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THE RESTRICTED PROBLEM OF THREE BODIES (III)

by

CURTIS A. WAGNER

DEPARTMENT OF PHYSICS
UNIVERSITY OF ILLINOIS
Urbana, Illinois

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Curtis A. Wagner
Department of Physics, University of Illinois
Urbana, Illinois

An exhaustive numerical search for symmetric solutions to the restricted problem of three bodies quickly reveals the dominance of classes of solutions associated with the so-called "periodic-asymptotic limiting orbits" of Strömgren.[1/] These orbits spiral asymptotically into and out from the two stable triangular libration points, L_4 and L_5. (We will not study the other asymptotic limiting orbits which approach the three unstable libration points on the line through the two finite masses.) If we wish to understand the properties of these periodic-asymptotic limiting orbits and their associated classes, we must study them over the complete range of mass ratio values. In this paper we shall report some results of this study.

Although Strömgren only located a few asymptotic limiting orbits which were symmetric with respect to the ξ-axis or the η-axis, it is not difficult to find more equally simple orbits using a digital computer. For Strömgren's case of equal finite masses a plot of 14 asymptotic limiting orbits (symmetric with respect to the ξ-axis) is shown as Figure 10 in the paper by Bartlett.[2/] Also included in that paper is a table of initial and final conditions plus a listing of the classes of periodic solutions associated with each of the limiting orbits. It was found that several small classes of solutions, for example the (k), (μ), and (β) classes, both began and ended on one of these 14 limiting orbits. The more extensive (g) class was found to begin at one of the masses and terminate on a hybrid combination of asymptotic orbits VII and VIII. This unexpected termination seemed to indicate the possibility of an infinite number of periodic-asymptotic hybrid limiting orbits and their associated classes of periodic solutions.

In studying the dependence of the simple symmetric classes upon the ratio of the two finite masses,[3/] it was found that many of the classes changed their topological structure above or below certain values of $\gamma = (m_1-m_2)/(m_1+m_2)$, the mass ratio parameter. For example the (k), (μ), and (β) classes changed from open classes terminating on spiral limiting orbits into closed classes that eventually shrank to points and vanished. The (g) class lost its spiral terminations on VII and VIII and developed another branch with different final symmetry properties. These behaviors suggested the need to study the periodic-asymptotic limiting orbits themselves as a function of the mass ratio.

The simple asymptotic-periodic limiting orbits which we studied all spiral out from the libration point L_4, located at ($E = -\pi/2$, $F = +\sinh^{-1}\sqrt{3} = +1.316958$), and strike either the E or F axes perpendicularly. In the neighborhood of L_4 (or L_5) the equations of motion can be simplified to yield the spiral trajectories when $-\gamma_o < \gamma < \gamma_o$, and closed elliptical motions when $1 \geq |\gamma| \geq \gamma_o$ (where $\gamma_o = \sqrt{23/27} = 0.922958$ is a critical or limiting mass ratio). Restricting ourselves to the spiral motions, it is then easy to calculate[3/] the slope of the trajectory where it intercepts the η-axis (i.e. the line $E = -\pi/2$ in the Thiele coordinates). Since the slope at this intercept is numerically identical in both coordinate systems, i.e.

$$\frac{d\eta}{d\xi}(\xi=0) = \frac{dF}{dE}(E = -\pi/2) ,$$

we can carry over these cartesian results immediately.

The equation for the slope of the outgoing spiral motion at the $E = -\pi/2$ intercept is[3]

$$\frac{dF}{dE}(E = -\pi/2) = p - \frac{\beta(p\theta + 5/4)}{(\theta^2 + 4q^2)}$$

where

$$\left\{ \begin{array}{l} \theta = 2p + \beta \\ \beta^2 = 27\gamma^2/16 \end{array} \right\}$$

and p,q are real numbers satisfying the equations

$$\left\{ \begin{array}{l} p^2 - q^2 = -1/2 \\ \text{and} \quad 4\,pq = \sqrt{\frac{27}{4}(1-\gamma^2) - 1} \end{array} \right\}.$$

Knowing the slope $dF/dE = (\dot{F}/\dot{E})$ at $E_i = -\pi/2$ and arbitrarily choosing an initial F_i coordinate near L_4, we can calculate both $\dot{E}_i$ and $\dot{F}_i$ by using the energy integral of the motion. This integral is given by[3]

$$\frac{\dot{E}^2+\dot{F}^2}{D} = -\frac{T}{2} + \frac{8}{D}(\text{ch}F - \gamma\cos E)$$

$$+ \frac{1}{4}(\text{ch}2F + \cos 2E) + \gamma\,\text{ch}F\cos E$$

where

$$\left\{ \begin{array}{l} 2D = \text{ch}2F - \cos 2E \\ T = K - \gamma^2 \end{array} \right\}$$

and K is the Jacobi constant of the motion. It has been convenient to use the parameter T as the energy integral for the following reason. If we evaluate the Jacobi constant at L_4 we obtain $K = 11 + \gamma^2$. Since all periodic-asymptotic limiting orbits begin at L_4, it would be helpful to have a constant of the motion which is independent of γ at L_4. If we therefore set $T = K-\gamma^2$ we see that this parameter will always have the constant value of 11 at L_4. Thus, for all values of γ our energy-position profiles (T vs E or T vs F, where E and F refer to the coordinates on the axis of symmetry) will have all the periodic-asymptotic limiting orbits located along the T=11 line. Their associated classes will spiral into them back and forth across this one line.[2/] This fixed energy at L_4 is a distinct advantage when we begin to vary the mass ratio γ.

Having initial conditions for the 14 periodic-asymptotic limiting orbits when $\gamma=0$,[2/] it is then an easy matter to trace out these orbits as continuous functions of γ. With a special computer subroutine to calculate the initial slope of the spiral at $E_i = -\pi/2$ as a function of γ, we can obtain $\dot{E}_i$ and $\dot{F}_i$ for any arbitrary F_i value (when T=11). Choosing a fixed value of γ close to zero, we can vary F_i from the original value until the resulting outgoing trajectory exactly meets the desired final boundary conditions. This process can then be continued until the full range $-\gamma_o < \gamma < \gamma_o$ has been scanned. Results of this γ vs F_i scan for 10 of the 14 original limiting orbits are plotted in Figure 1. We find a tightly nested set of curves whose F_i values lie within a band of width $\Delta F_i \cong 0.015$ for $\gamma=0$ but which increase rapidly and asymptotically as $\gamma \to \pm \gamma_o$. The present calculations were not carried much beyond $|\gamma| = 0.85$ because of the difficulty in obtaining accurate results. It should also be noted that we have plotted all the limiting orbits with their values of $F_i > 1.316958$, whereas some of the original tabulated values[2/] had

$F_i < 1.316958$. In those cases our plotted values correspond to the next intercept of the line $E = -\pi/2$ by the outgoing spiral trajectories.

A key feature of Figure 1 is the smooth coalescence of a number of pairs of the original limiting orbits which were topologically similar. For example, at $\gamma=0$ limiting orbits I and II are both similar and close to one another everywhere, even up to $E = -\pi$.[2/] They merge into one limiting orbit when $\gamma \cong -0.059$. On the same topological basis we would expect coalescence of limiting orbits III and IV, plus XI and XIV as well. This merging is indeed what we do find from our γ vs F_i scans shown in Figure 1. In addition we discover that limiting orbit VIII reaches an absolute minimum value of γ but then continues to exist for different F_i values as γ is increased again. At $\gamma=0$ we find that this new limiting orbit, called VIII* (the conjugate orbit of VIII), has a slightly larger F_i value than does the original orbit VIII. Thus VIII* is totally equivalent to orbit VIII, and as such is involved in the evolution of the (g) class. The other orbits (V, VI, and VII) plotted in Figure 1 are not topologically equivalent, and do not appear to join any conjugate limiting orbits. However, the scan has not been run close enough to $-\gamma_o$ to completely preclude their actual existence.

In order to get a more complete picture of the totality of the periodic-asymptotic orbits we make use of the eigensurface representation.[3/] Since T=11 for all of these limiting orbits we can obtain their complete representation by plotting mass ratio-position profiles (γ vs E or γ vs F) for T=11. These two profiles are shown respectively in Figures 2 and 3 for some of the original limiting orbits. In Figure 2 we observe that orbits III and IV coalesce at $E \cong 0.75$ when $\gamma \cong -0.24$. This coalescence coincides with the closing off of the (β) class, which begins and terminates only on orbits III and IV. The mechanism for this closing off is the touching and merging of the successive spiral loops in the T vs E profile of the (β) class as orbits III and IV approach one another (along the T=11 line as $\gamma \rightarrow -0.24$). The result is an infinite nested set of small closed classes which gradually shrink to points and vanish as γ is decreased further (e.g. the outermost member of the nested set for the (β) class disappears finally around $\gamma \cong -0.83$). In a similar fashion orbits I and II coalesce at $\gamma \cong -0.059$. This union coincides with the closing off of the (k) and (μ) classes, which begin and terminate only on orbits I and II.

The other orbits plotted in Figure 2 are the conjugate pair of limiting orbits VIII and VIII*. These orbits merge at E $\cong$ -2.27 when $\gamma \cong -0.70$. Since the (g) class terminates on a hybrid combination of orbits VII and VIII, but is not associated with VIII* (the class begins around one of the masses), we now find that the class still exists below $\gamma = -0.70$ but with different termination properties. In fact another branch of the (g) class appears, joins together, and then breaks apart in another mode with (f) class symmetry.[3/] The class associated with VIII* has not been traced out.

In Figure 3 we see the γ vs F profiles of orbits V, VI, XI, and XIV. Orbits V and VI cover almost the full range of γ, but do not join any conjugate pairs of similar orbits by $\gamma \cong -0.85$. However orbits XI and XIV do coalesce at F $\cong$ -0.35 when $\gamma \cong -0.582$. The associated (α-δ) class is affected in a complicated way by this coalescence. But after the breaking apart, touching and rejoining of portions of the class is accomplished, the class touches the zero velocity surface and thereby acquires new termination and symmetry properties.[3/]

So far we have been studying periodic-asymptotic limiting orbits which were symmetric with respect to the ξ-axis (i.e. our E- or F-axis). But Strömgren also displayed 6 periodic-asymptotic limiting orbits which were symmetric with respect to the η-axis[1/] (i.e. the line $E = \pm \pi/2$ in our coordinates). His second orbit, with $\eta_o = 1.7008$ for $\gamma=0$, is small and simple, encircling only the L_1 libration point before returning to spiral into L_4. The mirror image of this orbit, spiraling into L_5 instead of L_4, is plotted in Figure 4 as orbit number 1.

By starting on the line $E = -\pi/2$ just below L_5 and choosing a value of F_i approximately equal to the minimum value of F attained by orbit number 1, we can vary T about 11 until we locate a member of the associated class. This technique of locating a member of a class from a limiting orbit is based on the coiling property of the periodic-asymptotic classes.[2/] The class can be traced in the usual manner, and its T vs F profile is plotted in Figure 5. Two "extreme orbits" from this profile are also plotted in Figure 4 as number 2 (T=11.00) and number 3 (T=9.486659, very near the minimum). The class eigensurface is thus seen to be compact and simple. Because it both begins and ends on the <u>same</u> limiting orbit out of L_4, we would predict that the class exists over most of the range $-\gamma_o < \gamma < \gamma_o$. This prediction has not been checked.

It would be straightforward to trace out similar classes associated with Strömgren's other periodic-asymptotic orbits, or indeed locate entirely new classes and limiting orbits. But the unusual <u>hybrid</u> combinations of limiting orbits involved in the (g) and (α-δ) classes[3/] suggested that we try to locate a class associated exclusively with <u>two</u> or more periodic-asymptotic limiting orbits. In particular we tried to locate a class associated with a hybrid limiting orbit consisting of Strömgren's first two orbits symmetric with respect to the η-axis.[1/] (His second orbit, with η_o= 1.7008, has already been discussed; his first orbit, with $\eta_o \cong 1.72025$, crosses the ξ-axis outside the masses and then turns around to strike the η-axis perpendicularly well above L_4). Application of the coiling technique quickly located the desired hybrid class, and its relatively simple evolution is traced out in Figures 6 and 7.

In Figure 6 we see Strömgren's second limiting orbit plotted as orbit number 1 (T=11.0, γ=0). The other half of the hybrid limiting orbit, Strömgren's first orbit, is not plotted. It spirals out from L_5 to cross the F axis first above F = -1.0 and then above F = -2.0 as it turns back to the line E = - $\pi/2$ which it strikes perpendicularly around F $\cong$ -2.1. Orbit number 2, for T = 10.7753, has a large loop around both of the orginal spirals about L_5. As T decreases slightly and then increases, the loop shrinks (orbits 3 and 4, with T = 10.57482 and T = 11.45292), passes through a cusp, and changes into a smooth curve without a loop (orbit 5 with T = 12.45, very near the maximum T value).

In Figure 7 we see the whole process reversing itself as T decreases again. Orbit 6 at T = 12.373 is similar to orbit number 5, while orbit number 7 at T = 11.861 shows a cusp forming in the same region as the loop of orbit number 4. We also notice a small depression appearing where orbit 7 strikes the line E = - $\pi/2$ perpendicularly. By orbit number 8 (at T=11.093) this depression has passed through a singular cusp (infinite slope), formed a small loop, and consequently taken on new final boundary conditions. At the same time the original cusp near (E = -1.2, F = -1.1) has similarly become a large loop. In orbit number 9, at T = 10.77264, this loop has continued to swell in the direction of L_5, while the tiny loop at $E = -\pi/2$ has expanded to encircle L_5. As T continues to decrease and then increase again, the original loop passes and then also encircles L_5. More and more cusps and loops develop within these two encircling loops, and the class eventually terminates as it began, on the hybrid periodic-asymptotic limiting orbit.

Two T vs F eigensurface profiles of this hybrid class are shown in Figure 8. The right-hand branch has T plotted versus $|F|$ at $E = -\pi/2$ and corresponds to the absolute position of the lower portions of the orbits well below L_5. The spiral coils suggest that this class begins and terminates on the same periodic-asymptotic limiting orbits. The maximum value of T is seen to be about 12.5. The left-hand branch is for T vs F and corresponds to the position of the upper portions of the orbits just below L_4. The profile starts in a spiral around $F = 1.0$, rises to the maximum T value around 12.5, and then sharply retraces its earlier profile almost exactly. The difference is too small to appear in the figure. The dependence of this two-membered hybrid class on the mass ratio γ would probably be similar to that of the simpler class discussed, that is, this hybrid class would probably exist over a wide range of γ values (since the class again begins and ends on the same hybrid limiting orbit). This prediction also remains to be checked.

We have not searched for three- or four-membered hybrid classes, but it would appear that the sheer number of combinations and permutations possible for multi-membered periodic-asymptotic limiting orbits would be almost limitless. This possibility should apply to hybrid orbits symmetric with respect to either the η-axis or the ξ-axis, or combinations of both simultaneously and probably to all of the asymmetric limiting orbits as well. If the associated classes do indeed extend over large ranges of γ values, they would be by far the most abundant and important of all the classes of solutions to the problem. The problem would then be reduced to the study of the fundamental class generators, which would be the possible hybrid combinations of periodic-asymptotic limiting orbits.

REFERENCES

1. E. Strömgren, "Fortgesetzte Untersuchungen über asymptotische Bahnen im Probleme Restreint", Mat. Fys. Medd. Dan. Vid. Selsk. 10, Nr. 7, (1930).

2. J. H. Bartlett, "The Restricted Problem of Three Bodies", Mat. Fys. Skr. Dan. Vid. Selsk. 2, Nr. 7, (1964). Also NASA Technical Report No. 1, Grant NsG 280-63, (1964).

3. J. H. Bartlett and C. A. Wagner, "The Restricted Problem of Three Bodies (II)", Mat. Fys. Skr. Dan Vid. Selsk. 3, Nr. 1, (1965). Also NASA Technical Report No. 2, Grant NsG 280-63, (1965).

FIGURE CAPTIONS

Figure 1: Dependence of initial position F_i on the mass ratio parameter γ, for 11 periodic-asymptotic limiting orbits symmetric with respect to the ξ-axis ($T=11$, $E_i = -\pi/2$).

Figure 2: Dependence of final position E_f on the mass ratio parameter γ, for 4 periodic-asymptotic limiting orbits symmetric with respect to the ξ-axis ($T=11$, $E_i = -\pi/2$).

Figure 3: Dependence of final position F_f on the mass ratio parameter γ, for 4 periodic-asymptotic limiting orbits symmetric with respect to the ξ-axis ($T=11$, $E_i = -\pi/2$).

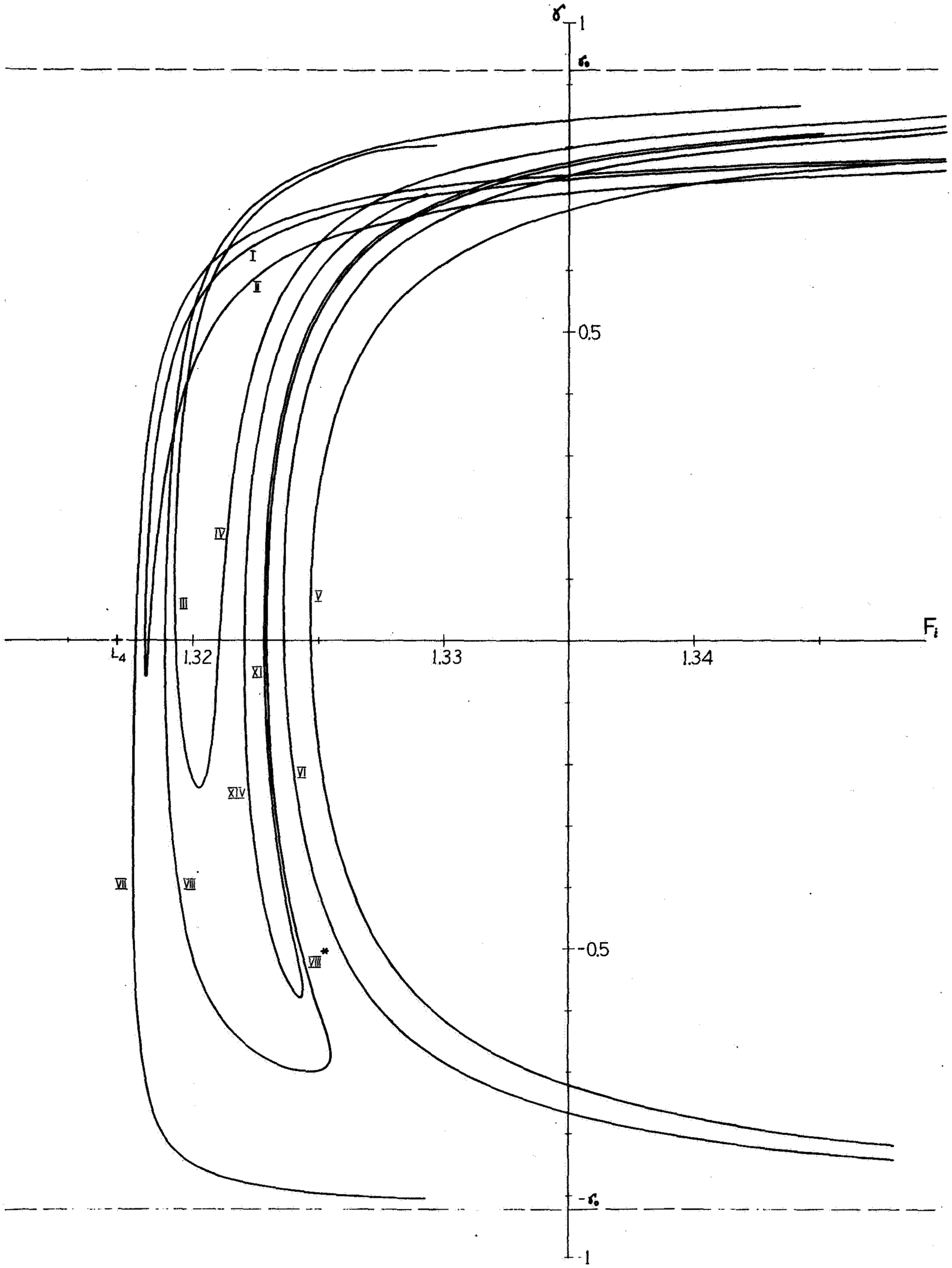
Figure 4: Strömgren's second limiting orbit symmetric with respect to the η-axis ($\eta_0 = 1.7008$, $T=11$, $\gamma=0$), and two "extreme orbits" of the associated class of periodic solutions.

Figure 5: Energy-position profile of the class associated with Strömgren's second limiting orbit symmetric with respect to the η-axis.

Figure 6: Initial evolution of a hybrid class of periodic solutions associated with a combination of Strömgren's first and second limiting orbits (symmetric with respect to the η-axis).

Figure 7: Final evolution of a hybrid class of periodic solutions associated with a combination of Strömgren's first and second limiting orbits (symmetric with respect to the η-axis).

Figure 8: Energy-position profile of a hybrid class associated with a combination of Strömgren's first and second limiting orbits (symmetric with respect to the η-axis).


γ
1
γ₀
0.5
-0.5
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-1
Fᵢ
L₄
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I
II
III
IV
V
VI
VII
VIII
VIII*
XI
XIV

Figure 1

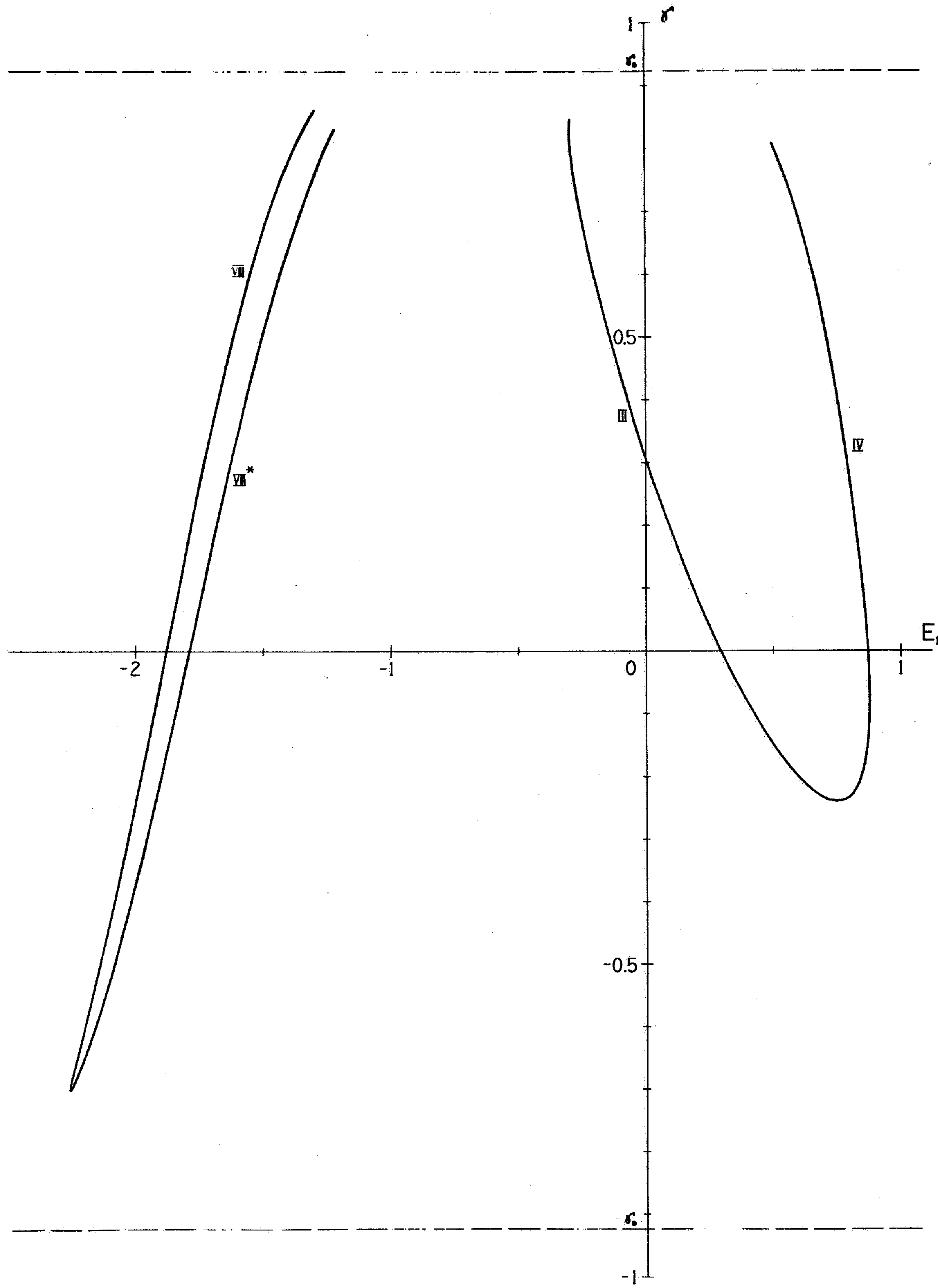

Figure 2

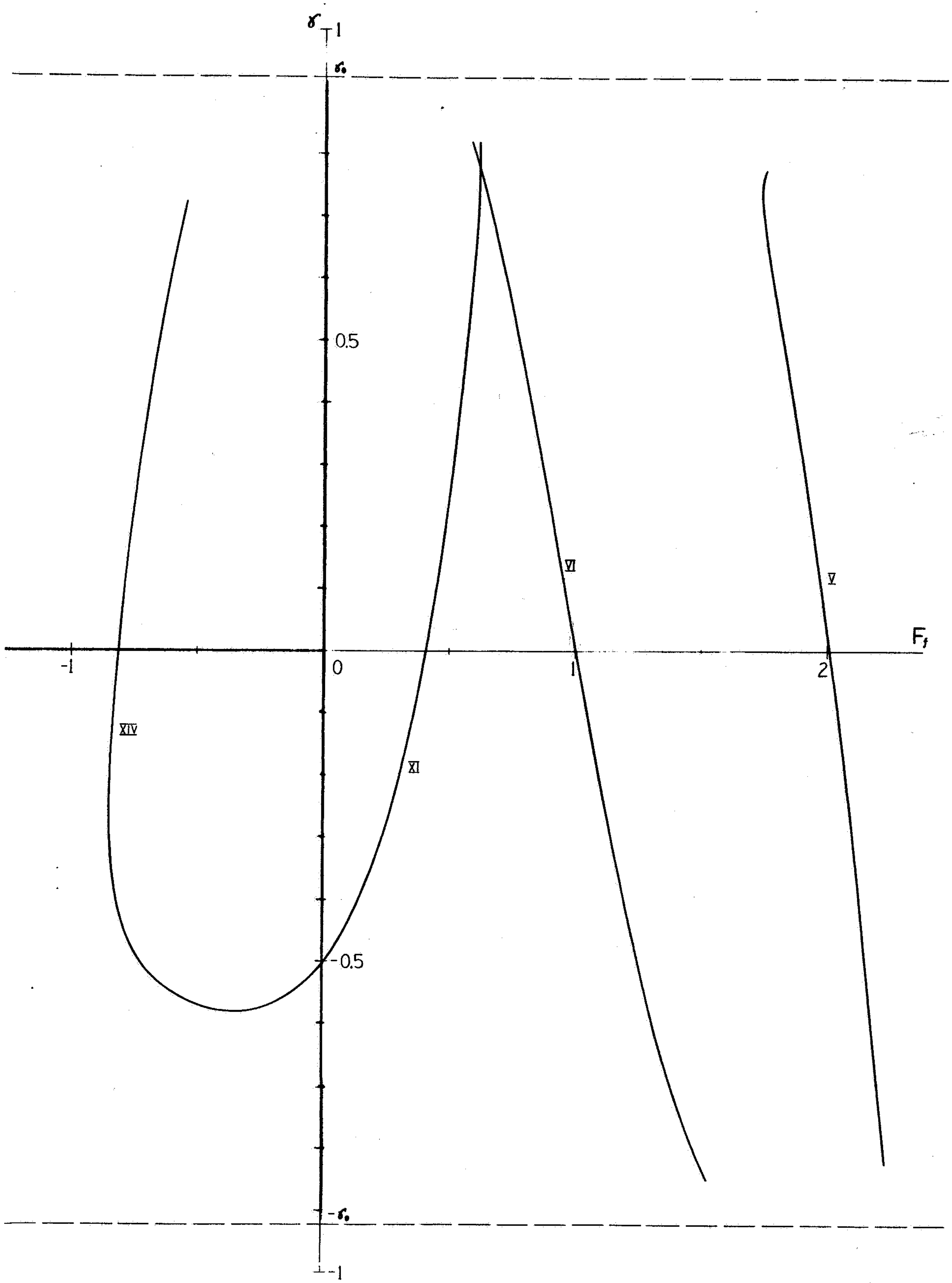

Figure 3

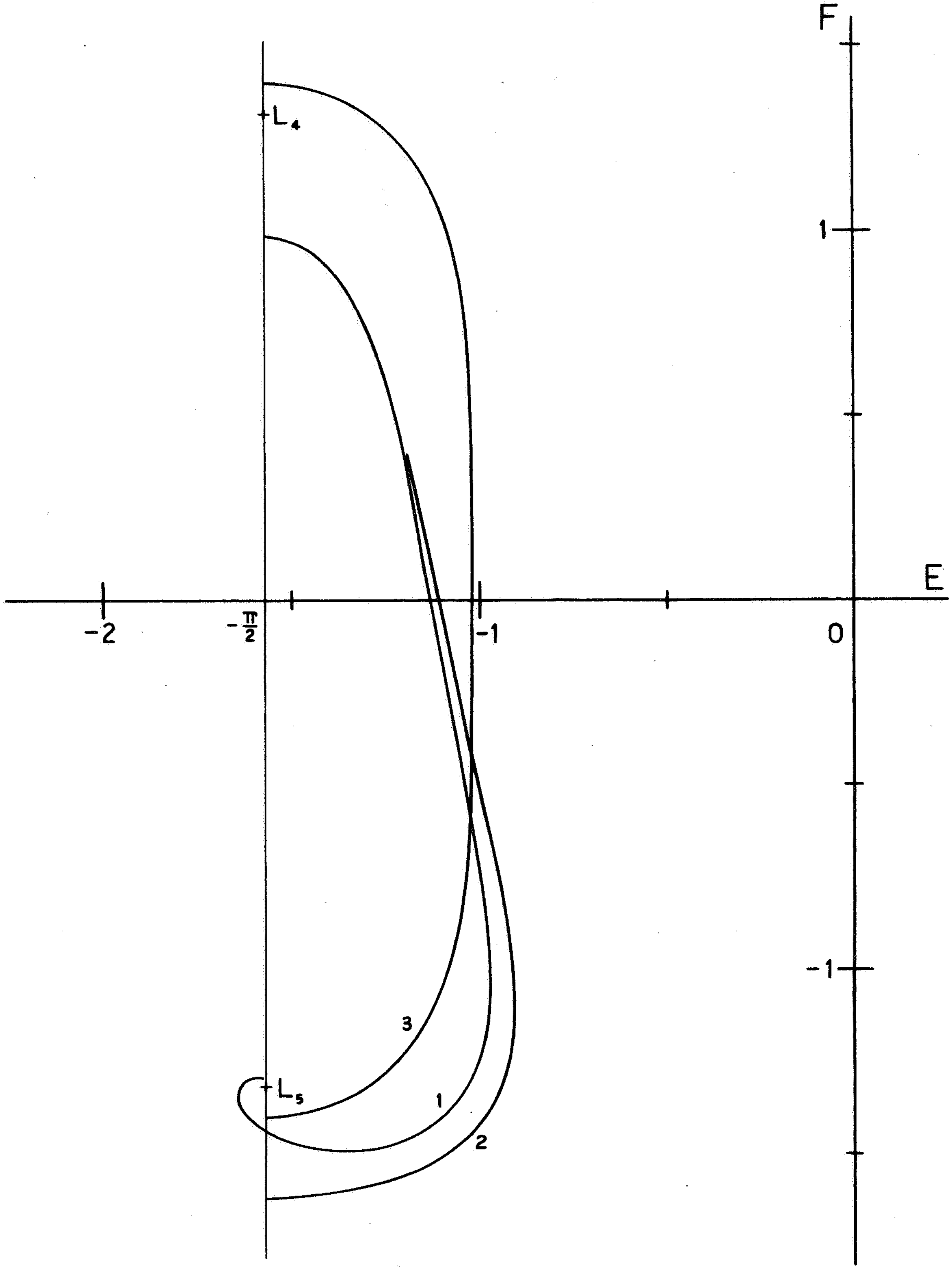

Figure 4

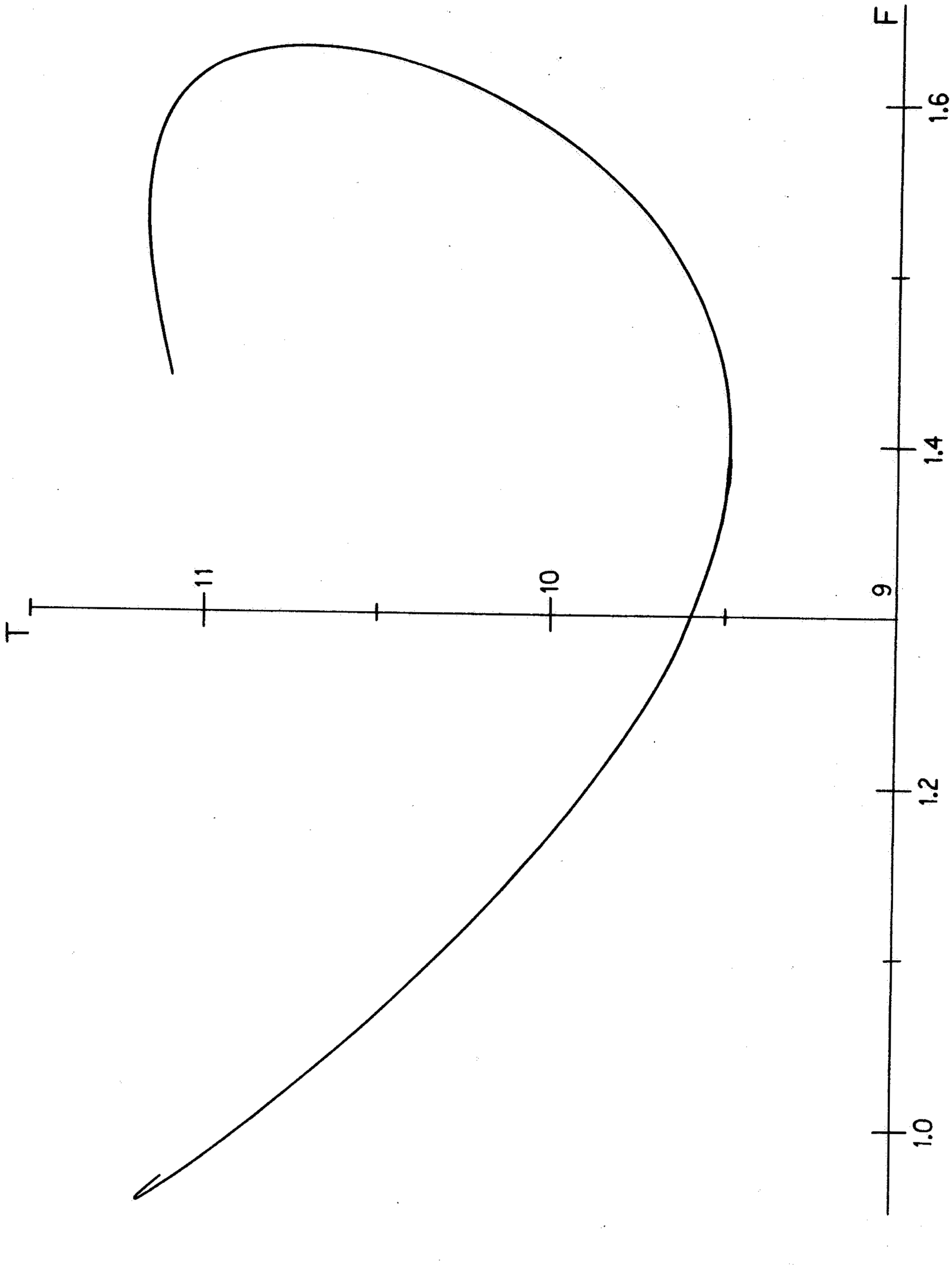

Figure 5

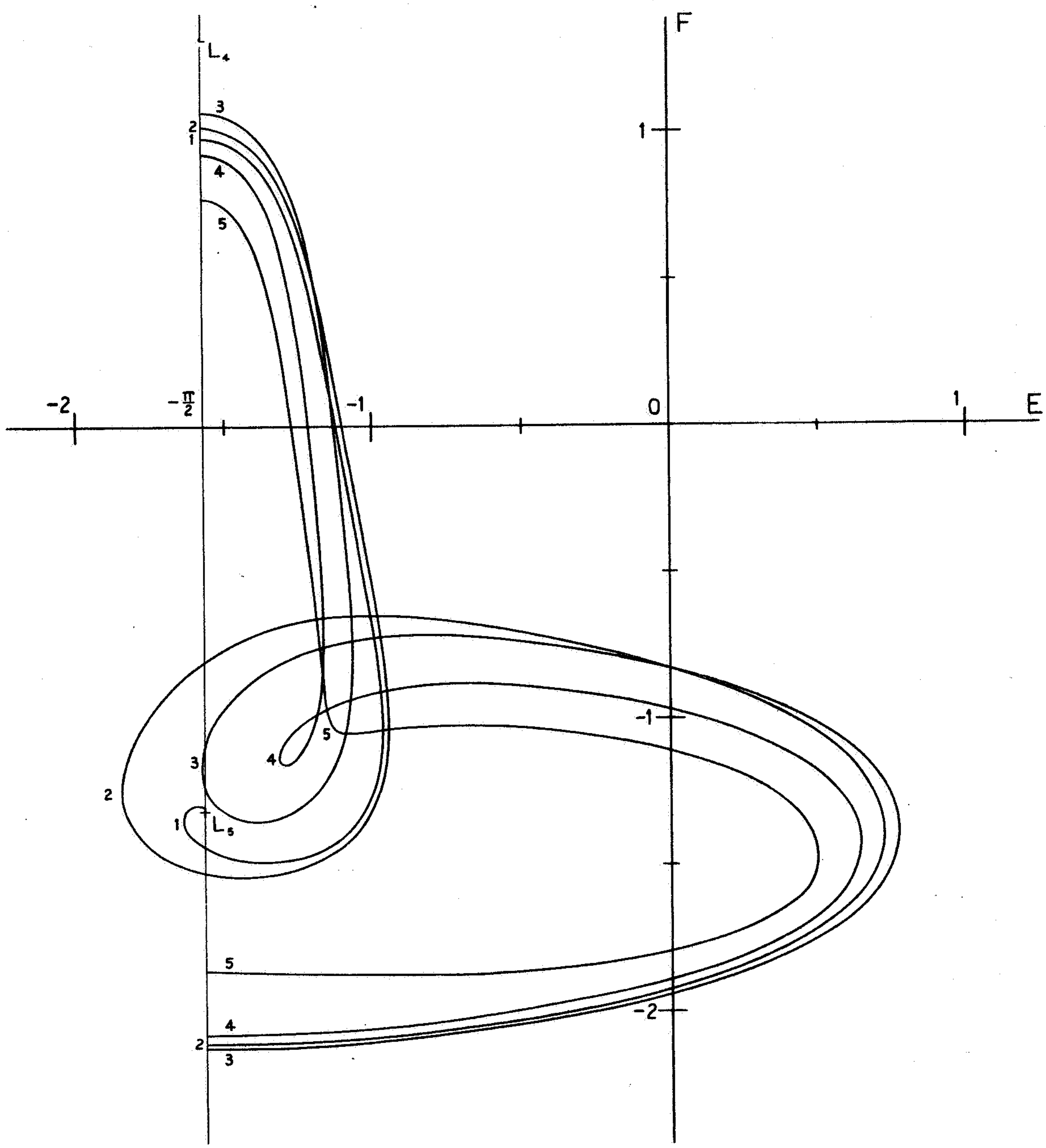

Figure 6

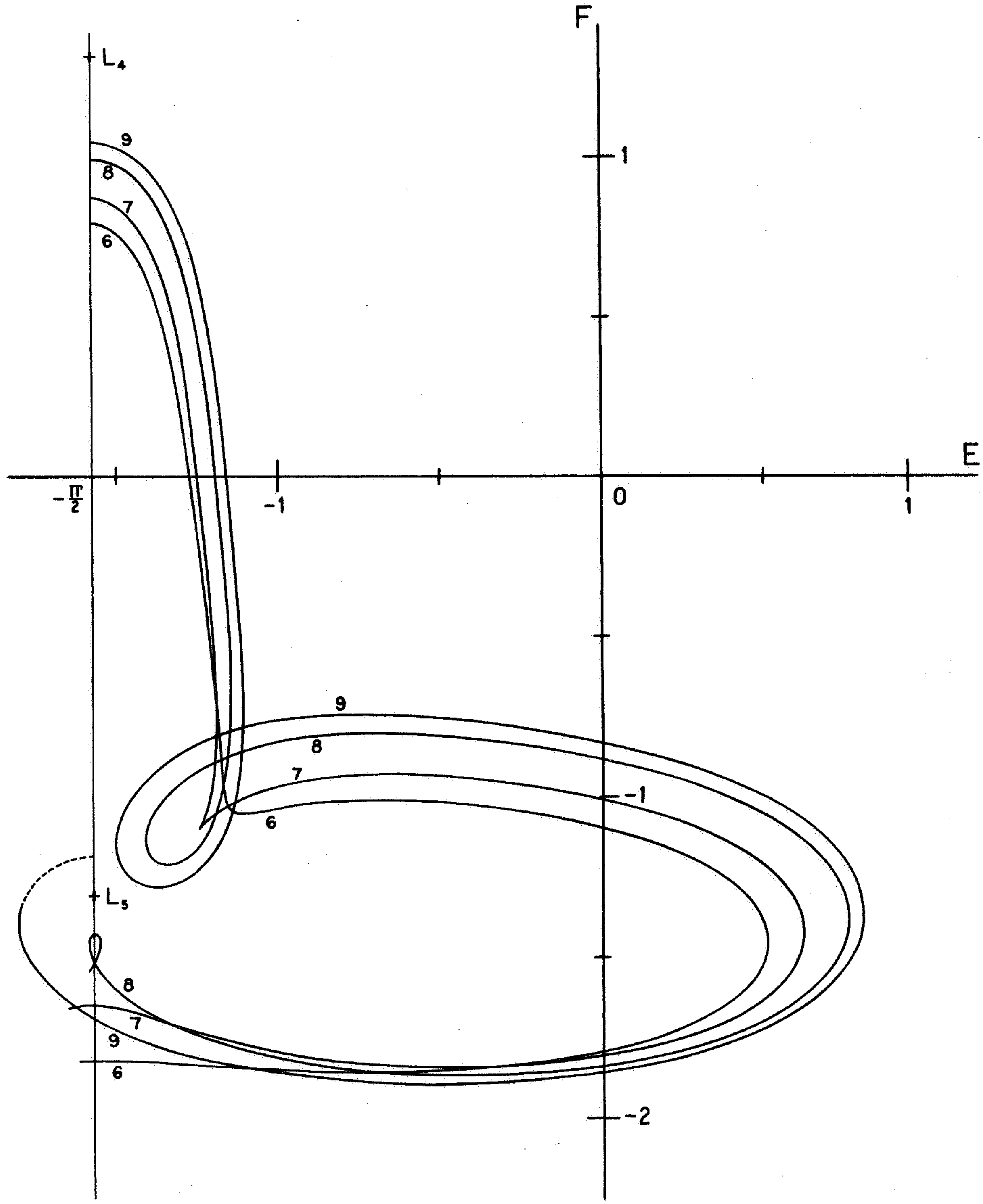

Figure 7

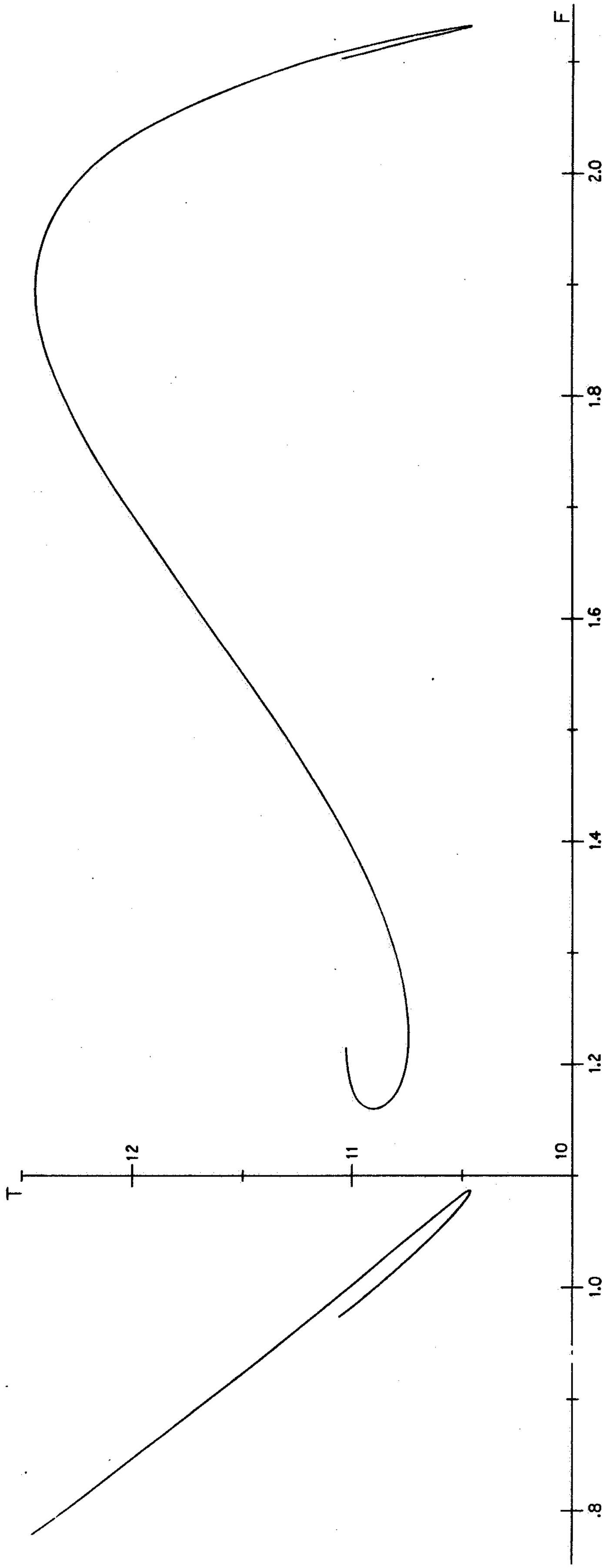

Figure 8

FINAL REPORT

NsG 280-63

Research on periodic solutions of the restricted 3-body problem has resulted in Technical Reports Nos. 1, 2 and 4. Research on stability of periodic solutions resulted in Technical Report No. 3, which is a sequel to other work partially performed under the grant and published in the Communications of the Danish Academy.

Technical Report No. 1, also published by the Danish Academy (Mat. Fys. Skr. Dan. Vid. Selsk. 2, 7 (1964)) surveyed most of the important classes of periodic solutions for the case of equal masses.

Technical Report No. 2, also published by the Danish Academy, co-author C. A. Wagner, (Mat. Fys. Skr. Dan. Vid. Selsk. 3, 1 (1965)) extended Report No. 1 to the case of unequal masses, generally over most of the allowed range.

Technical Report No. 4 shows how some limiting asymptotic orbits vary with mass ratio.

In order to provide a basis for discussion of stability of solutions of the restricted 3-body problem, motion under a periodic cubic force was studied in detail. The periodic solutions were located (Mat. Fys. Medd. Dan. Vid. Selsk. 36, 11 (1968)) and general comments on stability were made. This has been followed by work with C. A. Wagner which is detailed in Technical Report No. 3. It is intended to recast this as a journal article and to submit it for publication.

The chief assistant with all these computations has been C. A. Wagner. In addition, numerous others have been employed on a part-time basis as computers, programmers and draftsmen.

The grant was terminated on August 31, 1967, after two extensions.