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Contents

Figures

Abstract

Because of its direct relation to the reliability or the safe performance of mechanical and structural systems subjected to random external disturbances, the bounding technique of the first excursion probability is studied. In particular, the lower bound of the probability proposed previously by one of the present authors is improved. Numerical examples indicate that the improvement is significant.

The present method of improvement requires the knowledge of the joint density function of the random process at two arbitrary instants. Other than this, the method is universal; it can apply to stationary or nonstationary, Gaussian or non-Gaussian processes.

General expressions for lower and upper bounds are also derived in this study and their potential usefulness is pointed out.

On the Bound of First Excursion Probability

I. Introduction

In recent years, the first excursion probability or the first passage time problem in the area of random vibration attracted considerable attention among engineers (Refs. 1–8) because of its direct relation to the reliability or the safe performance of mechanical and structural systems subjected to random external disturbances.

In some of the previous studies, the response $X(t)$ of a single-degree-of-freedom system consisting of a mass connected to a Kelvin or Voigt model is considered under a white noise input $n(t)$:

$$\ddot{X}(t) + 2\zeta\,\omega_o\,\dot{X}(t) + \omega_o^2\,X(t) = n(t) \tag{1}$$

It is well known in this case that the response process and its time derivative are jointly Markovian and, hence, the Kolmogorov equation (for example, in the form of the Fokker-Planck equation) for the desired transition probability can be derived. Although the Kolmogorov equation undoubtedly provides certain useful information, the approach suffers from a rather serious limitation simply because the solution of the Kolmogorov equation associated with Eq. (1) under the initial and boundary conditions pertinent to the present first passage time problem is difficult to obtain and, in fact, has not been found as yet. Another limitation due to the restriction that the input be white is not a serious one, since the system in Eq. (1) is usually assumed to possess narrow-band characteristics.

These limitations suggest a need for alternative methods by which additional information and, hopefully, results of practical importance can be obtained, not only for the (stationary) white noise input, but also for more general cases involving nonstationary and nonwhite inputs.

In this respect, a bounding technique is presented in Ref. 6, where upper and lower bounds are derived for the (first excursion) probability $P_X\,(t_o;\ -\alpha,\ \beta)$ that the response process $X(t)$ (stationary or nonstationary) will exit at least once from an interval $[-\alpha,\ \beta]$ during a specified period of time $T \equiv [0,\ t_o)$, with $\alpha > 0$ and $\beta > 0$ (two-sided constant barrier problem). A possible use of this method in a practical problem is recently demonstrated in Ref. 7.

This bounding technique is further advanced in Ref. 8 so that the case can be dealt with, where at least one of the threshold values α and β is time dependent; $\alpha = \alpha(t) > 0$ and/or $\beta = \beta(t)$ (two-sided time-dependent barrier problem). In the following discussion, the first excursion probability is written as $P(T_0; -\alpha, \beta)$ for simplicity, whether α and β are functions of time as long as no confusion arises.

The main purpose of this report is to present (in Sections II and III) a method to improve the lower bound of the first excursion probability proposed previously in Refs. 6 and 8. This improvement can apply to the problem with either constant or time-dependent barriers. Section IV of this report is devoted to derivation of some general expressions for upper and lower bounds to suggest possibilities of further improvement on the bounds.

II. Improvement on Lower Bounds

The following technique of improving the lower bound is based on the methods by P. Whittle (Ref. 9) and by S. Gallot (Ref. 10). Consider the probability that the maximum of a set of random variables $X_1, X_2, \cdots, X_n$ is less than β when

$$P_j = P(X_j \geqq \beta) \tag{2}$$

and

$$P_{jk} = P(X_j \geqq \beta, X_k \geqq \beta) \tag{3}$$

are given, where: $P\,(E)$ is the probability of event E, and $P\,(E, F)$ is the probability of the joint occurrence of events E and F.

Let $\mathbf{P}$ and $\boldsymbol{\pi}$ denote the column vector of P_j and the matrix P_{ij}, respectively. It can then be shown that the lower bounds for $P\,(\max_j X_j \geqq \beta)$ are given as follows:

$$P\,(\max_j X_j \geqq \beta) > \mathbf{P}^T \boldsymbol{\pi}^{-1} \mathbf{P} \geqq L \tag{4}$$

where $\mathbf{P}^T$ indicates the transpose of $\mathbf{P}$, $\boldsymbol{\pi}^{-1}$ the inverse matrix of $\boldsymbol{\pi}$, and

$$L = \left(\sum_{j=1}^{n} P_j\right)^2 \Big/ \left(\sum_{j=1}^{n} \sum_{k=1}^{n} P_{jk}\right) \tag{5}$$

The expression $\mathbf{P}^T \boldsymbol{\pi}^{-1} \mathbf{P}$ is the best lower bound within this approach.

Since, however, the direct application of Eq. (4) is limited to the case of the one-sided constant barrier problem, the definitions of P_j and P_{jk} are modified here so that the two-sided time-dependent first excursion probability can also be dealt with.

Now, let E_j denote the event $(X_j < -\alpha_j)\; U\; (X_j > \beta_j)$ with $\alpha_j > 0$ and $\beta_j > 0$, where EUF denotes the occurrence of at least one of the events E and F; hence, E_j is the event that X_j takes a value outside the interval $[-\alpha_j, \beta_j]$. Define

$$P_j = P(E_j) \tag{6}$$

$$P_{jk} = P(E_j, E_k) \tag{7}$$

Then, in the same way in which Eq. (4) is derived (see Appendix), the probability

$$P\left(\overset{n}{\underset{j=1}{U}} E_j\right)$$

that at least one of $X_1, X_2, \cdots, X_n$ will take a value outside its specified interval can be shown to have lower bounds $\mathbf{P}^T \boldsymbol{\pi}^{-1} \mathbf{P}$ and L:

$$P\left(\overset{n}{\underset{j=1}{U}} E_j\right) > \mathbf{P}^T \boldsymbol{\pi}^{-1} \mathbf{P} \geqq L \tag{8}$$

Equation (8) has the desired form that can be applied directly to the two-sided constant and/or time-dependent barrier problem as described below.

If the first excursion probability is considered for random process $X(t)$ with a set of barriers $-\alpha(t)$ and $\beta(t)$ and if $X_j = X(t_j)$, $\alpha_j = \alpha(t_j)$, and $\beta_j = \beta(t_j)$ with $t_j \,\epsilon\, T$, then it follows that

$$P\,(t_o; -\alpha, \beta) > P\left(\overset{n}{\underset{j=1}{U}} E_j\right) \tag{9}$$

and, hence, by virtue of Eq. (8):

$$P\,(t_o; -\alpha, \beta) > \mathbf{P}^T \boldsymbol{\pi}^{-1} \mathbf{P} \geqq L \tag{10}$$

where, obviously $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ and $\beta_1 = \beta_2 = \cdots = \beta_n = \beta$, if α and β are not time dependent.

In the previous studies (Refs. 6 and 8), a lower bound for $P\,(t_o; -\alpha, \beta)$ based on a single time-point $t_j \,\epsilon\, T$ was obtained as $P\,(E_j)$ (henceforth referred to as "the lower bound based on a single point"). Evidently, $P\,(E_j)$ is, in general, a function of t_j; therefore, the best lower bound based on a single point is the maximum possible value $P\,(E^*)$ of $P\,(E_j)$ that occurs at, say, $t_j = t^* \,\epsilon\, T$.

It is shown in the Appendix that the lower bound $\mathbf{P}^T \pi^{-1} \mathbf{P}$ given in Eq. (10) is superior to $P(E^*)$ if one of $t_1, t_2, \cdots, t_n$ is chosen to be t^*.

The knowledge of the joint density function $f_{X_j X_k}(x, y)$ of X_j and X_k is sufficient to compute $\mathbf{P}^T \pi^{-1} \mathbf{P}$ or L, since

$$P_j = \int_{-\infty}^{-\alpha_j} f_{X_j}(x)\,dx + \int_{\beta_j}^{\infty} f_{X_j}(x)\,dx \tag{11}$$

$$\begin{aligned} P_{jk} = P_{kj} &= \int_{-\infty}^{-\alpha_j}\int_{-\infty}^{-\alpha_k} f_{X_j X_k}(x,y)\,dx\,dy \\ &+ \int_{\beta_j}^{\infty}\int_{\beta_k}^{\infty} f_{X_j X_k}(x,y)\,dx\,dy \\ &+ \int_{-\infty}^{-\alpha_j}\int_{\beta_k}^{\infty} f_{X_j X_k}(x,y)\,dx\,dy \\ &+ \int_{\beta_j}^{\infty}\int_{-\infty}^{-\alpha_k} f_{X_j X_k}(x,y)\,dx\,dy \qquad (j \neq j) \end{aligned} \tag{12a}$$

and

$$P_{jk} = P_j \qquad (j = k) \tag{12b}$$

where $f_{X_j}(x)$ is the density function of X_j, which can be obtained as the marginal density of X_j from $f_{X_j X_k}(x,y)$.

It is important to note that the computation of $\mathbf{P}^T \pi^{-1} \mathbf{P}$ or L involves the joint density functions only up to the second order, although any number of time points $t_1, t_2, \cdots, t_n$ can be considered. This fact makes the numerical computation for the improved bound manageable, since the evaluation of $\mathbf{P}^T \pi^{-1} \mathbf{P}$ or L on a computer presents no difficulty.

Some simplications can be achieved if one considers a stationary process $X(t)$ with constant barriers. Evidently, in this case, $P_1 = P_2 = \cdots = P_n$. If, furthermore, one chooses equally spaced time instants $t_1, t_2, \cdots, t_n$, then

$$P_{jk} = P_{j\pm l,\ k\pm l} \tag{13}$$

implying that those members of π parallel to the diagonal are identical. Recalling the fact that π is symmetric and that the diagonal members of π are $P_{kk} = P_k$, one can conclude from Eq. (13) that it is necessary to evaluate only $n-1$ joint probabilities given in Eq. (12a) to construct the matrix π numerically.

The probability P_j can be found from the table of normal distribution, whereas, P_{jk} from (Ref. 11). It is pointed out, however, that if the threshold values α and β are large compared with the standard deviation, an asymptotic expression for P_{jk} is analytically available (Refs. 11 and 12) as well as the well-known asymptotic expression for P_j. Such asymptotic forms can conveniently be used when the numerical computations are to be done in the computer.

III. Numerical Examples

A. Example 1: Stationary Process With Constant Barriers

Consider Eq. (1) with $\omega_o = 2$ rad/s and $\zeta = 0.02$. Let $\alpha = \beta = 3\sigma_X$, where $\sigma_X^2 = \pi S_o/(2\zeta\,\omega_o^3)$ with S_o being the mean square spectral density of $n(t)$.

Since, in this case, $X(t)$ is Gaussian, the Gaussian joint density function is used for $F_{X_j X_k}(x,y)$ with the correlation coefficient ρ_{jk}

$$\rho_{jk} = e^{-\zeta\omega_o\tau}\left[\cos(1-\zeta^2)^{1/2}\omega_o\tau + \frac{\zeta}{(1-\zeta^2)^{1/2}}\sin(1-\zeta^2)^{1/2}\omega_o\tau\right] \tag{14}$$

where

$$\tau = |t_j - t_k|$$

The lower bounds for $t_o = 1.0$, 10, 100, and 1000 s are evaluated by dividing the interval $[0, t_o)$ into 10 (curve I of Fig. 1), 100 (curve II), and 1000 (curve III) equal subintervals ($n = 11$, 101, or 1001) in each case. For example, when $t_o = 10$ s and $n = 11$, $t_1 = 0$ s, $t_2 = 1$ s, $\cdots$, $t_{11} = 10$ s.

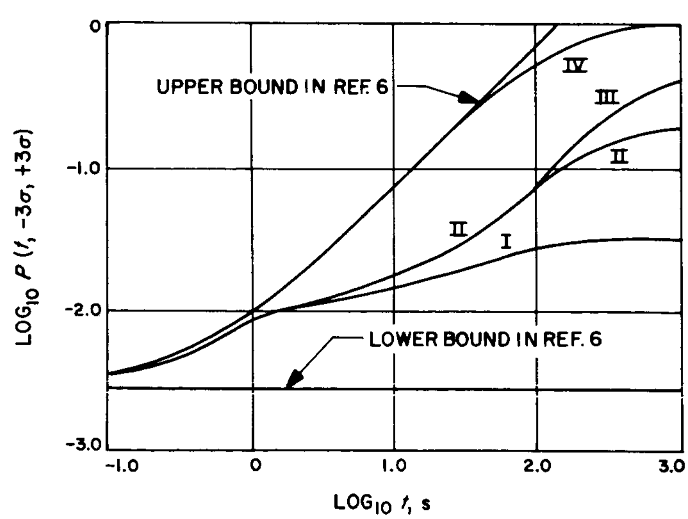

Fig. 1. Improved lower bound (stationary response)

The result of computations is shown in Fig. 1, where the upper and lower bounds due to the previous method (Ref. 6) are also given. It is evident from the diagram that the significant improvement on the lower bound is achieved as the number (n) of the time points to be considered is increased from 1 (the lower bound based on a single time-point) to 10 (curve I), 100 (curve II), and further to 1000 (curve III), although the improvement with 1000 intervals over that with 100 intervals is observable only after $t = 100$ s.

It should be mentioned that curve III ($n = 1000$) is obtained using Eq. (5) only. However, it is pointed out that the differences observed in the numerical values of $\mathbf{P}^T \pi^{-1} \mathbf{P}$ and L for $n = 10$ and 100 (curves I and II) are insignificant. Hence, one will not lose much by using L for lower bound instead of $\mathbf{P}^T \pi^{-1} \mathbf{P}$, which requires an inversion of π of order n. In the present example, where the process is stationary and the barriers are time independent, P_j and P_{jk} can be found on a computer using a subroutine for interpolation together with Ref. 11, and, hence, a large number of intervals, even as many as 1000, can be handled without any difficulty.

In Fig. 1, $1-\exp(-2\nu_\alpha T)$, an approximation sometimes used for $P(t_o; -\alpha,\alpha)$ (Ref. 13) is also plotted as curve IV where ν_α is the rate of crossing of α by $X(t)$ with positive slope.

B. Example 2: Nonstationary Process With Constant Barriers

Consider Eq. (1) subjected to an input of $f(t)$ instead of $n(t)$:

$$f(t) = \int_{-\infty}^{\infty} h_o(t-\tau)\,\psi(\tau)\,n(\tau)\,d\tau \tag{15}$$

where

$$h_o(t) = \left(1 + \frac{\mu_b^2}{\omega_b^2}\right)^{1/2} e^{-\mu_b t} \cos(\omega_b t + \delta)\,H(t) \tag{16}$$

$$\psi(t) = (e^{-at} - e^{-bt})\,H(t) \qquad (b > a > 0) \tag{17}$$

with $H(t)$ being the unit step function and

$$\delta = \tan^{-1} \frac{\mu_b}{\omega_b} \tag{18}$$

The response $X(t)$ to $f(t)$ is then

$$X(t) = \int_{-\infty}^{\infty} h_1(t-\tau)\,f(\tau)\,d\tau \tag{19}$$

where

$$h_1(t) = e^{-\zeta\omega_o t} \sin \omega_1 t \; H(t)/\omega_1 \tag{20}$$

with

$$\omega_1 = \omega_o (1 - \zeta^2)^{1/2} \tag{21}$$

The auto-covariance function of this process is

$$E[X(t)\,X(s)] = 2\pi S_o \int_{-\infty}^{\infty} \psi^2(v)\,h(t-v)\,h(s-v)\,dv \tag{22}$$

or

$$E[X(t)\,X(s)] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J_1(\omega_1, \omega_2)\, e^{i(\omega_1 t - \omega_2 s)}\, d\omega_1\, d\omega_2 \tag{23}$$

with

$$h(t-\tau) = H(t-\tau) \int_{-\infty}^{-\infty} h_1(t-v)\,h_o(v-\tau)\,dv \tag{24}$$

and

$$J_1(\omega_1, \omega_2) = 2\pi S_o H_o(\omega_1)\,\overline{H}_o(\omega_2)\,H_1(\omega_1)\,\overline{H}_1(\omega_2)\,\overline{\Psi}(\omega_1 - \omega_2) \tag{25}$$

where $E(\cdot)$ indicates the expected value; $H_o(\omega)$, $H_1(\omega)$ and $\overline{\Psi}(\omega)$ are the Fourier transform of $h_0(t)$, $h_1(t)$, and $\psi^2(t)$, respectively; and $\overline{H}_0(\omega)$ and $\overline{H}_1(\omega)$ denote the complex conjugates of $H_0(\omega)$ and $H_1(\omega)$, respectively.

$$H_o(\omega) = i\omega \frac{(\omega_b^2 + \mu_b^2 - \omega^2) - 2 i \mu_b \omega}{(\omega_b^2 + \mu_b^2 - \omega^2)^2 + 4 \mu_b^2 \omega^2} \tag{26}$$

$$H_1(\omega) = \frac{(\omega_o^2 - \omega^2) - 2 i \zeta \omega_o \omega}{(\omega_o^2 - \omega^2)^2 + 4 \zeta^2 \omega_o^2 \omega^2} \tag{27}$$

$$\overline{\Psi}(\omega) = \frac{1}{2a + i\omega} - \frac{2}{a + b + i\omega} + \frac{1}{2b + i\omega} \tag{28}$$

The discussion on the significance and possible applications of the nonstationary process $X(t)$, defined in

Eq. (19), was given in detail in Ref. 14; therefore, it is not repeated here, except for pointing out that $X(t)$ can be interpreted as the output of the filter with the impulse response function $h(t)$ to a nonstationary white noise $\psi(t)\ n(t)$, or equivalently as the resulting process when $\psi(t)\ n(t)$ is passed through two filters with the impulse response function $h_0(t)$ and $h_1(t)$.

In Ref. 14, the first excursion probability $P(\infty; -\alpha,\alpha)$ for this process was estimated as a function of α with the aid of the Monte Carlo technique; also, its upper and lower bounds were obtained using the method proposed previously (see Fig. 2). The use of this nonstationary process $X(t)$ is made here so that one can demonstrate, on the basis of numerical comparison, the significant improvement that the present method can produce.

Numerical values for the parameters involved to obtain Fig. 2 are as follows: $a = 0.25/\text{s}$, $b = 0.5/\text{s}$, $\omega_b = 12.3/\text{s}$, $\mu_b = 3.86/\text{s}$, $\omega_o = 31.4/\text{s}$, and $\zeta = 0.05$ and $S_o = 7.19 \times 10^4$ in.^{2}/s^5. The lower bounds $\mathbf{P}^T\ \pi^{-1}\ \mathbf{P}$ for $P(-\infty; -\alpha,\alpha)$ are evaluated for $\alpha = 0.1$, 0.2, and 0.3 in. with the following equally spaced time points:

(1) For $\alpha = 0.1$ in., $t_1 = 0.6$ s, $t_2 = 1.2$ s, $\cdots$, $t_{15} = 9.0$ s.

(2) For $\alpha = 0.2$ in., $t_1 = 1.2$ s, $t_2 = 1.8$ s, $\cdots$, $t_{11} = 7.2$ s.

(3) For $\alpha = 0.3$ in., $t_1 = 2.1$ s, $t_2 = 2.4$ s, $\cdots$, $t_{10} = 4.8$ s.

These points are chosen arbitrarily, except for assuring that one of the points coincides with $t^* = 3.0$ s at which the standard deviation takes a maximum value (see Fig. 1 of Ref. 14).

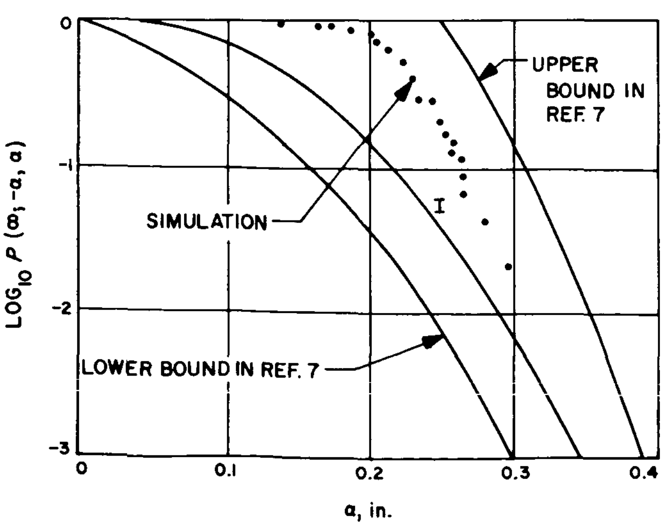

Fig. 2. Improved lower bound (nonstationary response)

To evaluate P_{jk}, the auto-covariance function of $X(t)$ has to be known first. For this purpose, Eq. (23) is used, because an efficient numerical method of double Fourier inversion has been developed (Ref. 15) and the computer program is available. Numerical results indicate again that the difference between $\mathbf{P}^T\ \pi^{-1}\ \mathbf{P}$ and L is insignificant.

The result is shown as curve I in Fig. 2. Here again, the improvement of lower bound is significant.

IV. General Expressions for Bounds

A. Lower Bounds

Consider a set of time points $t_1, t_2, \cdots, t_n \in T$. It can be shown that

$$P(t; -\alpha,\beta) > P(E_1) + P(E_1^c, E_2) + P(E_1^c, E_2^c E_3) + \cdots + P(E_1^c, E_2^c, \cdots, E_{n-1}^c, E_n) \tag{29}$$

where E_i^c denotes the complement of E_i and $P(E_1^c, E_2^c, \cdots, E_{n-1}^c, E_n)$ denotes the probability of the joint occurrence of $E_1^c, E_2^c, \cdots, E_{n-1}^c$, and E_n. Hence, the right-hand side of Eq. (29) is a lower bound of the first excursion probability involving a set of time points $t_1, t_2, \cdots, t_n$.

If $f_{X_1X_2\cdots X_k}(x_1, x_2, \cdots, x_k)$ denotes the joint density function of $X_1, X_2, \cdots, X_k$, then

$$P(E_1) = \int_{-\infty}^{-\alpha_1} f_{X_1}(x_1)\,dx_1 + \int_{\beta_1}^{\infty} f_{X_1}(x_1)\,dx_1 \tag{30}$$

$$P(E_1^c, E_2) = \int_{-\infty}^{-\alpha_2}\int_{-\alpha_1}^{\beta_1} f_{X_1X_2}(x_1, x_2)\,dx_1\,dx_2 + \int_{\beta_2}^{\infty}\int_{-\alpha_1}^{\beta_1} f_{X_1X_2}(x_1, x_2)\,dx_1\,dx_2 \tag{31}$$

$$P(E_1^c, E_2^c, E_3) = \int_{-\infty}^{-\alpha_3}\int_{-\alpha_2}^{\beta_2}\int_{-\alpha_1}^{\beta_1} f_{X_1X_2X_3}(x_1, x_2, x_3)\,dx_1\,dx_2\,dx_3 + \int_{\beta_3}^{\infty}\int_{-\alpha_2}^{\beta_2}\int_{-\alpha_1}^{\beta_1} f_{X_1X_2X_3}(x_1, x_2, x_3)\,dx_1\,dx_2\,dx_3 \tag{32}$$

and so forth.

The lower bound developed in Ref. 6 is a special case of the general expression in Eq. (29) with $n = 1$ and

$t_1 = t^*$. The difficulties are quite evident here. Even if the process considered is Gaussian, the probabilities on the right-hand side of Eq. (29) are not given in closed form; hence, they have to be evaluated numerically (except for the first two, for which numerical tables are available).

In general, these joint probabilities cannot be expressed in closed form while numerical work needed to evaluate the third term (in the right-hand side of Eq. 29) on will become significantly heavy, if the accuracy is demanded. However, the estimation of the order of magnitude for these joint probabilities, which is often quite useful, does not seem extremely difficult. In this connection, note that the domain of integration is rectangular and that disregarding a certain part of the domain of integration still produces a lower bound. Evidently, use of the Monte Carlo technique may be possible. However, it should be pointed out that simulations of an arbitrary, particularly non-Gaussian, random process is not always possible and that the number of member functions to be simulated becomes prohibitively large as the values of α and β increase much beyond the standard deviation of the process.

The advantage of the lower bound developed in Section II is now clear; it considers a set of time points $t_1, t_2, \cdots, t_n \in T$ without involving the integration of joint density functions of more than two events.

B. Upper Bounds

The upper bound developed previously is essentially based on the following inequality:

$$P(A, D) < P(C, D) \tag{33}$$

where A is the event that $X(t)$ is confined within the barriers with $t \in T$, C is the event that $-\alpha_o \equiv -\alpha(t_o) < X(t_o) < \beta(t_o) \equiv \beta_o$, and D is the event that $X(t)$ will cross the barriers in $(t_o, t_o + dt)$.

For the same reasoning for which Eq. (33) is obtained, one can derive:

$$P(A, D) < P(E_1^c, E_2^c, \cdots, E_n^c, C, D) < P(C, D) \tag{34}$$

Equation (34) indicates that $P(E_1^c, E_2^c, \cdots, E_n^c, C, D)$ is an improved upper bound of $P(A, D)$ over $P(C, D)$.

With a slight modification, the technique used in Ref. 8 leads to

$$P(E_1^c, E_2^c, \cdots, E_n^c, C, D) = \nu_n(t_o)\, dt$$
$$= dt \left[\underbrace{\int_{-\alpha_n}^{\beta_n} \cdots \int_{-\alpha_1}^{\beta_1}}_{n\text{-fold}} k(x_1, x_2, \cdots, x_n)\, dx_1 \cdots dx_n \right] \tag{35}$$

with

$$k(x_1, \cdots, x_n) = \int_{\dot{\beta}_o}^{\infty} (\dot{x}_o - \dot{\beta}_o) \times f_{x_1 x_2 \cdots x_n x_o \dot{x}_o}(x_1, x_2 \cdots, x_n, \beta_o, \dot{x}_o)\, d\dot{x}_o + \int_{-\infty}^{-\dot{\alpha}_o} (-\dot{\alpha}_o - \dot{x}_o) \times f_{x_1 x_2 \cdots x_n x_o \dot{x}_o}(x_1, x_2 \cdots, x_n, -\alpha_o, \dot{x}_o)\, dx_o \tag{36}$$

where

$$f_{x_1 x_2 \cdots x_n x_o \dot{x}_o}(x_1, x_2 \cdots, x_n, x_o, \dot{x}_o)$$

is the joint density function of $X(t_1)$, $X(t_2)$, $\cdots$, $X(t_n)$, $X(t_o)$, and $\dot{X}(t_o)$ with the dot indicating the time derivative, $\dot{\alpha}_o = \dot{\alpha}(t_o)$, and $\dot{\beta}_o = \dot{\beta}(t_o)$. Evidently, Eq. (36) and, therefore, Eq. (35) depend not only on t_o, but also on t_1, $t_2, \cdots, t_n$.

Since it can be shown that

$$P(t_o + dt; -\alpha, \beta) = P(t_o; -\alpha, \beta) + P(A, D)$$
$$< P(t_o; -\alpha, \beta) + P(E_1^c, E_2^c, \cdots, E_n^c, C, D) \tag{37}$$

it follows that

$$P(t_o; -\alpha, \beta) < \int_0^{t_o} \nu_n(t)\, dt + P_o \tag{38}$$

where P_o is the probability that $X(t)$ takes a value outside the interval $[-\alpha(0), \beta(0)]$ at $t = 0$; thus $P_o = P\{X(0) < -\alpha(0)\} + P\{X(0) > \beta(0)\}$.

When $n = 0$, $\nu_n(t)$ defined in Eq. (35) is nothing but the expected rate of crossing the barriers at time t, and

Eq. (38) indicates the same upper bound as developed in Ref. 8.

The same difficulty as encountered in the preceding discussion on the lower bound also exists here. A straightforward attempt to integrate Eqs.(35) and (36) indicates that, even dealing with a Gaussian process and even considering only one extra point ($n = 1$), no closed form expression is obtained for $\nu_1(t)$. To be more precise, the expression contains an integral involving the error function.

Nevertheless, the general upper bound derived in Eq. (38) with $\nu_n(t)$ is of considerable value, because it provides a foothold for future study and it permits numerical evaluation of the upper bound for $n = 1$ since then the bound can be obtained by performing a double integration (existence of an efficient subroutine computer program for the error function is justifiably assumed).

It seems worthwhile, therefore, to further pursue a possibility of finding a practical way of utilizing the method discussed above.

V. Conclusion

The lower bound of the first excursion probability proposed previously by one of the present authors has been improved on the basis of the recent work by S. Gallot (Ref. 10). The present method requires the knowledge of the joint density function of the random process at two arbitrary instants. Other than this, the method is universal; it can apply to stationary or nonstationary, Gaussian or non-Gaussian processes.

Numerical examples (Figs. 1 and 2) indicate that the improvement is significant for the work involved and that the difference between the upper and lower bounds has been narrowed considerably, in particular for the example involving the nonstationary process (Fig. 2).