A MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL PROBLEMS WITH FUNCTIONAL DIFFERENTIAL SYSTEMS

by

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In this note we present a maximum principle in integral form for optimal control problems with delay-differential system equations which also contain delays in the control. Recent related results for particular cases of the systems discussed below may be found in [1], [5], and [6]. Vector matrix notation will be used and we shall not distinguish between a vector and its transpose.

Let α_o and t_o be fixed in R^1 with $-\infty < \alpha_o < t_o$, $I = [\alpha_o, a)$ be a bounded interval containing $[\alpha_o, t_o]$, and put $I' = (t_o, a)$. For x continuous on I and t in I', the notation $F(x(\cdot), t)$ will mean that F is a functional in x, depending on any or all of the values $x(\tau)$, $\alpha_o \leqq \tau \leqq t$. $\bar{\Phi}$ will denote the class of absolutely continuous $n - 1$ vector functions on $[\alpha_o, t_o]$. Let Ω be a given convex subset of the class of all bounded Borel measurable functions u defined on I into R^r, and $\mathscr{T}$ be a given C^1 manifold in R^{2n-1}. The problem considered is that of minimizing

$$J[\bar{\varphi}, u, \bar{x}, t_1] = \int_{t_o}^{t_1} f^o(\bar{x}(\cdot), u(\cdot), t)\,dt$$

over $\bar{\Phi} \times \Omega \times C(I, R^{n-1}) \times I'$ subject to

$$\text{(i)}\quad \dot{\bar{x}}(t) = \bar{f}(\bar{x}(\cdot),u(\cdot),t) \qquad \text{a.e. on } [t_o,t_1]$$

$$\bar{x}(t) = \bar{\varphi}(t) \qquad \text{on } [\alpha_o,t_o]$$

$$\text{(ii)}\quad (\bar{x}(t_o),\bar{x}(t_1),t_1) \in \mathscr{T}.$$

We assume that $f = (f^o,\bar{f}) = (f^o,f^1,\ldots,f^{n-1})$ is an n-vector functional of the form

$$f^i(\bar{x}(\cdot),u(\cdot),t) = h^i(\bar{x}(\cdot),t) + \int_{\alpha_o}^{t} u(s)d_s\eta(t,s)g^i(\bar{x}(s),t) \tag{1}$$

$$\text{for } i = 0,1,\ldots,n-1,$$

where the integral is a Lebesgue-Stieltjes integral. Each $h^i(\bar{x}(\cdot),t)$ is assumed C^1 in $\bar{x}$ and measurable in t, and each $g^i(\bar{y},t)$ is C^1 in $(\bar{y},t)$ on R^n. The $r \times 1$ vector function $\eta(t,s)$ is measurable in t,s, and of bounded variation in s on $[\alpha_o,t]$. It is also assumed that the variation of η is dominated by an $L_1(I')$ function m. That is, $\bigvee_{s=\alpha_o}^{t}\eta(t,s) \leq m(t)$ for $t \in I'$. Finally, suppose that given $\bar{X}$ compact, $\bar{X} \subset R^{n-1}$, there exists an $\tilde{m}$ in $L_1(I')$ such that $h = (h^o,h^1,\ldots,h^{n-1})$ satisfies

$$|h(\bar{x}(\cdot),t)| \leq \tilde{m}(t)$$

and

$$|dh[\bar{x}(\cdot),t;\ \bar{\Psi}]| \leq \tilde{m}(t)\|\bar{\Psi}\|_t$$

for any $\bar{\Psi} \in C(I,R^{n-1})$ and $\bar{x} \in C(I,\bar{X})$, where $\|\bar{\Psi}\|_t =$ $\sup\{|\bar{\Psi}(s)|: s \in [\alpha_o,t]\}$ and dh is the Fréchet derivative of h with respect to $\bar{x}$. ($|A|$ denotes the Euclidean norm of A.)

If $(\bar{\varphi}^*,u^*,\bar{x}^*,t_1^*)$ is a solution of the above problem, we define the $n \times n - 1$ matrix function $\bar{\eta}^*$ for $t \in I'$, $s \in [\alpha_o,t]$ by

$$\bar{\eta}^*(t,s) = \bar{\eta}_1^*(t,s) + \bar{\eta}_2^*(t,s), \tag{2}$$

where $\bar{\eta}_1^*$ is such that

$$dh[\bar{x}^*(\cdot),t;\ \bar{\Psi}] = \int_{\alpha_o}^{t} d_s\bar{\eta}_1^*(t,s)\bar{\Psi}(s)$$

for $t \in I'$ and all $\bar{\Psi} \in C([\alpha_o,t],R^{n-1})$, and

$$\bar{\eta}_2^*(t,s) \equiv \begin{cases} -\int_s^t u^*(\beta)d_\beta\eta(t,\beta)g_{\bar{y}}(\bar{x}^*(\beta),t) & s < t \\ 0 & s \geq t. \end{cases}$$

(The existence of $\bar{\eta}_1^*$ is guaranteed by the Riesz theorem.) Then we have the following necessary conditions:

Theorem: Let $(\overline{\varphi}^*, u^*, \overline{x}^*, t_1^*)$ be a solution to the problem under the assumptions above. In addition, suppose that t_1^* is a Lebesgue point of $f(\overline{x}^*(\cdot), u^*(\cdot), t)$. Then there exists a non-trivial n-vector function $\lambda(t) = (\lambda^o(t), \overline{\lambda}(t))$ of bounded variation on $[t_o, t_1^*]$, continuous at t_1^*, satisfying:

(a) $\lambda^o(t) = \text{constant} \leq 0,\ \lambda(t_1^*) \not\equiv 0$

$$\overline{\lambda}(t) + \int_t^{t_1^*} \lambda(\beta)\,\overline{\eta}^*(\beta, t)\,d\beta = \overline{\lambda}(t_1^*) \quad \text{for} \quad t \in [t_o, t_1^*)$$

where $\overline{\eta}^*$ is defined by (2).

(b) $$\int_{t_o}^{t_1^*} \lambda(t) f(\overline{x}^*(\cdot), u^*(\cdot), t)\,dt \geq \int_{t_o}^{t_1^*} \lambda(t) f(\overline{x}^*(\cdot), u(\cdot), t)\,dt$$

for all $u \in \Omega$.

(c) The 2n-1 vector

$$\left(-\overline{\lambda}(t_o) + \int_{t_o}^{t_1^*} \lambda(\beta)\{\overline{\eta}^*(\beta, \alpha_o) - \overline{\eta}^*(\beta, t_o)\}\,d\beta,\ \overline{\lambda}(t_1^*),\ -\lambda(t_1^*)\cdot f^*(t_1^*)\right)$$

is orthogonal to $\mathscr{T}$ at $(\overline{x}^*(t_o), \overline{x}^*(t_1^*), t_1^*)$, where $f^*(t_1^*) \equiv f(\overline{x}^*(\cdot), u^*(\cdot), t_1^*)$.

The proof of this theorem involves showing that the class of functions $\mathscr{F} = \{F(\overline{x}(\cdot), t) : F(\overline{x}(\cdot), t) = f(\overline{x}(\cdot), u(\cdot), t),\ u \in \Omega\}$

is absolutely quasiconvex [2] and then using necessary conditions for extremals given in [2]. Absolute quasiconvexity is a generalization of ideas due to Gamkrelidze [4], who first obtained an integral maximum principle for control problems with ordinary differential system equations. The inequality in (b) is a maximum principle in integral form for the above described optimal control problem.

In many particular cases of the systems defined by (1), one can show that the multipliers $\overline{\lambda}$ are actually absolutely continuous and satisfy (a) in differentiated form. This differentiated form becomes the usual known multiplier equation for systems with simple time lags in the state variables (see [1]). The transversality conditions given in (c) also can be reduced to a simpler form for many special cases of (1).

Included in (1) are many integro-differential systems and time lag systems which appear in physical problems. For example, if one modifies slightly the biological population model formulated by Cooke in [3], one obtains the system equation

$$\dot{x}(t) = u(t-\tau)x(t-\tau) + \beta(t)u(t-\tau-\theta(t))x(t-\tau-\theta(t))$$

where $x(t)$ is the number in the population at time t, $u(t)$ is the birth rate at time t, and τ is the gestation period. Systems with

$$h(\overline{x}(\cdot),t) = \int_{\alpha_o}^{t} A(t,s)q(\overline{x}(s),t)ds,$$

which arise in the study of reactor dynamics [7], and with

$$h(\overline{x}(\cdot),t) = G(\overline{x}_t,t),$$

where $\overline{x}_t = \overline{x}(t+\theta)$, $\theta \in [-T,0]$, are also special cases of (1).

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