DEVELOPMENT OF A MOMENT METHOD TO SOLVE THE THREE-DIMENSIONAL BOUNDARY LAYER EQUATIONS

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J. A. FILLO

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DEPARTMENT OF MECHANICAL AND AERO-SPACE ENGINEERING
UNIVERSITY OF MASSACHUSETTS
AMHERST, MASSACHUSETTS 01002

INTRODUCTION

Problems in the theory of laminar incompressible and compressible three-dimensional boundary layers attract much interest. Comprehensive reviews of the subject appear in literature references.[1,2,3] The main objectives of the theory are to calculate the skin friction distribution, and to predict the location of the boundary layer separation surface on three-dimensional bodies, as well as to determine heat transfer effects during, say, the re-entry of space vehicles.

The three-dimensional boundary layer equations, either for "small" or "large" cross flows, are not easily amenable to mathematical analysis owing to the complexity of the set of partial differential equations governing the phenomena. Exact solutions, in general, are not found, and resort must be made to numerical procedures, or to approximate methods. With regards to the nature of solutions, the problem may be reformulated as follows: by exact solutions it is meant similar solutions and it is known that numerous and severe restrictions are imposed as flow conditions for similar solutions to exist. Considering the diverse applications of compressible and incompressible laminar boundary layer theory, it is seldom that all of the similarity conditions are met. Hence, the question: What approach should be used to predict the behaviour of the nonsimilar laminar boundary layer? In a recent review, Dewey and Gross [4] consider this question but with emphasis on two-dimensional flows. Their findings could also be applied to three-dimensional flows. In essence, it was shown that there are four basic types of approach: approximate techniques such as integral and series solutions containing free parameters; strictly numerical approach; locally similar methods; and, lastly, expansions about similar solutions.

Two of the above approaches are discussed in the present report extended to three-dimensional flows: expansions about similar solutions and an approximate integral technique.

Statement of the Problem

We consider a body (prevailingly convex) of finite dimensions, placed in a uniform air stream. Furthermore, we assume that the inviscid flow around the body is known and allows an extrapolation towards y = 0 (surface of the body). These extrapolated values at the wall are considered as the outer boundary values of the boundary layer which we suppose laminar over the whole surface.

The Navier-Stokes[1] equations, written in a system of curvilinear coordinates and based on the following three families of surfaces:

y = constant; surfaces parallel to that of the body (y = 0);

z = constant; normal surfaces along the streamlines of the inviscid flow at the wall of the body;

s = constant; normal surfaces along the orthogonal trajectories of the streamlines of the inviscid flow at the wall;

assume after the introduction of Prandtl's boundary layer hypothesis, the form:

$$\frac{\partial}{\partial s}(\rho u e_2) + \frac{\partial}{\partial y}(\rho v e_1 e_2) + \frac{\partial}{\partial z}(\rho w e_1) = 0 \tag{1}$$

$$\rho\left(\frac{u}{e_1}\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial y} + \frac{w}{e_2}\frac{\partial u}{\partial z} - K_2 u w + K_1 w^2\right) = -\frac{1}{e_1}\frac{\partial P}{\partial s} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\rho\left(\frac{u}{e_1}\frac{\partial w}{\partial s} - v\frac{\partial w}{\partial y} + \frac{w}{e_2}\frac{\partial w}{\partial z} - K_1 u w + K_2 u^2\right) = -\frac{1}{e_2}\frac{\partial P}{\partial z} + \frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) \tag{3}$$

$$\frac{\partial P}{\partial y} = 0 \tag{4}$$

$$\rho\left(\frac{u}{e_1}\frac{\partial H}{\partial s} + v\frac{\partial H}{\partial y} + \frac{w}{e_2}\frac{\partial H}{\partial z}\right) = \frac{\partial}{\partial y}\left(\frac{\mu}{\sigma}\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left[\mu\left(1-\frac{1}{\sigma}\right)\frac{\partial}{\partial y}\left(\frac{u^2+w^2}{2}\right)\right] \tag{5}$$

Here (u, v, w) are the components of the velocity vector $\vec{V}$ inside the boundary layer in the directions of (s, y, z); $e_1(s, z)$ and $e_2(s, z)$ are the metric elements along the inviscid flow streamlines and their orthogonal trajectories (e_3, along the outer normal of the body, being unity by definition). κ_1 and κ_2 are Gauss curvature terms, given by

$$K_1 = -\frac{1}{e_1 e_2}\frac{\partial e_2}{\partial s} \qquad\qquad K_2 = -\frac{1}{e_1 e_2}\frac{\partial e_1}{\partial z} \tag{6}$$

and

$$H = c_p T + \frac{u^2+v^2+w^2}{2} \sim c_p T + \frac{u^2+w^2}{2} \tag{7}$$

is the total energy of the flow. σ is the Prandtl number. The boundary conditions are:

at the wall

$$y=0: \quad u=v=w=0\,,\ H=H_w \tag{8a}$$

and at the outer edge of the boundary layer

$$y\rightarrow\infty: \ u\rightarrow u_e(s,z)\,,\quad w\rightarrow 0\,,\quad H\rightarrow H_e \tag{8b}$$

We assume now, as a first approximation, that $w \ll u$, $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial z}$ are derivatives parallel to the wall, and therefore, may be considered as operators which conserve the order of a term. The system becomes, after application of Euler's equations to $\frac{\partial P}{\partial s}$ and $\frac{\partial P}{\partial z}$:

$$\frac{1}{e_1}\frac{\partial}{\partial s}(\rho u e_2) - \frac{\partial}{\partial y}(\rho v e_2) = 0 \tag{9}$$

$$\rho\left(\frac{u}{e_1}\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial y}\right) = \rho\,\frac{u_e}{e_1}\frac{\partial u_e}{\partial s} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{10}$$

$$\rho\left\{\frac{u}{e_1}\frac{\partial w}{\partial s} + v\frac{\partial w}{\partial y} - K_1 u w + K_2 u_e^2\left[-\frac{\rho_e}{\rho} + \left(\frac{u}{u_e}\right)^2\right]\right\} = \frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) \tag{11}$$

$$\rho\left(\frac{u}{e_1}\frac{\partial H}{\partial s} + v\frac{\partial H}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\mu}{\sigma}\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left[\mu\left(1-\frac{1}{\sigma}\right)\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\right] \tag{12}$$

where Eqs. 9, 10, and 12 do not contain w, and depend on z only as a simple parameter. Furthermore, if we agree that all derivatives and all integrations are executed for z = constant, we may write

$$\frac{1}{e_1}\frac{\partial}{\partial s} = \frac{\partial}{\partial x} \qquad\qquad \text{where } dx = e_1 ds$$

is the length element of an inviscid flow streamline. (A formal perturbation procedure[5] has been recently employed to derive Eqs. 9-12, also).

Analysis

Applying the Levy-Lees transformation

$$\left.\begin{aligned} \xi(x) &= \int_0^x \rho_w \mu_w u_e e_2^2\, dx \\ \eta(x,y) &= \frac{u_e e_2}{\sqrt{2\xi}} \int_0^y \rho\, dy \end{aligned}\right\} \qquad z = \text{constant} \tag{13}$$

$$\frac{u}{u_e} = \frac{\partial f}{\partial \eta} = f_\eta\,, \quad \frac{H}{H_e} = g(\xi,\eta)\,, \quad \frac{w}{u_e} = \omega(\xi,\eta) \tag{14}$$

to Eqs. (10-12), we obtain

$$(c f_{\eta\eta})_\eta + f f_{\eta\eta} + \beta\left[\left(\frac{\rho_e}{\rho}\right) - f_\eta^2\right] = 2\xi\left[f_\eta f_{\eta\xi} - f_\xi f_{\eta\eta}\right] \tag{15}$$

$$(c\omega_\eta)_\eta + f\omega_\eta + \left(\beta_2^* - \frac{2\xi}{u_e}\frac{du_e}{d\xi}\right) - \beta_1^*\left[\frac{\rho_e}{\rho} - (f_\eta)^2\right] = 2\xi(f_\eta \omega_\xi - f_\xi \omega_\eta) \tag{16}$$

$$\left(\frac{c}{\sigma} g_\eta\right)_\eta + f g_\eta + 2\alpha\left[c\left(1 - \frac{1}{\sigma}\right) f_\eta f_{\eta\eta}\right]_\eta = 2\xi(f_\eta g_\xi - f_\xi g_\eta) \tag{17}$$

where

$$\beta_2^* = -\frac{2\xi}{e_2}\frac{\partial e_2}{\partial \xi} \qquad \beta_1^* = \frac{2\xi}{e_1}\left(\frac{\partial e_1}{\partial \psi}\right)_s = \text{constant}$$

$$\beta = \frac{2\xi}{u_e}\frac{du_e}{d\xi} \qquad \alpha = \frac{u_e^2}{2H_e} \tag{18}$$

$$c = \frac{\rho\mu}{\rho_e \mu_e}$$

The boundary conditions are:

$$\eta = 0 \;:\; f = f_\eta = 0 \qquad g = g_w \qquad \omega = 0 \tag{19a}$$

$$\eta \to \infty \;:\; f_\eta \to 1 \qquad g \to 1 \qquad \omega \to 0 \tag{19b}$$

Equations (15)-(17) subject to boundary conditions (19a) and (19b) are the equations on which we shall focus our attention.

Asymptotic Expansion of the Boundary Layer Equations

Of the four basic types of approach to the solution of the nonsimilar laminar boundary layer equations, it seems that expansions about similar solutions has received little, if any, attention for three-dimensional flows. The motivation for such an investigation arose out of the results reported in Refs. (5) and (6). In the former exact numerical calculations of the complete nonsimilar, small-cross flow, boundary layer equations were compared with locally similar solutions. While the results obtained from the two different methods for the main flow were in excellent agreement, the cross flow results, while not drastically different, did reveal that further analytical work is necessary.

In the latter reference only the method of utilizing locally similar solutions was used. As pointed out in this work, the application of locally similar methods to the secondary flow has no quantitative confirmation. With this in mind it was felt that some light could be shed on locally similar methods by the expansion methods. In addition, the use of integral or improved integral methods could be placed on surer footing if there are more reliable results to compare the integral results with.

The basis of the expansion method may be traced to the work of Meksyn.[7] The underlying premise is as follows: if the boundary layer is at all times very nearly described by a similar solution, then the direct effects of the nonsimilar terms may be calculated by asymptotically expanding the full boundary layer equations in terms of small parameters which measure the departure of the solutions from similarity. In this way, the accuracy of locally similar methods is explicitly determined by using the full nonsimilar equations.

To demonstrate the method for three-dimensional flows, we consider the incompressible momentum equations (obtained from Eqs. 15 and 16 by setting $\frac{\rho_e}{\rho} = 1$ and $C = 1$)

$$f_{\eta\eta\eta} + f f_{\eta\eta} + \beta(\xi)(1 - f_\eta^2) = (f_\eta f_{\xi\eta} - f_\xi f_{\eta\eta}) \tag{20}$$

$$w_{\eta\eta} + f w_\eta + \beta_1^*(f_\eta^2 - 1) + (\beta_2^* - \beta) f_\eta w = 2\xi(f_\eta w_\xi - w_\eta f_\xi) \tag{21}$$

subject to the boundary conditions

$$f(\xi,0) = f_\eta(\xi,0) = 0 \quad , \quad w(\xi,0) = 0 \tag{22a}$$

$$f_\eta(\xi,\infty) = 1 \quad , \quad w(\xi,\infty) = 0 \tag{22b}$$

It is to be noted that Eq. 20 is the transformed two-dimensional boundary layer equation. Merk[8] was the first to expand this complete nonsimilar equation. The key to the appropriate expansion was the inversion of independent variables; i.e., the change of variables from ξ, η to β, η. Adopting this change of variables for the crossflow equation, Eq. 21, we find that Eqs. 20 and 21 transform to

$$f_{\eta\eta\eta} + f f_{\eta\eta} + \beta(1 - f_\eta^2) = \epsilon(f_\eta f_{\beta\eta} - f_\beta f_{\eta\eta}) \tag{23}$$

$$w_{\eta\eta} + f w_\eta + \beta_1^*(f_\eta^2 - 1) + (\beta_2^* - \beta) f_\eta w = \epsilon(f_\eta w_\beta - w_\eta f_\beta) \tag{24}$$

and

$$f(\beta,0) = f_\eta(\beta,0) = 0 \quad , \quad w(\beta,0) = 0 \tag{25a}$$

$$f_\eta(\beta,\infty) = 1 \quad , \quad w(\beta,\infty) = 0 \tag{25b}$$

where

$$\epsilon(\beta) = 2\xi\,\beta'(\xi) = \frac{2\,\xi(\beta)}{\xi'(\beta)}$$

For small ϵ we perform an asymptotic expansion of f and w of the form

$$f(\beta,\eta) = f_0(\beta,\eta) + \epsilon(\beta) f_1(\beta,\eta) + \cdots \tag{26}$$

$$w(\beta,\eta) = w_0(\beta,\eta) + \epsilon(\beta) w_1(\beta,\eta) + \cdots \tag{27}$$

Substituting Eqs. 26 and 27 into Eqs. 23 and 24, as well as 25a and 25b, yields, on equating like powers of ϵ, the following sets of equations :

ϵ^0:

$$f_0''' + f_0 f_0'' + \beta[1-(f_0')^2] = 0 \tag{28}$$

$$f_0(\beta,0) = f_0'(\beta,0) = 0 \quad , \quad f_0'(\beta,\infty) = 1 \tag{29}$$

$$w_0'' + f_0 w_0' + \beta_1^*[(f_0')^2 - 1] + (\beta_2^* - \beta) f_0' w_0 = 0 \tag{30}$$

$$w_0(\beta,0) = 0 \; ; \quad w_0(\beta,\infty) = 0 \tag{31}$$

ϵ^1:

$$f_1''' + f_0 f_1'' - A_1 f_0' f_1' + A_2 f_0'' f_1 = \Phi_1(\beta,\eta) \tag{32}$$

$$f_1(\beta,0) = 0 \, , \quad f_1'(\beta,0) = 0 \, , \quad f_1'(\beta,\infty) = 0 \tag{33}$$

$$w_1'' + f_0 w_1' + B_1 f_0' w_1 = \chi_1(\beta,\eta) \tag{34}$$

$$w_1(\beta,0) = 0 \quad , \quad w_1(\beta,\infty) = 0 \tag{35}$$

where

$$A_1 = 2 + 2\beta + 2E \quad , \quad A_2 = 3 + 2E$$

$$E = E(\beta) = \xi(\beta) \frac{\beta''(\xi)}{\beta'(\xi)}$$

$$\Phi_1(\beta,\eta) = f_0' \, (f_0')_\beta - (f_0)_\beta \, f_0''$$

$$B_1 = \beta_2^* - \beta - 2(1+E)$$

$$\chi_1(\beta,\eta) = -\beta_1^* (2 f_0' f_1' - 1) - (\beta_2^* - \beta) w_0 f_1' + f_0'(w_0)_\beta - w_0'[(f_0)_\beta + 2(1+E) f_1]$$

Equations 28 and 29 represent the similar solutions of Falkner and Skan while Equations 32 and 33 represent the first correction f_1 to the main flow velocity profile which arises from nonsimilar terms. Equations 30 and 31 are the corresponding similar solutions for the corss flow and Equations 34 and 35 represent the first correction ω_1 to the secondary flow velocity profile because of nonsimilar terms. It is assumed that ϵ is not too large so that the series solution in powers of ϵ will converge satisfactorily.

Before proceeding with a discussion on the integration of the equations, a few comments on the question of similarity and local similarity of the cross flow question are pertinent. In the work of Geis[9] it was pointed out that for similar solutions if two velocity components that are different from zero and parallel to the wall are present in the external flow, the pertaining velocity U or W of the external flow must necessarily be chosen as the scale factor for u and w. If, on the other hand, no W-component is present in the external flow which corresponds to the use of streamline coordinates, an arbitrary function of the dimension of a velocity as scale factor may be chosen for w. While this discussion has been with respect to similar solutions, it is nevertheless valid for locally similar soultions as well. In the two previously cited references (5 and 6), where streamline coordinates were used, the question of arbitrariness in the scale factor is what lead to different small cross flow, locally similar equations. Two different scale factors were introduced. The present work is concerned with the dimensionless cross flow velocity component, w/v, and, hence, the reason why the leading terms, in ϵ°, check with the incompressible equations of Reference [5] and not with those in Reference [6].

Returning to the problem of integrating Equations 28-35, one finds that methods have been discussed [4,10] to integrate the equations for f_0 and f_1 (as well as subsequent higher order terms). The problem before us is to find ω_0 and ω_1.

As in the case of computing f_1, the primary difficulty arises in computing the inhomogeneous term $\chi_1(\beta,\eta)$ which contains derivatives of w_0 with respect to both η and β. This is a difficult term to calculate numerically because a number of similarity solutions in the neighborhood of β must be known with high precision. The problem is not insurmountable in light of the numerical methods discussed in [10] for the calculation of f_1. Some simplification of the equations may be achieved by eliminating the parameter β_1^* by defining new dependent variables

$$w_0^* = \frac{w_0}{\beta_1^*} \qquad , \qquad w_1^* = \frac{w_1}{\beta_1^*}$$

At present work is being carried on in the investigation of Equations 30, 31, 34 and 35.

Moment Integral Conditions

Returning to the compressible forms of the equations, Equations 15 - 19b , a discussion of the moment equations will follow. The basic formulation of the method has been presented in Reference [11] for two-dimensional flows and, in which, Equations 15 and 16 have been discussed in detail. What we shall be primarily interested in is Equation 17 .

Introducing a finite "edge", $\eta = \eta_e$ at which $w=0$ we obtain a series of integral conditions by multiplying Equation 17 by $\eta^m d\eta$ (m = 0, 1, 2,....) and by integrating termwise from 0 to η_e. It is assumed that the streamwise and cross flow boundary layers are described by the same thickness, a usual assumption for the three-dimensional case. The first term on the left hand side of Equation 17 becomes

$$\int_0^{\eta_e} \eta^m (c w_\eta)_\eta \, d\eta = -[c w_\eta]_w \qquad (36)$$

$$= -c_w \frac{[w_{\hat\eta}]_w}{\eta_e} \qquad m = 0 \quad (36a)$$

$$= -m \eta_e^{m-1} \int_0^1 \hat\eta^{m-1} c w_{\hat\eta} \, d\hat\eta \qquad m \geqq 1 \quad (46b)$$

For $m \geq 1$ the influence of the transport properties throughout the boundary layer is present. As in the case of the streamwise momentum equation, it is convenient to introduce a mean value of C, for $m>1$, denoted $\bar{\bar{C}}_{o,m}$ and varying with ξ and to consider the remainder to be a function of ξ and η, known from a previous step in the integration of the ordinary differential equations. Let

$$C(\xi,\eta) = \left[C(\xi,\eta) - \bar{\bar{C}}_{o,m}(\xi)\right] + \bar{\bar{C}}_{o,m}(\xi) \tag{37}$$

and for $m>1$, Equation 36b becomes

$$\int_0^{\eta_e} \eta^m (C w_\eta)_\eta \, d\eta = -m\eta_e^{m-1} \bar{\bar{C}}_{o,m}\Big[-(m-1)\int_0^1 \tilde{\eta}^{m-2} w \, d\tilde{\eta} + \int_0^1 \tilde{\eta}^{m-1}(C - \bar{\bar{C}}_{o,m})\, \bar{\bar{C}}_{o,m}^{-1} w_{\tilde{\eta}} \, d\tilde{\eta}\Big] \tag{38}$$

Selecting $\bar{\bar{C}}_{o,m}$ so that

$$\bar{\bar{C}}_{o,m} = \frac{\int_0^1 \tilde{\eta}^{m-1} C w_{\tilde{\eta}} \, d\tilde{\eta}}{(m-1)\int_0^1 \tilde{\eta}^{m-2} w \, d\tilde{\eta}} \tag{39}$$

Equation 38 assumes the form

$$\int_0^{\eta_e} \eta^m (C w_\eta)_\eta \, d\eta = -m(m-1)\eta_e^{m-1} \bar{\bar{C}}_{o,m} \int_0^1 \tilde{\eta}^{m-2} w \, d\tilde{\eta} \tag{40}$$

The remaining terms can be integrated by parts and the results are:

$$\int_0^{\eta_e} \eta^m f w_\eta \, d\eta = -\eta_e \int_0^1 w f_\eta \, d\tilde{\eta} \qquad m=0 \tag{41a}$$

$$= -\eta_e^m \Big[\eta_e \int_0^1 \tilde{\eta}^m w f_\eta \, d\tilde{\eta} + m\eta_e \int_0^1 w \Big(\int_0^{\tilde{\eta}} f_\eta \, d\tilde{\eta}\Big) d\tilde{\eta}\Big] \qquad m \geq 1 \tag{41b}$$

$$\int_0^{\eta_e} \eta^m (\beta_2^* - \beta) f_\eta w \, d\eta = (\beta_2^* - \beta)\eta_e \int_0^1 f_\eta w \, d\tilde{\eta} \qquad m=0 \tag{42a}$$

$$= (\beta_2^* - \beta)\eta_e^{m+1} \int_0^1 \tilde{\eta}^m f_\eta w \, d\tilde{\eta} \qquad m \geq 1 \tag{42b}$$

$$\int_0^{\eta_e} \eta^m \beta_1^* \left(\frac{\rho_e}{\rho} - f_\eta^2\right) d\eta = \beta_1^* \eta_e \int_0^1 \left(\frac{\rho_e}{\rho} - f_\eta^2\right) d\tilde{\eta} \qquad m = 0 \qquad (43a)$$

$$= \beta_1^* \eta_e^{m+1} \int_0^1 \tilde{\eta}^m \left(\frac{\rho_e}{\rho} - f_\eta^2\right) d\tilde{\eta} \qquad m \geq 1 \qquad (43b)$$

$$\int_0^{\eta_e} \eta^m 2\xi (f_\eta w_\xi - f_\xi w_\eta) d\eta = 2\xi \left\{ \frac{d\eta_e}{d\xi} \int_0^1 f_\eta w \, d\tilde{\eta} + \eta_e \frac{d}{d\xi} \int_0^1 f_\eta w \, d\tilde{\eta} \right\} \qquad m = 0 \qquad (44a)$$

$$= 2\xi \eta_e^m \left\{ (m+1) \frac{d\eta_e}{d\xi} \int_0^1 \tilde{\eta}^m f_\eta w \, d\tilde{\eta} + \eta_e \frac{d}{d\xi} \int_0^1 \tilde{\eta}^m f_\eta w \, d\tilde{\eta} + m \eta_e \int_0^1 \tilde{\eta}^{m-1} w \left(\int_0^{\tilde{\eta}} \frac{\partial f_\eta}{\partial \xi} d\tilde{\eta}' \right) d\tilde{\eta} \right\} \qquad m \geq 1 \qquad (44b)$$

Analogous to the case of the streamwise momentum equation, Equations 36 - 44b provide as many integral conditions for the cross flow velocity profile as is wanted by letting $m = 0, 1, 2, \ldots$

Summarizing, the integral conditions corresponding to $m = 0,1$ in Equations 36 - 44b take on the following forms:

$$m = 0 \qquad \frac{2\xi}{\eta_e} \frac{d}{d\xi}(\eta_e I_{11}) = \frac{C_w}{\eta_e^2}(w_{\tilde{\eta}})_w - I_{11} + (\beta_2^* - \beta) I_{13} + \beta_1^* I_2 \qquad (45)$$

$$m = 1 \qquad -\frac{2\xi}{\eta_e^2}\left[\frac{d}{d\xi}(\eta_e^2 I_{14} + \eta_e^2 I_{16}')\right] = \frac{\bar{\bar{C}}_{0.1}}{\eta_e^2} + I_{14} + (1 + \beta_2^* - \beta) I_{16} + (\beta_2^* - \beta) I_{18} - \beta_1^* I_5 \qquad (46)$$

where $I_2 = \int_0^1 (g - f_\eta^2) d\tilde{\eta}$

$I_5 = \int_0^1 \tilde{\eta}(g - f_\eta^2) d\tilde{\eta}$

$I_{11} = -\int_0^1 w f_\eta d\tilde{\eta}$

$I_{13} = \int_0^1 w_{\tilde{\eta}} \left(\int_0^{\tilde{\eta}} f_\eta d\tilde{\eta}'\right) d\tilde{\eta}$

$I_{14} = \int_0^1 \tilde{\eta} f_\eta w \, d\tilde{\eta}$

$I_{16} = \int_0^1 w \left(\int_0^{\tilde{\eta}} f_\eta d\tilde{\eta}'\right) d\tilde{\eta}$

$$I'_{16} = \int_0^1 \omega \left(\int_0^{\tilde{\eta}} \frac{\partial f_\eta}{\partial \xi} d\tilde{\eta}' \right) d\tilde{\eta}$$

$$I_{18} = \int_0^1 \tilde{\eta}\, \omega_{\tilde{\eta}} \left(\int_0^{\tilde{\eta}} f_\eta d\tilde{\eta}' \right) d\tilde{\eta}$$

The integral moment conditions for f and g may be found in Reference [11].

In order to integrate the above equations appropriate profiles for f_η , g and ω are needed. Since polynomials were utilized for f_η and g, it is convenient to express ω in terms of a polynomial. Of course, in doing so, one must recognize that appropriate parameters must be in the polynomial to account for change of sign in the cross flow profile. This is an important part of the study to be able to take account of and predict the change in sign of the secondary flow.

To gain a better understanding of the behaviour of the cross flow equations with regards to the moment method, the incompressible equations are being investigated. At present the example chosen is a problem with flow reversal.

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