CONTROLABILITY OF HIGHER ORDER LINEAR SYSTEMS

by

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<u>Introduction</u> We consider in this paper a dynamical system whose evolution in time is described by a second-order linear differential equation in a complex Banach space $E = \ u , v, \ldots$

$$u''(t) = Au(t) + Bf(t) \qquad (1)$$

Here A is a linear, possibly unbounded operator with domain $D(A) \subseteq E$ and range in E, B a bounded operator from another Banach space F to E. We shall assume $\rho(A)$, the resolvent set of A to be non-void, i.e. there exists a λ such that $R(\lambda ; A) = (\lambda I - A)^{-1}$ exists and is bounded. The state of the system at time t is given by the pair $(u(t) , u'(t))$ of elements of E; the F-valued function $f(\cdot)$ is the <u>input</u> or <u>control</u> by means of which we govern the system.

The problem of <u>complete controllability</u> consists, roughly speaking, in selecting a control f in a given class $\mathcal{L}$ in such a way that the systems evolutions from a given initial state to the vicinity of a given final state. If the initial state is taken to be $(0 , 0)$ then the problem is that of <u>null controllability</u>. We introduce in Section 1 some results on the theory of the equation (1) and apply them in Section 2 to show that the problem of complete controllability of the system (1) can be reduced to the corresponding one for the first-order system $u' = Au + Bf$ if A satisfies a certain condition (Condition (2.6). Finally, we examine in Section 3 the relation between null and complete controllability for first and second-order systems. An appendix refers a systems described by higher order equations $u^{(n)} = Au + Bf$, $n \geq 3$. We shall use without proofs some results on controllability of first order systems $u' = Au + Bf$; we refer to (<u>6</u>) for proofs and further details.

§1. Let $g(\cdot)$ be a E - valued, strongly continuous function defined for $t \geq 0$. We shall understand by a <u>solution</u> of

$$u''(t) = Au(t) + g(t) \qquad (1.1)$$

an E - valued function $u(\cdot)$ defined and with two continuous derivatives in $t \geq 0$, such that $u(t) \in D(A)$ and Eq. (1.1) is satisfied for all $t \geq 0$. [redacted]

We shall assume that the Cauchy problem for the homogenous equation

$$u''(t) = Au(t) \tag{1.2}$$

is <u>uniformly well posed</u> in $t \geq 0$, i.e. we shall suppose that

(a) There exists a dense subspace D of E such that if $u_0, u_1 \in D$ there exists a solution $u(\cdot)$ of (1.2) with $u(0) = u_0$, $u'(0) = u_1$.

(b) For each $t > 0$ there exists a constant $K_t < \infty$ such that

$$|u(s)| \leq K_t (|u(0)| + |u'(0)|), \quad 0 \leq s \leq t$$

for any solution $u(\cdot)$ of (1.2)

Let $u \in E$, $u(\cdot)$ (resp. $v(\cdot)$) be a solution of Eq (1.2) with $u(0) = u$, $u'(0) = 0$ (resp. $v(0) = 0$, $v'(0) = u$). Define

$$S(t)u = u(t) \quad (\text{resp. } T(t)u = v(t))$$

By virtue of (a) and (b) $S(t)$, $T(t)$ are well defined and bounded for all $t \geq 0$, at least in D. Thus they can be extended to bounded operators in E that we shall denote with the same symbols. It follows from a simple approximation argument that $S(\cdot)$, $T(\cdot)$ are strongly continuous functions of t. We shall call S, T the <u>solution operators</u> of Eq. (1.2)

We take from (<u>4</u>), Section 4 the following properties of A, $R(\lambda; A)$, $S(\cdot)$, $T(\cdot)$.

(c) For each $u \in E$, $t \geq 0$

$$T(t)u = \int_0^t S(s)u \, ds \tag{1.3}$$

(d) There exist constants $w < \infty$, $K < \infty$ such that

$$|S(t)| \leq K e^{wt}, \quad |T(t)| \leq K e^{wt}, \quad t \geq 0$$

(e) $\sigma(A)$, the spectrum of A is contained in the region $\{ \lambda ; \ \operatorname{Re}\lambda \leq w^2 - (\operatorname{Im}\lambda)^2/4w^2 \}$ and

$$R(\lambda^2; A)u = \lambda^{-1} \int_0^\infty e^{-\lambda t} S(t)u \, dt =$$

$$= \int_0^\infty e^{-\lambda t} T(t)u \, dt \tag{1.4}$$

for $\operatorname{Re}\lambda > w$.

With the help of $T(\cdot)$ we can construct solutions of the inhomogenous equation (1.1) . In fact, we have

<u>1.1 LEMMA</u> <u>Let</u> $g(\cdot)$ <u>be continuously differentiable . Then</u>

$$u(t) = \int_0^t T(t - s)g(s)ds \tag{1.5}$$

<u>is a solution of Eq.</u> (1.1) <u>with</u> $u(0) = u'(0) = 0$

Proof: Let $0 \leq t < t'$. We have

$$\frac{u(t') - u(t)}{t' - t} = \frac{1}{t' - t}\int_t^{t'} T(t' - s)g(s)ds +$$

$$+ \int_0^t \left[\frac{T(t' - s) - T(t - s)}{t - t'}\right] g(s)ds = I_1 + I_2$$

Using now Eq. (1.3) and the fact that $|S(\cdot)|$ is bounded on compacts of $[0, \infty)$ (a consequence of the principle of uniform boundedness, (1), Chapter I) we see that $|T(r)| = O(r)$ as $r \to 0$ and thus $I_1 \to 0$ as $t' - t \to 0$. Making use again of (1.3) the integrand in I_2 is seen to converge to $S(t - s)g(s)$ as $t' - t \to 0$; by the theorem of the mean of differential calculus is bounded in norm by

$$\sup_{0 \leq r \leq t'} |S(r)g(s)|$$

Thus u is continuously differentiable in $t \geq 0$ and

$$u'(t) = \int_0^t S(t - s)g(s)ds \tag{1.6}$$

Interchanging now s by $t - s$ in Eq. (1.6) and proceeding similarly as before we see that $u'(\cdot)$ is continuously differentiable in $t \geq 0$ and

$$u''(t) = S(t)g(0) + \int_0^t S(s)g'(t - s)ds \tag{1.7}$$

Let us now compute $Au(t)$. Integrating the expression (1.5) by parts we get

$$u(t) = \int_0^t T(s)g(0)ds + \int_0^t \left(\int_0^s T(r)dr\right) g'(t - s)ds$$

Using now the fact that

$$v_a = \int_0^a T(s)u\, ds$$

is an element of $D(A)$ for any $u \in E$ and $Av_a = S(a)u - u$ ((4) , Section 5) we see that $u(t) \in D(A)$ for all $t \geqslant 0$ and $Au(t) = u''(t) - g(t)$ as desired.

We close this section with another result on the equation (1.2) . Let $E^* = \{ u^* , v^* , \ldots \}$ be the dual space of E ; denote $\langle u^* , u \rangle$ or $\langle u , u^* \rangle$ the value of the functional u^* at the point $u \in E$.

<u>1.2 LEMMA</u> <u>Let</u> E <u>be reflexive. Then the Cauchy problem for the equation</u>

$$(u^*)''(t) = A^*u^*(t) \tag{1.8}$$

<u>is uniformly well posed. If</u> $S^*(\cdot)$, $T^*(\cdot)$ <u>are the solution operators of Eq.</u> (1.8) <u>we have</u> $S^*(t) = (S(t))^*$, $T^*(t) = (T(t))^*$, <u>where</u> S , T <u>are the solution operators of</u> (1.2)

The proof is a consequence of the characterization of operators A for which the Cauchy problem for Eq. (1.2) is well posed ((4), Theorem 5.9) . We shall assume throughout the rest of this paper that E is reflexive so that Lemma 1.2 applies.

§2. Let $E^2 = E \times E$ be the space of all pairs (u_o , u_1) of elements of E endowed with pointwise operations and any of its natural norms, for instance $|(u_o , u_1)| = |u_o| + |u_1|$. The dual space $(E^2)^*$ can be identified algebraically and topologically with the space $(E^*)^2$, application of the functional $u^* = (u_o^* , u_1^*)$ to the element $u = (u_o , u_1)$ being given by

$$\langle u^* , u \rangle = \langle u_o^* , u_o \rangle + \langle u_1^* , u_1 \rangle$$

Let the linear control system

$$u''(t) = Au(t) + Bf(t) \tag{2.1}$$

(we we shall denote by L) be given . We shall assume the class $\mathcal{L}$ of controls to consist of all F - valued infinitely differentiable functions defined in $[0, \infty)$. Call $K_t(L)$, $t \geqslant 0$ the subspace of E^2 consisting of all the pairs (u_o , u_1) ,

$$u_o = \int_0^t T(t - s)Bf(s)ds \ , \quad u_1 = \int_0^t S(t - s)Bf(s)ds \tag{2.2}$$

In wiew of Lemma 1, $K_t(L)$ can be described as the subspace of all pairs $(u(t), u'(t))$, $u(\cdot)$ a solution of Eq. (2.1) with $u(0) = u'(0) = 0$, $f \in \mathcal{L}$ or simply as the subspace of all possible states of the system at time t - the initial state being $(0,0)$ for $t = 0$. We also define $K(L) = \bigcup_{t>0} K_t(L)$. We shall say that the system L is <u>null controllable</u> if $\mathrm{Cl}\, K(L) = E^2$, <u>null controllable at time</u> t_o if $\mathrm{Cl}\, K_{t_o}(L) = E^2$. It is a consequence of the Hahn-Banach theorem that $\mathrm{Cl}\, K(L) = E^2$ if and only if $K(L)^{\perp} = \{(u_o^*, u_1^*) \in (E^*)^2 \mid \langle (u_o^*, u_1^*), (u_o, u_1)\rangle = 0$ for all $(u_o, u_1) \in K(L)\} = \{0\}$, $\mathrm{Cl}\, K_t(L) = E$ if $K_t(L)^{\perp} = 0$, $K_t(L)^{\perp}$ similarly defined.

Our first results are analogous to Proposition 2.1 and Corollary 2.2 of (6)

<u>2.1 LEMMA</u> $(u_o^*, u_1^*) \in K(L)^{\perp}$ $(K_t(L)^{\perp})$ <u>if and only if</u>

$$B^*(T^*(s)u_o^* + S^*(s)u_1^*) = 0 \ , \ 0 \leq s \ (0 \leq s \leq t) \quad (2.3)$$

<u>Proof</u> Assume $(u_o^*, u_1^*) \in K(L)^{\perp}$. Then, for any $f \in \mathcal{L}$

$$0 = \left\langle u_o^*, \int_0^t T(t-s)Bf(s)ds \right\rangle + \left\langle u_1^*, \int_0^t S(t-s)Bf(s)ds \right\rangle =$$

$$\int_0^t \langle B^*(T^*(t-s)u_o^* + S^*(t-s)u_1^*), f(s)\rangle \, ds$$

Taking now $f(s) = y(s)u$, u any element of E, $y(\cdot)$ any scalar-valued function we easily see that (2.3) holds. The reverse implication is clear. The proof is similar for $K_t(L)$.

Let us denote $\rho_o(A)$ the connected component of $\rho(A)$ that contains the half-plane $\mathrm{Re}\,\lambda > w^2$ (w the constant in (d), Section 1)

<u>2.2 COROLLARY</u> $(u_o^*, u_1^*) \in K(L)$ <u>if and only if</u>

$$B^*R(\lambda; A^*)(u_o^* + \lambda^{\frac{1}{2}}u_1^*) = 0 \quad \text{for } \lambda \in \rho_o(A) \quad (2.4)$$

($\arg \lambda^{\frac{1}{2}} = \frac{1}{2}\arg\lambda$, $-\pi < \arg\lambda < \pi$.

<u>Proof:</u> We obtain (2.4) for λ real, $\lambda^{\frac{1}{2}} > w$ integrating $\exp(-\lambda^{\frac{1}{2}}t)B^*(T^*(s)u_o^* + S^*(s)u_1^*)$ in $(0, \infty)$ and applying (2.3) and Lemma (2.1). For $\rho_o(A)$ the result follows from an analytic continuation argument. The reverse implication is, as in (6), Corollary 2.2 a consequence of uniqueness of Laplace transforms.

We shall also consider in what follows the first order system M,

$$u'(t) = Au(t) + Bf(t) \tag{2.5}$$

Now $K_t(M)$ is defined as the subspace of E consisting of all values (at time t) of solutions of (2.5) such that $u(0) = 0$, $f \in \mathcal{L}$, $K(M)$, $K(M)^{\perp}$, $K_t(M)^{\perp}$ are defined in a way similar to that for second-order systems (see (6) for more details)

2.3 THEOREM *Assume* A *satisfies the condition*

> *there exists a simple closed curve* C
> *entirely contained in* $\rho_0(A)$ *and such that* (2.6)
> *the origin is contained in the interior of* C

Then $K(L) = \{ (u_0, u_1) ; u_0, u_1 \in K(M) \}$

Proof Obviously we only have to prove $K(L)^{\perp} = \{ (u_0^*, u_1^*) ; u_0^*, u_1^* \in K(M)^{\perp} \}$. We shall use the following characterization of the elements of $K(M)^{\perp}$ (see (6), Corollary 2.2); $u^* \in K(M)^{\perp}$ if and only if $B^*R(\lambda ; A^*)u^* = 0$, $\lambda \in \rho_0(A)$. This makes clear that if $u_0^*, u_1^* \in K(M)^{\perp}$ then $(u_0^*, u_1^*) \in K(L)^{\perp}$. Conversely, assume $(u_0^*, u_1^*) \in K(L)^{\perp}$. Consider (2.4) for a given $\lambda \in C$. As λ turns once around the origin and returns to its original value, $\lambda^{\frac{1}{2}}$ changes sign. Adding up the two versions of (2.4) so obtained we get $B^*R(\lambda ; A^*)u_0^* = 0$, $B^*R(\lambda ; A^*)u_1^* = 0$ for $\lambda \in C$; by analytic continuation this holds as well for all $\lambda \in \rho_0(A)$, which ends the proof.

2.4 COROLLARY *Assume* A *satisfies Condition* (2.6). *Then the control system* L *is null controllable if and only if* M *is null controllable*

2.5 REMARK If Condition (2.6) is not satisfied then Theorem 2.3 may fail to hold. We construct in what follows an example of this situation.

Let $E = L^2 = L^2_y(-\infty, \infty) = \{ u(y), v(y), \ldots \}$ Recall that the space H^2 of the upper half-plane consists of all those functions in L^2 that are boundary values of functions $u(y + if)$, holomorphic in the upper half-plane and such that

$$\sup_{f > 0} \int |u(y + if)|^2 \, dy < \infty$$

(all integrals hereafter shall be taken on $(-\infty, \infty)$) By the Paley-Wiener theorem ((2), Chapter 8) H^2 consists of all those functions on L^2 whose Fourier-Plancherel transform vanishes for $t \geq 0$, i.e. of those $u(\cdot)$ in L^2 such that

$$\hat{u}(t) = (2\pi)^{-\frac{1}{2}} \int u(y)\, e^{iyt} dy = 0 \text{ for } t \geq 0$$

We shall make use of the following

2.6 LEMMA *Let* $m \in L^2$, $m \geq 0$, $m \neq 0$. *There exists* $u \in H^2$ *such that* $|u(y)| = m(y)$ *if and only if*

$$\int |\log m(y)|\, (1 + y^2)^{-1} dy < \infty$$

For a proof for H^2 of the unit circle see (3), Theorem 7.33; it can be adapted to the case of the half-plane by using the results in (2), Chapter 8 .

2.7 COROLLARY *Let* $\{a_{ij}(y)\}$, $i,j = 1,2$ *be* a 2 x 2 matrix of functions in L^2 . Assume

$$\int |\log |\det\{a_{ij}(y)\}||\, (1 + y^2) dy < \infty \tag{2.8}$$

Then there exist v_1 , v_2 , both different from zero almost everywhere and such that

$$v_1 a_{11} + v_2 a_{12} = v_1 a_{21} + v_2 a_{22} \in H^2 \tag{2.9}$$

Proof Let $w_1(y) = b(y)(a_{22}(y) - a_{12}(y))$, $w_2(y) = b(y)(a_{11} - a_{21}(y))$ where $b(y)^{-1} = \text{sgn} \det\{a_{ij}(y)\}$. We have

$$w_1 a_{11} + w_2 a_{12} = w_1 a_{21} + w_2 a_{22} = \det\{a_{ij}\}$$

In view of Lemma 2.6 there exists $u \in H^2$ such that $|\det\{a_{ij}(y)\}| = |u(y)|$ Thus if we set $v_i = w_i \text{ sgn } u$, $i = 1.2$ v_1 , v_2 satisfy (2.9)

Let us now pass to the example proper. Let $E = L^2 = L^2_x(\infty, \infty) = \{u(x), v(x), \dots\}$, A_r the (self adjoint) operator defined by

$$(A_r u)(x) = u''(x) + ru(x) , \tag{2.10}$$

$D(A_r) = \{u \in L^2 ;\ u'' \in L^2\}$ (u'' understood in the sense of distributions) , $F = C^2 = \{(y_1,y_2), \dots\}$ two dimensional unitary space, $f(t) = (f_1(t), f_2(t))$, $B(y_1,y_2)(x) = y_1 g_1(x) + y_2 g_2(x)$, g_1 , g_2 elements of L^2 to be determined later.

The Fourier-Plancherel transform $u(x) \longleftrightarrow \hat{u}(s)$ defines an isometric isomorphism of L^2 onto itself under which the operator A_r transforms into the multiplication operator

$$(A_r u)(s) = (-s^2 + r)u(s) , \tag{2.11}$$

$D(A_r) = \{ u \in L^2 \mid s^2 u(s) \in L^2 \}$. Thus we may consider $E = L^2 = L^2_s(-\infty, \infty) = \{ u(s) , v(s) , \dots \}$ A_r defined by (2.11) , $B(y_1, y_2)(s) = y_1 h_1(s) + y_2 h_2(s)$, $h_1 = \hat{g}_1$, $h_2 = \hat{g}_2$. The adjoint of B is given by $B^* u = (y_1(u) , y_2(u))$,

$$y_i(s) = \int u(s) k_i(s) ds , \quad i = 1,2$$

(where we have set $k_i(s) = \bar{h}_i(s)$) . It is not difficult to see that the Cauchy problem for $u'' = A_r u$ is uniformly well posed for any r , the propagators being given by

$$S_r(t)u(s) = a(r,s,t)u(s) , \quad T_r(t)u(s) = b(r,s,t)u(s) ,$$

$$a(r,s,t) = \begin{cases} \cosh (r - s^2)^{\frac{1}{2}} t & \text{if } r \geqslant 0 \text{ and } |s| \leq r^{\frac{1}{2}} , \\ \cos (s^2 - r)^{\frac{1}{2}} t & \text{if } r < 0 \text{ or if } r \geqslant 0 \text{ and } |s| \geqslant r^{\frac{1}{2}} \end{cases}$$

$$b(r,s,t) = \begin{cases} (r - s^2)^{-\frac{1}{2}} \sinh (r - s^2)^{\frac{1}{2}} t & \text{if } r \geqslant 0 \text{ and } |s| \leq r^{\frac{1}{2}} , \\ (s^2 - r)^{-\frac{1}{2}} \sin (s^2 - r)^{\frac{1}{2}} t & \text{if } r < 0 \text{ or if } r \geqslant 0 \text{ and } |s| \geqslant r^{\frac{1}{2}} . \end{cases}$$

The spectrum of A_r consists of the half-line $(-\infty , r]$; thus if $r < 0$ condition (2.6) is satisfied. By Theorem 2.3 the system

$$u''(t) = A_r u(t) + Bf(t) \tag{2.12}$$

is null controllable if and only if

$$u'(t) = A_r u(t) + Bf(t) \tag{2.13}$$

is null controllable. For $r = 0$ the system (2.13) has been considered in (5) , Section 4; it is null controllable if and only if

$$h_1(s)h_2(-s) - h_1(-s)h_2(s) \neq 0 \quad \text{a.e.} \tag{2.14}$$

It is easy to see that the same result holds for any r (null controllability is "translation-invariant" for first-order systems) . Condition (2.14) holds for instance when $g_1(x) =$

$= \exp(-|x|)$, $g_2 = \exp(-|x+1|)$; then $h_1(s) = 2(1+s^2)^{-1}$, $h_2(s) = 2\exp(-is)(1+s^2)^{-1}$ and the expression on the left-hand side of (2.14) reduces to $8i(1+s^2)^{-2}\sin s$.

Let us now examine (2.12) for $r \geqslant 0$, with the same choice of g_1 , g_2 . Let u_o , $u_1 \in L^2_s$. It is plain that Eq. (2.3) will hold for them if and only if

$$\int (a(r,s,t)u_1(s) + b(r,s,t)u_o(s))k_i(s)ds = 0 \ , \ t \geqslant 0 \qquad (2.15)$$

It is easy to see by means of simple changes of variable that the part of the integral in the left-hand side of (2.15) extending over $|s| \geqslant r^{\frac{1}{2}}$ can be written

$$\int (\cos yt\,(K_r u_1)(y) + y^{-1}\sin yt\,(K_r u_o)(y))(K_r k_i)(y)dy \qquad (2.16)$$

where K_r is the isometric isomorphism of $L^2(|s| \geqslant r^{\frac{1}{2}})$ onto L^2_y given by

$$(K_r u)(y) = |y|^{\frac{1}{2}}(y^2+r)^{-\frac{1}{4}}\,u((y^2+r)^{\frac{1}{2}}\,\mathrm{sgn}\,y)$$

If $u_o(y)/y$ is summable at the origin we can write (2.16) as follows:

$$\tfrac{1}{2}\int e^{iyt}(v_1(y)\tilde{k}_i(y) + v_2(y)\tilde{k}_i(-y))dy \ , \qquad (2.17)$$

$v_1(y) = \tilde{u}_1(y) - i\tilde{u}_o(y)/y$, $v_2(y) = \tilde{u}_1(-y) - i\tilde{u}_o(-y)/y$ (here we have written $\tilde{u}_i = K_r u_i$, $\tilde{k}_j = K_r k_j$) . Call now $a_{ij}(y) = k_i((-1)^j y)$, $i.j = 1.2$. The matrix $\{a_{ij}\}$ so defined satisfies the assumptions in Corollary 2.7 and thus there exist v_1 , v_2 in L^2 , $v_i \neq 0$ such that

$$v_1 a_{21} + v_2 a_{22} \in H^2 \ , \ i = 1,2 \qquad (2.18)$$

It is plain that (2.18) holds as well for $w_i(y) = v_i(y)(i+y)^{-1}$, $i = 1,2$. If we now define $\tilde{u}_1(y) = \frac{1}{2}(w_1(y) + w_2(-y))$, $\tilde{u}_2(y) = \frac{1}{2}iy(w_1(y) - w_2(-y))$ then u_1 , $u_2 \in L^2$ and (2.17) - a fortiori (2.16) - vanishes for $t \geqslant 0$. Taking now $u_i = K_r^{-1}\tilde{u}_i$ which are defined for $|s| \geqslant r^{\frac{1}{2}}$ and extending them to the entire real line by setting $u_i = 0$ in $|s| < r^{\frac{1}{2}}$ we obtain two non-vanishing elements of L^2_s such that (2.15) holds, which shows that the system (2.12) is not completely controllable for $r \geqslant 0$.

2.6 REMARK Our results on density of K(L) generalize to other topologies in $E \times E$. We show briefly in what follows how this can be done.

Let E_o, E_1 be Banach spaces with norms $|\cdot|_o$, $|\cdot|_1$ such that $E_o \subseteq E$, $E_1 \subseteq E$ (no relation between the topologies of E, E_1, E_2 is postulated) Let $m > 0$; introduce in $D(A^m)$, the domain of A^m the topology given by the norm $|u|_{D(A^m)} = |u|_E + |A^m u|_E$ or the equivalent one $|u|'_{D(A^m)} = |(\lambda - A)^m u|_E$, λ any element of $\rho(A)$. We shall assume that:

$$D(A^m) \subseteq E_o \quad , \; D(A^{m-1}) \subseteq E_1$$

both inclusions being continuous (we shall always consider $D(A^m)$ endowed with the topology given before, $D(A^{m-1})$ with the similar topology obtained replacing m by m-1); moreover we suppose

$$B(F) \subseteq D(A^{m-1}) \;,\; S(t)E_i \subseteq E_i \;,\; T(t)E_i \subseteq E_i \;,\; t \geq 0 \;,\; i = 0,1$$

$S(\cdot)$, $T(\cdot)$ are strongly continuous functions in the topologies of E_1, E_2. Under all this conditions it is easy to show that if $u(\cdot)$ is a solution of (2.1) with $u(0) = u'(0) = 0$, $u(t) \in D(A^m)$, $u'(t) \in D(A^{m-1})$ for all $t \geq 0$. It is then natural to ask when K(L) will be dense in $E_o \times E_1$, i.e. when the system (2.1) will be null controllable <u>in the topology of</u> $E_o \times E_1$.

Let E_o^*, E_1^* be the dual spaces of E_o, E_1, application of a functional $u_i^* \in E_i^*$ to an element $u_i \in E_i$ being indicated

$$\langle u_i^* , u_i \rangle_i \quad , \; i = 0,1$$

Assume (2.1) is not null controllable in $E_o \times E_1$ and let $(u_o^*, u_1^*) \in K(L)^{\perp}$. Then we have, in view of (1.5)

$$\langle u_o^* , \int_0^t T(s)Bf(s)ds \rangle_o + \langle u_1^* , \int_0^t S(s)Bf(s)ds \rangle_1 = 0 \quad (2.19)$$

for all $t \geq 0$, $f \in \mathcal{L}$. Setting $f(s) = \exp(-\lambda^{\frac{1}{2}} s)u$, $u \in F$, $\lambda^{\frac{1}{2}} > w$ (w the constant in (d), Section 1) and letting $t \to \infty$ we get from (2.19) that

$$\langle u_o^* , R(\lambda;A)Bu \rangle_o + \langle u_1^* , \lambda^{\frac{1}{2}} R(\lambda;A)Bu \rangle_1 = 0$$

for $\lambda > w^2$ and a fortiori for all $\lambda \in \rho_o(A)$. Assume now A satisfies Condition (2.6). Then, by using the same trick in the proof of Theorem (2.3) we can show that

$$\langle u_o^* , R(\lambda ;A)Bu\rangle_o = \langle u_1^* , R(\lambda ;A)Bu\rangle_1 = 0 \qquad (2.19')$$

for all $\lambda \in \rho_o(A)$. Assume now the first-order system M is null controllable. Then, if $B^*R(\lambda ;A^*)u^* = 0$ for some $u^* \in E^*$ and all $\lambda \in \rho_o(A)$, $u^* = 0$ or, what amounts to the same thing, the subspace of E generated by all elements of the form

$$R(\lambda ;A)Bu \qquad (2.20)$$

$u \in F$, $\lambda \in \rho_o(A)$ is dense in E. Let us see that the same thing happens with the subspace of E generated by the elements

$$(\mu - A)^m R(\lambda ;A)Bu \qquad (2.21)$$

μ a fixed element of $\rho(A)$, $u \in F$, $\lambda \in \rho_o(A)$. In fact, assume this is not true. Then there exists $u^* \in E^*$ such that

$$\langle u^* , (\mu - A)^m R(\lambda ;A)Bu\rangle = 0 \qquad (2.22)$$

for all $u \in F$, $\lambda \in \rho_o(A)$. Adding up (2.22) for two different elements λ_o , λ_1 of $\rho_o(A)$ and using the first resolvent equation we get

$$\langle u^* , (\mu - A)^m R(\lambda_o;A)R(\lambda_1;A)Bu\rangle = 0 \qquad (2.23)$$

Differentiating (2.23) with respect to λ_1 m-1 times we get

$$\langle u^* , (\mu - A)^m R(\lambda_o;A)R(\lambda_1;A)^m Bu\rangle = 0$$

for all $u \in F$, $\lambda_o \in \rho_o(A)$. Then

$$B^*R(\lambda_o;A^*)(\mu - A^*)^m R(\lambda_1;A^*)^m u^* = 0$$

which, in view that M is completely controllable, implies

$$(\lambda - A^*)^m R(\lambda_1;A^*) = 0 ,$$

a fortiori, $u^* = 0$.

Let us observe next that to assert that the subspace generated by all elements of the form (2.21) is dense in E is equivalent to assert that the subspace generated by all elements of the form

(2.20) is dense in $D(A^m)$. But then it will also be dense in E_o; thus, in wiew of (2.19') , $u_o^* = 0$. The second term in (2.19') can be treated in the same way than the first. Collecting all our observations we have

2.7 THEOREM Let E_o , E_1 be Banach spaces satisfying all the conditions in Remark 2.6. Assume the first-order control system (2.5) is null controllable, and assume A satisfies Condition (2.6) . Then the system (2.1) is null controllable in the topology of $E_o \times E_1$

§3. Let us call the system (2.1) completely controllable if, given u_o , $u_1 \in D$, v_o , $v_1 \in E$, $\epsilon > 0$ there exists $f \in \mathcal{L}$ such that the solution of Eq. (2.1) with $u(0) = u_o$, $u'(0) = u_1$ satisfies

$$|u(t) - v_o| \leq \epsilon \ , \ |u'(t) - v_1| \leq \epsilon$$

for some $t > 0$. It is plain that complete controllability of L implies null controllability. The reverse implication is also true; this follows from the fact that the solutions of Eq. (2.1) can be translated and inverted in time, i.e. if $u(\cdot)$ is a solution of (2.1) for some $f(\cdot) \in \mathcal{L}$ then $v(t) = u(a - t)$ is also a solution of Eq. (2.1) for $g(t) = f(a - t)$. Thus to steer the system from (u_o , u_1) to the vicinity of (v_o , v_1) we only have to steer first to the vicinity of the origin (using null controllability and the inversion property just mentioned) and then from the origin to the vicinity of (v_o , v_1) .

The situation is diferent for first-order systems; in fact a first-order system may be null controllable without being completely controllable. There are, however, two important particular cases where the equivalence holds; these are (a) the case where A generates an analytic semigroup and (b) the case where A generates a group and $\rho_o(A) = \rho_o(-A)$, this last condition meaning that we can unite the points $+\infty$ and $-\infty$ of the real axis by means of a curve that does not meet the spectrum of A.

§4. Problems similar to the ones we have considered for the systems $u^{(n)} = Au + Bf$, $n = 1 , 2$ may also be considered for $n \geqslant 3$. However, the interest of these generalizations is limited by the fact that the assumption of well posedness of the homogenous problem $u^{(n)} = Au$ implies the boundedness of A ((4), Section 3), thus precluding applications to partial differential equations. The results are as follows: if L is the n-th order system

$$u^{(n)}(t) = Au(t) + Bf(t) \qquad (2.24)$$

and M, as usual, is the first-order system

$$u'(t) = Au(t) + Bf(t) \qquad (2.25)$$

then the four notions, null controllability, null controllability at time t_0, complete controllability, complete controllability at time t_0 are equivalent for the system L and equivalent to the corresponding notions for the system M. The proof is a consequence of the fact that the solution operators of Eq. (2.24) - and also of equation (2.25) - are analytic when A is bounded.

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