RANDOM INDEPENDENT SPLITTING MODEL
FOR THE MASS SPECTRUM OF PROTOSTARS
AND INTERSTELLAR CLOUDS

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ABSTRACT

It is often assumed that stars are formed when a large initial cloud splits up into smaller fragments, each of which may in turn split into smaller pieces, and so on. This continues until the final stage before the cloud condenses into a visible star, at which stage the cloud fragment may be called a protostar. We present a simple mathematical model for the expected number density (i.e., "mass spectrum") of cloud fragments which does not require detailed knowledge of the physics of the fragmentation process, but uses merely some simple and plausible phenomenological hypotheses.

The present state of empirical evidence about the initial mass spectrum of protostars and interstellar clouds is first reviewed. We then describe and expand upon a mathematical model proposed by Filippov (1960) for the mass spectrum of objects formed by the repeated random independent splitting of an initial object. Assuming that both the rate of splitting and the "one-shot" splitting distribution are power functions of fragment mass, it is possible to reproduce either the "fractional exponential" or inverse-power laws which have been proposed as descriptions of the observed mass spectrum of protostars and interstellar clouds.

In the inverse-power law description, the Filippov model cannot produce an index larger than 1.0, whereas the index estimated from field stars larger than one solar mass is on the order of 1.33. This difference can be explained by assuming that present field stars were formed in different initial clouds, thus, the observed mass spectrum is a mixture of Filippov-type spectra.

The index 2/3 observed in the inverse-power law mass spectrum of small stars (less than one solar mass) can be explained by a one-shot splitting law in which gravitational potential energy of fragments of a cloud is distributed uniformly over equal ranges of fragment mass. This seems to be more realistic than the other one-shot splitting laws proposed by various authors.

RANDOM INDEPENDENT SPLITTING MODEL FOR THE MASS SPECTRUM OF PROTOSTARS AND INTERSTELLAR CLOUDS

1.0 INITIAL PROTOSTELLAR MASS SPECTRUM

Can we learn something about the processes governing the formation of stars and interstellar clouds from the present stellar luminosity function? The present distribution of stellar absolute magnitudes can be related to theoretical models of star formation by means of the initial protostellar mass spectrum, $q(m,t)$. Denote the present epoch by T, and the expected (normalized) number of stars whose absolute visual magnitudes are between M_v and $M_v - dM_v$ by $\Psi(M_v,T)dM_v$; the function $\psi(M_v,T)$ is known as the present luminosity function.

Suppose that luminosity M_v is a one-to-one function of the stellar mass m, $M_v(m)$(see, e.g., Limber (1960)). Then the expected number of stars of mass m to m + dm at the present time T is $q(m,T)dm$, where

$$q(m,T) = \psi(M_v(m),T)\,|dM_v(m)/dm| \tag{1}$$

is the present mass spectrum, i.e., expected number density.

We are interested in the stellar mass spectrum at the time t when the large cloud from which stars had allegedly formed was split into many small fragments or protostars which had not yet commenced thermonuclear element-burning. Unfortunately the initial protostar spectrum $q(m,t)$ is probably not equal to the present stellar mass spectrum $q(m,T)$. First of all, it is possible that not all of the mass of the protostar contracted into a presently observable star (Kiang, 1966a) and that the fraction of mass lost varied greatly with the initial protostellar mass. A physical process exhibiting this effect would be the loss of gas by radiation pressure when the central condensation of the protostar grew hot before the outlying regions did. In the rest of this paper we will ignore this effect.

Secondly, we no longer observe old massive stars. The old massive stars have already evolved off the main sequence into relatively invisible white dwarfs. This is often taken into account by means of an "evolution correction" of $\phi(m,T,t)$ which is the fraction of stars of mass m which are still visible at time T, i.e., the proportion of stars of mass m whose main sequence ages are less than the ages T - t of the star system. Therefore

$$q(m,t) = q(m,T)/\phi(m,T,t) \tag{2}$$

This correction was first introduced explicitly by Salpeter (1955).

Salpeter's assumption is that stars are "born" (we interpret this as "reaching the main sequence") uniformly in time, and thereafter evolve at a rate which is independent of the time of birth. Thus

$$\begin{aligned} \phi(m,T,t) &= \tau_{ms}(m)/(T-t) && \text{if } T - t > \tau_{ms}(m) \\ \phi(m,T,t) &= 1 && \text{if } T - t \leq \tau_{ms}(m) \end{aligned} \tag{3}$$

where $\tau_{ms}(m)$ is the lifetime on the main sequence of a star whose mass is m. Even if this were correct the function $\tau_{ms}(m)$ is not well known. Schmidt (1959) uses

$$\tau_{ms}(m) = k(m)\ m/L \tag{4}$$

where k(m) is a slowly varying function of m, and m/L is the mass-to-light ratio. It can be said in general that $\tau_{ms}(m)$ [therefore $\phi(m,T,t)$] rapidly decreases with increasing m, exceeding the presumed age of the galaxy (perhaps some 8×10^9 years) only for $m < 0.75\ m_\odot$, roughly. ($m_\odot$ is the mass of the sun.)

The basic difficulty in applying formula (3) is that we cannot say what the distribution of birthdays is without making some very strong and, in general, unverifiable assumptions about the physical processes of star and protostar formation. Some attempts to compute the rate of star formation have been made by Mathis (1959), Salpeter (1959) and Schmidt (1959); the rate of protostar and stellar cloud formation has been estimated by Field and Saslaw (1965).

The above-named astronomers readily point out the speculative nature of their conclusions.

In summary, while some sort of evolution correction is needed in order to estimate the initial protostellar mass spectrum for relatively large stars, the precise nature of the correction is unknown. We must therefore restrict our attention to the mass spectrum of relatively small stars ($m < 0.75\ m_\odot$).

Unfortunately, the present spectrum of small stars is not well determined empirically, a major reason being selection effects for low-visibility stars. Nevertheless, there appear to be no visible stars with $m < 0.035\ m_\odot$ and only three with $m < 0.075\ m_\odot$, (O'Leary, 1966). For $0.15\ m_\odot < m < 0.75\ m_\odot$ the mass spectrum of field stars is empirically of the form of an inverse power law with index α,

$$q(m,T) = \text{constant}\ m^{-\alpha-1} \tag{5}$$

There is considerable disagreement about the numerical value of α. Brown (1964) and O'Leary (1966) concentrate on low-mass ($m < 0.5\ m_\odot$) stars, obtaining respectively $\alpha = 0.61$ and $\alpha = 0.74$. Warner (1961 a,b) verified Salpeter's (1955) value $\alpha = 1.35$ for $0.25\ m_\odot < m < 10\ m_\odot$; however, this estimate was based on an "evolution corrected" mass spectrum over most of the range of m, and we have already discussed some of the difficulties in applying this correction (3). Using a somewhat different evolution correction, Limber (1960) obtains a spectrum of inverse power law type with $\alpha = 1.55$ for $1.5\ m_\odot < m < 40\ m_\odot$. Reddish (1966) concludes that $\alpha = 1.5\ (\pm 0.3)$ for $0.1\ m_\odot < m < 100\ m_\odot$, and $\alpha = 1.33$ over most of this range; these counts include field stars locally and in the small Magellenic cloud, as well as the galactic cluster h-Persei, Pleiades, Hyades, and Praesepe (Reddish, 1962). However, the mass spectrum is definitely somewhat flatter for $m < 0.5\ m_\odot$ than the $\alpha = 1.33$ inverse power law. If Wanner's (1964) luminosity function for the solar neighborhood is correct, i.e., $\psi(M_V,T)$ is fairly constant for $6 < M_B < 16$, then $\alpha \gtrsim 0$ for small m. In summary: The mass spectrum of small stars, for which no evolutionary correction is needed, is too poorly determined to furnish reliable information about the initial protostellar mass spectrum.

The mass spectrum of very small and very massive stars falls off much faster than the above-mentioned power laws. Kiang (1966 a) has fitted the upper end of the "corrected" stellar mass spectrum by a fractional exponential law with characteristic exponent n,

$$q(m,t) = (\text{constant})\ m^{b}\ e^{-cm^{n}}$$

where n = 1/5 for Salpeter's (1955) spectrum. These empirical results and the theoretical arguments at the end of this paper suggest that we consider the family of initial mass spectra

$$q(m,t) = (\text{constant})\ m^{-\alpha-1}\ e^{-cm^{n}} \qquad (6)$$

which exhibits both inverse power law and fractional exponential law behavior, according to the values of the constants, α, c and n.

2.0 MASS SPECTRUM OF INTERSTELLAR CLOUDS (REVIEW)

Scheffler (1967) has attempted to estimate the index α in an assumed inverse power law mass spectrum. The index α is determined empirically in two ways:

(1) From the relation between visual absorption values and cloud diameters, and the distribution of these absorption values.

(2) From the frequency distribution of absorption values and optical depths at 21 cm wavelength.

These are the only determinations of the cloud mass spectrum known to the author. We will briefly review Scheffler's arguments.

He first notes that, on the average, the dust spatial density ρ_s, gas density ρ, diameter Δ and absorption ∂ (in magnitudes) of an interstellar cloud are related by

$$\rho_s \sim \rho^{\beta}$$

$$\rho \sim \Delta^{\varepsilon} \qquad (7)$$

$$\partial \sim \rho_s \Delta^{1+\varepsilon}$$

The mass $m(\partial)$ of a cloud with absorption ∂ is

$$m(\partial) \sim \rho\Delta^3 \sim \partial^{\xi} \tag{8}$$

with

$$\xi = (3\beta + \varepsilon)/\beta(1 + \varepsilon) \tag{9}$$

The probability density of absorption (i.e., attenuation of light) along a line of sight, say $\bar{g}(\partial)$, is related to the true (spatial number) density function of clouds $h(\partial)$ by

$$\bar{g}(\partial) \sim \Delta^2(\partial)h(\partial) \sim \partial^{\frac{2}{1+\varepsilon}}h(\partial) \tag{10}$$

Scheffler observes, for large ∂,

$$\bar{g}(\partial) \sim \partial^{-\delta} \tag{11}$$

Thus the mass spectrum $q(m)$ is

$$q(m) \sim h(\partial)d\partial/dm \sim m^{-\alpha-1} \tag{12}$$

for large m, where

$$\alpha = \frac{\delta - 1}{\xi} + \frac{2}{\xi(1 + \varepsilon)} \tag{13}$$

The parameters δ, ε, ξ are not well determined numerically. We are likely to have, however,

$\beta \gtrsim 1$ (dust density proportional to or increasing slightly with increased gas density)

$\varepsilon \lesssim 0$ (larger clouds may have slightly higher gas density due to self-compression)

$\xi \gtrsim 3$ (from the preceding) (14)

$\delta > 1$ ($\delta \sim 3$) (observed)

$\alpha \simeq \frac{\delta + 1}{3} > 2/3$ ($\alpha \sim 4/3$) (from the preceding)

Scheffler attempts to estimate α from estimated actual sizes of absorbing clouds, but the physical uncertainties leave α very poorly determined--$2/3 \le \alpha \le 4/3$ for $m < 5000\ m_\odot$ and $\alpha \approx 1/2$ for $m > 5000\ m_\odot$.

The distribution of the optical depth τ for 21 cm radiation leads to a more consistent estimate. Assuming that optical depth τ and cloud diameter Δ are related by

$$\tau \sim \Delta^{1 + \epsilon/\beta} \tag{15}$$

then the mass $m(\tau)$ of a cloud with optical depth τ is

$$m(\tau) \sim \tau^{\xi^*} \tag{16}$$

where

$$\xi^* = (3\beta + \epsilon)/(\beta + \epsilon) \tag{17}$$

The observed line of sight probability density of τ, $g^*(\tau)$, is assumed to be a power law for large τ,

$$g^*(\tau) \sim \tau^{-\delta^*} \tag{18}$$

Proceeding as above, we derive

$$\alpha = \frac{\delta^* - 1}{\xi^*} + \frac{2}{\xi^*(1 + \epsilon/\beta)} \tag{19}$$

Scheffler obtains the empirical value $\delta^* \approx 2$, thus $\alpha \approx 1$, using Clarks' (1965) data. With the same data the author obtains (Fig. 1) $\delta^* \sim 3.0$ for $\tau > 0.75$, thus $\alpha \sim 4/3$. The power law assumed for $g^*(\tau)$ cannot, of course, be verified with such a small number of observations.

To summarize, independent lines of investigation tenuously support the hypothesis that the mass spectrum of interstellar clouds is roughly an inverse power law with index $\alpha \ge 1$ ($\alpha \approx 4/3$).

3.0 FILIPPOV'S MODEL OF INDEPENDENT SPLITTING (REVIEW)

Suppose that at time $t = 0$ there is a single particle of positive mass m_o. This particle breaks up into a finite or countably infinite number of smaller particles whose masses are positive numbers $m_1, m_2, m_3, \ldots$. The splitting conserves mass:

$$\sum_{i=1}^{\infty} m_i = m_o \tag{20}$$

Each particle existing at time t can split in the interval of time $(t, t + \Delta t)$ (independent of its past and of the fate of other particles) with probability $p(m)\Delta t + o(\Delta t)$ where m is the mass of the particle and the function p(m) depends only on m and is bounded in any positive interval $D_1 < m < D_2$, $D_1 > 0$, $D_2 < \infty$. The number and masses of the particles arising from one splitting of m_o are random variables. The fraction of mass in particles whose mass is less than m is

$$\frac{1}{m_o} \sum_{m_i < m} m_i = \Phi(m_o, m) \tag{21}$$

Thus $m_o \, \Phi(m_o, m)$ is the mass of all particles whose masses are less than m, formed by a single splitting of the mass m_o. Assume that the mathematical expectations

$$\begin{aligned} &E\left\{\Phi\left(m_o, m\right)\right\} = F\left(m_o, m\right) \\ &E\left\{\Phi\left(m_o, m\right)\Phi\left(m_o, \bar{m}\right)\right\} = B\left(m_o, m, \bar{m}\right) \end{aligned} \tag{22}$$

depend only on the indicated arguments. In general

$$\begin{aligned} &F\left(m_o, +0\right) = 0 \\ &B\left(m_o, +0, \bar{m}\right) = 0 \qquad B\left(m_o, m, +0\right) = 0 \\ &\Phi\left(m_o, m_o\right) = 1 \end{aligned} \tag{23}$$

These functions are sufficient to describe a very general splitting process (Filippov, 1961). Some further specific assumptions are needed to compute a limit distribution. They are:

$$\begin{aligned} &(a) \quad p(m) = m^n , \qquad n \geq 0 \\ &(b) \quad F\left(m_o, m\right) \end{aligned} \tag{24}$$

depends only on the value of m/m_o, thus

$$F\left(m_o, \lambda m_o\right) = f(\lambda) \tag{25}$$

for any $0 \leq \lambda \leq 1$ and m_o. Similarly,

$$B\left(m_o, \lambda_1 m_o, \lambda_2 m_o\right) = b\left(\lambda_1, \lambda_2\right)$$

for any $0 \leq \lambda_1, \lambda_2 \leq 1$ and m_o

$$(c) \quad \int_o^1 f(\lambda) d\lambda/\lambda < \infty$$

$$(d) \quad f'(\lambda) > 0$$

for a set of λ of positive measure.

Assumption (a) says that the probability per unit time (rate) of splitting is a non-decreasing function of the mass of the particle, i.e., large particles tend to break up faster than small particles, which seems physically plausible. That p(m) is a power function is also plausible, since we would suspect that the splitting rate might be proportional to the radius (n = 1/3) or radius squared (n = 2/3) or mass (n = 1) of the cloud. Since the exact mechanism governing the fragmentation of gas clouds is not known, it seems prudent to leave the parameter n unspecified. Note that the unit of time is that which a particle of unit mass will survive with probability 1/e.

The similarity assumption (b) says that the breakup mechanism operates in the same manner whatever the size of the initial particle. This is likely to be untrue for m_o sufficiently large or sufficiently small, but may well hold for a very large range of values of m_o.

Assumption (c) says, in effect, that not too much mass is lost in small (zero-mass) particles. Assumption (d) says that a positive quantity of mass goes into fragments with strictly different sizes. These assumptions are also mathematically necessary.

Using the above assumptions, Filippov derives integro-differential equations for the first and second moments of the proportion of mass m(x,t) contained in particles of mass $\leq$ x at time t. In particular, for $m_o = 1$

$$\frac{\partial}{\partial t} E\{m(x,t)\} = -E\{m(x,t)\} + \int_0^1 E\left\{m\left(\frac{x}{\lambda}, t\lambda^n\right)\right\} df(\lambda) \tag{26}$$

$$= \int_x^\infty \mu^n f\left(\frac{x}{\mu}\right) d_\mu E\{m(\mu,t)\}$$

Filippov establishes that (26) has an asymptotic solution ($t \to \infty$) of the form

$$E\{m(x,t)\} = G\left[(t+1)x^n\right] \tag{27}$$

In (26), letting $u = (t+1)x^n$, $v = (t+1)\mu^n$, and $f(\lambda) = f_1(\lambda^n)$, he obtains

$$uG'(u) = \int_u^\infty f_1\left(\frac{u}{v}\right) v dG(v)$$

$$G(0) = 0 \quad , \quad G(\infty) = 1 \tag{28}$$

or letting $uG'(u) = cg(u)$ for c constant,

$$g(u) = \int_u^\infty f_1\left(\frac{u}{v}\right) g(v) dv \tag{29}$$

Thus

$$\lim_{t \to \infty} \max_{0 \leq x < \infty} \left| E\{m(x,t)\} - G\left(tx^n\right) \right| = 0$$

Where G(u) may be found from (28) or (29). He also establishes a stronger result, the convergence in probability of m(x,t) to G.

An important special case in which G may be evaluated explicitly is a power law for splitting.

$$f(\lambda) = \lambda^{k} \quad , \quad k > 0 \tag{30}$$

In that case

$$G(u) = \frac{1}{\Gamma\left(\frac{k}{n}\right)} \int_{0}^{u} v^{\frac{k}{n} - 1} e^{-v} dv \tag{31}$$

$$E\{m(x,t)\} = te^{-t} \int_{0}^{x} F\left(1 + \frac{k}{n} , 2 , \left(1 - v^{\frac{n}{k}}\right)t\right) dv \tag{32}$$

for $x \leq m_o = 1$ and $n > 0$ where $F(\alpha,\gamma,z) = M(\alpha,\gamma,z)$ is the confluent hypergeometric function (Jahnke and Emde, 1945). Also

$$E\{m(x,t)\} = te^{-t} \int_{0}^{x^k} \frac{1}{\sqrt{-t \log v}} I_1\left(2\sqrt{-t \log v}\right) dv \tag{33}$$

for $x \leq 1$, $n = 0$ where I_1 is a modified Bessel function of order 1 (this case n = 0 was discussed by Kolmogorov (1940)). We are really interested in the <u>mass spectrum</u> or <u>expected number density</u>

$$q(x,t) = \frac{1}{x} \frac{\partial}{\partial x} E\{m(x,t)\} \tag{34}$$

where the number of particles of mass x to x + dx expected at time t from a single initial particle of unit mass is q(x,t)dx. In general the function q(x,t) cannot be normalized to a probability density. The total expected number of particles N(t) at time t is

$$E\{N(t)\} = \int_{0}^{1} q(x,t)dx + e^{-t} \tag{35}$$

and from (32) and (34) we obtain

$$E\{N(t)\} = F\left(\frac{k}{n}\ ,\ \frac{k-1}{n}\ ,\ t\right) \qquad \text{for } n > 0,\ k > 1$$

$$= \infty \qquad \text{for } n > 0,\ k \leq 1$$

For n = 0 and <u>any</u> $f(\lambda)$ we obtain

$$E\{N(t)\} = e^{(a-1)t} \tag{36}$$

where $a = \int_0^1 f'(\lambda)d\lambda/\lambda$ may be infinite (this happens under (30) for $k \leq 1$). Assuming (24) and (30) and t sufficiently large for the asymptotic solutions (32, 33) to apply, we obtain

$$q(x,t) = kt\ e^{-t}\ x^{k-2}F\left(1 + \frac{k}{n}\ ,\ 2\ ,\ \left(1-x^n\right)t\right) \qquad \text{for } n > 0 \tag{37}$$

$$q(x,t) = kt\ e^{-t}\ x^{k-2}I_1\left(2\sqrt{-kt\ \log x}/\sqrt{-kt\ \log x}\right) \qquad \text{for } n = 0 \tag{38}$$

Independent of the asymptotic properties, it can be shown that as $t \to 0$,

$$q(x,t) \sim kt\ x^{k-2} \tag{39}$$

which makes sense, since for small t only the initial particle has had time to break up, and the mass spectrum of fragments from the initial particle is just $f'(x)/x = kx^{k-2}$. As $z \to \infty$ we have for the asymptotic solution

$$F(\alpha,\gamma,z) \sim \frac{\Gamma(\gamma)}{\Gamma(a)}\ z^{\alpha-\gamma}e^{z} \tag{40}$$

$$F(\alpha,\gamma,-z) \sim \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\ z^{-\alpha} \tag{41}$$

$$I_1(z) \sim e^{z}/\sqrt{2\pi z} \tag{42}$$

thus

$$q(x,t) \sim \frac{kt^{k/n}}{\Gamma\left(1+\frac{k}{n}\right)} x^{k-2} e^{-tx^n} \left(1-x^n\right)^{\frac{k}{n}-1} \qquad \text{for } n>0 \qquad (43)$$

$$q(x,t) \sim \frac{(kt)^{1/4}}{2x^2\sqrt{\pi}\,(-\log x)^{3/4}} e^{-\left(\sqrt{-k\log x}-\sqrt{t}\right)^2} \qquad \text{for } n=0 \qquad (44)$$

$$E\{N(t)\} \sim \frac{\Gamma((k-1)/n)}{\Gamma(k/n)} t^{1/n} \qquad \text{for } k>1 \text{ and } n>0 \qquad (45)$$

4.0 FILIPPOV MODEL: NUMERICAL CALCULATIONS

The author has studied in greater detail the case

$$\frac{k}{n} = r + 1 \quad , \qquad (46)$$

r a non-negative integer. We will derive simple explicit formulas for $q(x,t)$ from (32). If k/n is not an integer, it may still be possible to calculate $q(x,t)$ by interpolating between values derived for integer k/n. It is easily shown that

$$F(\alpha+r,\alpha,z) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+r)} \frac{1}{\alpha-1} \frac{d^r}{dz^r}\left(z^{\alpha+r-1}e^z\right) = Q_r(z)e^z \qquad (47)$$

where $Q_r(z)$ is a polynomial of degree r in z. Therefore

$$q(x,t) = nt\, x^{n-2} e^{-tx^n} \qquad \text{for } k=n \qquad (48)$$

$$= 2nt\left[1+\left(1-x^n\right)\frac{t}{2}\right] x^{2n-2} e^{-tx^n} \qquad \text{for } k=2n \qquad (49)$$

$$= 3nt\left[1+\left(1-x^n\right)t+\left(1-x^n\right)^2\frac{t^2}{6}\right] x^{3n-2} e^{-tx^n} \qquad \text{for } k=3n \qquad (50)$$

and so on.

We see for large t and x<<1 (the initial cloud is extensively fragmented) the mass spectrum of fragments will be of the form

$$q(x,t) \sim \frac{kt^{k/n}}{\Gamma\left(1 + \frac{k}{n}\right)} x^{k-2} e^{-tx^n} \tag{51}$$

which has aspects of __both__ the inverse power law $\left(x^{k-2}\right)$ and fractional exponential law with exponent $n\left(e^{-tx^n}\right)$.

The fractional exponential behavior will be noticed only for sufficiently large x, say

$$tx^n > 1/5 \quad , \quad x > (5t)^{-1/n} \quad .$$

Even when n = o, the mass spectrum will be roughly the same form, as is seen by rewriting (44) to read

$$q(x,t) \sim \frac{(kt)^{1/4}}{2\sqrt{\pi}} e^{-t} \frac{x^{k-2}}{(-\log x)^{3/4}} e^{2\sqrt{-kt \log x}} \tag{52}$$

The exact asymptotic mass spectrum q is sketched in Figures 2-10, for n = 1/3, 2/3, k = 1/3, 2/3, 1, 4/3, t = 1, 2, 3, and for k = n=1, t = 1, 2, 3. k < 1 corresponds to the shattering of a cloud into a multitude of small pieces. k > 1 corresponds to the splitting of a cloud into one large piece and a few very small ones, i.e., to the "shedding" of a few small fragments.

It is important to notice that if q(x,t) behaves essentially like an inverse power law, its index α must obey

$$\alpha = 1 - k < 1 \tag{53}$$

since k > 0. This is a consequence of mass conservation at each splitting. However, we have shown that for some real distributions, $\alpha > 1$. The author, therefore, extended Filippov's model to cover these observations.

5.0 EFFECT OF MIXING SPECTRA FROM CLOUDS OF DIFFERENT AGES OR INITIAL SIZES

In the previous section we obtain asymptotic formulae for the expected number density $q(x,t)$ of cloud fragments of mass x at time t from a single initial cloud of mass $x = 1$ at time $t = 0$. In practice the mass spectrum of field stars (in the Solar neighborhood, for example) will have come from clouds with different ages or, what amounts to the same thing, with different initial masses m_o. There is also the possibility of density variations within a large initial cloud.

Mixing together protostars from clouds with different ages t will yield an observed mass spectrum $q_T(x)$ which is a mixture of spectra $q(x,t)$, say

$$q_T(x) = \int_o^T q(x,u)\; f_t(u)du \tag{54}$$

where T is the present epoch and $f_t(u)$ is the probability density function of the birthdays t of the initial clouds contributing observable protostars. A simple calculation will illustrate qualitatively the effect of time averaging. Suppose t is distributed uniformly over the interval $(0,T_o)$, i.e.,

$$f_t(u) = 1/T_o \qquad \text{for } o < u < T_o \tag{55}$$

$$f_t(u) = 0 \qquad \text{otherwise}$$

and further assume that T_o is very large,

$$T_o >> k/n \quad , \quad T_o >> 1$$

This kind of hypothesis is assumed explicitly by Salpeter (1955). Since T_o is large, we adopt the asymptotic formula (43) and write

$$q(x,t) = \frac{kt^{k/n}}{\Gamma\left(1 + \frac{k}{n}\right)}\; x^{k-2} \left(1-x^n\right)^{\frac{k}{n} - 1} e^{-tx^n} \tag{56}$$

Therefore

$$q_T(x) = \frac{k}{\Gamma\left(1 + \frac{k}{n}\right)} x^{k-2} \left(1-x^n\right)^{\frac{k}{n} - 1} \int_0^{T_0} t^{k/n} e^{-tx^n} dt/T_0$$

$$\sim k\, x^{-n-2} \left(1-x^n\right)^{\frac{k}{n} - 1}/T_0 \tag{57}$$

i.e., the mass spectrum is (asymptotically) an inverse power law with index n + 1 (>1).

In the same way let us consider a mixture in m_0. Let the initial (t = 0) cloud mass be m_0, and suppose that m_0 has a probability density $f_{m_0}(m)$. Then the observed mass spectrum at time t is

$$q_{m_0}(m;t) = \int_0^\infty q(m/\mu,t)\, f_{m_0}(\mu)d\mu/\mu \tag{58}$$

where q(x,t) is the normalized spectrum derived as in the preceding section. Assuming that for large enough t the asympotic formula (43) is exactly correct, we obtain

$$q_{m_0}(m,t) = \frac{k}{\Gamma\left(1 + \frac{k}{n}\right)} t^{k/n} m^{k-2} \int_m^\infty \mu^{1-k} \left[1 - \left(m\mu\right)^n\right]^{\frac{k}{n} - 1} e^{-t\left(m/\mu\right)^n} f_{m_0}(\mu)d\mu \tag{59}$$

Suppose, for the sake of definiteness, that m_0 has an inverse power law probability density

$$f_{m_0}(\mu) = \alpha s^\alpha \mu^{-\alpha-1} \qquad \text{for } \mu > s > 0$$

$$= 0 \qquad \text{for } \mu < s \tag{60}$$

where $\alpha > 0$ and s is "small". Then for $m > s$,

$$q_{m_o}(m,t) = \frac{\alpha\Gamma\left(\frac{\alpha+k-1}{n}\right)}{\Gamma\left(\frac{\alpha+2k-1}{n}\right)} t^{k/n} s^{\alpha} m^{-\alpha-1} F\left(\frac{\alpha+k-1}{n}, \frac{\alpha+2k-1}{n}, -t\right) \quad (61)$$

assuming as well that $\alpha > 1 - k$. Since, for large t, the confluent hypergeometric function F is approximately

$$F(a,b,-t) \sim \frac{\Gamma(b)}{\Gamma(b-a)} t^{-a}$$

we have, for very large t, and $\alpha \neq 1$,

$$q_{m_o}(m,t) \sim \frac{\alpha\Gamma\left(\frac{\alpha+k-1}{n}\right)}{\Gamma\left(\frac{k}{n}\right)} t^{\frac{1-\alpha}{n}} s^{\alpha} m^{-\alpha-1} \quad (62)$$

We see that, in a certain limited sense, the inverse power law type of spectrum reproduces itself under Filippov's independent splitting mechanism. If q(m,t) is both time-mixed by (55) and initial-mass-mixed by (60), the initial-mass mixing effect (index α) will predominate.

We will not pursue this line of argument further. It is clear that mixtures of protostar populations from different initial clouds can lead to an inverse power law type of mass spectrum with index > 1, which is strictly impossible for the unmixed spectrum. Local field stars, which probably were formed in many different initial clouds, may exhibit a mixed spectrum of this kind.

6.0 SOME OTHER MODELS

Previous theoretical models for q(m,t) are based on "one-shot" once-and-forever splitting mechanisms, rather than on repeated mechanisms. For example, one very simple model for the fragmentation of a large initial volume of gas has been proposed by Auluck and Kothari (1965) (see also Kushwaha and Kothari, 1960). The assumed physical process is that the volume of interest is cut into parallelipipeds by three orthogonal systems of parallel planes, the position of each plane along a perpendicular axis being given by a Poisson (purely random) process. The mass spectrum derived is approximately

$$q(m) \sim (\text{constant}) \left(m/\bar{m}\right)^{-2/3} \left\{3\left(m/\bar{m}\right)^{1/3} - 1\right\} e^{-3\left(m/\bar{m}\right)^{1/3}}$$

$$\sim (\text{constant})\ m^{-1/3}\ e^{-cm^{1/3}} \qquad (63)$$

for $m >> \bar{m}$ = average mass of fragment. This is similar to (6) with $\alpha = -2/3$ and $n = 1/3$. It does not fit the observed spectrum very well. Further, the underlying physical model does not appear to be very plausible.

A much more plausible model has been suggested by Kiang (1966 b). Let points (condensation centers) be distributed Poissonwise (at random) in space, and suppose that each center captures all the matter closer to itself than to any other center. The region of interest will then be divided into a set of irregular polyhedra known as Voronoi polyhedra. The probability density of the volume (equivalently, the mass) of these polyhedra is approximately of the form of a gamma density with exponent 6, or

$$q(m) \approx (\text{constant})\ m^5\ e^{-cm} \qquad (64)$$

the determination of this density being the result of a Monte Carlo calculation (Kiang, 1966b). However, it can be shown quite rigorously (Gilbert, 1962) that this random cell model produces a number density which for large m is of the order of e^{-cm}, or more precisely

$$\int_m^\infty q(m')dm' \sim (\text{constant})\ e^{-cm} \qquad (65)$$

for some c and for m sufficiently large. The present stellar number density (equivalently, mass spectrum) does not show an exponential tail, which leads Kiang (1966a) to suggest that only a fraction of the protostellar mass contracts into an observable star, the fraction being roughly proportional to m^n for some $n < 1$.

Gilbert also discusses a plausible generalization of the cell model, known as the Johnson-Mehl model. In the Johnson-Mehl model condensation centers are inserted into the region at random times as well as in random positions, and cells grow outward from these centers at a uniform velocity until they encounter other cell walls. Gilbert (1962) establishes that the tail of the mass spectrum will fall off even faster than exponentially; more precisely,

$$\int_m^\infty q(m')dm'' \sim (\text{constant})\ e^{-cm^{4/3}} \qquad (66)$$

for large m, and some constant c. Therefore this model also cannot describe the observations.

Kruszewski (1961) has computed the mass spectrum with the assumption that linear perturbations in density have a "white noise" spectrum, and then invoking a parallelipiped approximation (as did Auluck and Kothari). Assuming that the minimum stellar mass is $\mu = 0.07\ m_\odot$, he obtains an initial stellar mass spectrum

$$q(m) = (\text{constant}) \left[\log\left|m/\mu\right|\right]^2 m^{-3} \quad \text{for } m>\mu \qquad (67)$$

which is actually in rather good agreement with Limber's (1960) initial mass spectrum for $0.10\ m_\odot < m < 50\ m_\odot$. However, the physical basis of Kruszewski's results requires further elaboration in particular; it would be interesting to see if the parallelipiped approximation can be removed.

A coalescence and breakup model for interstellar clouds, originally proposed by Oort, has been analyzed by Field and Saslaw (1965). This model assumes that small interstellar clouds grow by collision and coalescence into larger and larger clouds, until finally the cloud mass exceeds some critical value and the large cloud breaks up into very small clouds, which begin repeating the coalescence process. The coalescence and breakup mechanisms tend to produce a steady-state mass spectrum which, for constant coalescence kernel, is approximately of the form

$$q(m) \simeq (\text{constant})\ m^{-3/2}\ \beta^m \qquad (68)$$

where β, a constant ≥ 1, depends on the critical unstable cloud mass and rate of breakup of large unstable clouds. This is also of the same form as (6), with $\alpha = 1/2$ and $n = 1$, $c \leq o$. For a coalescence kernel proportional to $m^{2/3}$ (a geometric collision factor, and physically plausible) q(m) will be roughly of the form of an inverse power law with $\alpha = 2/3$.

Reddish (1962, 1966) has proposed a "one-shot" splitting mechanism based on equipartition of gravitational potential energy among cloud fragments. Since a variation of his arguments will play an important role in our interpretation of the data, we will next discuss this splitting process.

7.0 THE REDDISH MODEL

Reddish (1962, 1966) makes the following argument: Suppose that a cloud (of initial mass m_o and radius r_o) breaks up into spherical fragments of uniform density. The gravitational potential energy of a fragment of radius r is

$$\Omega(r) = (\text{constant})\ m^2/r = (\text{constant})\ r^5 \tag{69}$$

where m is the mass of the fragment. Let $\overline{f}(r)$ be the <u>radius spectrum</u> resulting from this splitting: i.e., the expected number of fragments whose radii are between r and r + dr is $\overline{f}(r)dr$. If gravitational potential energy is divided equally among fragments of different radius, we have

$$\begin{aligned} \Omega(r)\overline{f}(r) &= \text{constant} \\ \overline{f}(r) &= (\text{constant})\ r^{-5} \left(r_1 < r < r_o\right) \end{aligned} \tag{70}$$

Reddish assumes that this splitting occurs only once, and that the process does not produce fragments smaller than radius r_1(mass s) in which case (reverting to the notation of Section 3) the "one-shot" splitting spectrum is given by

$$\begin{aligned} \left(1/m\right)\frac{d}{dm}\ F\left(m,m_o\right) &= \overline{f}\left(\left(3m/4\pi\rho\right)^{1/3}\right)\left(36\pi\rho\right)^{-1/3}\ m^{-2/3} \\ &= (\text{constant})\ m^{-7/3} \qquad \text{for } s < m < m_o \end{aligned} \tag{71}$$

thus

$$F\ m,m_o = \int_s^m (\text{constant})\ m^{-4/3}\ dm$$

or

$$\begin{aligned} f(\lambda) = \int_{s/m_o}^{\lambda} (\text{constant})\ x^{-4/3}\ dx &= (\text{constant})\left[\left(m_o/s\right)^{1/3}\right. \\ &\left. - \lambda^{-1/3}\right] \qquad \text{for } 1 > \lambda > s/m_o \end{aligned} \tag{72}$$

To let $s \rightarrow o$ introduces a mathematical singularity in $f(\lambda)$; the physical interpretation of this singularity is that most of the mass of the initial particle is found in the smallest (zero-mass) fragments. We therefore cannot use the hypothesis (70) advanced by Reddish to determine a single-splitting fragment size distribution $f(\lambda)$ which is of the form required in Filippov's analysis, i.e., a power law (30) with $s = 0$.

A simple reformulation of Reddish's argument produces a theoretical single-splitting fragment mass distribution which is applicable to Filippov's repeated splitting theory, and which has empirical justification. The gravitational potential energy as a function of fragment mass m is

$$\Omega(m) = (\text{constant})\ m^{5/3} \tag{73}$$

Suppose now that gravitational potential energy is distributed evenly among fragments of different mass. The mass spectrum from a single splitting of a particle of mass m_o is, as before, $(1/m)\,\frac{d}{dm}\,F(m,m_o)$, thus the analogy of (70) - (72) with $s = 0$ is

$$\Omega(m)\ \left(1/m\right)\frac{d}{dm}\ F\left(m,m_o\right) = \text{constant}$$

$$F\left(m,m_o\right) = \left(m/m_o\right)^{1/3} \tag{74}$$

$$f(\lambda) = \lambda^{1/3}$$

in other words

$$k = 1/3$$

$$\alpha = 1 - k = 2/3 \tag{75}$$

The prediction from (74) is, therefore, that the initial mass spectrum of small stars from a single cloud, which have not yet evolved off the main sequence, is an inverse power law with index 2/3. This is supported by Brown (1964) and O'Leary (1966).

8.0 SOME INTERPRETATIONS OF THE STELLAR MASS SPECTRUM

If the splitting process leading to the "initial" protostellar mass spectrum was very extended in time, Filippov's treatment seems the most plausible and the most tractable. For an initial mass spectrum which is the result of a single cloud fragmentation, the approach suggested by Reddish seems most plausible. The author's application of Reddish's method yields k = 1/3, that is, α = 2/3 for small stars.

We interpret the index α = 4/3 (which may apply to large field stars) in the following way. The observed mass spectrum is in fact a mixture of spectra from clouds with different ages and initial masses. It was shown in Section 2 that the distribution of cloud masses is roughly an inverse power law with index 4/3, which would produce an initial-mass mixed spectrum of protostellar masses of about the same shape (62). On the other hand, a time-mixed spectrum could produce an inverse power law with index 1 + n (see (57)) which in this case would imply n = 1/3, i.e., the rate of splitting is roughly proportional to the radius of the fragment. It is not clear which of these interpretations is correct.

Probably the most useful tool in clearing up these theoretical uncertainties would be an improved, unevolved luminosity function or mass spectrum, determined either for young clusters or for small ($m < 0.5\ m_{\odot}$) field stars.

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2015-AHM-kse A. H. Marcus

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CAPTIONS TO FIGURES

Figure 1. Number of interstellar clouds with optical depth τ (in intervals of length 0.5), after Clark (1965)

Figures 2-10. Expected number density q(x,t) of fragments of mass x at time t from an initial particle of unit mass.

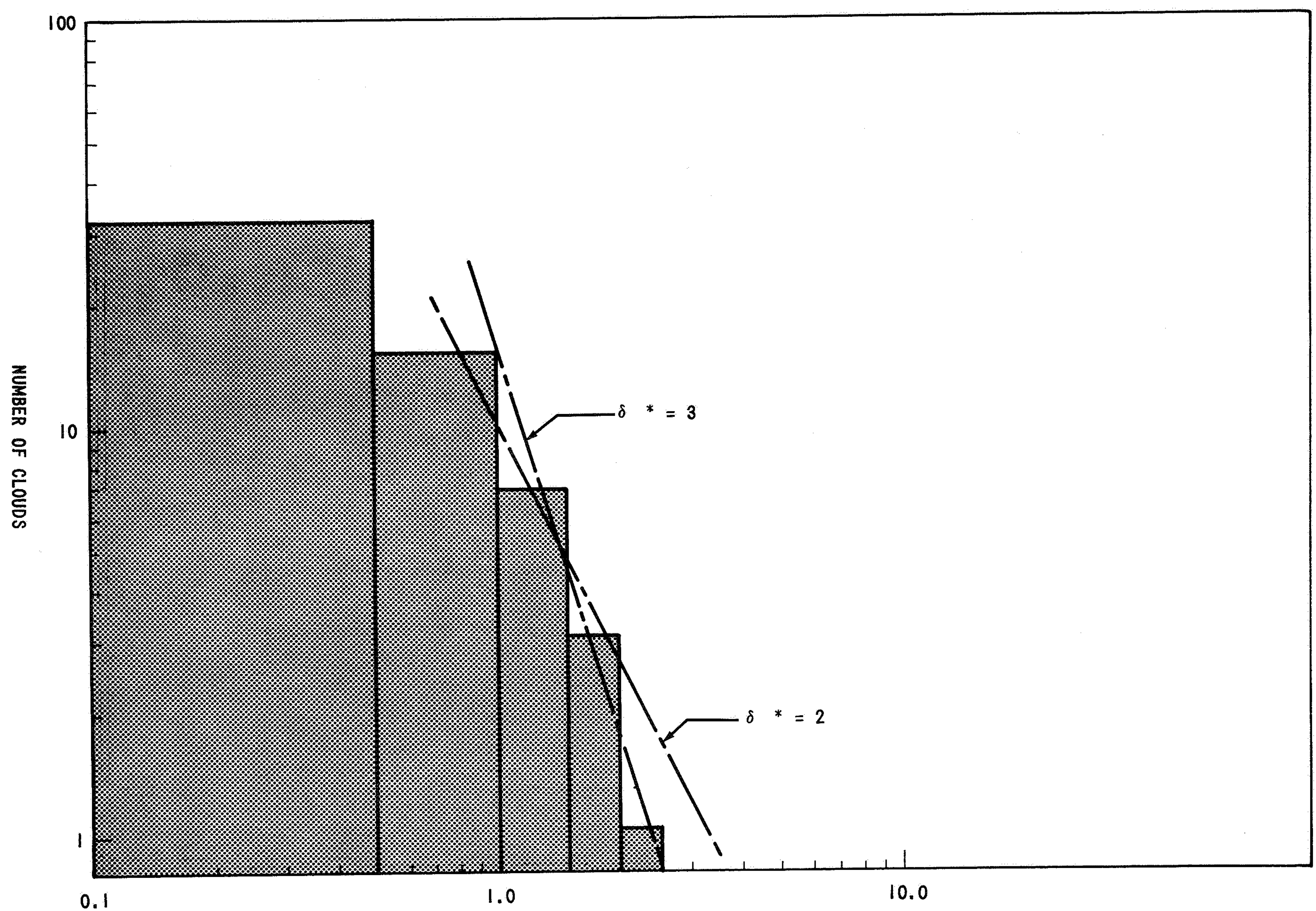
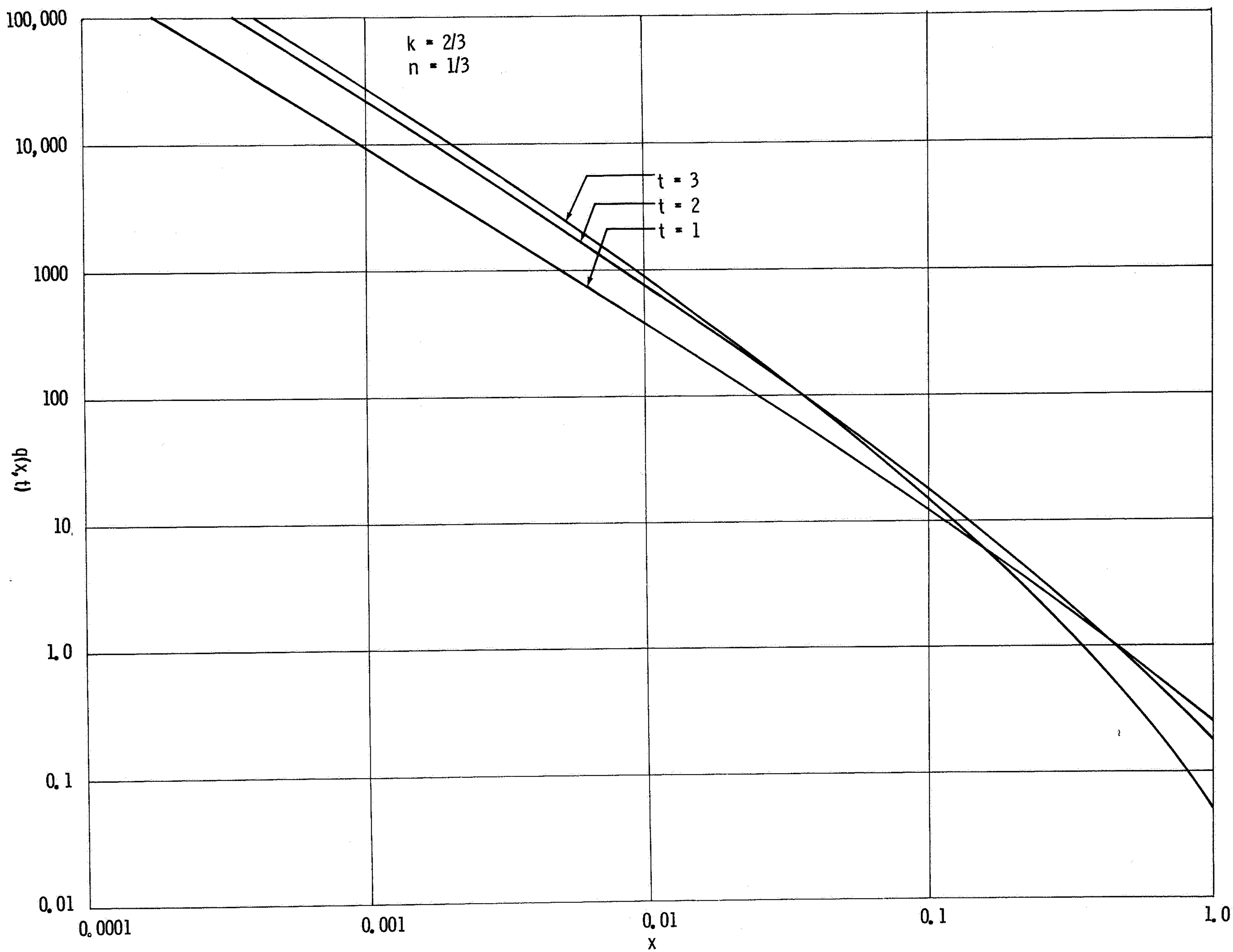


FIGURE I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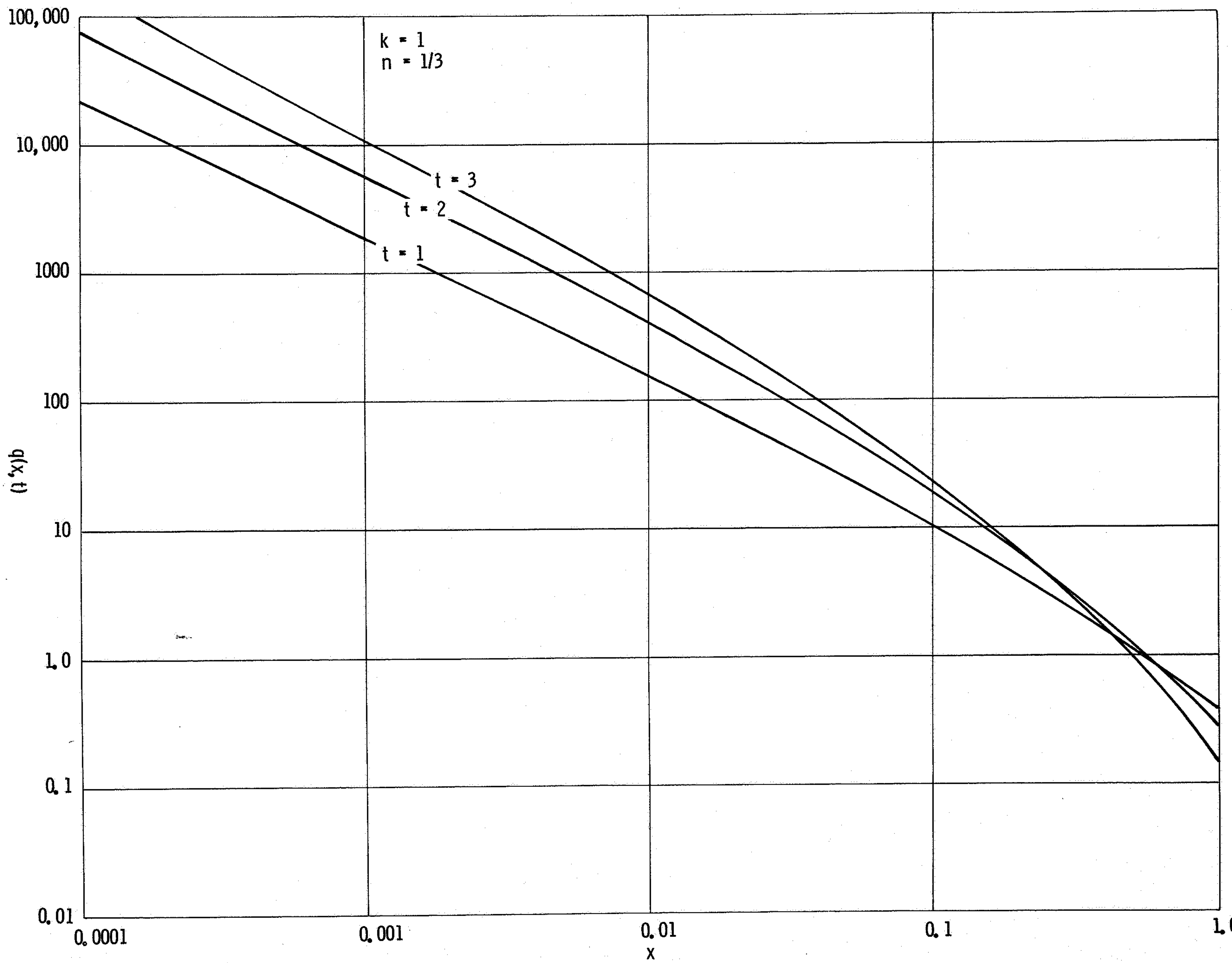
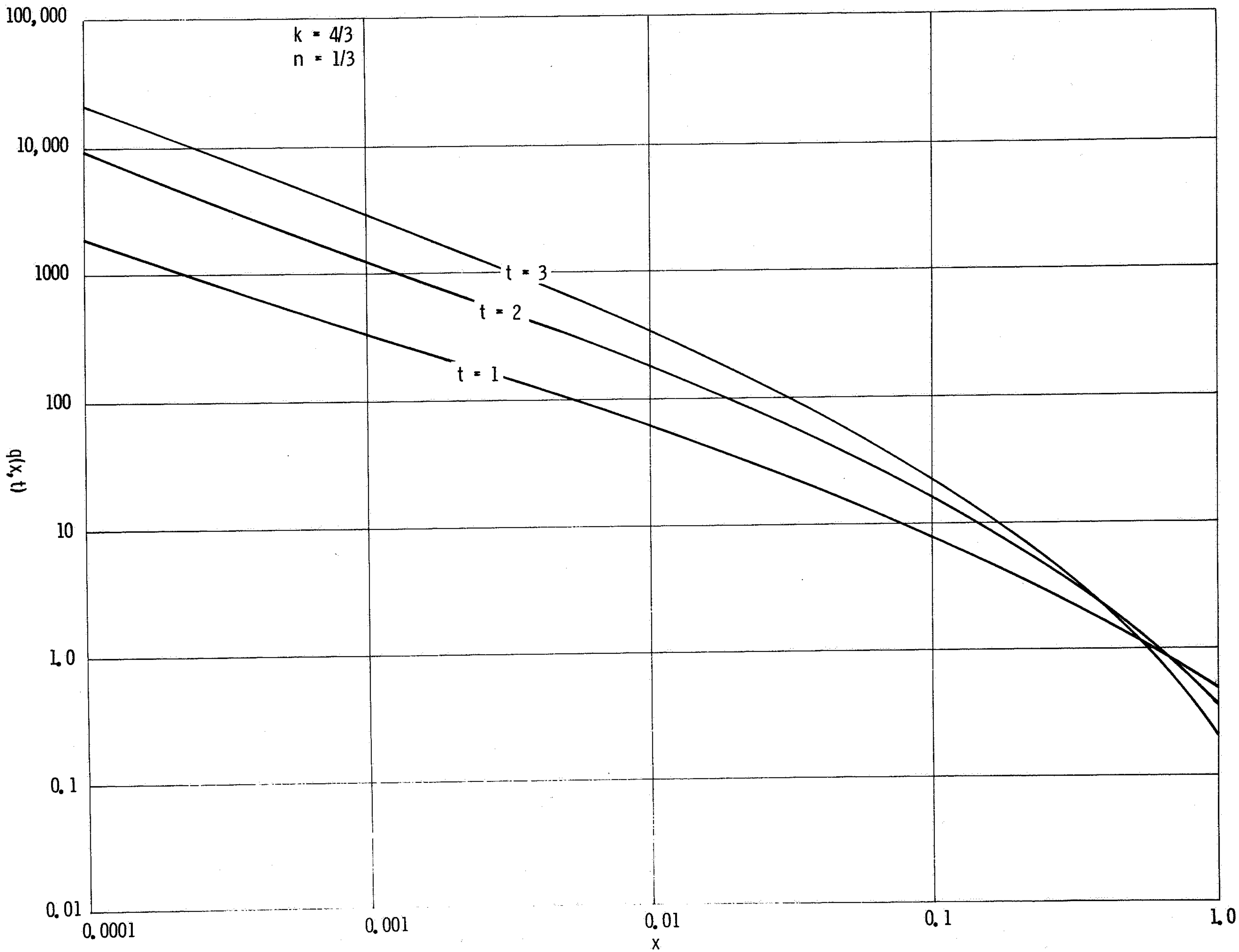
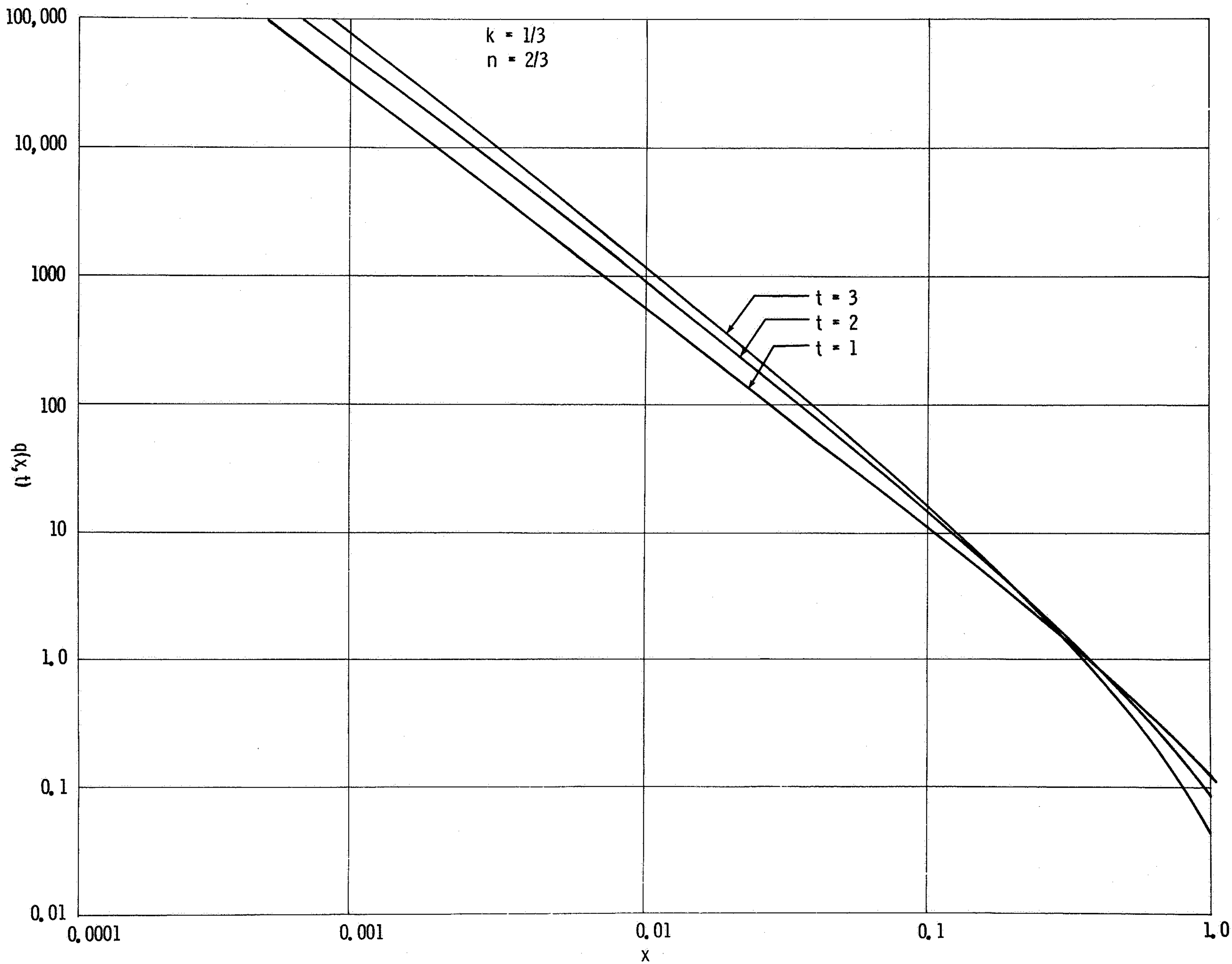
n = 1/3
t = 3
t = 2
t = 1
100,000
10,000
1000
100
10
1.0
0.1
0.01
q(x, t)
0.0001
0.001
0.01
0.1
1.0
x
FIGURE 2



100,000
10,000
1000
100
10
1.0
0.1
0.01
q(x, t)
k = 1
n = 1/3
t = 3
t = 2
t = 1
0.0001
0.001
0.01
0.1
1.0
x

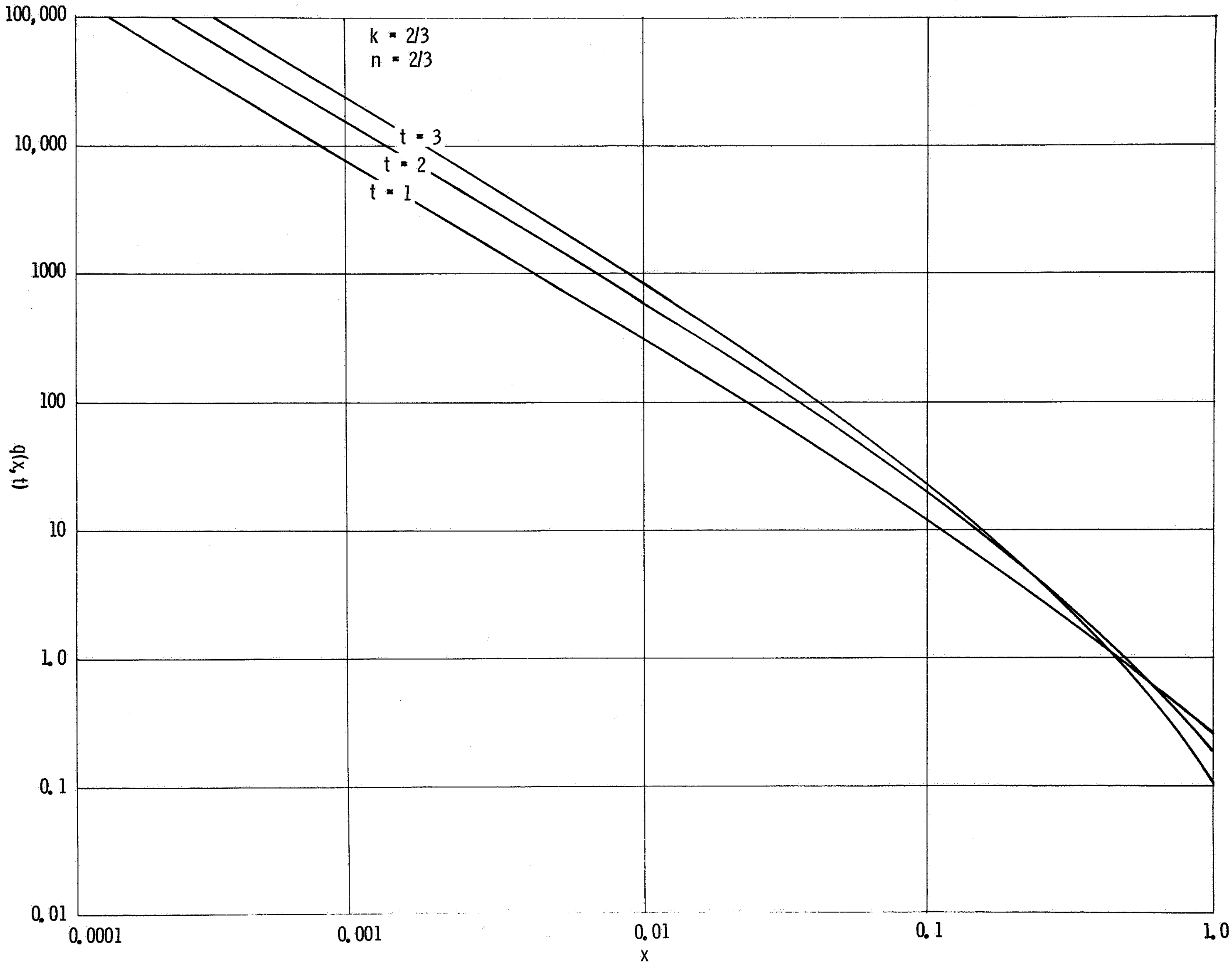
FIGURE 3


100,000
10,000
1000
100
10
1.0
0.1
0.01
q(x,t)
k = 4/3
n = 1/3
t = 3
t = 2
t = 1
0.0001
0.001
0.01
0.1
1.0
x
FIGURE 4



k = 1/3
n = 2/3
t = 3
t = 2
t = 1
100,000
10,000
1000
100
10
1.0
0.1
0.01
q(x, t)
0.0001
0.001
0.01
0.1
1.0
x

FIGURE 5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n = 2/3
t = 3
t = 2
t = 1
100,000
10,000
1000
100
10
1.0
0.1
0.01
q(x, t)
0.0001
0.001
0.01
0.1
1.0
x

FIGURE 6

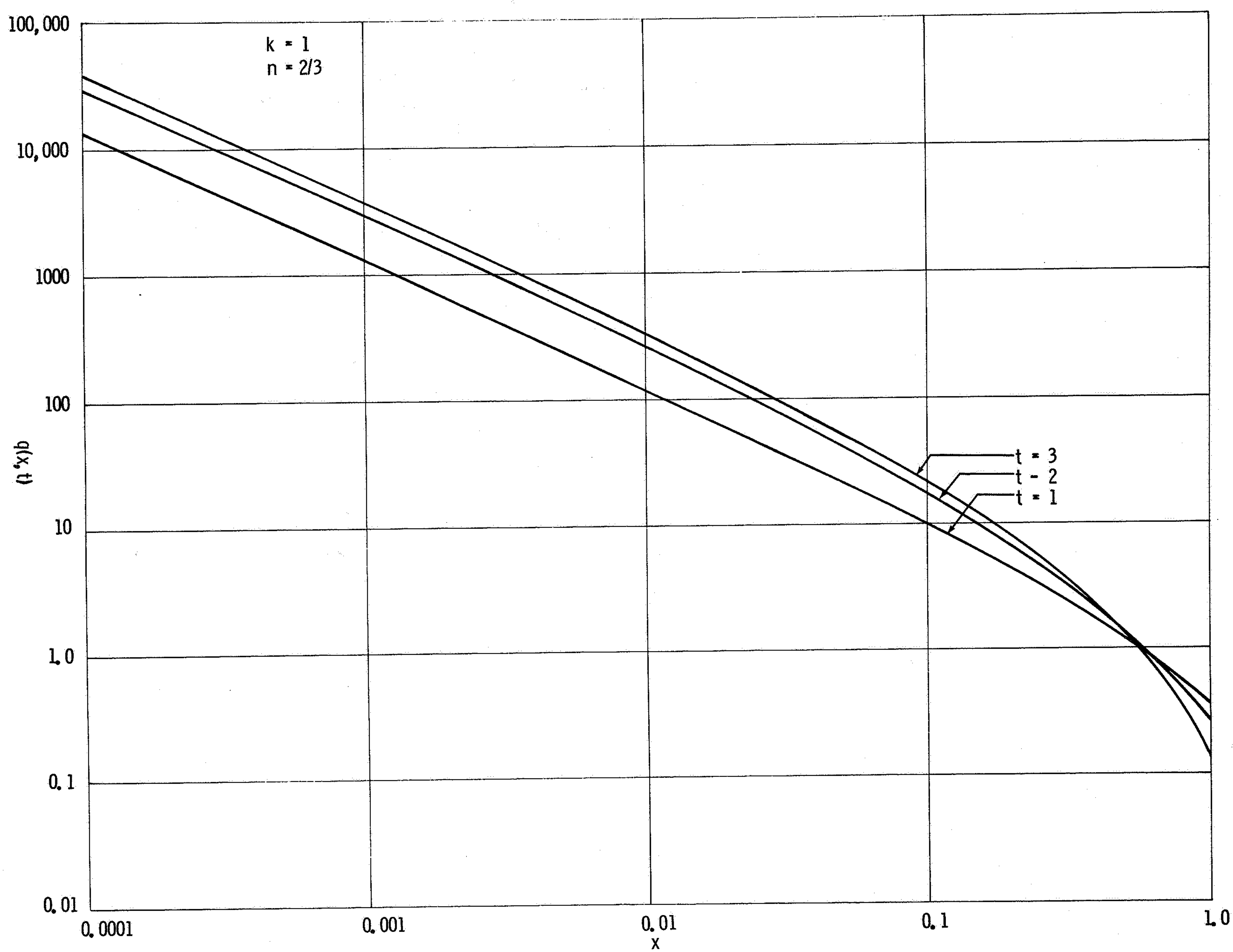
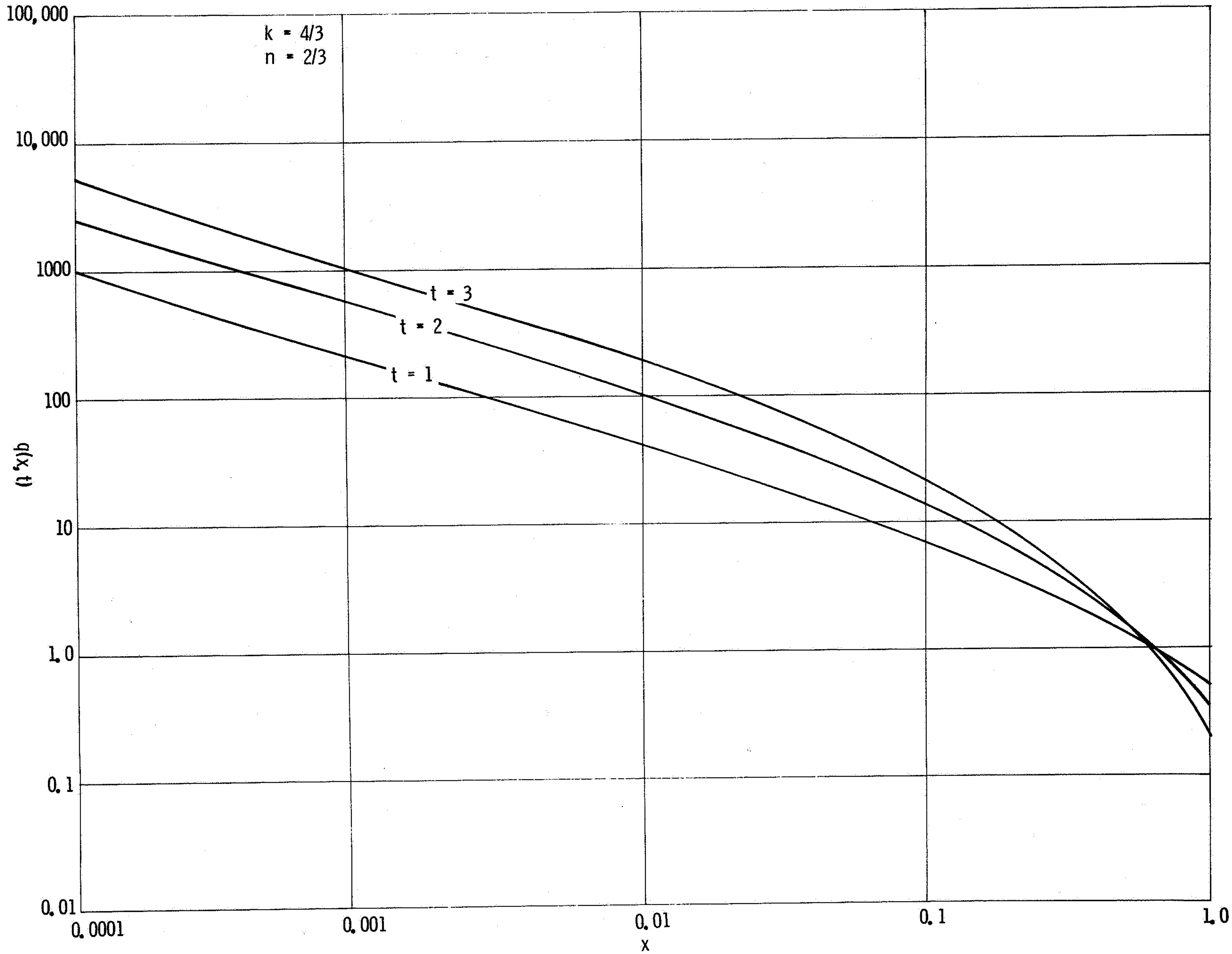

100,000
10,000
1000
100
10
1.0
0.1
0.01
q(x, t)
k = 1
n = 2/3
t = 3
t - 2
t = 1
0.0001
0.001
0.01
0.1
1.0
x

FIGURE 7



k = 4/3
n = 2/3
t = 3
t = 2
t = 1
100,000
10,000
1000
100
10
1.0
0.1
0.01
q(x, t)
0.0001
0.001
0.01
0.1
1.0
x

FIGURE 8

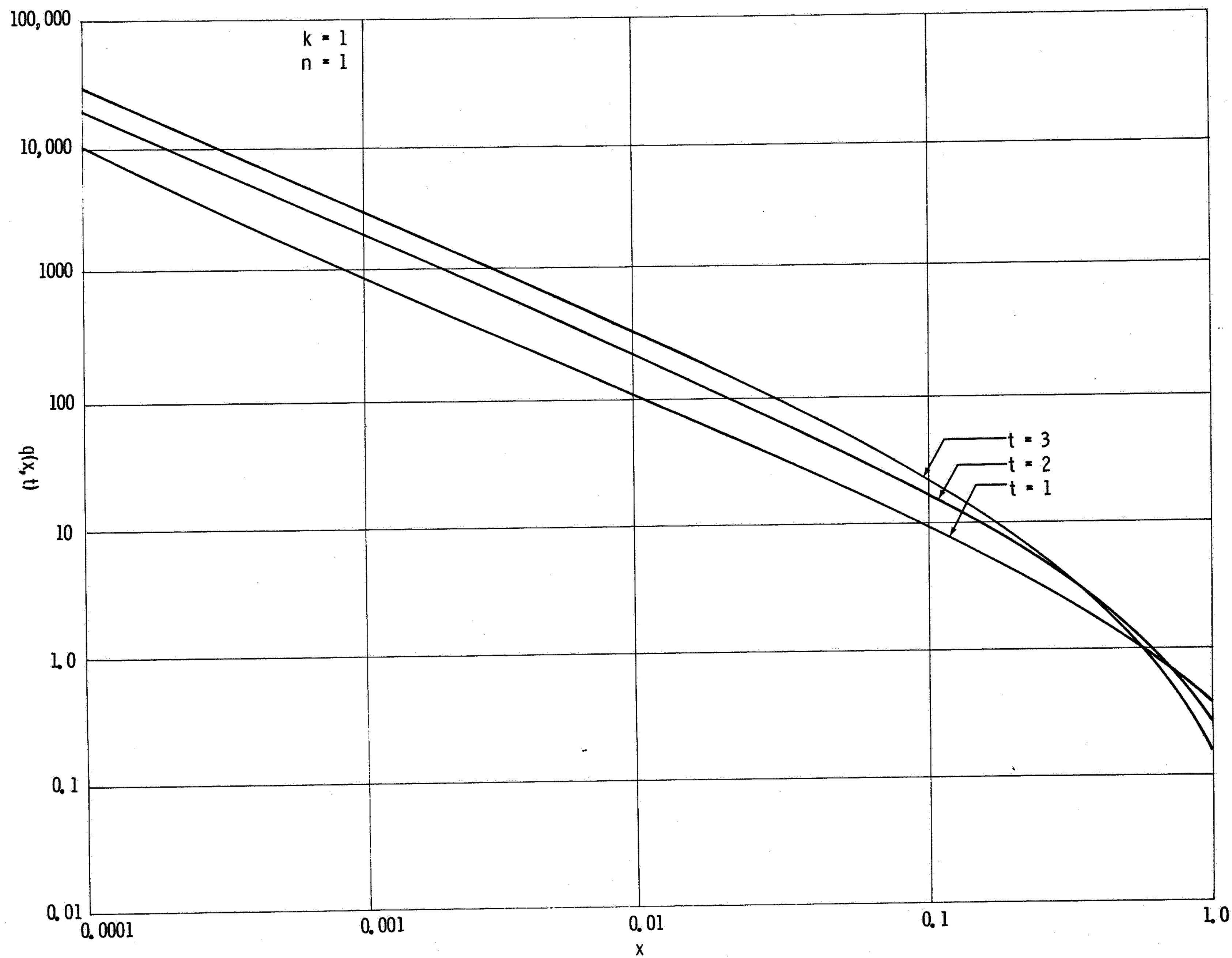
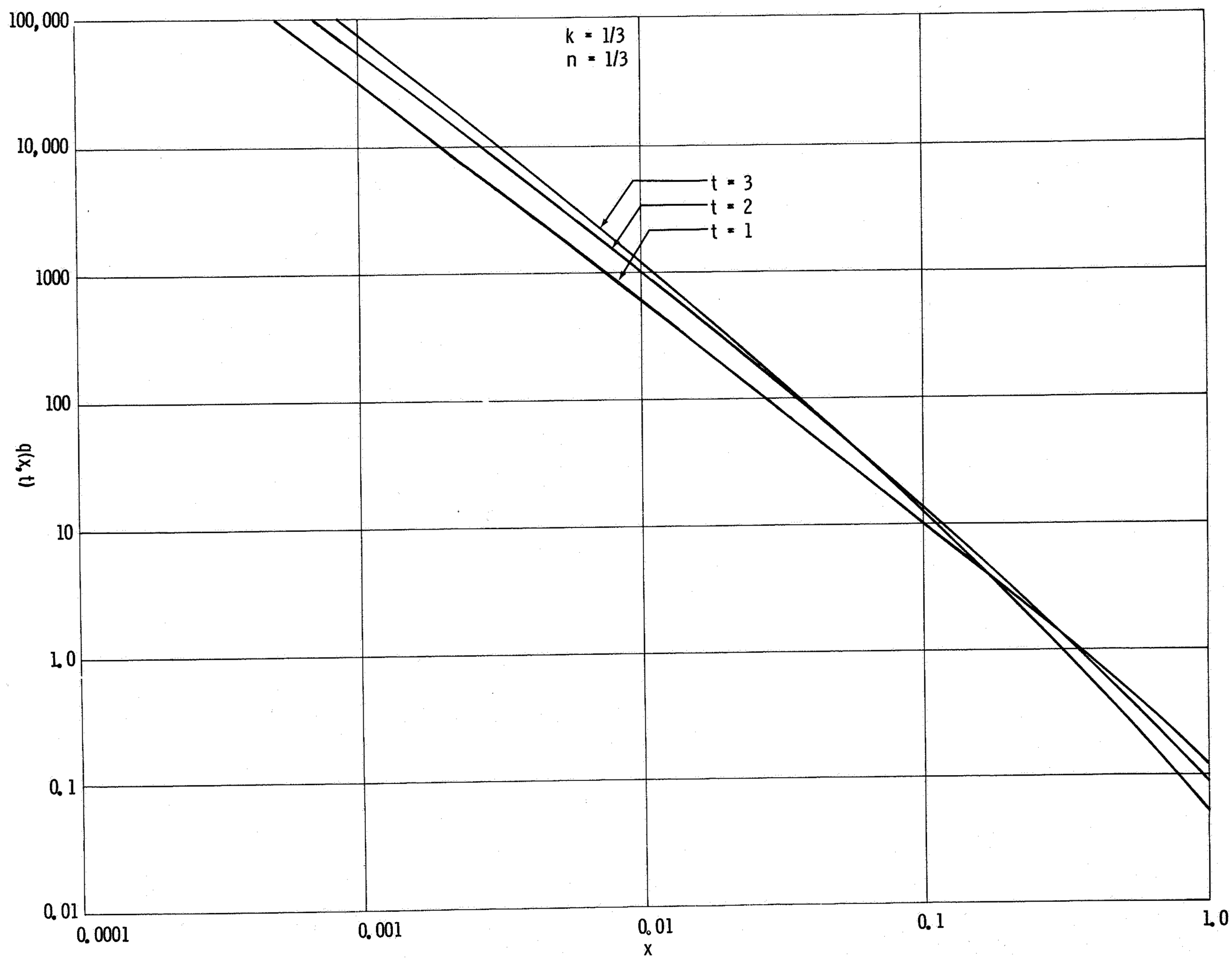

100,000
10,000
1000
100
10
1.0
0.1
0.01
q(x, t)
0.0001
0.001
0.01
0.1
1.0
x
k = 1
n = 1
t = 3
t = 2
t = 1

FIGURE 9



100,000
10,000
1000
100
10
1.0
0.1
0.01
q(x, t)
k = 1/3
n = 1/3
t = 3
t = 2
t = 1
0.0001
0.001
0.01
0.1
1.0
x

FIGURE 10

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