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METHOD OF PARTICULAR SOLUTIONS FOR LINEAR, TWO-POINT BOUNDARY-VALUE PROBLEMS

PART 1 - PRELIMINARY EXAMPLES

by

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Method of Particular Solutions for Linear, Two-Point Boundary-Value Problems

Part 1 - Preliminary Examples[1]

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Abstract. The methods commonly employed for solving linear, two-point boundary value problems require the use of two sets of differential equations: the original set and the derived set. This derived set is the adjoint set if the method of adjoint equations is used, the Green's functions set if the method of Green's functions is used, and the homogeneous set if the method of complementary functions is used.

With particular regard to high-speed digital computing operations, this report explores an alternate method, the method of particular solutions, in which only the original, nonhomogeneous set is used. As a preliminary example, a second-order system is considered, and the boundary-value problem is solved by combining linearly several particular solutions of the original, nonhomogeneous set. Both the case of an uncontrolled system and that of a controlled system are considered.

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1. Introduction

In recent years, considerable attention has been devoted to the solution of the two-point boundary-value problem for linear differential systems. Among the techniques available, we mention (a) the method of adjoint equations (Refs. 1-2), (b) the method of Green's functions (Refs. 3-6), and (c) the method of complementary functions (Refs. 7-10). Other techniques involve the use of series expansions, for instance, Fourier series (Ref. 11) and Chebyshev series (Ref. 12). With reference to (b), the determination of the Green's functions has been the object of several recent papers (see, for example, Refs. 13-15).

Methods (a) through (c) have one common characteristic. Each requires the solution of two differential sets, namely, the original set plus the derived set. This derived set is the adjoint set in Case (a), the Green's functions set in Case (b), and the homogeneous set in Case (c). With particular regard to high-speed digital computing operations, it has occurred to this writer that programming can be made simpler if one employs the original set only.

This technique, a modification of (c), consists of combining linearly several particular solutions of the original, nonhomogeneous set. For this reason, it can be termed the method of particular solutions. It has the following advantages with respect to the previous techniques: (i) it makes use of only one differential system, namely, the original, nonhomogeneous set; (ii) each particular solution can be made to satisfy the same prescribed initial conditions; and (iii) in a physical problem, each particular solution represents a physically possible trajectory, even though it satisfies only the initial conditions and not the final conditions.

For the particular case of a linear differential equation of the second order, the idea of combining particular solutions of the original, nonhomogeneous equation was employed by Fox in Chapter 8 of Ref. 8. However, this idea was abandoned in favor of (c) in order to reduce the number of undetermined constants by one. In the opinion of this writer, this minor advantage is more than offset by considerations (i) through (iii), especially for complex physical systems.

This report is an introduction to the method of particular solutions. Several preliminary examples are treated in terms of two second-order systems, one of the uncontrolled type and one of the controlled type. In a subsequent report, the general theory is presented for a system of any order (Ref. 16).

2. Uncontrolled System

In this section, we consider the following linear, nonhomogeneous system of order two[3]:

$$\begin{aligned} \dot{x} &= ax + by + c \\ \dot{y} &= ex + fy + g \end{aligned} \tag{1}$$

in which t is the independent variable, x and y are the dependent variables, and the dot sign denotes a derivative with respect to t. We assume that the coefficients a, b, c, and e, f, g are time-dependent and continuous. We also assume that the following boundary conditions must be satisfied:

$$x(0) = \alpha \tag{2}$$

$$x(\tau) = \gamma \tag{3}$$

where α, γ, τ are prescribed constants. Then, we formulate the following problem: Find the functions

$$x = x(t), \qquad y = y(t) \tag{4}$$

which satisfy the differential system (1), the initial condition (2), and the final condition (3).

In order to solve this problem, we integrate Eqs. (1) forward twice from $t = 0$ using two different sets of initial conditions and the stopping condition $t = \tau$. In the

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[3] The system (1) can be called uncontrolled in that its trajectory in the txy-space is completely determined once the initial conditions are given.

first integration (subscript 1), we employ the initial conditions

$$x_1(0) = \alpha, \qquad y_1(0) = \beta_1 \tag{5}$$

and obtain the particular solution

$$x_1 = x_1(t), \qquad y_1 = y_1(t) \tag{6}$$

In the second integration (subscript 2), we employ the initial conditions

$$x_2(0) = \alpha, \qquad y_2(0) = \beta_2 \tag{7}$$

and obtain the particular solution

$$x_2 = x_2(t), \qquad y_2 = y_2(t) \tag{8}$$

In each integration, the initial condition for the x-variable is identical with (2); the initial condition for the y-variable is arbitrary and can be changed, if necessary.

Next, we introduce the undetermined constants k_1 and k_2 and form the linear combinations

$$\begin{aligned} x &= k_1x_1 + k_2x_2 \\ y &= k_1y_1 + k_2y_2 \end{aligned} \tag{9}$$

Then, we inquire whether, by an appropriate choice of the constants, these linear combinations can satisfy the differential equations (1), the initial condition (2), and the final condition (3).

By substituting (9) into (1) and rearranging terms, we obtain the relations

$$\begin{aligned} k_1(\dot{x}_1 - ax_1 - by_1) + k_2(\dot{x}_2 - ax_2 - by_2) &= c \\ k_1(\dot{y}_1 - ex_1 - fy_1) + k_2(\dot{y}_2 - ex_2 - fy_2) &= g \end{aligned} \tag{10}$$

Since each pair of functions (6) and (8) is a solution of (1), Eqs. (10) become

$$\begin{aligned} k_1c + k_2c &= c \\ k_1g + k_2g &= g \end{aligned} \tag{11}$$

and are satisfied providing the constants are such that

$$k_1 + k_2 = 1 \tag{12}$$

Substitution of (9-1) into the initial condition (2) leads to the relation

$$k_1x_1(0) + k_2x_2(0) = \alpha \tag{13}$$

In the light of (5-1) and (7-1), Eq. (13) can be rewritten as

$$k_1\alpha + k_2\alpha = \alpha \tag{14}$$

and is satisfied providing the constants are consistent with (12).

Finally, substitution of (9-1) into the final condition (3) leads to the relation

$$k_1x_1(\tau) + k_2x_2(\tau) = \gamma \tag{15}$$

which, together with (12), determines the constants k_1 and k_2. In this way, the proposed problem is solved in principle.

2.1. Remarks. The following comments are pertinent to the previous discussion:

(a) The particular solutions (6) and (8) must be linearly independent. This is precisely the case, since the initial condition (5-2) differs from (7-2).

(b) Because of the arbitrariness of the initial conditions for the particular solutions, it is conceivable that the matrix of the coefficients in Eqs. (12) and (15) may be ill-conditioned. Should this situation arise, corrective steps can be taken by changing (5-2) or (7-2).

(c) Thus far, the continuity of the coefficients a, b, c and e, f, g has been assumed. If this restriction is removed, that is, if the coefficients exhibit a finite number of discontinuities, the previous results are still valid. The only difference is that, in the continuous case, $\dot{x}$ and $\dot{y}$ are continuous functions of time; while, in the discontinuous case, $\dot{x}$ and $\dot{y}$ exhibit discontinuities even though x and y are continuous.

2.2. Relation to the Method of Complementary Functions. Here, we establish a connection between the method of particular solutions and the method of complementary functions. First, we solve Eq. (12) in terms of the constant k_2 as follows:

$$k_2 = 1 - k_1 \tag{16}$$

Next, we rewrite Eqs. (9) in the form

$$\begin{aligned} x &= k_1 v_1 + x_2 \\ y &= k_1 w_1 + y_2 \end{aligned} \tag{17}$$

where, by definition,

$$v_1 = x_1 - x_2, \qquad w_1 = y_1 - y_2 \tag{18}$$

We note that the functions

$$v_1 = v_1(t), \qquad w_1 = w_1(t) \tag{19}$$

are solutions of the following homogeneous system derived from (1):

$$\begin{aligned} \dot{v} &= av + bw \\ \dot{w} &= ev + fw \end{aligned} \tag{20}$$

We also note that the following initial conditions must be employed[4]:

$$v_1(0) = 0, \qquad w_1(0) = \beta_1 - \beta_2 \tag{21}$$

and that the constant k_1 must be determined from the final condition

$$k_1 v_1(\tau) + x_2(\tau) = \gamma \tag{22}$$

Therefore, in the method of complementary functions, the solution of (1) can be obtained by combining linearly the solution (19) of the homogeneous system (20) and the solution (8) of the complete system (1). However, different initial conditions must be used; specifically, conditions (21) apply to the homogeneous system and conditions (7) to the complete system.

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[4] Since (5-2) and (7-2) are arbitrary, the initial condition (21-2) is arbitrary and can be changed, if necessary.

2.3. Final Time Unspecified. It is now assumed that the final time τ is unspecified and that the differential system (1) is subject to the boundary conditions

$$x(0) = \alpha \tag{23}$$

$$x(\tau) = \gamma, \qquad y(\tau) = \delta \tag{24}$$

where α, γ, δ are prescribed constants and τ is to be determined.

Once more, we integrate Eqs. (1) forward twice from $t = 0$. In the first integration, the initial conditions (5) are employed, and (6) is the corresponding solution. In the second integration, the initial conditions (7) are employed, and (8) is the corresponding solution. We note that the linear combinations (9) satisfy the differential equations (1) and the initial condition (23) providing the constants k_1 and k_2 are consistent with (12).

Next, we turn our attention to the final conditions. By substituting (9) into (24), we obtain the relations

$$\begin{aligned} k_1 x_1(\tau) + k_2 x_2(\tau) &= \gamma \\ k_1 y_1(\tau) + k_2 y_2(\tau) &= \delta \end{aligned} \tag{25}$$

which are compatible with (12) if, and only if,

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1(\tau) & x_2(\tau) & \gamma \\ y_1(\tau) & y_2(\tau) & \delta \end{vmatrix} = 0 \tag{26}$$

This equation is the stopping condition of the integration process and supplies the final time τ. Once τ is known, the constants k_1 and k_2 can be obtained from (25).

3. Controlled System

Here, we consider the following modification of the previous system[5]:

$$\dot{x} = ax + by + c + du$$
$$\dot{y} = ex + fy + g + hu \qquad (27)$$

where u is a control and where the coefficients a, b, c, d and e, f, g, h are time-dependent. We assume that the following boundary conditions must be satisifed:

$$x(0) = \alpha, \qquad y(0) = \beta \qquad (28)$$

$$x(\tau) = \gamma \qquad (29)$$

where $\alpha, \beta, \gamma, \tau$ are prescribed constants. Then, we formulate the following problem: Find a set of functions

$$u = u(t), \qquad x = x(t), \qquad y = y(t) \qquad (30)$$

which satisfy the differential system (27), the initial conditions (28), and the final condition (29). We emphasize that (27) subject to (28)-(29) admits an infinite number of solutions. Nevertheless, we are concerned here with finding only one among these infinite solutions.

In order to solve this problem, we integrate Eqs. (27) forward twice from $t = 0$ using the initial conditions (28), the stopping condition $t = \tau$, and two different time-histories of the control. In the first integration, the control employed is $u_1(t)$ and

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[5] The system (27) can be called controlled in that its trajectory in the txy-space depends not only on the initial conditions but also on the time-history of the control u(t).

the corresponding solution of Eqs. (27) is denoted by

$$u_1 = u_1(t), \qquad x_1 = x_1(t), \qquad y_1 = y_1(t) \tag{31}$$

In the second integration, the control employed is $u_2(t)$ and the corresponding solution of Eqs. (27) is denoted by

$$u_2 = u_2(t), \qquad x_2 = x_2(t), \qquad y_2 = y_2(t) \tag{32}$$

Next, we introduce the undetermined constants k_1 and k_2 and form the linear combinations

$$\begin{aligned} u &= k_1u_1 + k_2u_2 \\ x &= k_1x_1 + k_2x_2 \\ y &= k_1y_1 + k_2y_2 \end{aligned} \tag{33}$$

Then, we inquire whether, by an appropriate choice of the constants, these linear combinations can satisfy the differential equations (27), the initial conditions (28), and the final condition (29).

By substituting (33) into (27) and rearranging terms, we obtain the relations

$$\begin{aligned} k_1(\dot{x}_1 - ax_1 - by_1 - du_1) + k_2(\dot{x}_2 - ax_2 - by_2 - du_2) &= c \\ k_1(\dot{y}_1 - ex_1 - fy_1 - hu_1) + k_2(\dot{y}_2 - ex_2 - fy_2 - hu_2) &= g \end{aligned} \tag{34}$$

Since each triplet of functions (31) and (32) is a solution of Eqs. (27), Eqs. (34) become

$$\begin{aligned} k_1 c + k_2 c &= c \\ k_1 g + k_2 g &= g \end{aligned} \tag{35}$$

and are satisfied providing the constants are such that

$$k_1 + k_2 = 1 \tag{36}$$

Substitution of (33-2) and (33-3) into the initial conditions (28) leads to the relations

$$\begin{aligned} k_1 x_1(0) + k_2 x_2(0) &= \alpha \\ k_1 y_1(0) + k_2 y_2(0) &= \beta \end{aligned} \tag{37}$$

Since each particular solution satisfies the initial conditions (28), Eqs. (37) can be rewritten as

$$\begin{aligned} k_1 \alpha + k_2 \alpha &= \alpha \\ k_1 \beta + k_2 \beta &= \beta \end{aligned} \tag{38}$$

and are satisfied providing the constants are consistent with (36).

Finally, substitution of (33-2) into the final condition (29) leads to the relation

$$k_1 x_1(\tau) + k_2 x_2(\tau) = \gamma \tag{39}$$

which, together with (36), determines the constants k_1 and k_2. In this way, the proposed problem is solved in principle.

3.1. Final Time Unspecified. It is now assumed that the final time τ is unspecified and that the differential system (27) is subjected to the boundary conditions

$$x(0) = \alpha, \qquad y(0) = \beta \tag{40}$$

$$x(\tau) = \gamma, \qquad y(\tau) = \delta \tag{41}$$

where $\alpha, \beta, \gamma, \delta$ are prescribed constants and τ is to be determined.

Once more, we integrate Eqs. (27) forward twice from $t = 0$ using the initial conditions (40) and two different time-histories of the control. In the first integration, the control employed is $u_1(t)$ and (31) is the corresponding solution. In the second integration, the control employed is $u_2(t)$ and (32) is the corresponding solution. We note that the linear combinations (33) satisfy the differential equations (27) and the prescribed initial conditions (40) providing the constants k_1 and k_2 are consistent with (36).

Next, we turn our attention to the final conditions. By substituting (33-2) and (33-3) into (41), we obtain the relations

$$\begin{gathered} k_1 x_1(\tau) + k_2 x_2(\tau) = \gamma \\ k_1 y_1(\tau) + k_2 y_2(\tau) = \delta \end{gathered} \tag{42}$$

which are compatible with (36) if, and only if,

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1(\tau) & x_2(\tau) & \gamma \\ y_1(\tau) & y_2(\tau) & \delta \end{vmatrix} = 0 \qquad (43)$$

This equation is the stopping condition of the integration process and supplies the final time τ. Once τ is known, the constants k_1 and k_2 can be obtained from (42).

4. Discussion and Conclusions

In the previous sections, the boundary-value problem associated with linear differential systems has been solved by means of the method of particular solutions. Several preliminary examples are presented for both uncontrolled and controlled systems of the second order. The basic ideas are (i) to perform all the integrations in terms of the original, nonhomogeneous system, (ii) to combine linearly several particular solutions, and (iii) to use the same prescribed initial conditions for all of the particular solutions. For each particular boundary-value problem, the required number of integrations equals the number of prescribed final conditions plus one (the stopping condition). The main result is that a linear combination of particular solutions consistent with (i), (ii), and (iii) automatically satisfies the differential system and the initial conditions as long as the sum of the constants is one.

It is of interest to compare the present method with (a) the method of adjoint equations, (b) the method of Green's functions, and (c) the method of complementary functions. Techniques (a), (b), (c) require using two differential sets, namely, the original set plus the derived set. This derived set is the adjoint set in Case (a), the Green's functions set in Case (b), and the homogeneous set in Case (c). The comparison shows that the present method is conceptually simpler than (a), (b), or (c) because it makes use of only one differential system, because each particular solution can be made to satisfy the same prescribed initial conditions, and because, in a physical problem, each particular solution represents a physically possible trajectory, even though it satisfies only the initial conditions and not the final conditions.

The generalization of the present point of view to a system of any order is described in a subsequent report (Ref. 16).

APPENDIX A

General Solution for an Uncontrolled System

The technique derived in Section 2 can also be employed to find the general solution of (1) in the closed interval $[0, \tau]$. To do so, we integrate the differential system (1) three times from $t = 0$ using three different sets of initial conditions, for instance,

$$x_1(0) = \alpha_1, \qquad y_1(0) = \beta_1 \tag{44}$$

$$x_2(0) = \alpha_2, \qquad y_2(0) = \beta_2 \tag{45}$$

$$x_3(0) = \alpha_3, \qquad y_3(0) = \beta_3 \tag{46}$$

where $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ are arbitrary. By doing so, we obtain the particular solutions[6]

$$x_1 = x_1(t), \qquad y_1 = y_1(t) \tag{47}$$

$$x_2 = x_2(t), \qquad y_2 = y_2(t) \tag{48}$$

$$x_3 = x_3(t), \qquad y_3 = y_3(t) \tag{49}$$

in which the subscripts 1, 2, 3 denote first, second, and third integration, respectively.

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[6] The initial conditions (44)-(46) are assumed to be such that the particular solutions (47)-(49) are linearly independent.

Next, we introduce the undetermined constants k_1, k_2, k_3 and form the linear combinations

$$\begin{aligned} x &= k_1 x_1 + k_2 x_2 + k_3 x_3 \\ y &= k_1 y_1 + k_2 y_2 + k_3 y_3 \end{aligned} \tag{50}$$

Then, we inquire whether, by an appropriate choice of the constants, this linear combination can satisfy the differential equations (1). Simple manipulations, omitted for the sake of brevity, show that this is precisely the case providing the constants are such that

$$k_1 + k_2 + k_3 = 1 \tag{51}$$

A.1. Relation to the Method of Complementary Functions. Here, we establish a connection between the method of particular solutions and the method of complementary functions. First, we combine Eqs. (50) and (51) to obtain

$$\begin{aligned} x &= k_1 v_1 + k_2 v_2 + x_3 \\ y &= k_1 w_1 + k_2 w_2 + y_3 \end{aligned} \tag{52}$$

where, by definition,

$$\begin{aligned} v_1 &= x_1 - x_3, \qquad w_1 = y_1 - y_3 \\ v_2 &= x_2 - x_3, \qquad w_2 = y_2 - y_3 \end{aligned} \tag{53}$$

We note that each pair of complementary functions

$$v_1 = v_1(t), \qquad w_1 = w_1(t) \tag{54}$$

$$v_2 = v_2(t), \qquad w_2 = w_2(t) \tag{55}$$

is a solution of the homogeneous system (20) derived from (1). Therefore, Eqs. (52) express a well-known theorem: The general solution of a linear, nonhomogeneous system is the sum of the general solution of the corresponding homogeneous system and a particular solution of the complete system.

A.2. Remark. The general solution (50) of Eq. (1) contains three independent solutions. On the other hand, in the boundary-value problem represented by Eqs. (1)-(3), two independent solutions were employed. This apparent anomaly is now explained. If Eq. (50-1) is combined with the initial condition (2) and the final condition (3), the following relations are obtained:

$$\begin{aligned} k_1x_1(0) + k_2x_2(0) + k_3x_3(0) &= \alpha \\ k_1x_1(\tau) + k_2x_2(\tau) + k_3x_3(\tau) &= \gamma \end{aligned} \tag{56}$$

and, together with (51), determine the constants k_1, k_2, k_3.

Assume now that (47)-(49) satisfy the initial condition (2), that is,

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha \tag{57}$$

Under these conditions, Eq. (56-1) becomes

$$k_1\alpha + k_2\alpha + k_3\alpha = \alpha \tag{58}$$

and, therefore, is identical with (51). Since the system composed of Eqs. (51) and (56) admits an infinite number of solutions, it is entirely permissible to set

$$k_3 = 0 \tag{59}$$

that is, integrate the system (1) only twice.[7] This was precisely done in Section 2.

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[7] Clearly, only two independent solutions satisfying the initial condition (2) exist.

References

1. BLISS, G.A., Mathematics for Exterior Ballistics, John Wiley and Sons, New York, 1944.
2. GOODMAN, T.R., and LANCE, C.N., The Numerical Integration of Two-Point Boundary-Value Problems, Mathematical Tables and Other Aids to Computation, Vol. 10, No. 54, 1956.
3. BELLMAN, R., Introduction to the Mathematical Theory of Control Processes, Vol. 1: Linear Equations and Quadratic Criteria, Academic Press, New York, 1967.
4. MILLER, K.S., Linear Differential Equations in the Real Domain, W.W. Norton and Company, New York, 1963.
5. GURA, I.A., State Variable Approach to Linear Systems, Instruments and Control Systems, Vol. 40, No. 10, 1967.
6. INCE, E.L., Ordinary Differential Equations, Dover Publications, New York, 1956.
7. TIFFORD, A.N., On the Solution of Total Differential, Boundary-Value Problems, Journal of the Aeronautical Sciences, Vol. 18, No. 1, 1951.
8. FOX, L., The Numerical Solution of Two-Point Boundary Problems in Ordinary Differential Equations, The Clarendon Press, Oxford England, 1957.
9. FOX, L., Editor, Numerical Solution of Ordinary and Partial Differential Equations, Addison-Wesley Publishing Company, Reading, Massachusetts, 1962.
10. BOYCE, W.E., and DiPRIMA, R.C., Elementary Differential Equations and Boundary Value Problems, John Wiley and Sons, New York, 1965.
11. DENNIS, S.C.R., and POOTS, G., The Solution of Linear Differential Equations, Proceedings of the Cambridge Philosphical Society, Vol. 51, No. 3, 1955.

12. CLENSHAW, C.W., The Numerical Solution of Linear Differential Equations in Chebyshev Series, Proceedings of the Cambridge Philosophical Society, Vol. 53, No. 1, 1957.

13. KAGIWADA, H.H., and KALABA, R.E., A Practical Method for Determining Green's Functions Using Hadamard's Variational Formula, Journal of Optimization Theory and Applications, Vol. 1, No. 1, 1967.

14. KAGIWADA, H.H., KALABA, R.E., SCHUMITZKY, A., and SRIDHAR, R., Cauchy and Fredholm Methods for Euler Equations, Journal of Optimization Theory and Applications, Vol. 2, No. 4, 1968.

15. KAGIWADA, H.H., and KALABA, R.E., Derivation and Validation of an Initial-Value Method for Certain Nonlinear Two-Point Boundary-Value Problems, Journal of Optimization Theory and Applications, Vol. 2, No. 6, 1968.

16. MIELE, A., Method of Particular Solutions for Linear, Two-Point Boundary-Value Problems, Part 2, General Theory, Rice University, Aero-Astronautics Report No. 49, 1968.