AERO-ASTRONAUTICS REPORT NO. 50

USE OF THE METHOD OF PARTICULAR SOLUTIONS IN NONLINEAR, TWO-POINT BOUNDARY-VALUE PROBLEMS

PART 1 - UNCONTROLLED SYSTEMS

by

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(THRU) 1
23
(PAGES)
(CODE)
CR-95726
(NASA CR OR TMX OR AD NUMBER)
19
(CATEGORY)
FACILITY FORM 602

RICE UNIVERSITY

1968

Use of the Method of Particular Solutions In Nonlinear, Two-Point Boundary-Value Problems Part 1 - Uncontrolled Systems[1]

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Abstract. Several nonlinear, two-point boundary value problems are considered in this report. First, quasilinearization techniques are employed and the system of nonlinear equations is replaced by one that is linear. Then, the method of particular solutions is employed in order to solve the linear problem. The procedure is employed iteratively, and it is shown to converge rapidly to the desired solution in the following typical cases: (a) the Blasius equation of boundary-layer theory, (b) the Falkner-Skan equation of boundary-layer theory, and (c) a particular fifth-order system.

[1] This research was supported by the NASA-Manned Spacecraft Center, Grant No. NGR-44-006-089.

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1. Introduction

In Ref. 1, Miele developed the method of particular solutions for solving linear, two-point boundary-value problems. He treated a system of order n, subjected to p initial conditions and q final conditions, with $p + q = n$. He proved that $q + 1$ particular solutions of the original, nonhomogeneous system satisfying the initial conditions but not the final conditions can be combined linearly so as to satisfy simultaneously the original, nonhomogeneous system and the initial conditions, providing the sum of the constants of the linear combination is one. This relation and the q prescribed final conditions constitute a system of $q + 1$ linear algebraic equations in the $q + 1$ unknown constants.

As mentioned by Miele in Ref. 1, the method of particular solutions can also be used to solve nonlinear, two-point boundary-value problems. First, quasilinearization techniques must be employed and the nonlinear system must be replaced by one that is linear in the perturbations about a nominal curve (see, for example, Ref. 2). To this linear system, Miele's method can be applied to find the perturbations leading to a new nominal curve. Then, the procedure is employed iteratively.

To illustrate the versatility of the method described in Ref. 1, the following nonlinear problems are treated: (a) the Blasius equation of boundary-layer theory, (b) the Falkner-Skan equation of boundary-layer theory, and (c) a particular fifth-order system. We note that, for comparison purposes, power-series solutions are available for Cases (a) and (b) and an analytical solution is available for Case (c). We also note that Systems (a), (b), (c) are uncontrolled: their behavior is fully determined if a complete set of initial conditions is specified.

2. Blasius Equation

For the laminar, incompressible boundary layer over a flat plate, the velocity profile is governed by the Blasius equation (Ref. 3)

$$2\dddot{x} + x\ddot{x} = 0 \tag{1}$$

for which the boundary conditions are

$$x(0) = \dot{x}(o) = 0, \quad \dot{x}(\infty) = 1 \tag{2}$$

In Eq. (1), the independent variable t is proportional to the distance from the flat plate and the dependent variable x is such that its derivative $\dot{x}$ is proportional to the velocity.

After the auxiliary variables y, z defined by

$$\dot{x} - y = 0, \quad \dot{y} - z = 0 \tag{3}$$

are introduced, Eq. (1) can be rewritten as

$$2\dot{z} + xz = 0 \tag{4}$$

and the boundary conditions (2) become[3]

$$x(0) = y(0) = 0, \quad y(10) = 1 \tag{5}$$

Therefore, the problem of solving the Blasius equation consists of finding the functions x(t), y(t), z(t) which solve the nonlinear system (3)-(4) subject to the boundary

[3] For computational purposes, the boundary condition at $t = \infty$ is replaced by a boundary condition at $t = 10$.

conditions (5). Note that the auxiliary variable y is proportional to the velocity.

As a first step, Eqs. (3)-(4) are linearized about a nominal curve $x_*(t)$, $y_*(t)$, $z_*(t)$ which is not a solution of (3)-(4) but satisfies the boundary conditions (5). The linearization leads to the perturbation equations

$$\begin{aligned} \delta\dot{x} - \delta y + (\dot{x} - y)_* &= 0 \\ \delta\dot{y} - \delta z + (\dot{y} - z)_* &= 0 \\ 2\,\delta\dot{z} + z_*\delta x + x_*\delta z + (2\dot{z} + xz)_* &= 0 \end{aligned} \tag{6}$$

which are subject to the boundary conditions

$$\delta x(0) = \delta y(0) = 0, \qquad \delta y(10) = 0 \tag{7}$$

In Eqs. (6)-(7), the symbols δx, δy, δz denote the perturbations of x, y, z at a constant station t, that is,

$$\begin{aligned} \delta x &= x(t) - x_*(t) \\ \delta y &= y(t) - y_*(t) \\ \delta z &= z(t) - z_*(t) \end{aligned} \tag{8}$$

Having linearized the third-order system corresponding to the Blasius equation, we now apply the method of particular solutions. Since $p = 2$, $q = 1$, a forward integration is desirable. Since $q + 1 = 2$, two particular solutions are required and are designated with the subscripts 1, 2, respectively. In the first integration, we employ the initial conditions

$$\delta x_1(0) = \delta y_1(0) = 0, \qquad \delta z_1(0) = 1 \tag{9}$$

and obtain the functions

$$\delta x_1(t), \quad \delta y_1(t), \quad \delta z_1(t) \tag{10}$$

In the second integration, we employ the initial conditions

$$\delta x_2(0) = \delta y_2(0) = 0, \qquad \delta z_2(0) = 2 \tag{11}$$

and obtain the functions

$$\delta x_2(t), \quad \delta y_2(t), \quad \delta z_2(t) \tag{12}$$

As shown in Ref. 1, the linear combinations

$$\begin{aligned} \delta x &= k_1 \delta x_1 + k_2 \delta x_2 \\ \delta y &= k_1 \delta y_1 + k_2 \delta y_2 \\ \delta z &= k_1 \delta z_1 + k_2 \delta z_2 \end{aligned} \tag{13}$$

satisfy the differential system (6) and the initial conditions (7-1)-(7-2) provided

$$k_1 + k_2 = 1 \tag{14}$$

and the final condition (7-3) provided

$$k_1 \delta y_1(10) + k_2 \delta y_2(10) = 0 \tag{15}$$

If the constants k_1, k_2 are consistent with (14)-(15), then the linear combinations (13) are the desired solutions to (6). Once the perturbation functions are known, the approximate trajectory of the system is given by

$$\begin{aligned} x &= x_* + \delta x \\ y &= y_* + \delta y \\ z &= z_* + \delta z \end{aligned} \tag{16}$$

and, in this way, the first iteration is completed. Next, the functions x(t), y(t), z(t) given by Eqs. (16) are employed as the nominal functions for the second iteration, and the procedure is repeated.

Computations were performed with an IBM 7040 computer. The following nominal curve was chosen for the first iteration:

$$\begin{aligned} x_* &= t^2/10 - t^3/300 \\ y_* &= t/5 - t^2/100 \\ z_* &= 1/5 - t/50 \end{aligned} \tag{17}$$

At the terminal points, Eqs. (17) yield

$$x_*(0) = y_*(0) = 0, \qquad y_*(10) = 1 \tag{18}$$

and, hence, the boundary conditions (5) are satisfied by the nominal curve.

Convergence was rapid. For instance, at station $t = 4$, no change occurred in the 5th significant figure after four iterations. The results are plotted in Figs. 1-3, in which the symbol n denotes the iteration number. Therefore, $n = 0$ is the zeroth iteration [the nominal curve (17)], $n = 1$ is the first iteration, $n = 2$ is the second iteration, and $n = 3$ is the third iteration. The curve $n = 4$ is so close to the curve $n = 3$ that the relative differences cannot be detected in the scale of Figs. 1-3.

3. Falkner-Skan Equation

For the laminar, incompressible boundary layer over a double wedge of included angle $\pi\beta$, the velocity profile is governed by the Falkner-Skan equation (Ref. 3)

$$2\dddot{x} + x\ddot{x} + \beta(1 - \dot{x}^2) = 0 \tag{19}$$

for which the boundary conditions are

$$x(0) = \dot{x}(0) = 0, \qquad \dot{x}(\infty) = 1 \tag{20}$$

After the auxiliary variables y, z defined by Eqs. (3) are introduced, Eq. (19) can be rewritten as

$$2\dot{z} + xz + \beta(1 - y^2) = 0 \tag{21}$$

and the boundary conditions (20) reduce to (5). Therefore, the problem of solving the Falkner-Skan equation consists of finding the functions x(t), y(t), z(t) which solve Eqs. (3) and (21) subject to the boundary conditions (5).

We linearize Eqs. (3) and (21) about a nominal curve $x_*(t)$, $y_*(t)$, $z_*(t)$ which is not a solution of (3) and (21) but satisfies the boundary conditions (5). We obtain the perturbation equations

$$\begin{aligned} \delta\dot{x} - \delta y + (\dot{x} - y)_* &= 0 \\ \delta\dot{y} - \delta z + (\dot{y} - z)_* &= 0 \\ 2\delta\dot{z} + z_*\delta x - 2\beta y_*\delta y + x_*\delta z + [2\dot{z} + xz + \beta(1 - y^2)]_* &= 0 \end{aligned} \tag{22}$$

which are subject to the boundary conditions (7).

Having linearized the third-order system corresponding to the Falkner-Skan equation, we now apply the method of particular solutions. As with the Blasius equation, only two particular solutions are needed. The first particular solution (10) is obtained by integrating Eqs. (22) forward with the initial conditions (9). The second particular solution (12) is obtained by integrating Eqs. (22) forward with the initial conditions (11). Then, these particular solutions are combined as in Eq. (13) to give the desired solution to Eqs. (22); once more, the constants k_1, k_2 must satisfy Eqs. (14)-(15). Once the perturbation functions are known, the approximate trajectory of the system is given by Eqs. (16) and, in this way, the first iteration is completed. Next, the functions x(t), y(t), z(t) given by Eqs. (16) are employed as the nominal functions for the second iteration, and the procedure is repeated.

Computations were performed with an IBM 7040 computer for $\beta = 1$, corresponding to stagnation flow. The nominal curve (17) satisfying the boundary conditions (18) was chosen for the first iteration. Once more, convergence was rapid. For instance, at station t = 4, no change occurred in the 4th significant figure after four iterations. The results are plotted in Figs. 4-6, which give the nominal functions (17) as well as the functions obtained after the first, second, and third iteration. The curve n = 4 is not plotted: it is so close to the curve n = 3 that the relative differences cannot be detected in the scale of Figs. 4-6.

4. Particular Fifth-Order System

In this section, we consider the following nonlinear fifth-order system:

$$
\begin{aligned}
&\dot{x} - y = 0 \\
&\dot{y} - z = 0 \\
&\dot{z} + z^2uv/6 = 0 \\
&\dot{u} - v = 0 \\
&\dot{v} + yv^3/2 = 0
\end{aligned}
\tag{23}
$$

subject to the boundary conditions

$$x(1) = u(1) = 1, \quad v(1) = -1, \quad x(2) = 16, \quad u(2) = 1/2 \tag{24}$$

Once more, we linearize Eqs. (23) about a nominal curve $x_*(t)$, $y_*(t)$, $z_*(t)$, $u_*(t)$, $v_*(t)$ which is not a solution of (23) but satisfies the boundary conditions (24). The linearization leads to the perturbation equations

$$
\begin{aligned}
&\delta\dot{x} - \delta y + (\dot{x} - y)_* = 0 \\
&\delta\dot{y} - \delta z + (\dot{y} - z)_* = 0 \\
&\delta\dot{z} + (zuv/3)_*\delta z + (z^2v/6)_*\delta u + (z^2u/6)_*\delta v + (\dot{z} + z^2uv/6)_* = 0 \\
&\delta\dot{u} - \delta v + (\dot{u} - v)_* = 0 \\
&\delta\dot{v} + (v^3/2)_*\delta y + (3yv^2/2)_*\delta v + (\dot{v} + yv^3/2)_* = 0
\end{aligned}
\tag{25}
$$

which are subject to the boundary conditions

$$\delta x(1) = \delta u(1) = \delta v(1) = 0, \quad \delta x(2) = \delta u(2) = 0 \tag{26}$$

In Eqs. (25)-(26), the symbols δx, δy, δz, δu, δv denote the perturbations of x, y, z, u, v at a constant station t, that is,

$$\begin{aligned} \delta x &= x(t) - x_*(t) \\ \delta y &= y(t) - y_*(t) \\ \delta z &= z(t) - z_*(t) \\ \delta u &= u(t) - u_*(t) \\ \delta v &= v(t) - v_*(t) \end{aligned} \tag{27}$$

Having linearized the above fifth-order system, we now apply the method of particular solutions. Since $p = 3$, $q = 2$, a forward integration is desirable. Since $q + 1 = 3$, three particular solutions are required and are designated with the subscripts 1, 2, 3, respectively. In the first integration, we employ the initial conditions

$$\delta x_1(1) = \delta u_1(1) = \delta v_1(1) = 0, \quad \delta y_1(1) = 2, \quad \delta z_1(1) = 0 \tag{28}$$

and obtain the functions

$$\delta x_1(t), \quad \delta y_1(t), \quad \delta z_1(t), \quad \delta u_1(t), \quad \delta v_1(t) \tag{29}$$

In the second integration, we employ the initial conditions

$$\delta x_2(1) = \delta u_2(1) = \delta v_2(1) = 0, \quad \delta y_2(1) = 1, \quad \delta z_2(1) = 1 \tag{30}$$

and obtain the functions

$$\delta x_2(t), \quad \delta y_2(t), \quad \delta z_2(t), \quad \delta u_2(t), \quad \delta v_2(t) \tag{31}$$

And, in the third integration, we employ the initial conditions

$$\delta x_3(1) = \delta u_3(1) = \delta v_3(1) = 0, \qquad \delta y_3(1) = -1, \qquad \delta z_3(1) = 2 \tag{32}$$

and obtain the functions

$$\delta x_3(t), \qquad \delta y_3(t), \qquad \delta z_3(t), \qquad \delta u_3(t), \qquad \delta v_3(t) \tag{33}$$

As shown in Ref. 1, the linear combinations

$$\begin{aligned}
\delta x &= k_1 \delta x_1 + k_2 \delta x_2 + k_3 \delta x_3 \\
\delta y &= k_1 \delta y_1 + k_2 \delta y_2 + k_3 \delta y_3 \\
\delta z &= k_1 \delta z_1 + k_2 \delta z_2 + k_3 \delta z_3 \\
\delta u &= k_1 \delta u_1 + k_2 \delta u_2 + k_3 \delta u_3 \\
\delta v &= k_1 \delta v_1 + k_2 \delta v_2 + k_3 \delta v_3
\end{aligned} \tag{34}$$

satisfy the differential system (25) and the initial conditions (26-1)-(26-3) provided

$$k_1 + k_2 + k_3 = 1 \tag{35}$$

and the final conditions (26-4)-(26-5) provided

$$\begin{aligned}
k_1 \delta x_1(2) + k_2 \delta x_2(2) + k_3 \delta x_3(2) &= 0 \\
k_1 \delta u_1(2) + k_2 \delta u_2(2) + k_3 \delta u_3(2) &= 0
\end{aligned} \tag{36}$$

If the constants k_1, k_2, k_3 are consistent with (35)-(36), then the linear combinations (34) are the desired solutions to (25). Once the perturbation functions are known, the approximate trajectory of the system is given by

$$
\begin{aligned}
x &= x_* + \delta x \\
y &= y_* + \delta y \\
z &= z_* + \delta z \\
u &= u_* + \delta u \\
v &= v_* + \delta v
\end{aligned}
\tag{37}
$$

and, in this way, the first iteration is completed. Next, the functions x(t), y(t), z(t) u(t), v(t) given by Eqs. (37) are employed as the nominal functions for the second iteration, and the procedure is repeated.

Once more, computations were performed with an IBM 7040 computer. The following nominal curve was chosen for the first iteration:

$$
\begin{aligned}
x_* &= 7t^2 - 6t \\
y_* &= 14t - 6 \\
z_* &= 14 \\
u_* &= t^2/2 - 2t + 5/2 \\
v_* &= t - 2
\end{aligned}
\tag{38}
$$

At the terminal points, Eqs. (38) yield

$$x_*(1) = u_*(1) = 1, \quad v_*(1) = -1, \quad x_*(2) = 16, \quad u_*(2) = 1/2 \tag{39}$$

and, hence, the boundary conditions (24) are satisfied by the nominal curve.

Convergence was rapid. For instance, at station $t = 1.5$, no change occurred in the 5th significant figure after four iterations. The results are plotted in Figs. 7-11, which give the nominal functions (38) as well as the functions obtained after the first, second, and third iteration. For the functions $x(t)$, $y(t)$, $z(t)$ the curve $n = 4$ is not plotted, since it is extremely close to the curve $n = 3$. Analogously, for the functions $u(t)$, $v(t)$, the curve $n = 3$ is not plotted, since it is extremely close to the curve $n = 2$. It can be verified that the nonlinear system (23) subject to (24) admits the analytical solution

$$x = t^4, \quad y = 4t^3, \quad z = 12t^2, \quad u = 1/t, \quad v = -1/t^2 \tag{40}$$

5. Conclusions

Two mathematical techniques, quasilinearization and the method of particular solutions, were combined to solve three nonlinear, two-point boundary-value problems: (1) the Blasius equation, (2) the Falkner-Skan equation, and (3) a particular fifth-order system. Specifically, the nonlinearity of these problems was removed by quasilinearization, and the resulting linear, two-point boundary-value problem was solved by the method of particular solutions employed iteratively.

Computational results were obtained using an IBM 7040 computer. The analyses show that the combination of quasilinearization with the method of particular solutions can be a powerful tool in solving nonlinear, two-point boundary-value problems. Provided the initial guess used in the iteration procedure is chosen with discretion, convergence to a solution is quite rapid, and the accuracy of the solution is limited only by the integration step size and the integration technique employed.

References

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3. SCHLICHTING, H., Boundary Layer Theory, McGraw-Hill Book Company, New York, 1960.

List of Captions

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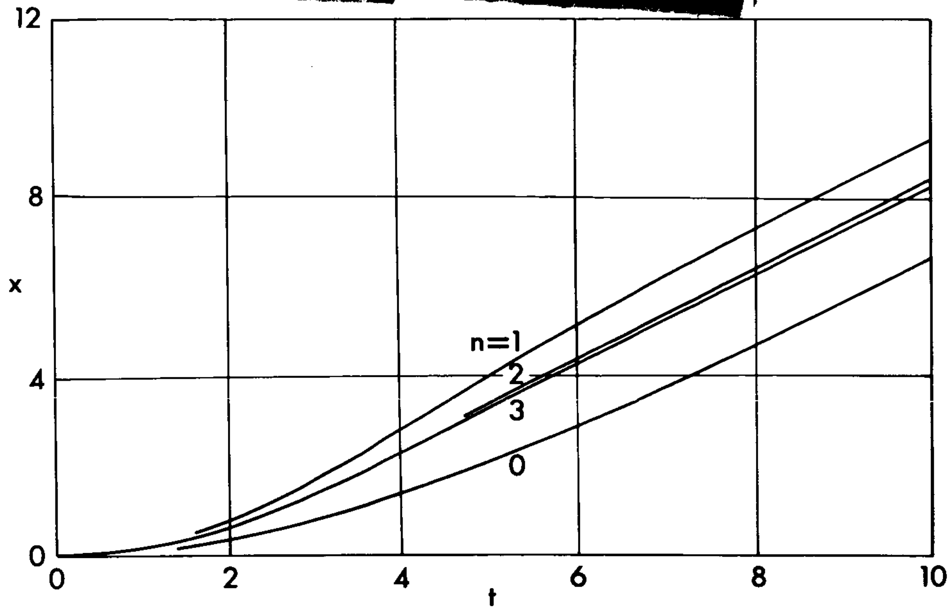

Fig. 1 The function x(t).

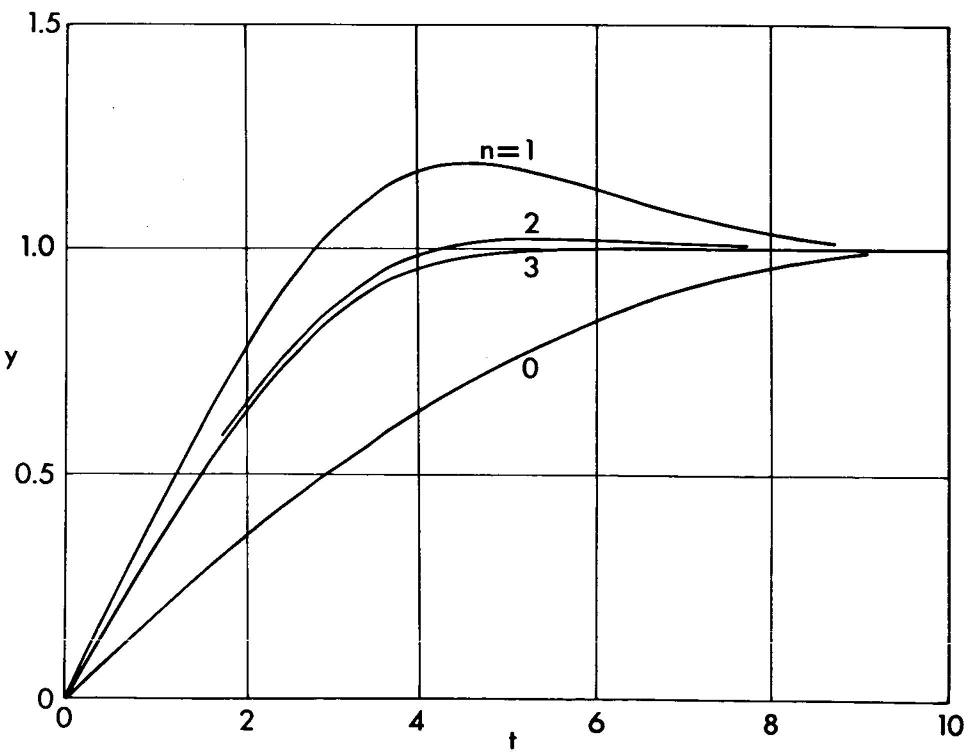

Fig. 2 The function y(t).

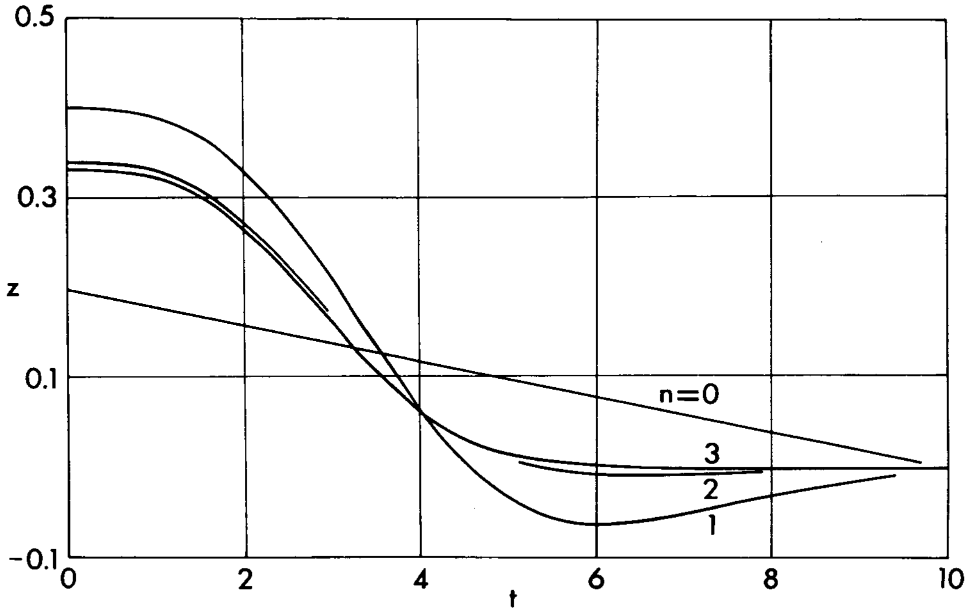

Fig. 3 The function z(t).

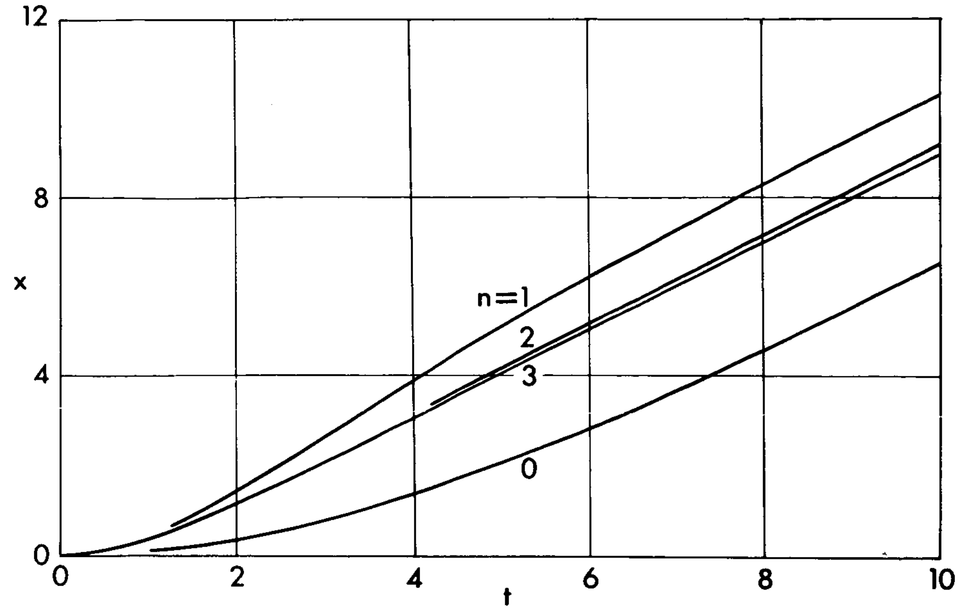

Fig. 4 The function x(t).

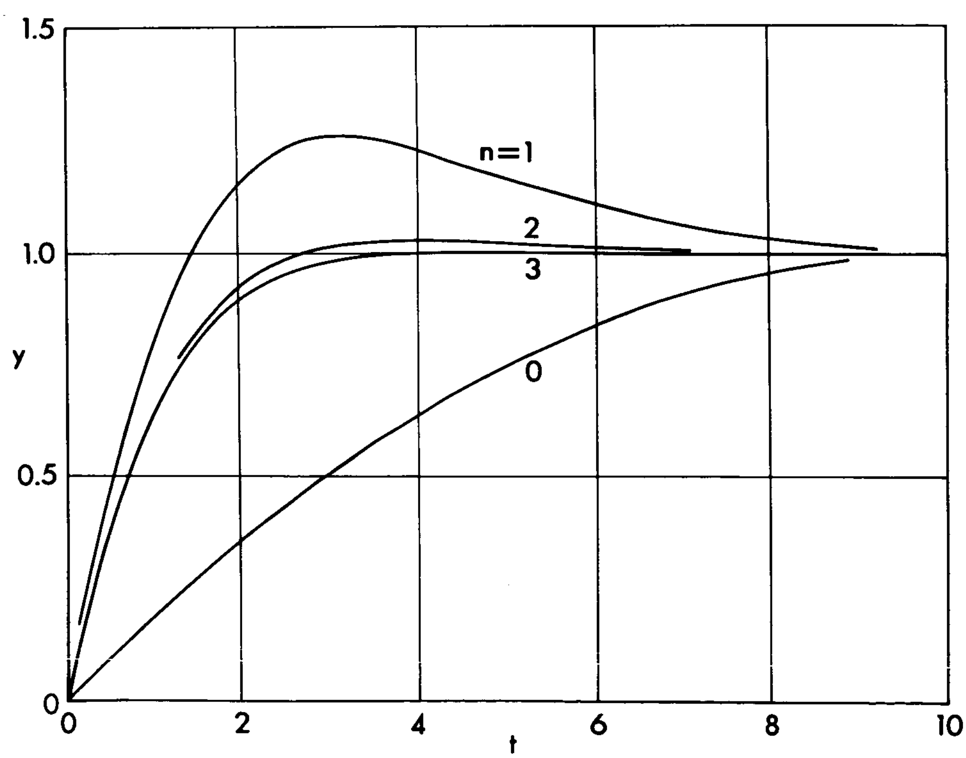

Fig. 5 The function y(t).

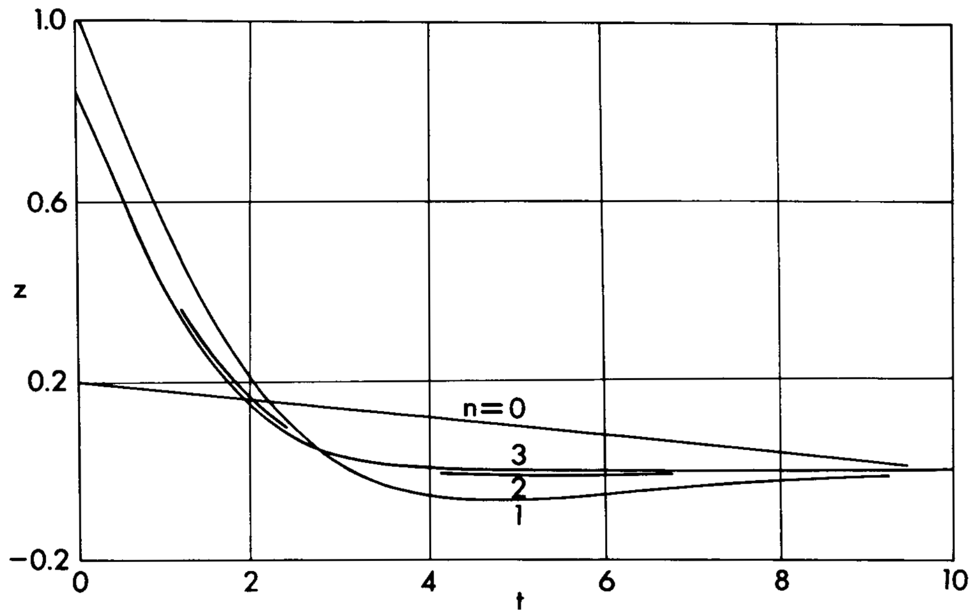

Fig. 6   The function z(t).

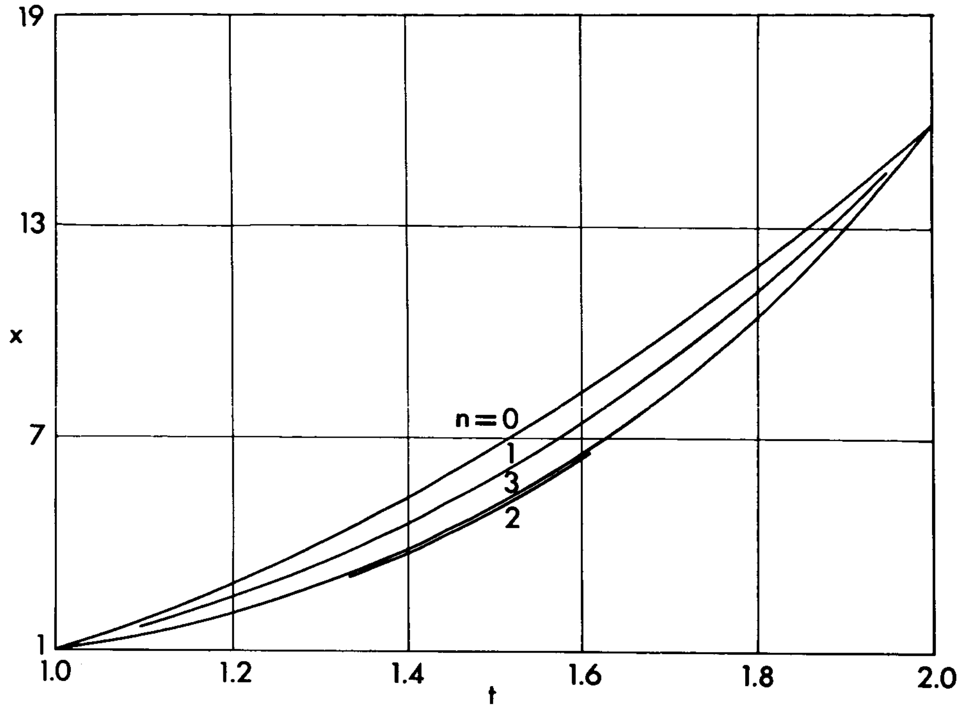

Fig. 7 The function x(t).

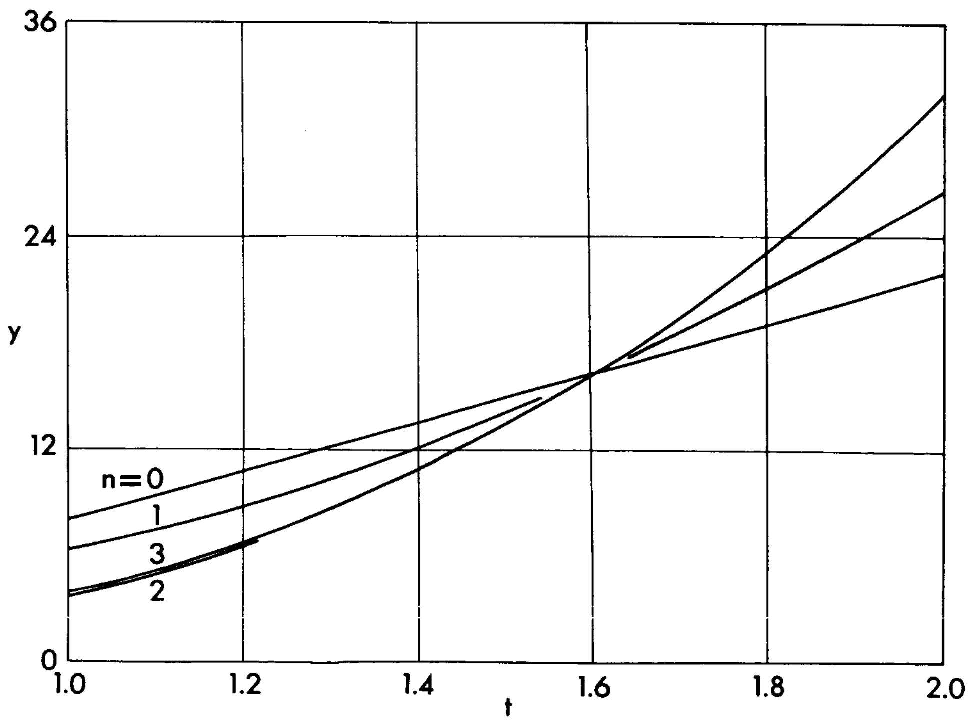

Fig. 8 The function y(t).

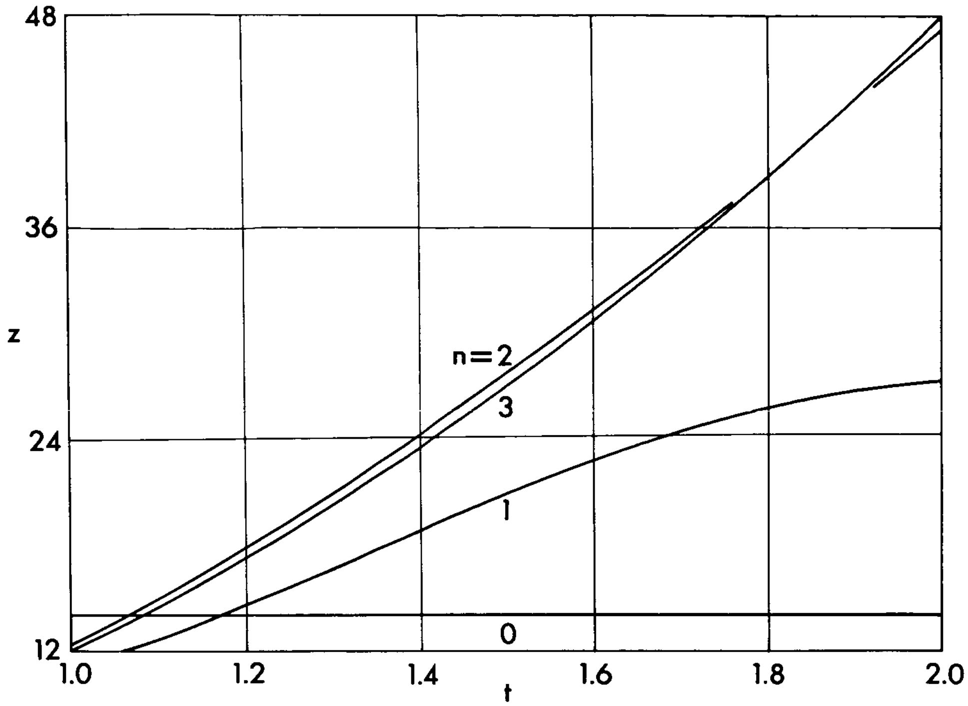

Fig. 9 The function z(t).

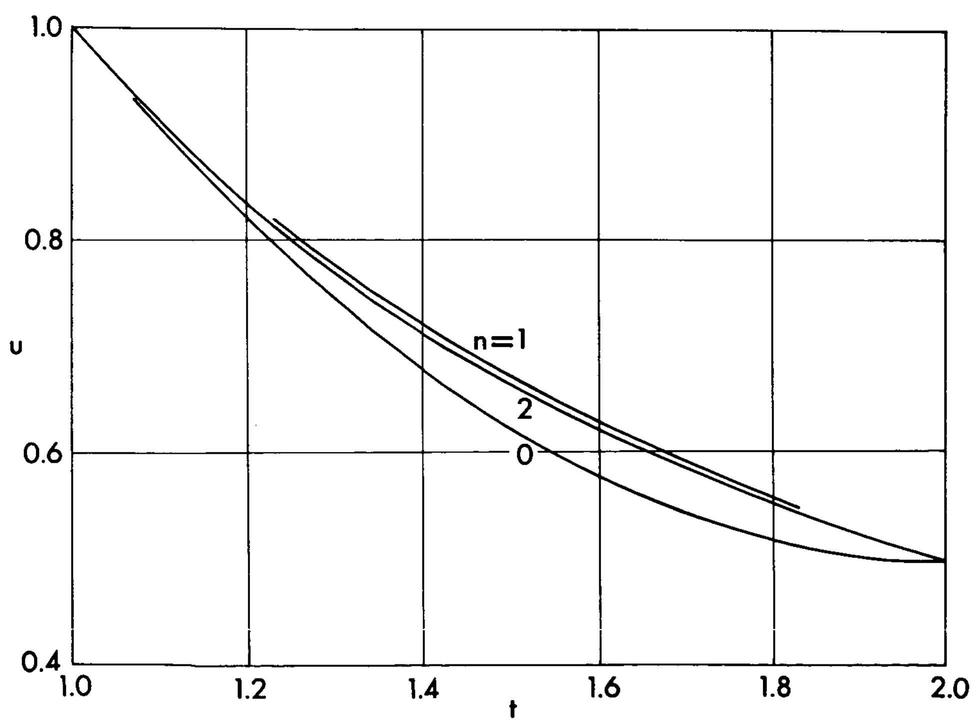

Fig. 10 The function u(t).

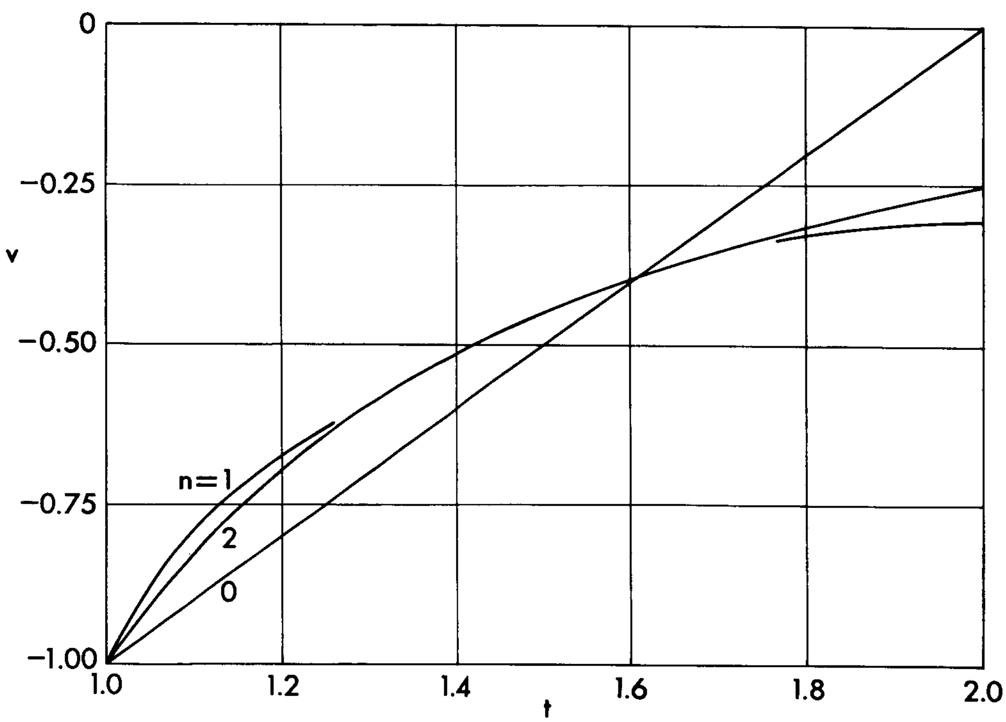

Fig. 11 The function v(t).