}\, u_i^2 + \frac{1}{3!} V_{ijk}\, u_i\, u_j\, u_k + \frac{1}{4!} V_{ijkl}\, u_i\, u_j\, u_k\, u_l +$$

$$p\left[\frac{1}{2!} V_{pij}\, u_i\, u_j + \ldots\right. \qquad (31)$$

Consider now the possibility that in the local orthogonal coordinate system the second stability coefficient $(\lambda_2 + \delta\lambda_2)$ vanishes. The local stability coefficients at the second critical point are the roots of the equation

$$\begin{vmatrix} V_{11} + \delta V_{11} - (\lambda_i + \delta\lambda_i) & \delta V_{12} \\ \delta V_{21} & V_{22} + \delta V_{22} - (\lambda_i + \delta\lambda_i) \end{vmatrix} = 0 \qquad (32)$$

where $V_{11} = 0$. On the P-axis the first order variations are

$\delta V_{11}=V_{11p}\Delta P+\ldots$; $\delta V_{22}=V_{22p}\Delta P+\ldots$; and $\delta V_{12}=\delta V_{21}=V_{12p}\Delta P+\ldots$

where $V_{11p} = \partial^3 V/\partial u_1^2 \partial p$, etc. and ΔP is the parameter difference between the two critical states. Putting i = 2,

$$\begin{vmatrix} \delta V_{11} - (\lambda_2 + \delta\lambda_2) & \delta V_{12} \\ \delta V_{21} & \delta V_{22} - \delta\lambda_2 \end{vmatrix} = 0$$

If all the variations δV_{ij} are of the same order, the first order solution yields

$$-V_{22}\ (\delta V_{22} - \delta\lambda_2) = 0$$

Hence

$$\delta\lambda_2 \doteq \delta V_{22} \doteq V_{p22}\ \Delta P + \ldots \tag{33a}$$

and similarly

$$\delta\lambda_1 = \delta V_{11} = V_{p11}\ \Delta P + \ldots \tag{33b}$$

Be definition, at the second critical point

$$\lambda_2 + \delta\lambda_2 = V_{22} + V_{p22}\Delta P + \ldots = 0;$$

therefore,

$$V_{22} = -\ V_{p22}\ \Delta P + \ldots \tag{34}$$

The approximate equilibrium equations of the two-degree of freedom system may be deduced from (31) using (11) and the last result, i.e.

$$\begin{aligned} u_1 \left[\frac{1}{3!} (V_{1111}\ u_1^2 + 3V_{1122}\ u_2^2) + V_{p11}\ p \right] &= 0 \\ u_2 \left[\frac{1}{3!} (V_{2222}\ u_2^2 + 3V_{1122}\ u_1^2) + V_{p22}\ (p - \Delta P) \right] &= 0 \end{aligned} \tag{35}$$

Eliminating p,

$$(V_{1111}V_{p22} - 3V_{1122}V_{p11})u_1^2 + (3V_{1122}V_{p22} - V_{2222}V_{p11})u_2^2 = - 3!\ V_{p11}V_{p22}\ \Delta P \tag{36}$$

Since now the local eigenvectors are orthogonal, elimination of u_1 from these equations yields the orthogonal projection of the coupled path in the (p,u_2) plane. Thus

$$p(V_{1111}V_{p22} - 3V_{1122}\ V_{p11}) = V_{1111}\ V_{p22}\Delta P - \frac{1}{3!}(V_{1111}\ V_{2222} - 9V_{1122}^2)\ u_2^2 \tag{37a}$$

and similarly the orthogonal projection in the (p, u_1) plane,

$$(p - \Delta P)(3V_{1122} V_{p22} - V_{2222} V_{p11}) = V_{2222} V_{p11} \Delta P + \frac{1}{3!} (V_{1111} V_{2222} - 9V_{1122}{}^2) u_1{}^2 \qquad (37b)$$

Equilibrium equations obtained in this case by Chilver and Supple using Cartesian analysis and coordinate axes with varying obliquity are the same to the first order accuracy on account of symmetry, V_{p12} being fortuitously zero. The uncoupled equilibrium paths are obtained from (35) on setting alternatively $u_2 = 0$ and $u_1 = 0$.

6. STABILITY OF EQUILIBRIUM IN THE COUPLED MODES OF BUCKLING

Stability of an equilibrium state at constant value of the load parameter depends on the local change Δv in the total potential energy with respect to the local variations h_i in the generalized coordinates u_i. Generally the local equilibrium is non-critical; therefore it suffices to represent this change by the dominant quadratic form in the localized neighbourhood of the equilibrium point. If Δv assumes a negative value for at least one independent variation h_i, then by (3)bis conversion of the total potential into kinetic energy is locally possible and therefore equilibrium at the given point is unstable. Thus

$$\Delta v = \frac{1}{2!}(h_1 \quad h_2)\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \dots \qquad (38)$$

The coefficients v_{ij} may be found from (31). Then, at point (p, u_1, u_2),

$$v_{11} = V_{1122}\, u_2^{\,2} + V_{1111}\, u_1^{\,2} + V_{p11}\, p$$

$$v_{12} = v_{21} = V_{1122}\, u_1\, u_2 \qquad (39)$$

$$v_{22} = V_{1122}\, u_1^{\,2} + V_{2222}\, u_2^{\,2} + (p - \Delta P)\, V_{p22}$$

We diagonalize this form by introducing the principal coordinates ζ_i, referred to the base of <u>orthogonal unit eigenvectors $\mathbf{b}_j$</u> at the point (p, u_1, u_2). If the columns of $\mathbf{B}$ are the vectors $\mathbf{b}_j$, then $\mathbf{B}^T\mathbf{B} = \mathbf{I}$. The corresponding stability coefficients λ_j are the roots of the equation

$$\begin{vmatrix} v_{11} - \lambda_j & v_{12} \\ v_{21} & v_{22} - \lambda_j \end{vmatrix} = 0 \qquad (40)$$

i.e.

$$\lambda_1 = \tfrac{1}{2}(v_{11} + v_{22}) - \sqrt{\tfrac{1}{4}(v_{11} + v_{22})^2 - v_{11}v_{22} + v_{12}^2} \qquad (41a)$$

and

$$\lambda_2 = \tfrac{1}{2}(v_{11} + v_{22}) + \sqrt{\tfrac{1}{4}(v_{11} + v_{22})^2 - v_{11}v_{22} + v_{12}^2} \qquad (41b)$$

The vectors $\mathbf{b}_j$ and the eigenvalues λ_j are then related to the local stability matrix $[v_{ij}]$ by (15), i.e. $\mathbf{B}^T [v_{ij}] \mathbf{B} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \Lambda$. Putting $\mathbf{h} = \mathbf{B}\boldsymbol{\zeta}$, the expression for Δv becomes

$$\Delta v = \frac{1}{2!} \mathbf{h}^T [v_{ij}] \mathbf{h} = \frac{1}{2!} \boldsymbol{\zeta}^T \Lambda \boldsymbol{\zeta} = \frac{1}{2!} (\lambda_1 \zeta_1^2 + \lambda_2 \zeta_2^2) \qquad (42)$$

Then, if the system is in equilibrium at the point (p, u_1, u_2) this equilibrium is unstable if either λ_1 or λ_2 or both are negative.

According to (41a) and (41b), three possibilities arise:

(i) $-v_{11}v_{22} + v_{12}^2 > 0$; then $\lambda_1 < 0$, whether $(v_{11} + v_{22})$ is positive or negative.

(ii) $-v_{11}v_{22} + v_{12}^2 < 0$; then $\lambda_1 < 0$ and $\lambda_2 < 0$, provided that the discriminant is positive. This requires

$$(v_{11} + v_{22}) < -2\left|\sqrt{v_{11}\, v_{22} - v_{12}^2}\right|$$

(iii) $-v_{11}v_{22} + v_{12}^2 < 0$; then $\lambda_1 > 0$ and $\lambda_2 > 0$ if

$$(v_{11} + v_{22}) > +2\left|\sqrt{v_{11}\, v_{22} - v_{12}^2}\right|$$

Otherwise λ_1 and λ_2 are not real. To determine the instability of an equilibrium path conditions (i) and (ii) must be satisfied on that path. Similarly, on a stable path the stability condition (iii) must be fulfilled. Substituting for the coefficients v_{ij}, to be evaluated on the paths, the three conditions become:

$$\text{(i)}\quad \left(\frac{1}{2!} V_{1111}u_1^{\,2} + \frac{1}{2!} V_{1122}u_2^{\,2} + V_{p11}p\right)\left[(p - \Delta P)V_{p22} + \frac{1}{2!} V_{2222}u_2^{\,2} + \frac{1}{2!} V_{1122}u_1^{\,2}\right] < V_{1122}^{\,2}u_1^{\,2}u_2^{\,2}$$

Substituting further the equilibrium equations (35) into this criterion we obtain on the coupled paths that

$$u_1^{\,2}u_2^{\,2}\,(V_{1111}V_{2222} - 9V_{1122}^{\,2}) < 0 \qquad (43)$$

where u_1 and u_2 are the coordinates of the path. Therefore, if u_1 and u_2 are non-zero, the first instability condition is independent of the coordinates of the coupled paths and it depends only on the constants V_{1111}, V_{2222} and V_{1122} of the system. If $u_1 = 0$ and $u_2 \neq 0$ the last inequality becomes an identity which implies that at the particular point on the path $\lambda_1 = 0$ and $\lambda_2 = v_{11} + v_{22} = \frac{1}{3} V_{2222}u_2^{\,2} \neq 0$. Therefore at the points of intersection of the coupled path with the (p,u_2) plane λ_1 is the critical stability coefficient. These points correspond to the secondary branching points on the uncoupled paths in the (p,u_2) plane. A similar conclusion follows for the branching points on the uncoupled path in the (p,u_1) plane, when $\lambda_1 = 0$ and $\lambda_2 = \frac{1}{3} V_{1111}u_1^{\,2}$. Otherwise λ_1 is always negative on the coupled path since (43) is then of one sign. This means λ_1 vanishes in the secondary critical states but does <u>not</u> change sign. Therefore, the coupled path has stationary values at these points which represent coincident equilibrium states. So far nothing is said about the vanishing of λ_2.

(ii) The second instability condition evaluated on the paths requires

$$u_1^2 u_2^2 \, (V_{1111}V_{2222} - 9V_{1122}) > 0 \qquad (44)$$

and

$$\tfrac{1}{3}(V_{1111}u_1^2 + V_{2222}u_2^2) < -\tfrac{2}{3}\left|\sqrt{V_{1111}V_{2222} - 9V_{1122}}\sqrt{u_1^2 u_2^2}\right| \qquad (45)$$

where u_1 and u_2 are again the coordinates of the coupled paths. On completing the squares this expression transforms to

$$V_{1111}\left[\left(u_1 + u_2\sqrt{\frac{V_{2222}}{V_{1111}} - 9\,\frac{V_{1122}^2}{V_{1111}^2}}\right)^2 + 9\,\frac{V_{1122}^2}{V_{1111}^2}\,u_2^2\right] < 0 \qquad (46)$$

or to

$$V_{2222}\left[\left(u_2 + u_1\sqrt{\frac{V_{1111}}{V_{2222}} - 9\,\frac{V_{1122}^2}{V_{2222}^2}}\right)^2 + 9\,\frac{V_{1122}^2}{V_{2222}^2}\,u_1^2\right] < 0$$

In the equilibrium state $u_1 = 0$ and $u_2 \neq 0$ on the path $\lambda_1 = 0$ and $\lambda_2 = \frac{1}{3}V_{2222}u_2^2 < 0$. Therefore λ_1 is again the critical stability coefficient at the points of intersection of the coupled path with the (p,u_2) plane. When $u_2 = 0$ and $u_1 \neq 0$, similarly, $\lambda_1 = 0$ and $\lambda_2 = \frac{1}{3}V_{1111}u_1^2 < 0$. These points correspond to the four branching states on the uncoupled paths in the (p,u_2) and (p,u_1) planes.* Otherwise λ_1 and λ_2 are always negative on the coupled path which is therefore everywhere unstable. λ_2 may vanish if (45) becomes an identity and if (46) vanishes, but the last condition can yield no

---

*The proof that the coupled and uncoupled paths meet and that uncoupled paths may exchange their stability at secondary branching points is omitted here.

real solutions for u_1 and u_2 on the path.

(iii) The stability condition written on the path is

$$u_1^2 u_2^2 \, (V_{1111}V_{2222} - 9V_{1122}^2) > 0 \qquad (44)\text{bis}$$

and

$$\frac{1}{3}(V_{1111}u_1^2 + V_{2222}u_2^2) > + \frac{2}{3}\left|\sqrt{V_{1111}V_{2222} - 9V_{1122}}\sqrt{u_1^2 u_2^2}\right| \qquad (47)$$

On completing the squares this becomes

$$V_{1111}\left[\left(u_1 - u_2\sqrt{\frac{V_{2222}}{V_{1111}} - 9\,\frac{V_{1122}^2}{V_{1111}^2}}\right)^2 + 9\,\frac{V_{1122}^2}{V_{1111}^2}\,u_2^2\right] > 0$$

or

$$V_{2222}\left[\left(u_2 - u_1\sqrt{\frac{V_{1111}}{V_{2222}} - 9\,\frac{V_{1122}^2}{V_{2222}^2}}\right)^2 + 9\,\frac{V_{1122}^2}{V_{2222}^2}\,u_1^2\right] > 0 \qquad (48)$$

Conclusions regarding the secondary branching points when $\lambda_1 = 0$ are similar as in the last case only now $\lambda_2 = \frac{1}{3}V_{2222}u_2^2 > 0$, at $u_1 = 0$, $u_2 \neq 0$, and $\lambda_2 = \frac{1}{3}V_{1111}u_1^2 > 0$ at $u_2 = 0$, $u_1 \neq 0$. Otherwise λ_1 and λ_2 are positive everywhere on the coupled path which is therefore stable.

The last three conditions lead to

<u>Theorem (iii)</u>: If $(V_{1111}V_{2222} - 9V_{1122}^2) > 0$, then excluding the critical states on the path, equilibrium of a two-degree-of-freedom system in the coupled mode of buckling is stable everywhere on the path provided $V_{1111} > 0$ and unstable if $V_{1111} < 0$, where V_{1111} and V_{2222} must have the same sign.

We consider next several possible forms of coupled buckling.

CASE (i) It is readily verified by (36) that the orthogonal projection of the coupled paths in the (u_1, u_2) plane is an ellipse if

$$3\,V_{1122}V_{p11} - V_{1111}V_{p22} > 0$$

$$3\,V_{1122}V_{p22} - V_{2222}V_{p11} < 0$$

The projection of the coupled paths in the (p,u_2) plane is convex upwards, (37a), if $V_{1111}V_{2222} - 9V_{1122}^2 < 0$. Then, in the (p,u_1) plane this projection is convex downwards. Since this condition is synonymous with the first instability condition, such a path is unstable. When $V_{1111}V_{2222} - 9V_{1122}^2 > 0$, the projection of the coupled path in the (p,u_2) plane is convex downwards and in the (p,u_1) plane upwards, (37a) and (37b). If $V_{1111} < 0$ and $V_{2222} < 0$ the path is unstable by the second condition, and if $V_{1111} > 0$ and $V_{2222} > 0$ it is stable by the third condition.

Theorem (iv): If the higher uncoupled path of the system in case (i) is rising, i.e. if $V_{2222} > 0$, the lower uncoupled path must also be rising, i.e. $V_{1111} > 0$. The closed coupled path is stable if

$$V_{1111}V_{2222} - 9V_{1122}^2 > 0; \quad u_1, u_2 \neq 0$$

and unstable if

$$V_{1111}V_{2222} - 9V_{1122}^2 < 0$$

Figures 3 and 4.

Theorem (v): If the higher uncoupled path of the system in case (i) is falling, $V_{2222} < 0$, and the lower uncoupled path rising, $V_{1111} > 0$, the closed coupled path is always unstable since

$$V_{1111}V_{2222} - 9V_{1122}^2 < 0; \quad u_1, u_2 \neq 0$$

Theorem (vi): If both uncoupled paths of the system in case

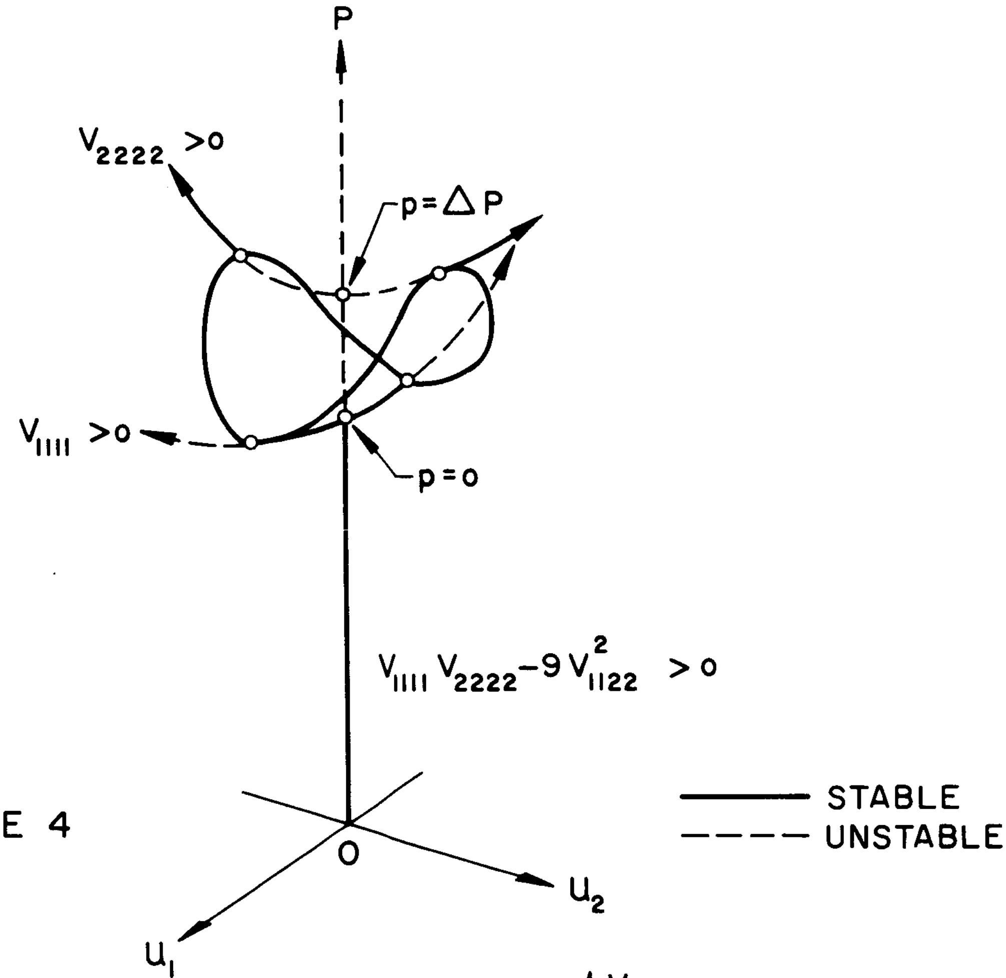

FIGURE 4

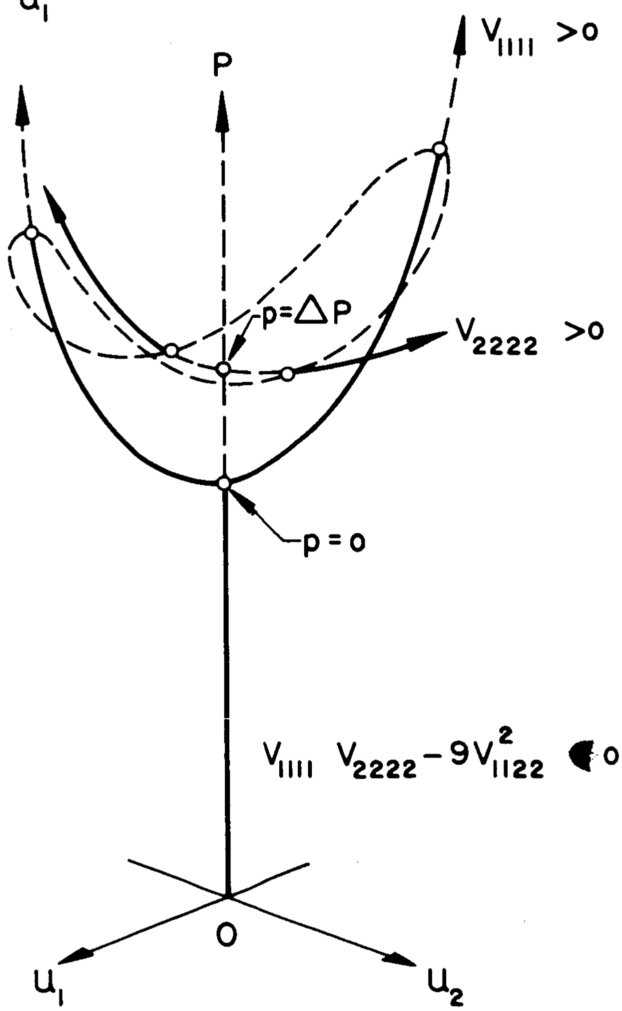

FIGURE 3

(i) are falling, the closed coupled path is always unstable.

CASE (ii): The projection of the coupled paths in the (u_1,u_2) plane is a hyperbola cutting the u_1-axis. Then

$$3V_{1122}V_{p11} - V_{1111}V_{p22} > 0$$

$$3V_{1122}V_{p22} - V_{1111}V_{p11} > 0$$

In conjunction with (37a) and (37b) we deduce:

<u>Theorem (vii)</u>: If both uncoupled paths are rising, i.e. $V_{1111} > 0$, $V_{2222} > 0$, and $V_{1122} \gtrless 0$, the coupled path branching from the lower uncoupled path is also rising if $V_{1111}V_{2222} - 9V_{1122}^2 > 0$ and falling if this quantity is negative. According to Condition (iii) the rising path is stable, critical states excepted, and the falling path is clearly unstable.

<u>Theorem (viii)</u>: If both uncoupled paths are falling, $V_{1111}<0$, $V_{2222} < 0$ and therefore $V_{1122} < 0$, the coupled path branching from the lower uncoupled path is falling. According to (43) this path is unstable.

<u>Theorem (ix)</u>: If one uncoupled path is falling and the other rising, the coupled paths branching from the lower path are always falling. Then condition (i), (43), is always satisfied and the coupled paths are clearly unstable.

CASE (iii): The orthogonal projection in the (u_1,u_2) plane of the coupled paths is a hyperbola cutting the u_2-axis. Then

$$3V_{1122}V_{p11} - V_{1111}V_{p22} < 0$$

$$3V_{1122}V_{p22} - V_{2222}V_{p11} < 0$$

From these inequalities and the equilibrium conditions (36, 37) we deduce:

<u>Theorem (x)</u>: If both uncoupled paths are rising, $V_{1111} > 0$, $V_{2222} > 0$ and $V_{1122} > 0$, the coupled path, branching from the higher uncoupled path, is always rising. Then, $V_{1111}V_{2222} - 9V_{1122}^2 < 0$, and the rising coupled paths are always unstable.

<u>Theorem (xi)</u>: If both uncoupled paths are falling, $V_{1111} < 0$, $V_{2222} < 0$, and $V_{1122} \gtrless 0$, the coupled path, branching from the higher uncoupled path, is rising if $V_{1111}V_{2222} - 9V_{1122}^2 < 0$ and falling if $V_{1111}V_{2222} - 9V_{1122}^2 > 0$. According to conditions (i) and (ii) these paths are always unstable.

<u>Theorem (xii)</u>: If one uncoupled path is falling and the other rising, i.e., if $V_{1111} < 0$, $V_{2222} > 0$ or $V_{1111} > 0$, $V_{2222} < 0$, the coupled paths, branching from the higher uncoupled path, are always rising. According to criterion (i) these paths are always unstable.

These results demonstrate the existence of <u>stable and unstable</u> coupled equilibrium paths and the occurrence of secondary branching points in the proximity of two (several) critical states in a system with two (several) degrees of freedom. An exchange of stabilities does not necessarily take place at secondary branching points on the coupled paths, as it may on the uncoupled ones. Similar conclusions regarding the rising and falling character of coupled paths in a two-degree-of-freedom system were reached independently by Supple. The author, however, does not analyse the stability of equilibrium in the coupled modes of elastic buckling.

The instability or stability of the uncoupled paths, or any other equilibrium paths in more complex systems, may be found similarly according to the method outlined in this section.

7. EXAMPLE

The theory is illustrated on a plane two-degree-of-freedom system consisting of two elastic bars of initial length L, Fig. 5. The moment of inertia I of the cross-section of bar (2) is slightly higher than that of bar (1). Thus, $EI_2 = EI_1 + E\Delta I$. The system is loaded with a vertical load P. The first critical state occurs when the Euler load is realized in bar (1) and the second, slightly higher, when this occurs in bar (2). Then coupled buckling may result on account of the geometrical discrepancy ΔI which may be regarded as the "<u>coupling imperfection</u>" in the system. A similar situation would occur if the initial lengths of the bars were slightly different. The system may be analysed similarly as in reference 4. (The reader is referred to reference 4 for details of this analysis.)

It may be shown that the variation v in the total potential energy from the first critical state equals

$$\bar{v} = v/S_1 = -\frac{7\pi^2}{192}u_1^4 + \frac{16\pi^2}{192}u_1^2u_2^2 - \frac{7\pi^2}{192}u_2^4 + \frac{\sqrt{3}}{12}\Delta\bar{P}\,u_2^2 + \frac{\sqrt{3}}{3}\bar{p}\left[-\frac{1}{4}u_1^2 - \frac{1}{4}u_2^2 - \frac{5}{128}u_1^4 + \frac{1}{12}u_1^2u_2^2 - \frac{5}{128}u_2^4 + \ldots\right. \qquad (49)$$

where

$S_1 = \frac{EI}{L}$ (for bar 1)

$\Delta P = \sqrt{3}\,\frac{\pi^2 E\Delta I}{L^2}$.....change in P between the two critical states

$\Delta\bar{P} = L\Delta P/S_1$

p .... increment above $P^0 = \sqrt{3}\,\frac{\pi^2 EI}{L^2}$

$\bar{p} = pL/S_1$

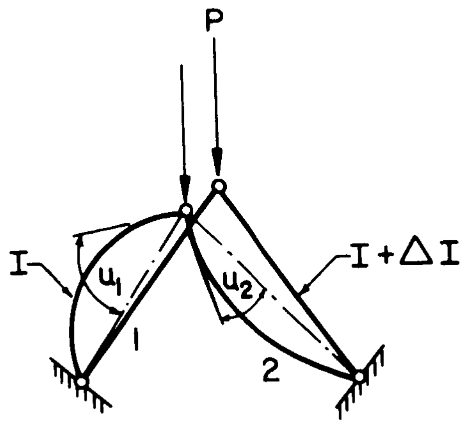

FIGURE 5

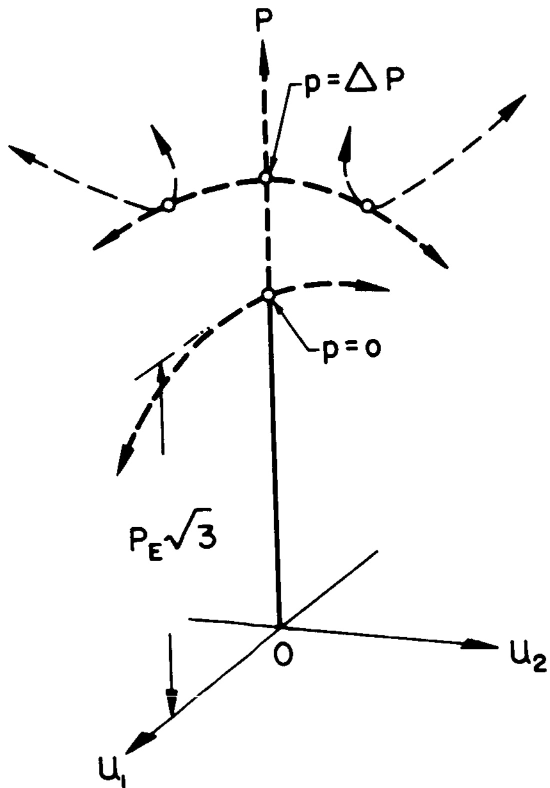

FIGURE 6

Comparing this with (14), (31) or (35), we observe that now

$$V_{11} = 0, \qquad V_{22} = \frac{\sqrt{3}}{6}\Delta P, \quad V_{p22} = -\frac{\sqrt{3}}{6}, V_{p11} = -\frac{\sqrt{3}}{6}$$

$$V_{1111} = -\frac{7\pi^2}{8} \quad V_{2222} = -\frac{7\pi^2}{8}, V_{1122} = \frac{\pi^2}{3}, V_{p1111} = -\frac{5\sqrt{3}}{16} \ldots \text{etc.} \tag{50}$$

The coefficients satisfy the inequalities in case (iii); therefore the coupled paths are hyperbolic. Since in this case

$$V_{1111}V_{2222} - 9V_{1122}^{\;2} = -\frac{15}{64}\pi^4 < 0, \tag{51}$$

then according to Theorem (xi), the coupled paths are rising and unstable. The result is shown in Fig. 6. The uncoupled paths are given by

$$\bar{p} = -\frac{7\pi^2\sqrt{3}}{24}u_1^{\;2}$$
$$\bar{p} = \Delta\bar{P} - \frac{7\pi^2\sqrt{3}}{24}u_2^{\;2} \tag{52}$$

or in terms of the flectural contractions e_1 and e_2 and p by

$$p = -\frac{7\sqrt{3}}{6}P_E\frac{e_1}{L}$$
$$p = \Delta P - \frac{7\sqrt{3}}{6}P_E\frac{e_2}{L} \tag{53}$$

where $e/L = u^2/4$.

In practice the coupled modes in this case may be realized only under controlled conditions. Usually the system develops unstable motion in the lower uncoupled mode.

CONCLUSIONS

In the case of a single load parameter several equilibrium paths may exist near an all-critical state of a completely symmetrical system. Uncoupled and coupled paths are approximately parabolic in form when this state is realized simultaneously by all generalized coordinates. Rising paths, in this case, may also be unstable.

In the presence of distinct types of geometrical imperfections within the system accumulation of several critical states may occur. This gives rise to other equilibrium states which form continuous open or closed paths connected to the uncoupled paths at the secondary branching points. Occasionally coupled paths may be stable, but more often they appear to be unstable. These conclusions are based on the study of the two-degree-of-freedom-systems. Geometrical deviations from perfection which may induce coupling between two or several freedoms of deformation are referred to as coupling imperfections. These should be distinguished from other imperfections which only influence the uncoupled modes of elastic buckling (e.g. references 6 and 7).

The stability analysis outlined in section (6) may be extended to more complex systems in the same way, although in the absence of order disparities in the coordinate space, algebraic evaluation of local stability coefficients and the corresponding instability zones may become involved beyond systems with three degrees of freedom. However, numerical techniques may be envisaged in that respect. The method is easily verifiable on simple model systems such as that in section (7).

In the presence of a required number of load parameters, equilibrium states in the vicinity of an all-critical state may be realized within the physical coordinate space of the system in an infinite variety of ways. Then, under controlled conditions and negligible losses theorem (i) may be used to measure the conversion of the total potential energy within the system and thereby to establish experimentally its local instability in a particular mode of elastic deformation.

REFERENCES

1. Chilver, A. H. 1967 J. Mech. Phys. Solids, 15, 15.
2. Supple, W. J. 1967 Int. J. Mech. Sci., 9, 97.
3. Britvec, S. J. 1960 Dissertation, I., Cambridge University.
4. Britvec, S. J. 1963 Ingenieur Archiv, 32, 6, 443.
5. Britvec, S. J. 1963 Int. J. Mech. Sci., 5, 447.
6. Roorda, J. 1965 J. Eng. Mech. Div. ASCE, 91, 4230.
7. Britvec, S. J. 1965 Int. J. Mech. Sci., 7, 661.