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DIFFERENTIAL SYSTEMS

by Robert Bradley McNeill

Prepared by
THE PENNSYLVANIA STATE UNIVERSITY
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I INTRODUCTION AND PRELIMINARIES

1. Introduction

Since the turn of the century, the "qualitative" theory of differential equations and systems has been a major source of activity for the mathematician. With the works of Lyapunov, Perron and Poincaré serving as the superstructure, a whole new system has been devised for the purpose of determining the behavior (stability, boundedness, etc.) of a solution of a given differential system (or equation) without explicitly finding that solution. This approach is justified (if, indeed, being mathematics is not justification enough!) since scientists and engineers have found many of the results obtained to be quite use-able.

There are at present two major ways of answering the questions of behavior of a given differential system: conversion to an integral equation and construction of a Lyapunov function. This paper will be concerned with the first method only.

Chapter II deals entirely with algebraic properties of linear systems and, for the most part, is independent of the remaining ones. Here the question is asked: If a given system has only solutions which are bounded or tend to zero, then under what pertubations will the new system have the same properties?

In Chapter III, results similar to Lagrange's variation of parameters formula are noted, and examples are given. Some of these results are then extended.

Finally, in Chapter IV, certain oscillation theorems are presented because of their connection with some of the results in Chapter III.

Chapter V is devoted to a summary and conclusions.

2. Preliminaries

All necessary preliminary notations, definitions and theorems are collected in this section, and, unless stated otherwise, the assumptions made here will prevail throughout.

Those differential systems and equations considered here will meet the following two restrictions:

(1) They will involve real-valued functions and/or matrices of a real variable, which here will be denoted by t. Furthermore, all the elements of any matrix considered will be continuous functions of t.

(2) Solutions are assumed to exist. Uniqueness will not be assumed unless stated otherwise. It should be noted, however, that the assumptions in (1) guarantee unique solutions for the linear systems (1.1) and (1.2).

Capital letters will denote $n \times n$ matrices, while small letters, unless stated otherwise, will represent $n \times 1$ vectors, $1 \times n$ vectors, and scalars; it will be clear from context as to which use is being employed. $\frac{d}{dt}$ and $'$ will be the usual differentiation symbols, and $\|\cdot\|$ will denote any convenient norm.

Let

$$L(t_o, t) = \{B(t): \int_{t_o}^{t} \|B(s)\| \, ds < \infty\}$$

and

$$L_1(t_o, t) = \{f(t): \int_{t_o}^{t} |f(s)| \, ds < \infty\}.$$

With respect to the systems

(1.1) $\quad x' = B(t)x$

and

(1.2) $\quad y' = -y\,B(t)$,

the following sets are defined:

$\mathcal{B} = \{B(t):$ all solutions of (1.1) are bounded$\}$

$\mathcal{B}_a = \{B(t):$ all solutions of (1.2) are bounded$\}$

$\mathcal{J} = \{B(t):$ all solutions of (1.1) tend to zero as $t \to \infty\}$

$\mathcal{B}_a$ = {B(t): all solutuions of (1.2) tend to zero as $t \to \infty$}.

The following definitions will be used in the sequel.

A vector $x(t)$ is said to be <u>bounded</u> if there exists a real number $M > 0$ such that $\| x(t) \| < M, \quad t \geq t_o$.

If $\Phi(t)$ is a matrix whose n columns are n linearly independent solutions of an n^{th} order differential system, then $\Phi(t)$ is said to be a <u>solution matrix</u> for that system. If the system is linear, then $\Phi(t)$ is said to be a <u>fundamental matrix</u>. If $\Phi(t)$ is a fundamental matrix and $\Phi(t_o) = I$, then $\Phi(t)$ is said to be the <u>Fundamental matrix</u>.

Finally, the following theorems will serve as a guide in what follows.

<u>Theorem 1.1</u>: Let $u(t) > 0$, $v(t)$, $w(t) \geq 0$, and c be a positive constant. If

$$u(t) \leq c + \int_{t_o}^{t} v(s)\, w(s)\, ds$$

for all $t \geq t_o$, then

$$u(t) \leq c \exp\left(\int_{t_o}^{t} \frac{v(s)\, w(s)}{u(s)}\, ds \right).$$

Proof: Dividing both sides of the inequality by $(c + \int_{t_o}^{t} v(s)\, w(s)\, ds)$, multiplying by $\frac{v(t)w(t)}{u(t)}$, and integrating from t_o to t yields

$$\ln\left[c + \int_{t_o}^{t} v(s)w(s)ds\right] - \ln c \leq \int_{t_o}^{t} \frac{v(s)w(s)}{u(s)}\, ds.$$

Hence, taking exponentials

$$c + \int_{t_o}^{t} v(s)w(s)ds \leq c\, \exp\left(\int_{t_o}^{t} \frac{v(s)w(s)}{u(s)}\, ds\right)$$

and the result follows.

If $u(t) \equiv v(t)$, then Theorem 1.1 reduces to the " fundamental lemma " of Bellman [1, p. 35]. This is also known as "Gronwall's Inequality."

Theorem 1.2: If $Y(t)$ is a matrix of functions satisfying

(1.3) $Y' = B(t)Y$

then det $Y(t)$ satisfies

(1.4) $(\det Y)' = (\operatorname{tr} B(t))(\det Y)$

where $\operatorname{tr} B(t) = \sum_{i=1}^{n} b_{ii}(t)$. Hence

(1.5) $\det Y(t) = \det Y(t_o) \exp\left(\int_{t_o}^{t} \operatorname{tr} B(s)\, ds\right)$.

Proof. The proof can be found in Coddington and Levinson [3, p. 28] and goes as follows. Let y_{ij}, b_{ij} be the elements in the i^{th} row and j^{th} column of

Y and B, respectively. Then (1.3) becomes

(1.6) $$y'_{ij}(t) = \sum_{k=1}^{m} b_{ik}(t)\, y_{kj}(t).$$

The derivative of det Y is a sum of n determinants

(1.7) $$(\det Y(t))' = \begin{vmatrix} y'_{11} & y'_{12} & \cdots & y'_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} + \ldots + \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ y'_{n1} & y'_{n2} & \cdots & y'_{nn} \end{vmatrix}.$$

Using (1.6) in the first determinant in (1.7), that determinant becomes

$$\begin{vmatrix} \sum_k b_{1k} y_{k1} & \sum_k b_{1k} y_{k2} & \cdots & \sum_k b_{1k} y_{kn} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix}$$

and this determinant is unchanged if from the first row there is subtracted b_{12} times the second row plus b_{13} times the third row up to b_{1n} times the n^{th} row. This gives

$$\begin{vmatrix} b_{11}y_{11} & b_{11}y_{12} & \cdots & b_{11}y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix}$$

which is $b_{11}\det Y(t)$. Applying a similar procedure to the remaining determinants in (1.7) gives (1.4). A direct integration of (1.4) leads to (1.5), and the proof is complete.

Hence, if $\det Y(t_o) \neq 0$, then $Y^{-1}(t)$ exists if and only if $\int_{t_o}^{t} \operatorname{tr} B(s)\, ds > -\infty$.

<u>Theorem 1.3</u>: If $Y(t)$ is a fundamental matrix for (1.1), then $Y^{-1}(t)$ is a fundamental matrix for (1.2).

<u>Proof</u>. The proof can be found in Coddington and Levinson [3, p. 70] and proceeds as follows. $Y^{-1}(t)$ exists by Theorem 1.2, and

(1.8) $\quad 0 = (YY^{-1})' = Y(Y^{-1})' + Y'Y^{-1}$.

Substituting (1.3) in (1.8) yields

$$0 = Y(Y^{-1})' + (BY)Y^{-1} = Y(Y^{-1})' + B.$$

Hence

$$(Y^{-1})' = -Y^{-1}B(t).$$

Since $(\det Y^{-1}(t)) = (\det Y(t))^{-1} \neq 0$, the proof is complete.

Theorem 1.4: If $B(t) \in L(t_o, \infty)$, then $B(t) \in \mathcal{B}$.

Proof. The proof is in Cesari [2, p. 37] and is the following. Integrating (1.1) yields

$$x(t) = \int_{t_o}^{t} B(s)\, x(s)\, ds + x(t_o).$$

Taking norms

$$(1.9) \qquad \| x(t) \| \leq \| x(t_o) \| + \int_{t_o}^{t} \| B(s) \| \cdot \| x(s) \| ds.$$

Applying Theorem 1.1 to (1.9) gives

$$\| x(t) \| \leq \| x(t_o) \| \exp\left(\int_{t_o}^{t} \| B(s) \| \, ds\right)$$

$$\leq \| x(t_o) \| \exp\left(\int_{t_o}^{\infty} \| B(s) \| \, ds\right) < \infty$$

and the theorem is proved.

Theorem 1.5: (Lagrange's variation of parameters formula) If $X(t)$ is the Fundamental matrix for (1.1), then any solution $x(t)$ of

$$(1.10) \quad x' = B(t)x + w(t)$$

satisfies

$$x(t) = X(t)x(t_o) + \int_{t_o}^{t} X(t)X^{-1}(s)w(s)\, ds.$$

Proof. Since $(X^{-1})' = -X^{-1}B(t)$, it is clear from (1.10) that

$$(X^{-1}x)' = X^{-1}w(t).$$

Integrating

$$X^{-1}x = x(t_o) + \int_{t_o}^{t} X^{-1}(s)w(s)ds,$$

$$x(t) = X(t)x(t_o) + \int_{t_o}^{t} X(t)X^{-1}(s)w(s)ds.$$

Theorem 1.6: If $X(t)$ is the Fundamental matrix for (1.1), then every solution $y(t)$ of

(1.11) $y' = -yB(t) + u(t)$

satisfies

$$y(t) = y(t_o)X^{-1}(t) + \int_{t_o}^{t} u(s)X(s)X^{-1}(t)ds.$$

Proof. For any solution $y(t)$ of (1.11)

$$(yX)' = u(t)X.$$

Therefore

$$(yX)(t) = (yX)(t_o) + \int_{t_o}^{t} u(s)X(s)\, ds,$$

and

$$y(t) = y(t_o)X^{-1}(t) + \int_{t_o}^{t} u(s)X(s)X^{-1}(t)ds.$$

II LINEAR SYSTEMS

1. Introduction

In this chapter, the linear differential system

(2.1) $x' = B(t)x$

is studied relative to the question: If the solutions of (2.1) have certain properties, under what pertubetions of $B(t)$ will the solutions of the new system retain those properties?

2. Boundedness

The simplest results are obtained in the case when $B(t) \equiv B$, a constant matrix.

Theorem 2.1: If B is a constant matrix such that $B \in \mathfrak{B}$, and $C(t) \in L(t_o, \infty)$, then $B + C(t) \in \mathfrak{B}$.

Proof. The proof is in Cesari [2, p. 37] and goes as follows. Let $X(t)$ be the Fundamental matrix for (2.1). Then by Theorem 1.5, any solution $x(t)$ of

$$x' = [B + C(t)]x$$

satisfies

$$x(t) = X(t)x(t_o) + \int_{t_o}^{t} X(t)X^{-1}(s)C(s)x(s)ds.$$

However, since B is a constant matrix, $X(t)X^{-1}(s) = X(t-s)$. This follows since $X(t-s)|_{s=t_o} = X(t)X^{-1}(t_o)$ and both are solutions of (2.1), by uniqueness, they must be equal for all $t \geq t_o$. Therefore

$$x(t) = X(t)x(t_o) + \int_{t_o}^{t} X(t-s)C(s)x(s)\, ds.$$

Applying norms

$$\| x(t) \| \leq \| X(t) \| \cdot \| x(t_o) \| + $$

$$+ \int_{t_o}^{t} \|X(t-s)\| \cdot \|C(s)\| \cdot \|x(s)\| \, ds$$

$$\| x(t) \| \leq K \|x(t_o)\| + \int_{t_o}^{t} K \|C(s)\| \cdot \|x(s)\| \, ds.$$

By Theorem 1.1

$$\| x(t) \| \leq K \|x(t_o)\| \exp[K \int_{t_o}^{t} \| C(s) \| ds] < \infty$$

and the result is shown.

This result is false, however, if B is allowed to be a variable matrix, as can be seen from the following example taken from Bellman [1, p. 42].

<u>Example 2.1</u>: Consider the system

$$(2.2) \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -a & 0 \\ 0 & \sin \ln t + \cos \ln t - 2a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

whose general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-at} \\ c_2 \exp(t \sin \ln t - 2at) \end{pmatrix}$$

If $a > \frac{1}{2}$, every solution of (2.2) approaches zero as $t \to \infty$ (and hence every solution is bounded) since

$$e^{t \sin \ln t} \le e^{t|\sin \ln t|} \le e^{t}$$

for all $t \ge 0$. Choose as the pertubating matrix

$$C(t) = \begin{pmatrix} 0 & 0 \\ e^{-at} & 0 \end{pmatrix} .$$

Then the perturbed system becomes

$$(2.3) \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' = \begin{pmatrix} -a & 0 \\ e^{-at} & \sin \ln t + \cos \ln t - 2a \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

The solution of this system is

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-at} \\ \exp(t \sin \ln t - 2at)[c_2 + c_1 \int_0^t \exp(-s \sin \ln s) ds] \end{pmatrix}$$

Let $t = e^{(2n + \frac{1}{2})\pi}$, n a positive integer. Hence

$$\int_0^t \exp(-s \sin \ln s) ds > \int_{te^{-\pi}}^{te^{-2\pi/3}} \exp(-s \sin \ln s) ds$$

and

$$z_2 > e^{t-2at}[c_2 + c_1 t(e^{-2\pi/3} - e^{-\pi})\exp(\frac{e^{-\pi}t}{2})]$$

$$= c_2 e^{t-2at} + c_1 t(e^{-2\pi/3} - e^{-\pi})\exp([1 - 2a + \frac{e^{-\pi}}{2}]t).$$

Hence, if $1 < 2a$ and $1 - 2a + \frac{e^{-\pi}}{2} > 0$, i.e., if

$$1 < 2a < 1 + \frac{e^{-\pi}}{2}$$

then the solutions of (2.3) will be bounded if and only if $c_1 = 0$. Hence, not all solutions of (2.3) are bounded.

For variable $B(t)$ the following theorem, which can be found in Bellman [1, p. 43], is easily shown.

<u>Theorem 2.2</u>: If $B(t) \in \mathcal{B}$, $C(t) \in L(t_o, \infty)$, and

$$\varliminf_{t\to\infty} \int_{t_o}^{t} \operatorname{tr} B(s)\, ds > -\infty, \quad \text{then} \quad B(t) + C(t) \in \mathcal{B}.$$

<u>Proof</u>. Let $X(t)$ be the Fundamental matrix for (2.1). Then by Theorem 1.5 any solution $x(t)$ of

(2.4) $\quad x' = [B(t) + C(t)]x$

satisfies

$$x(t) = X(t)x(t_o) + \int_{t_o}^{t} X(t)X^{-1}(s)C(s)x(s)\, ds.$$

Therefore

$$\| x(t) \| \leq \|X(t)\| \cdot \|x(t_o)\| +$$

$$+ \int_{t_o}^{t} \|X(t)\| \cdot \|X^{-1}(s)\| \cdot \|C(s)\| \cdot \|x(s)\| ds.$$

Since $B(t) \in \mathcal{B}$, there exists a positive real number K_1 such that $\| X(t) \| \leq K_1$ for all $t \geq t_o$. Also, there exists another positive real constant K_2 such that $\|X^{-1}(t)\| \leq K_1$ for all $t \geq t_o$ since the elements of $X^{-1}(t)$ are formed from the (bounded) elements of $X(t)$ times $[\det X(t)]^{-1}$, and $\det X(t) \neq 0$ for all $t \geq t_o$ by Theorem 1.2. Hence

$$\| x(t) \| \leq K_1 \|x(t_o)\| + \int_{t_o}^{t} K_1 K_2 \|C(s)\| \cdot \|x(s)\| ds.$$

Applying Theorem 1.1

$$\| x(t) \| \leq K_1 \|x(t_o)\| \exp[K_1 K_2 \int_{t_o}^{t} \|C(s)\| ds] < \infty$$

and $B(t) + C(t) \in \mathcal{B}$.

Actually, Theorem 2.2 is only a partial result and can be obtained directly from the following theorem.

Theorem 2.3: If $B(t) \in \mathcal{B} \cap \mathcal{B}_a$, and $C(t) \in L(t_o, \infty)$, then $B(t) + C(t) \in \mathcal{B} \cap \mathcal{B}_a$.

Proof. This theorem is due to Conti [19] and the proof is the following. Let $X(t)$ be the Fundamental matrix for (2.1). Again, by Theorem 1.5 any solution $x(t)$ of (2.4) satisfies

$$x(t) = X(t)x(t_o) + \int_{t_o}^{t} X(t)X^{-1}(s)C(s)x(s)ds.$$

Thus

$$\|x(t)\| \le \|X(t)\| \cdot \|x(t_o)\| +$$

$$+ \int_{t_o}^{t} \|X(t)\| \cdot \|X^{-1}(s)\| \cdot \|C(s)\| \cdot \|x(s)\| ds.$$

By hypotheses, there exist positive real numbers K_1, K_2 such that $\| X(t) \| \le K_1$, $\| X^{-1}(t) \| \le K_2$ for all $t \ge t_o$. Hence

$$\| x(t) \| \le K_1 \| x(t_o) \| + \int_{t_o}^{t} K_1 K_2 \| C(s) \| \cdot \| x(s) \| ds.$$

By Theorem 1.1

$$\| x(t) \| \le K_1 \| x(t_o) \| \exp[K_1 K_2 \int_{t_o}^{t} \| C(s) \| ds] < \infty$$

and $B(t) + C(t) \in \mathfrak{B}$.

By Theorem 1.6, any solution $y(t)$ of

(2.5) $\quad y' = -y[B(t) + C(t)]$

satisfies

$$y(t) = y(t_o) X^{-1}(t) - \int_{t_o}^{t} y(s) C(s) X(s) X^{-1}(t) ds.$$

Therefore, taking norms

$$\| y(t) \| \le \| y(t_o) \| \cdot \| X^{-1}(t) \| +$$

$$+ \int_{t_o}^{t} \| y(s) \| \cdot \| C(s) \| \cdot \| X(s) \| \cdot \| X^{-1}(t) \| ds$$

$$\| y(t) \| \le \| y(t_o) \| \cdot K_1 + \int_{t_o}^{t} \| y(s) \| \cdot \| C(s) \| \cdot K_1 K_2 ds.$$

Applying Theorem 1.1

$$\|y(t)\| \le K_1 \|y(t_o)\| \exp[K_1 K_2 \int_{t_o}^{t} \|C(s)\| ds] < \infty$$

and $B(t) + C(t) \in \mathcal{B}_a$. Hence $B(t) + C(t) \in \mathcal{B} \cap \mathcal{B}_a$, and the proof is complete.

Theorem 2.2 follows from Theorem 2.3 since the conditions that $B(t) \in \mathcal{B}$ and $\underline{\lim}_{t\to\infty} \int_{t_o}^{t} \text{tr } B(s)ds > -\infty$ together imply that $B(t) \in \mathcal{B}_a$ by Theorems 1.2 and 1.3.

Corollary: If $B(t) \in \mathcal{B} \cap \mathcal{B}_a$, and $C(t) - B(t) \in L(t_o, \infty)$, then $C(t) \in \mathcal{B} \cap \mathcal{B}_a$.

Proof. Writing $C(t)$ in the form

$$C(t) = B(t) + (C(t) - B(t))$$

and the result follows from the above theorem.

These theorems appear to be completely theoretical in nature, but the following remarks are in order.

Remark 1: When investigating a linear system, it may be advantageous to add pieces which are absolutely integrable and work with that system instead.

Remark 2: It may be possible to decompose $B(t)$ into two parts, $B_1(t)$ and $B_2(t)$, in such a way that $B_1(t) \in \mathcal{B} \cap \mathcal{B}_a$ (or $B_1 \in \mathcal{B}$ if B_1 is a constant matrix) and $B_2(t) \in L(t_o, \infty)$.

These remarks are illustrated by the following example.

Example 2.2: Consider the system

$$(2.6)\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1 & \varphi(t) \\ \psi(t) & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$= \left[\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & \varphi(t) \\ \psi(t) & 0 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $\psi,\ \varphi \in L_1(t_o, \infty)$. The matrix $B_1 \equiv \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \in \mathcal{B}$

since the general solution of

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 & e^{-t} \\ c_2 & e^{-2t} \end{pmatrix} .$$

The matrix $B_2(t) \equiv \begin{pmatrix} 0 & \varphi(t) \\ \psi(t) & 0 \end{pmatrix} \in L(t_o, \infty)$ by hypotheses. Therefore, $B_1 + B_2 \in \mathcal{B}$ by Theorem 2.1, and the system (2.6) has only bounded solutions.

The major result of this section is the following theorem.

Theorem 2.4: If $B(t) \in \mathcal{B}$ and $Y(t)$ is a fundamental matrix for (2.1) such that $Y^{-1}(t)C(t)Y(t) \equiv H(t) \in \mathcal{B}$, then $B(t) + C(t) \in \mathcal{B}$.

Proof. Considering

$$(Y^{-1})' = -Y^{-1}B(t)$$

and

(2.4) $$x' = [B(t) + C(t)]x$$

it is clear that

$$(Y^{-1}x)' = Y^{-1}Cx = H(t)(Y^{-1}x).$$

Since $H(t) \in \mathcal{B}$, there exists a real positive number M such that $\|Y^{-1}x\| \leq M < \infty$ for all $t \geq t_o$. But, $x = Y(Y^{-1}x)$; therefore

$$\|x(t)\| \leq \|Y\| \cdot \|Y^{-1}x\| \leq M \cdot \|Y\| < \infty$$

and $B(t) + C(t) \in \mathcal{B}$. The theorem is complete.

Corollary: If $B(t) \in \mathcal{B}$ is such that $Y(t)$ is a diagonal matrix, then $B(t) + C(t) \in \mathcal{B}$ for any diagonal matrix $C(t) \in \mathcal{B}$.

Proof. If $Y(t)$ is diagonal, then so is $Y^{-1}(t)$. Since diagonal matrices commute, $Y^{-1}(t)C(t)Y(t) \equiv C(t) \in \mathcal{B}$, the conditions of Theorem 2.4 are satisfied, and the result follows.

Corollary: If $B(t) \in \mathcal{B}$, and $Y(t)$ is a fundamental matrix for (2.1) such that

$$Y^{-1}(t)(C(t) - B(t))Y(t) \equiv H(t) \in \mathcal{B},$$

then $C(t) \in \mathcal{B}$.

Proof. The result is clear since $C(t) = B(t) + (C(t) - B(t)) \in \mathcal{B}$ by Theorem 2.4.

Remark 1: Theorem 2.3 is easily obtained from Theorem 2.4, since if $B(t) \in \mathcal{B} \cap \mathcal{B}_a$ and

$C(t) \in L(t_o, \infty)$, then

$$\int_{t_o}^{\infty} \| Y^{-1}CY \| \leq \int_{t_o}^{\infty} \|Y^{-1}\| \cdot \|C\| \cdot \|Y\|$$

$$\leq K \int_{t_o}^{\infty} \|C\| < \infty$$

and, hence, $Y^{-1}BY \in \mathcal{R}$ by Theorem 1.4.

Remark 2: Coddington and Levinson [3, p. 70] have shown that if $Y(t)$ is a fundamental matrix for (2.1), then every fundamental matrix has the form $Y(t)D$ for some non-singular constant matrix D. It is easily seen that if $Y(t)$ is a fundamental matrix and D is a non-singular constant matrix, then

$$(Y(t)D)' = Y(t)'D = A(t)(Y(t)D)$$

i.e., $Y(t)D$ satisfies the system. Moreover, since

$$\det(Y(t)D) = (\det Y(t))(\det D) \neq 0$$

$Y(t)D$ is a fundamental matrix. Conversely, if $Z(t)$ is another fundamental matrix for (2.1), define $W(t) = Y^{-1}(t)Z(t)$. Hence $Z(t) = Y(t)W(t)$. Differentiating this equation gives $Z' = YW' + Y'W$. Hence

$$AZ = YW' + AYW = YW' + AYY^{-1}Z.$$

This implies $YW' = 0$, or since Y is non-singular,

$W' = 0$. Therefore, $W = D$, a constant matrix. D is non-singular since Y and Z are.

Also, it is easily seen that $B(t) \in \mathcal{B}$ if and only if $D^{-1}B(t)D \in \mathcal{B}$ for every non-singular constant matrix D. For, if $B(t) \in \mathcal{B}$ and

(2.7 $\quad z' = D^{-1}B(t)(Dz)$

then

$$(Dz)' = B(t)(Dz)$$

and $w(t) \equiv Dz(t)$ is bounded. Hence, $z(t) = D^{-1}w(t)$ is bounded and $D^{-1}B(t)D \in \mathcal{B}$. Conversely, if $D^{-1}B(t)D \in \mathcal{B}$, then applying the transformation $x(t) = Dz(t)$ to (2.7) shows that $B(t) \in \mathcal{B}$.

Hence, the second condition of Theorem 2.4 holds either for all $Y(t)$ or for none of them. As a matter of application then, it is enough to answer as to whether $Y^{-1}(t)C(t)Y(t) \in \mathcal{B}$ for a given $Y(t)$.

The following example shows that Theorem 2.4 is indeed a <u>stronger</u> result that Theorem 2.3.

<u>Example 2.3</u>: Consider the system (2.1) with

$$B(t) = \begin{pmatrix} -t & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$Y(t) = \begin{pmatrix} e^{-t^2/2} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

is a fundamental matrix, and

$$Y^{-1}(t) = \begin{pmatrix} e^{t^2/2} & 0 \\ 0 & e^{t} \end{pmatrix} .$$

Let $C(t)$ be any 2x2 diagonal matrix subject to the further restriction that $C(t) \in L(t_0, \infty)$. By the first Corollary to Theorem 2.4, $B(t) + C(t) \in \mathcal{B}$.

Remark 1: Theorem 2.3 is not applicable since $B(t) \notin \mathcal{B}_a$; i.e., $Y^{-1}(t)$ becomes unbounded as $t \to \infty$.

Remark 2: If the roles of $B(t)$ and $C(t)$ are interchanged, then the first part of the hypothesis of Theorem 2.3 is satisfied (by $C(t)$), but the second part is not; i.e., $B(t) \notin L(t_0, \infty)$.

Theorem 2.5: If $B(t) \in \mathcal{B}_a$, and $B(t) + C(t) \in \mathcal{B}$, then $Y^{-1}CY \in \mathcal{B}$ for any fundamental matrix $Y(t)$ of (2.1).

Proof. Considering (2.1) and

(2.8) $\quad x' = [B(t) + C(t)]x$

it is clear that

$$(Y^{-1}x)' = Y^{-1}Cx.$$

Let $x = Yw$. Then

$$w' = (Y^{-1}CY)w$$

and $Y^{-1}x = w$. Hence, w is bounded if Y^{-1} and x are bounded; therefore, $Y^{-1}CY \in \mathcal{B}$.

Theorem 2.6: Let $B(t) \in \mathcal{B} \cap \mathcal{B}_a$. Then $B(t) + C(t) \in \mathcal{B}$ if and only if $Y^{-1}CY \in \mathcal{B}$ for any fundamental matrix $Y(t)$ of (2.1).

Proof. If $B(t) \in \mathcal{B} \cap \mathcal{B}_a$ and $Y^{-1}CY \in \mathcal{B}$, then $B(t) + C(t) \in \mathcal{B}$ by Theorem 2.4. Conversely, if $B(t) \in \mathcal{B} \cap \mathcal{B}_a$ and $B(t) + C(t) \in \mathcal{B}$, then $Y^{-1}CY \in \mathcal{B}$ by Theorem 2.5.

Note: Since the n^{th} order linear differential equation

$$(2.9) \qquad u^{(n)} + a_1(t)u^{(n-1)} + \dots + a_{n-1}(t)u' + a_n(t)u = 0$$

can be converted by the transformation

$$\begin{cases} x_1 = u \\ x_2 = u' \\ \vdots \\ x_n = u^{(n-1)} \end{cases}$$

into the system (2.1) with

$$(2.10) \qquad B(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & & \dots & 1 \\ -a_n & -a_{n-1} & & \dots & -a_1 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

the above results are applicable and easily give such theorems as the following one due to Bellman [5].

Theorem 2.7: If the equation (2.9) with $a_1 \equiv 0$ has only solutions, which together with their first (n-1) derivatives are uniformly bounded for $t \geq 0$, and if $a_K(t) - b_K(t) \in L_1(t_o, \infty)$, then all solutions of

$$(2.11) \quad u^{(n)} + b_1(t)u^{(n-1)} + \dots + b_{n-1}(t)u' + b_n(t)u = 0$$

are bounded, together with their first (n-1) derivatives.

Proof. Since $a_1 \equiv 0$, $\operatorname{tr} B \equiv 0$, and the result follows from Theorem 2.3.

The following example shows that the converse of Theorem 2.4 is false.

Example 2.4: Consider the system (2.1) with

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} .$$

Then

$$Y(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix}$$

is a fundamental matrix with

$$Y^{-1}(t) = \begin{pmatrix} e^{t} & 0 \\ 0 & 1 \end{pmatrix} .$$

Let

$$C(t) = \begin{pmatrix} -1 - [1/t^2] & 0 \\ e^{-t/2} & 0 \end{pmatrix}$$

Then

$$Y^{-1}[C(t) - B]Y = \begin{pmatrix} -1/t^2 & 0 \\ e^{t/2} & 0 \end{pmatrix} \notin \mathcal{B}$$

and, hence, Theorem 2.4 is not applicable, while $C(t) \in \mathcal{B}$.

3. Solutions Which Tend to Zero

The following theorems are extensions to the set $\mathcal{Z}$ of the major results of Section 2.

Theorem 2.8: If $B(t) \in \mathcal{Z}$ and $Y(t)$ is a fundamental matrix for (2.1) such that $Y^{-1}CY \equiv H(t) \in \mathcal{B}$, then $B(t) + C(t) \in \mathcal{Z}$.

Proof. Considering (2.1) and

$$x' = [B(t) + C(t)]x$$

it is clear that

$$(Y^{-1}x)' = Y^{-1}Cx = H(t)(Y^{-1}x).$$

Since $H(t) \in \mathcal{B}$, there exists a positive real number M such that $\|Y^{-1}x\| \leq M$. But $x = Y(Y^{-1}x)$; therefore,

$$\|x\| \leq \|Y\| \cdot \|Y^{-1}x\| \leq M \cdot \|Y\| \rightarrow 0$$

and the theorem is complete.

Example 2.4 shows that the converse of this theorem is false.

<u>Theorem 2.9</u>: If $B(t) \in \mathcal{Z}_a$ and $B(t) + C(t) \in \mathcal{B}$ or if $B(t) \in \mathcal{B}_a$ and $B(t) + C(t) \in \mathcal{Z}$, then $Y^{-1}CY \in \mathcal{Z}$ for any fundamental matrix $Y(t)$ of (2.1).

<u>Proof</u>. Applying the transformation $z = Y^{-1}x$ to the system

$$z' = Y^{-1}CYz$$

gives

$$(Y^{-1}x)' = Y^{-1}C(t)x. \tag{2.12}$$

Since $Y^{-1\prime} = -\,Y^{-1}B(t)$, (2.12) becomes

$$x' = [B(t) + C(t)]x.$$

But $z = Y^{-1}x$; $\|z\| \leq \|Y^{-1}\| \cdot \|x\|$, and the results follow.

There is no theorem for the set $\mathcal{Z}$ which corresponds to Theorem 2.6 since if $B(t) \in \mathcal{Z}$, then $B(t) \notin \mathcal{Z}_a$ as can be seen from the inequality

$$1 \leq \|I\| = \|Y(t)Y^{-1}(t)\| \leq \|Y(t)\| \cdot \|Y^{-1}(t)\|$$

for if $\|Y(t)\| \to 0$, then $\|Y^{-1}(t)\| \to \infty$.

III NONLINEAR SYSTEMS

1. Introduction

Although relatively little is known for general linear systems, there is even less general theory concerning nonlinear ones. Here, the nonlinear system

(3.1) $x' = B(t)x + f(t,x)$

is studied relative to

(3.2) $z' = B(t)z$

and their "adjoint" system

(3.3) $y' = -yB(t)$.

2. Elementary Theory

The most important theorem concerning (3.1) is the following one, which is a restatement of Theorem 1.5.

Theorem 3.1: Let $Y(t)$ be the Fundamental matrix for (3.2). Then every solution $x(t)$ of (3.1) satisfies

$$x(t) = Y(t)x(t_o) + \int_{t_o}^{t} Y(t)Y^{-1}(s)f(s,x)ds.$$

Theorem 3.2: If $B(t) \in \mathcal{B} \cap \mathcal{B}_a$, and $\|f(t,x)\| \leq \lambda(t)\|x\|$ with $\lambda(t) \in L_1(t_o,\infty)$ for all solutions $x(t)$ of (3.1), then all solutions of (3.1) are bounded.

Proof. The following proof was taken from Bellman [1, p. 91]. By Theorem 3.1

$$x(t) = Y(t)x(t_o) + \int_{t_o}^{t} Y(t)Y^{-1}(s)f(s,x)ds.$$

Taking norms

$$\|x(t)\| \leq \|Y(t)\| \cdot \|x(t_o)\| + $$
$$+ \int_{t_o}^{t} \|Y(t)\| \cdot \|Y^{-1}(s)\| \cdot \|f(s,x)\| \, ds.$$

Hence

$$\|x(t)\| \leq K_1\|x(t_o)\| + \int_{t_o}^{t} K_1 \cdot K_2\lambda(s) \|x(s)\| \, ds.$$

Applying Theorem 1.1

$$\|x(t)\| \leq K_1 \|x(t_o)\| \exp[K_1K_2 \int_{t_o}^{t} \lambda(s)ds] < \infty .$$

The following theorem is a sort of converse to Theorem 3.2.

Theorem 3.3: Let $X(t)$ be a solution matrix for (3.1) such that $X^{-1}(t)$ exists. Further suppose that $X(t)$ and $X^{-1}(t)$ are bounded by some real number m. Under the above assumptions, if $\|f(t,x)\| \leq \lambda(t)\|x(t)\|$

with $\lambda(t) \in L_1(t_o, \infty)$ for all solutions $x(t)$ of (3.1), then $B(t) \in \mathcal{B} \cap \mathcal{B}_a$.

Proof. If $y(t)$ satisfies (3.3), then

$$(yX)' = yf(t,X)$$

where $f(t,X)$ is a matrix. Hence

$$(yX)(t) = (yX)(t_o) + \int_{t_o}^{t} y(s)f(s,X)ds$$

$$y(t) = (yX)(t_o)X^{-1}(t) + \int_{t_o}^{t} y(s)f(s,X)X^{-1}(t)ds.$$

Therefore

$$\|y(t)\| \leq \|y(t_o)\| \cdot \|X(t_o)\| \cdot \|X^{-1}(t)\| + $$
$$+ \int_{t_o}^{t} \|y(s)\| \cdot \|f(s,X)\| \cdot \|X^{-1}(t)\| \, ds$$

$$\|y(t)\| \leq \|y(t_o)\| \cdot \|X(t_o)\| \cdot m + $$
$$+ \int_{t_o}^{t} \|y(s)\| \cdot \lambda(s) \, \|X(s)\| \cdot m \, ds$$

$$\|y(t)\| \leq \|y(t_o)\| \cdot \|X(t_o)\| \cdot m + \int_{t_o}^{t} \|y(s)\| \cdot \lambda(s) \cdot m \cdot m ds.$$

Applying Theorem 1.1 gives

$$\|y(t)\| \leq \|y(t_o)\| \cdot \|X(t_o)\| \cdot m \exp[m^2 \int_{t_o}^{t} \lambda(s)ds]$$
$$< \infty.$$

Hence $B(t) \in \mathcal{B}_a$.

It remains to show that $B(t) \in \mathcal{B}$. Since $(XX^{-1})' = 0$, it is clear that $X^{-1}(t)$ satisfies

$$(X^{-1})' = -X^{-1}B(t) - X^{-1}f(t,X)X^{-1}.$$

Hence from (3.2)

$$(X^{-1}z)' = - X^{-1}f(t,X)X^{-1}z$$

$$(X^{-1}z)(t) = (X^{-1}z)(t_o) - \int_{t_o}^{t} X^{-1}(s)f(s,X)X^{-1}(s)z(s)ds.$$

Therefore

$$z(t) = X(t)(X^{-1}z)(t_o) - \int_{t_o}^{t} X(t)X^{-1}(s)f(s,X)X^{-1}(s)z(s)ds$$

$$\|z(t)\| \leq \|X(t)\|\cdot\|(X^{-1}z)(t_o)\| +$$

$$+ \int_{t_o}^{t} \|X(t)\|\cdot\|X^{-1}(s)\|\cdot\|f(s,X)\|\cdot \|X^{-1}(s)\|\cdot\|z(s)\|ds$$

$$\|z(t)\| \leq m\,\|(X^{-1}z)(t_o)\| + \int_{t_o}^{t} m\cdot m\cdot\lambda(s)\cdot m\cdot m\|z(s)\|ds.$$

By Theorem 1.1

$$\|z(t)\| \leq \|(X^{-1}z)(t_o)\|\cdot m \exp[m^4 \int_{t_o}^{t} \lambda(s)ds] < \infty.$$

Hence $B(t) \in \mathcal{B}$, and the theorem is complete.

The following definition is now needed.

Definition: An nx1 vector $x(t)$ is said to be bounded with respect to a 1xn vector $y(t)$ if there exists a real number m such that $|(yx)(t)| \leq m$ for all $t \geq t_o$.

<u>Theorem 3.4</u>: If $x(t)$ is a solution of (3.1) with the property that $|yf(t,x)| \leq \lambda(t)|(yx)(t)|$ with $\lambda(t) \in L(t_o,\infty)$ for some solution $y(t)$ of (3.3), then $x(t)$ is bounded with respect to $y(t)$. Furthermore, under the above hypothesis, if $(yx)(t_o) = 0$, then $(yx)(t) \equiv 0$.

<u>Proof</u>. If $x(t)$ and $y(t)$ are solutions of (3.1) and (3.3) respectively, then

$$(yx)' = yf(t,x)$$

or

$$(yx)(t) = (yx)(t_o) + \int_{t_o}^{t} y(s)f(s,x)ds.$$

Hence

$$|(yx)(t)| \leq |(yx)(t_o)| + \int_{t_o}^{t} |y(s)f(s,x)|ds$$

$$|(yx)(t)| \leq |(yx)(t_o)| + \int_{t_o}^{t} \lambda(s)|(yx)(s)|ds.$$

Applying Theorem 1.1

$$|(yx)(t)| \leq |(yx)(t_o)| \exp\left[\int_{t_o}^{t} \lambda(s)ds\right]$$

$$|(yx)(t)| \leq |(yx)(t_o)| \exp\left[\int_{t_o}^{\infty} \lambda(s)ds\right]$$

$$\leq |(yx)(t_o)|\cdot m$$

and the results follow.

<u>Theorem 3.5</u>: If $\|f(t,x)\| \leq \lambda_1(t)\|x(t)\|$ for all solutions $x(t)$ of (3.1), and

$\| y(t) \| \cdot \| x(t) \| \leq \lambda_2(t) |(yx)(t)|$ for some solution $y(t)$ of (3.3) with $\lambda_1(t) \cdot \lambda_2(t) \in L_1(t_o, \infty)$, then all solutions $x(t)$ of (3.1) are bounded with respect to $y(t)$.

<u>Proof</u>. Clearly

$$(yx)' = yf(t,x)$$

$$(yx)(t) = (yx)(t_o) + \int_{t_o}^{t} y(s)f(s,x)ds.$$

Therefore

$$|(yx)(t)| \leq |(yx)(t_o)| + \int_{t_o}^{t} \|y(s)\| \cdot \|f(s,x)\| \, ds$$

$$|(yx)(t)| \leq |(yx)(t_o)| + \int_{t_o}^{t} \lambda_1(s) \, \|y(s)\| \cdot \|x(s)\| ds$$

$$|(yx)(t)| \leq |(yx)(t_o)| + \int_{t_o}^{t} \lambda_1(s)\lambda_2(s) |(yx)(s)| ds.$$

Applying Theorem 1.1

$$|(yx)(t)| \leq |(yx)(t_o)| \exp[\int_{t_o}^{t} \lambda_1(s)\lambda_2(s)ds] < \infty .$$

<u>Theorem 3.6</u>: If there exists a solution $y(t)$ of (3.3) such that $yf(t,x) = \lambda(t)(yx)(t)$ for all solutions $x(t)$ of (3.1), then

$$\|y(t)\| \cdot \| x(t) \| \geq |(yx)(t_o)| \exp \left(\int_{t_o}^{t} \lambda(s)ds \right).$$

<u>Proof</u>. Since

$$(yx)' = yf(t,x) = \lambda(t)(yx)(t)$$

$$(yx)(t) = (yx)(t_o) \exp\left(\int_{t_o}^{t} \lambda(s)ds\right).$$

Therefore

$$\|y(t)\| \cdot \|x(t)\| \geq \|(yx)(t)\| = |(yx)(t)| =$$

$$= |(yx)(t_o)| \exp\left(\int_{t_o}^{t} \lambda(s)ds\right).$$

The following examples illustrate the uses of the above theorems.

Example 3.1: Consider the first order nonlinear differential equation

(3.4) $\quad x' = a(t)x + (h(t) \cos^n x)x$

where $h(t) \in L_1(t_o, \infty)$ and n is a positive integer, and its "adjoint" system

(3.5) $\quad y' = - a(t)y.$

Integrating (3.4) gives

$$x(t) = x(t_o) + \int_{t_o}^{t} [a(s) + h(s)\cos^n x]x(s)ds.$$

Hence

$$|x(t)| \leq |x(t_o)| + \int_{t_o}^{t} [|a(s)| + |h(s)|]|x(s)| \, ds.$$

Applying Theorem 1.1

$$|x(t)| \leq |x(t_o)| \exp\left[\int_{t_o}^{t} |a(s)|ds\right] \exp\left[\int_{t_o}^{t} |h(s)|ds\right]$$

$$|x(t)| \leq |x(t_o)| \exp[\int_{t_o}^{t} |a(s)|ds] \exp[\int_{t_o}^{\infty} |h(s)|ds]$$

$$(3.6) \qquad |x(t)| \leq m|x(t_o)| \exp[\int_{t_o}^{t} |a(s)|ds]$$

where $m = \exp[\int_{t_o}^{\infty} |h(s)|ds]$.

Applying Theorem 3.4 to (3.4) with $f(t,x) = (h(t)\cos^n x)x$ gives

$$|(yx)(t)| \leq |(yx)(t_o)|m.$$

In this particular case

$$y(t) = y(t_o) \exp\left(\int_{t_o}^{t} -a(s)ds\right).$$

Let $y(t_o) = 1$. Then

$$|x(t)| \exp\left(\int_{t_o}^{t} -a(s)ds\right) \leq m|x(t_o)|$$

or

$$(3.7) \qquad |x(t)| \leq m|x(t_o)| \exp\left(\int_{t_o}^{t} a(s)ds\right).$$

Clearly, (3.7) is a much better bound than (3.6) for the solutions of (3.1).

<u>Example 3.2</u>: Let $f(t,x) = \text{diag}\{\alpha(t)\}x(t)$ in (3.1) where

$$\text{diag}\{\alpha(t)\} = \begin{pmatrix} \alpha(t) & 0 & \cdots & 0 \\ 0 & \alpha(t) & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \alpha(t) \end{pmatrix}.$$

If $(yx)(t_0) \neq 0$ for some solution $y(t)$ of (3.3), then the following remarks follow from Theorem 3.6.

<u>Remark 1</u>. If $B(t) \in \mathcal{B}_a$, then

$$\|x(t)\| \geq m \exp\left(\int_{t_0}^{t} \alpha(s)ds \right).$$

Hence, if $\lim_{t\to\infty} \int_{t_0}^{t} \alpha(s)ds = \infty$, then $\|x(t)\| \to \infty$.

Also, if $\lim_{t\to\infty} \int_{t_0}^{t} \alpha(s)ds > -\infty$, then $\|x(t)\|$ is bounded away from zero.

<u>Remark 2</u>. If $B(t) \in \mathcal{B}_a$, then $B(t) + \operatorname{diag}\{\alpha(t)\} \equiv C(t) \in \mathcal{B}$ only if

$\lim_{t\to\infty} \int_{t_0}^{t} \alpha(s)ds < \infty$.

<u>Example 3.3</u>: Consider the system (3.1) with

$$B(t) = \begin{pmatrix} 0 & \alpha(t) \\ 0 & \beta(t) \end{pmatrix} \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$f(t,x) = \lambda(t) \begin{pmatrix} f_1(t,x) \\ x_2 \end{pmatrix}.$$

Then

$$(3.8) \qquad y(t) = \left(0,\ \exp\left(\int_{t_0}^{t} -\beta(s)ds \right)\right)$$

is a solution of (3.3). Furthermore

$$yf(t,x) = \lambda(t)(yx)(t).$$

Hence

$$\|y(t)\| \cdot \|x(t)\| \geq k \exp\left(\int_{t_o}^{t} \lambda(s)ds\right)$$

for any solution $y(t)$ of (3.3). Using that $y(t)$ given by (3.8) gives

$$\|x(t)\| \geq k \exp\left(\int_{t_o}^{t} [\lambda(s) + \beta(s)]ds\right).$$

It should be pointed out that the idea of studying the system (3.1) relative to (3.2) and (3.3) can lead to direct integration if $B(t)$ is "reasonable" enough. For instance, in the last example, using the $y(t)$ given in (3.8) yields

$$\left(0, \exp\left(\int_{t_o}^{t} -\beta(s)ds\right)\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (0,1)\begin{pmatrix} x_1(t_o) \\ x_2(t_o) \end{pmatrix}\exp\left(\int_{t_o}^{t} \lambda(s)ds\right).$$

Therefore

$$x_2(t)\exp\left(\int_{t_o}^{t} -\beta(s)ds\right) = x_2(t_o)\exp\left(\int_{t_o}^{t} \lambda(s)ds\right)$$

$$x_2(t) = x_2(t_o) \exp\left(\int_{t_o}^{t} [\lambda(s) + \beta(s)]ds\right).$$

Depending on $f_1(t,x)$, it may also be possible to integrate

$$x_1'(t) = \alpha(t)x_2(t) + \lambda(t)f_1(t,x).$$

3. Extensions

In this section several results obtained in the last section are generalized. As a first stip in this direction, consider the system

$$(3.9) \qquad w' = B(t)w + f(t,w) + g(t,w)$$

and the system (3.1). Suppose that there exists a solution matrix $X(t)$ of (3.1) such that $X^{-1}(t)$ exists for all $t \geq t_o$. Clearly

$$(X^{-1}w)' = X^{-1}[f(t,w) - f(t,X)X^{-1}w + g(t,w)]$$

$$(X^{-1}w)(t) = (X^{-1}w)(t_o) + \int_{t_o}^{t} X^{-1}(s)[f(s,w)-f(s,X)X^{-1}w + g(s,w)]ds.$$

Therefore

$$(3.10) \quad w(t) = X(t)X^{-1}(t_o)w(t_o) + \int_{t_o}^{t} X(t)X^{-1}(s)[f(s,w)-f(s,X)X^{-1}w+g(s,w)]ds.$$

Remark 1. If $f(t,w) \equiv 0$ in (3.1), then $X^{-1}(t)$ exists by Theorem 1.3, and if $X(t_o) = 1$, then (3.10) reduces to the usual variation of parameters formula.

Remark 2. If $g(s,w) \equiv 0$, then (3.10) becomes

$$(3.11)\quad w(t) = X(t)X^{-1}(t_o)w(t_o) + $$

$$+ \int_{t_o}^{t} X(t)X^{-1}(s)[f(s,w) - f(s,X)X^{-1}w]ds.$$

Hence, under the above assumptions, all solutions of (3.1) satisfy (3.11).

<u>Remark 3</u>. If $f(t,w) \equiv 0 \equiv g(t,w)$, or if $g(t,w) \equiv 0$ and $f(t,w) \equiv B(t)w$, then (3.10) reduces to

$$(3.12)\quad w(t) = X(t)X^{-1}(t_o)w(t_o)$$

which is the usual result for linear systems.

<u>Theorem 3.7</u>: Let $X(t)$ be a solution matrix for (3.1) such that $X^{-1}(t)$ exists for all $t \geq t_o$. Further suppose that both $X(t)$ and $X^{-1}(t)$ are bounded and that

$$\|f(s,w) - f(s,X)X^{-1}w + g(s,w)\| \leq \lambda(t)\|w\|$$

for all solutions $w(t)$ of (3.9) with $\lambda(t) \in L_1(t_o,\infty)$. Then, all solutions of (3.9) are bounded.

<u>Proof</u>. Any solution $w(t)$ of (3.9) satisfies (3.10)

$$w(t) = X(t)X^{-1}(t_o)w(t_o) + $$

$$+ \int_{t_o}^{t} X(t)X^{-1}(s)[f(s,w) - f(s,X)X^{-1}w + g(s,w)]ds.$$

Taking norms

$$\|w(t)\| \leq \|X(t)\| \cdot \|X^{-1}(t_o)\| \cdot \|w(t_o)\| +$$

$$\int_{t_o}^{t} \|X(t)\| \cdot \|X^{-1}(s)\| \cdot \|f(s,w) - f(s,X)X^{-1}w + g(s,w)\| ds$$

$$\|w(t)\| \leq m \cdot \|X^{-1}(t_o)\| \cdot \|w(t_o)\| +$$

$$+ \int_{t_o}^{t} m \cdot m \cdot \lambda(s) \|w(s)\| ds.$$

Applying Theorem 1.1

$$\|w(t)\| \leq m \|X^{-1}(t_o)\| \cdot \|w(t_o)\| \exp[m^2 \int_{t_o}^{t} \lambda(s) ds] < \infty$$

and the theorem is complete.

The following examples illustrate some of the possible uses of the results of this section.

<u>Example 3.4</u>: Consider the differential equation

(3.13) $$w' = w + \left(\frac{e^t}{1 - e^t} + h(t)\right) w$$

where $h(t) \in L_1(t_o, \infty)$. Rewriting (3.13) gives

(3.14) $$w' = w + w^2 + \left(\frac{e^t}{1 - e^t} + h(t)\right) w - w^2 \quad .$$

In this case, (3.1) becomes

(3.15) $$x' = x + x^2.$$

Now, $x(t) = \frac{e^t}{1 - e^t}$ is a solution of (3.15). Since $x(t) < 0$ for all $t \geq t_o > 0$, $x^{-1} = \frac{1 - e^t}{e^t}$ exists for all $t \geq t_o > 0$. By (3.10)

$$w(t) = \left(\frac{e^t}{1-e^t}\right)\left(\frac{1-e^{t_o}}{e^{t_o}}\right)w(t_o) +$$

$$+ \int_{t_o}^{t} \left(\frac{e^t}{1-e^t}\right)\left(\frac{1-e^s}{e^s}\right)[w^2 - \frac{e^s}{1-e^s}w - w^2 + \frac{e^s}{1-e^s}w + h(s)w]ds.$$

Hence

$$w(t) = k\frac{e^t}{1-e^t} + \int_{t_o}^{t}\left(\frac{e^t}{1-e^t}\right)\left(\frac{1-e^s}{e^s}\right)h(s)w(s)ds.$$

Therefore, since $\frac{e^t}{1-e^t}$ and $\frac{1-e^t}{e^t}$ are bounded for

$t \geq t_o > 0$

$$|w(t)| \leq km + \int_{t_o}^{t} m^2|h(s)|\ |w(s)|\ ds.$$

Applying Theorem 1.1

$$|w(t)| \leq km\ \exp[m^2 \int_{t_o}^{t}|h(s)|ds] < \infty$$

and all solutions of (3.13) are bounded. This also follows from Theorem 2.3 since (3.13) is linear.

Example 3.5: Consider the system (3.1) with $f(t,x) = h(x)x$. Let $X(t)$ be a solution matrix for the above system with components $x_{ij}(t)$, and let $x^i(t)$ be the i^{th} column vector of $X(t)$. Then

$$(3.16)\quad (\det X(t))' = \begin{vmatrix} x'_{11} & x'_{12} & \cdots & x'_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \\ x'_{n1} & x'_{n2} & \cdots & x'_{nn} \end{vmatrix} .$$

The first determinant in (3.16) becomes

$$\begin{vmatrix} \sum_k b_{1k}x_{k1} + h(x^1)x_{11} & \sum_k b_{1k}x_{k2} + h(x^1)x_{12} & \cdots & \sum_k b_{1k}x_{kn} + h(x^1)x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}$$

This determinant is unchanged if from the first row there is subtracted b_{12} times the second row plus b_{13} times the third row up to b_{1n} times the n^{th} row. This gives

$$\begin{vmatrix} b_{11}x_{11}+h(x^1)x_{11} & b_{11}x_{12}+h(x^1)x_{12} & \cdots & b_{11}x_{1n}+h(x^1)x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}$$

which is

$$b_{11} \det X(t) + h(x^1) \det X(t).$$

Applying a similar procedure to the remaining determinants in (3.16) gives

$$(3.17) \quad (\det X(t))' = (\operatorname{tr} B)\det X(t) + \sum_{i=1}^{n} h(x^i)\det X(t).$$

Integrating gives

$$(3.18) \quad \det X(t) = \det X(t_o) \exp\left(\int_{t_o}^{t} [\operatorname{tr} B(s) + \sum_{i=1}^{n} h(x^i(s))]ds \right).$$

(If $h(x) \equiv 0$, then (3.17) and (3.18) become (1.4) and (1.5), respectively.) Hence, if

$$\lim_{t\to\infty} \int_{t_o}^{t} [\operatorname{tr} B(s) + \sum_{1}^{n} h(x^i(s))ds] > -\infty , \quad \text{then} \quad X^{-1}(t)$$

exists for $t \geq t_o$, and the results of this section are applicable if $X(t)$ is bounded.

Example 3.6: Consider the first order nonhomogeneous differential equation

$$y' + q(t)y = p(t).$$

Let $z(t)$ satisfy

$$z' = - q(t)z \qquad z(t_o) = 1.$$

Then, by (3.10)

$$y(t) = z(t)y(t_o) +$$

$$+ \int_{t_o}^{t} z(t) \cdot \frac{1}{z(s)} [q(s)y - q(s)y + p(s)]ds.$$

Since

$$z(t) = \exp\left(\int_{t_o}^{t} - q(u)\, du \right)$$

$$y(t) = \exp\left(- \int_{t_o}^{t} q(u) du \right) \left[y(t_o) + \right.$$

$$\left. + \int_{t_o}^{t} [p(s) \exp\left(- \int_{t_o}^{s} q(u) du \right)] ds \right] \quad .$$

IV OSCILLATIONS

1. Introduction

This chapter deals with the zeros of various differential systems. There are two principal reasons for considering these results here.

1. These results can give sufficient results for a solution matrix $Y(t)$ being unbounded; i.e., if $\|Y^{-1}(t)\| \to 0$ as $t \to t^*$, then $\|Y(t)\| \to \infty$ as $t \to t^*$.

2. If $X(t)$ is a solution matrix for the system

$$(3.1) \qquad x' = B(t)x + f(t,x)$$

then the question of the existence of $X^{-1}(t)$ reduces to the study of the zeros of $u(t) \equiv$ $\equiv \det X(t)$ if $X(t)$ is bounded. Let x^i be the i^{th} column vector of $X(t)$ and f_k be the k^{th} element of $f(t,x)$. Let

$$(4.1) \qquad \varphi(t,X) = \sum_{k=1}^{m} \det(X)_k$$

where $(X)_k = X$, except for the k^{th} row which has been replaced by $f_k(x^i)$. Then by (3.17)

$$(4.2) \qquad u' = (\operatorname{tr} B)u + \varphi(t,X).$$

Combining (3.1) and (4.2) leads to the $(n+1)^{st}$

order differential system

$$(4.3) \qquad \begin{cases} x' = B(t)x + f(t,x) \\ u' = (\text{tr } B)u + \varphi(t,X) \end{cases} .$$

Hence, if a solution of (4.3) has no zeros, then the results of Section 3.3 may be applicable.

2. Oscillations

Following Butewski [16], a vector $x(t)$ is said to be *oscillatory*, if given T $(0 \leq T < \infty)$, there exists a finite $t^* \geq T$ such that $\|x(t^*)\| = 0$. Otherwise, $x(t)$ is said to be *non-oscillatory*.

The first result applies to linear systems only.

Theorem 4.1: Consider the system (2.1) where the components $b_{ij}(t)$ of B satisfy

$$(4.4) \qquad \begin{cases} b_{12}(t) > 0 & b_{i1}(t) > 0 \qquad 2 \leq i \leq n \\ b_{ij}(t) \geq 0 & i \neq j. \end{cases}$$

(The reason for this requirement is to insure that there is at least one non-diagonal element in each row which is strictly positive.) If $x(t)$ is a solution of (2.1) such that $x_i(t_o) > 0$, $1 \leq i \leq n$, then $x_i(t) > 0$, $t \geq t_o$. Furthermore, if $b_{ii}(t) \geq 0$, $1 \leq i \leq n$, then $x_i'(t) > 0$, $t \geq t_o$.

Proof. Define

(4.5) $y_i(t) = x_i(t) \exp\left(-\int_{t_o}^{t} b_{ii}(s)ds\right)$

(4.6) $s_{ij}(t) = (1-\delta_{ij})b_{ij}(t) \exp\left(\int_{t_o}^{t}[b_{jj}(s)-b_{ii}(s)]ds\right)$

where $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases}$ and let

(4.7) $y' = S(t)y.$

Then (2.1) is equivalent to (4.7) with y_i and s_{ij} given by (4.5) and (4.6), respectively. This is easily seen, for since

$$y_i' = \sum_{j=1}^{n} s_{ij}y_j = \sum_{\substack{j=1 \\ j\neq i}}^{n} b_{ij}\exp\left(\int_{t_o}^{t} b_{jj}-b_{ii}\right)x_j\exp\left(-\int_{t_o}^{t} b_{jj}\right)$$

then

$$\left[x_i \exp\left(-\int_{t_o}^{t} b_{ii}\right)\right]' = \sum_{\substack{j=1 \\ j\neq i}}^{n} b_{ij}x_j \exp\left(-\int_{t_o}^{t} b_{ii}\right)$$

or

$$x_i' \exp\left(-\int_{t_o}^{t} b_{ii}\right) - b_{ii}x_i \exp\left(-\int_{t_o}^{t} b_{ii}\right) =$$

$$= \sum_{\substack{j=1 \\ j\neq i}}^{n} b_{ij}x_j \exp\left(-\int_{t_o}^{t} b_{ii}\right).$$

Hence

$$x_i' = \sum_{j=1}^{n} b_{ij}x_j.$$

Let $y(t)$ be a solution of (4.7). Then $y_i(t_o) = x_i(t_o) > 0$ by (4.4) and (4.5). By (4.4) and (4.6)

$$(4.8) \quad \begin{cases} s_{12}(t) > 0 & s_{i1}(t) > 0 \quad i = 2,\ldots,n \\ s_{ij}(t) \geq 0 & i \neq j \\ s_{ii}(t) \equiv 0 & 1 \leq i \leq n. \end{cases}$$

Using these relations, $y_i'(t_o) > 0$, $1 \leq i \leq n$. Therefore, there exists an $\varepsilon > 0$ such that

$$(4.9) \quad \begin{cases} y_i(t) > 0 \\ y_i'(t) > 0 \end{cases} \qquad t \in [t_o, t_o+\varepsilon).$$

Actually, (4.9) holds for all $t \geq t_o$, for suppose the contrary. Without loss of generality, it may be assumed that $y_1'(t)$ is the first of the $y_i(t)$ that vanishes and that T_1 is its first zero. Hence, $y_1'(T_1) = 0$ and $y_i(t) > 0$ in $[t_o, T_1]$ for $1 \leq i \leq n$. However, by (4.7) and (4.8), $y_1'(T_1) > 0$, a contradiction. Hence, T_1 does not exist, and (4.9) is valid for $t \geq t_o$. Finally, by (4.5) and (4.9), $x_i(t) > 0$, and the first assertion is proved.

The second half of the theorem is easily shown by differentiating (4.5)

$$y_i'(t) = (x_i' - b_{ii}x_i) \exp\left(- \int_{t_o}^{t} b_{ii}\right).$$

Since $y_i'(t) > 0$

$$x_i' > b_{ii}x_i$$

and the theorem is complete.

Example 4.1: Consider the n^{th} order equation (2.9) where the $a_i(t) < 0$, $t \geq t_o$. Transform (2.9) into the system (2.1) with $B(t)$ given by (2.10). Theorem 4.1 is now applicable; hence, any solution $u(t)$ of (2.7) which satisfies $u^{(j)}(t_o) > 0$, $0 \leq j \leq n-1$, also satisfies $u^{(j)}(t) > 0$, $t \geq t_o$. That is, under the above assumptions, the solutions of (2.9) are non-oscillatory.

Theorem 4.2: Consider the system

$$(4.10) \quad x' = f(t,x)$$

where the components f_i of f are continuous for $t \geq t_o > 0$ and $-\infty < x_i < \infty$, $1 \leq i \leq n$. Let a unique solution of (4.10) pass through each point in (t,x) space. Let $x(t)$ be any nontrivial solution of (4.10). If for each fixed i, $1 \leq i \leq n$

$$f_i(t,x) \neq 0 \quad \text{when} \quad |x_i| < \infty \quad \text{and} \quad \max_{\substack{1 \leq j \leq n \\ j \neq i}} |x_j| \neq 0$$

and

$$f_i(t,x) = 0 \quad \text{when} \quad |x_i| < \infty \quad \text{and} \quad \max_{\substack{1 \leq j \leq n \\ j \neq i}} |x_j| = 0$$

then the following statements are valid.

i) If one of the x_i's is identically zero in $[\alpha, \beta]$, then all of them are identically zero in (α, β).

ii) If one of the x_i's is non-oscillatory, then all of then are non-oscillatory.

iii) The zeros of each $x_i(t)$ are simple.

iv) The zeros of the x_j, $1 \leq j \leq n$, separate one another.

v) Each $x_i(t)$ possesses only a finite number of zeros in any finite interval $[\alpha, \beta]$.

Proof. i) Suppose $x_1(t) \equiv 0$ for $t \in [\alpha, \beta]$. Then $x_1'(t) \equiv 0$ for $t \in (\alpha, \beta)$ implies that the remaining $x_i(t) = 0$ for $t \in (\alpha, \beta)$.

ii) Suppose that $x_1(t) \neq 0$ for $t \geq T$. Then

$$x_i'(t) = f_i(t, x_1, \ldots, x_n) > 0 \quad \text{for} \quad t \geq T.$$

Hence, for $2 \leq i \leq n$, $x_i(t)$ is monotone for $t \geq T$.

iii) Suppose that $x_1(t^*) = 0 = x_1'(t^*)$. If this is the case, then $x_2(t^*) = \ldots = x_n(t^*) = 0$. By the assumption of uniqueness, this would imply that $x(t) \equiv 0$.

iv) Suppose that $x_1(t_1) = x_1(t_2) = 0$, and $x_1(t) \neq 0$ for $t_1 < t < t_2$. By Rolle's Theorem, there exists T, $t_1 < T < t_2$ such that $x_1'(T) = 0$. Hence $x_i(T) = 0$, $2 \leq i \leq n$. Suppose that there exists two zeros of some $x_j(t)$ between t_1 and t_2 at the points T_1 and T_2. Then $x_j(T_i) = 0$, i=1,2. If this were the case, then by Rolle's Theorem there would exist τ, $t_1 < T_1 < \tau < T_2 < t_2$, such that $x_j'(\tau) = 0$. Hence, $f_j(\tau,x) = 0$, and $x_1(\tau) = 0$, a contradiction.

v) Suppose that $x_1(t) = 0$ for $t \in \{s_\nu\} \subset [\alpha,\beta]$. Then, there exists $s \in [\alpha,\beta]$ such that $s_\nu \to s$. By continuity, $x_1(s) = 0$, and

$$\frac{x_1(s_\nu) - x_1(s)}{s_\nu - s} = 0.$$

Therefore, $x_1'(s) = 0$. Then by iii), $x(t) \equiv 0$, and the theorem is complete.

Theorem 4.3: In addition to the hypotheses of the previous theorem, further suppose that for some $j \neq i$

$$f_i(t,x) > 0 \quad \text{for} \quad t \geq t_o > 0 \quad \text{and} \quad \begin{cases} |x_i| < \infty & \\ 0 \leq x_k < \infty & 1 \leq k \leq n,\ k \neq i,j \\ 0 < x_j < \infty & \end{cases}$$

and

$$f_i(t,x) < 0 \quad \text{for} \quad t \geq t_o > 0 \quad \text{and} \quad \begin{cases} |x_i| < \infty & \\ -\infty < x_k \leq 0 & 1 \leq k \leq n,\ k \neq i,j \\ -\infty < x_j < 0 & \end{cases}.$$

Then the system (4.10) is non-oscillatory.

Proof. If $x_1(t)$ is non-oscillatory, then by ii) of Theorem 4.2, the system (4.10) is non-oscillatory.

Hence, it may by assumed that given $t \geq t_o > 0$, there exist $\tau_1, \tau_2 \geq t$ such that $x_1(\tau_1) = x_1(\tau_2) = 0$. By Rolle's Theorem, there exists T, $\tau_1 < T < \tau_2$, such that $x_1'(T) = 0$. Hence, by the above assumptions, $x_j(T) = 0$ for $2 \leq j \leq n$.

<u>Case 1</u>. Let $x_1(T) > 0$ and $x_j(T) = 0$ for $2 \leq j \leq n$. Since $x_1(T) > 0$, it is clear that $x_j'(T) > 0$ for $2 \leq j \leq n$. Hence, by continuity, there exists a $\tau > T$ such that

$$\begin{cases} x_k(\tau) > 0 \\ x_k'(\tau) > 0 \end{cases} \qquad \text{for } 1 \leq k \leq n$$

Let $x_{i_o}'(t)$ be the first of the $x_i'(t)$ which vanishes and τ_o be its first zero. Hence

$$\begin{cases} x_{i_o}'(\tau_o) = 0 & \\ x_j'(\tau_o) > 0 & j \neq i_o \\ x_k(\tau_o) > 0 & 1 \leq k \leq n. \end{cases}$$

But $x_{i_o}'(\tau_o) = f_{i_o}(\tau_o, x) > 0$, a contradiction. Hence, τ_o does not exist and

$$\begin{cases} x_k(t) > 0 \\ x_k'(t) > 0 \end{cases} \qquad 1 \leq k \leq n \qquad t \geq \tau.$$

<u>Case 2</u>. Let $x_1(T) < 0$ and $x_j(T) = 0$ for $2 \leq j \leq n$. The proof of this case is the same as that for case 1 with $x_i(t)$ and $x_i'(t)$ replaced by

$-x_i(t)$ and $-x_i'(t)$, respectively. Hence the theorem is proved.

<u>Application</u>: Let $X(t)$ be a solution matrix for (3.1), and consider the system (4.3). Let

$$B^*(t) = \begin{pmatrix} B(t) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \ \dots \ 0 & \operatorname{tr} B(t) \end{pmatrix}$$

and

$$f^*(t,w) = \begin{pmatrix} f_1(t,w) \\ \vdots \\ f_n(t,w) \\ \varphi(t,X(t)) \end{pmatrix} .$$

Then (4.3) can be expressed as the $(n+1)^{st}$ order system

(4.11) $w' = B^*(t)w + f^*(t,w).$

If (4.11) satisfies the conditions of Theorem 4.3, then there is a τ such that $X^{-1}(t)$ exists for $t > \tau$, and the results of Chapter three are applicable for the system (3.1)

V SUMMARY AND CONCLUSIONS

1. Summary

In this paper an attempt was made to secure a comprehensive theory for linear and nonlinear differential systems under most general conditions, with the exception of Chapter four. Because of this lack of specialization, some of the results obtained may appear to be pathelogical; however, their application to a specific problem should prove to be quite fruitful.

In addition to the necessary preliminaries, a generalized form of Bellman's "fundamental lemma" [1, p. 35] and the "correct" proof of Lagrange's variation of parameters formula were presented in Chapter one.

In Chapter two, linear systems were studied with respect to the sets $\mathcal{B}$ and $\mathcal{Z}$ (see Chapter one). Here an important necessary and sufficient theorem of Conti [19] was generalized. Moreover, a stronger sufficiency theorem was shown, even though one of the original conditions was dropped. This allowed an obvious extension to the set $\mathcal{Z}$, which was unattainable via the original theorem.

Nonlinear systems were studied in Chapter three by means of their accompaning integral equations. A converse theorem to a standard result was given (see Theorem 3.3). The concept of one vector being bounded with respect to another vector was introduced, and a series of theorems concerning this definition followed. Sharper upper bounds were obtained for some cases, while heretofore unknown lower bounds were derived. Finally, Lagrange's variation of parameters formula was extended to a certain class of nonlinear systems.

Chapter four concerned oscillation theorems which, though important in their own right, were presented in order to strenghten Section 3.3.

Throughout this paper, an effort was made to illustrate the theory with simple, yet meaningful, examples.

2. Conclusions

It seems unreasonable to expect to obtain a strong theory for differential systems by way of the methods used in this thesis. However, the following remarks are in order.

1. Although a generalization of Bellman's " fundamental lemma " was given in Chapter one, only the weaker form was used. Perhaps sharper results could be obtained through the stronger form. Bihari [13] has given a different generalization of the above theorem which also could be brought to bear on the problem.

2. Bellman [9, 10, 11] has obtained results which guarantee the existence of a matrix of (bounded) solutions for certain nonlinear systems. These theorems, when used in conjunction with Chapter three, should prove useful.

3. It is conjectured that a clever application of some of the results of Chapter three (in particular, see the proof of Theorem 3.6) could lead to a direct integration of the linear differential equation (2.9). Some special cases have been solved, but the general result is still unknown.

4. Finally, most of the results in this thesis should carry over into the fields of differential-difference equations and control theory.

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