NASA CONTRACTOR REPORT

NASA CR-1175

THE RESPONSE OF A SIMPLY SUPPORTED PLATE TO TRANSIENT FORCES

Part I - The Effect of N-Waves at Normal Incidence

by Anthony Craggs

Prepared by
UNIVERSITY OF SOUTHAMPTON
Southampton, England
for Langley Research Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • SEPTEMBER 1968

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THE RESPONSE OF A SIMPLY SUPPORTED PLATE TO TRANSIENT FORCES

PART I: THE EFFECT OF N-WAVES AT NORMAL INCIDENCE

By Anthony Craggs
Institute of Sound and Vibration Research
University of Southampton, England

SUMMARY

A numerical method is presented for determining the response of a structure to transient forces of arbitrary form. It is used to evaluate the response of a simply supported plate to an 'n' wave at normal incidence. Three response parameters are considered: the displacement, the acceleration and the stress at a particular point on the plate. It is shown that the fundamental mode dominates the displacement and stress responses in all cases. However, there is a significant contribution from the higher modes in the accelerations. The maximum response occurs when the duration of the wave is equal to the fundamental period of the plate.

INTRODUCTION

This report describes a preliminary investigation into the effects of Sonic Booms on parts of building structures.

There are several papers which deal with the response of plates and beams to 'n' waves, notably by BENVENISTE and CHENG (1) and CHENG*, using a Fourier Integral transform technique. The problem treated was that of an 'n' wave with zero rise time; this gave a substantial saving in the amount of algebraic manipulation required. One of the inherent difficulties with transform methods of solving transient problems, and methods involving a direct integration of the Duhammel Integral is the degree of labour required even for pulses with fairly simple shapes. Damping can also increase the amount of work considerably. Further, once the formal algebraic solution has been written out it is still necessary to compute numerical values from this formula, and this is only for a special case. A new formula is required for a different shaped pulse. In this report a numerical method is presented, which is based on the principle of superposition and it is used to compute the effects of rise time and pulse duration on the response. The superposition process is concisely expressed in a matrix form. One of the advantages of the method is the ease with which it may be applied to solving response problems to a forcing function of any arbitrary form.

*Unpublished work submitted by D. H. Cheng, City College of the City University of New York, prepared during temporary assignment at NASA Langley Research Center.

The time history of a Sonic Boom is classically represented by a capital 'N'. Some actual characteristics are given in reports by HUBBARD and MAGLIERI (2) also WEBB and WARREN (3). There the time history is shown to vary from a single 'N' to two 'N's superimposed on one another. Nearly all of the traces are distorted in some way and show wide variation in the rise and fall time. Since there are these differences in the shape of the pulse it is essential that the method used to study the structural response is flexible.

For continuous structures, problems of a transient nature may be solved by using a series of normal modes, so that the nature of the loading and the response must be such that a reasonable accuracy may be obtained by using only a finite number of modes. BENVENISTE and CHENG (1) points out that for conditions in which the ratio of the period of the boom to the fundamental period of the plate is in a range from 1/2 to 4, the response is dominated by the fundamental mode. The plate and beam that were considered were then idealised to single degree of freedom systems. However, only the series for the displacements were given when examining the rate of convergence and this is not particularly sensitive to the higher terms.

The response to the loading may be displacement, velocity, acceleration or stress, and, depending upon what criterion is to be used, each may be important. However, their time histories are different and it is necessary, therefore, to treat them individually. The stresses and accelerations contain a greater contribution from the higher modes and consequently more modes may be needed when evaluating the response.

The object of the present work is to study the variation in the response parameters: displacement, acceleration and stress under various loading conditions. The different load conditions to be investigated are realised by changing the rise time and duration of the boom.

SYMBOLS

w	displacement
$\ddot{w}$	acceleration
σ	stress
ρ	mass density
ω	natural frequency
E	Young's Modulus

th	plate thickness
$p, \hat{p}$	pressure, maximum pressure
h(t)	Unit impulse response function
[H]	Integrating matrix containing unit impulse functions
[R]	Integrating matrix for unit impulse functions of the principal co-ordinates
a,b	Overall plate dimensions
β	Aspect ratio = b/a
d	dimensionless quantities having the same time dependence as the principal displacement
$\ddot{d}$	dimensionless quantities having the same time dependence as the principal acceleration
R_{mn}	$(\beta^2 m^2 + n^2)/(\beta^2 + 1)$
ω_{mn}	natural frequency of m, n^{th} mode
λ	dimensionless time factor
t	time
T_1	periodic time of fundamental mode
τ	duration of pulse
ν	damping coefficient

THE RESPONSE OF A SIMPLE UNDAMPED OSCILLATOR TO AN IMPULSE

In order to obtain a numerical solution to an arbitrary forcing it is intended to break up the forcing function into a finite number of segments of equal duration. Each segment is then treated as an impulse and the net response is built up by the process of superposition. There is, however, a limitation on the width of each segment if the response is to be governed almost entirely on its area, i.e. the net impulse, and not, to any marked degree, on its shape. One restriction is that the duration is small when it is compared with the period of the system. A more quantitative bound may be found heuristically by comparing the responses of a simple oscillator to different shaped pulses.

Fundamental Results

For free vibration the displacement of a simple oscillator is dependent only on the initial conditions and it is given by:

$$x = x_i \cos \omega t + \frac{v_i}{\omega} \sin \omega t \tag{1}$$

where x_i and v_i are the initial displacement and velocity respectively and ω is the natural frequency.

In response to a pulse having any time dependence, $f(\tau)$, and duration, T, the solution for the displacement when the system is left vibrating freely, may be obtained from the Duhammel Integral and it is given by:-

$$x(t) = \int_0^T \frac{f(\tau)}{\omega} \cdot \sin \omega(t - \tau) d\tau \quad t > T \tag{2}$$

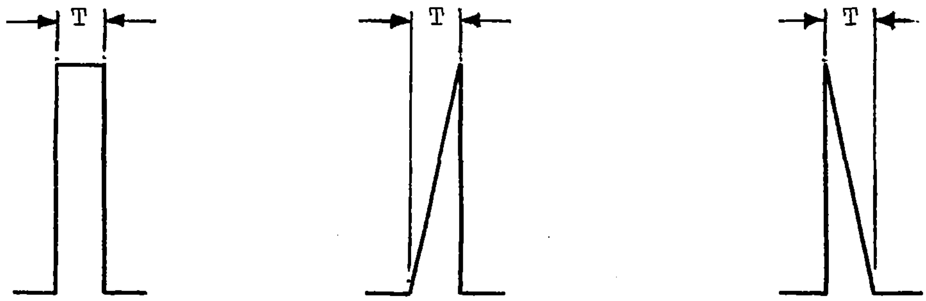
and because the limits are independent of t, then

$$\dot{x}(t) = \frac{dx}{dt} = \int_0^T \frac{f(\tau)}{\omega} \frac{d}{dt} (\sin \omega(t - \tau)) d\tau$$

$$= \int_0^T f(\tau) \cos \omega(t - \tau)\, d\tau \tag{3}$$

These results will now be applied to the pulse shown in Fig. 1.

Rectangular Pulse. $f(\tau) = F \quad 0 < \tau < T; \quad f(\tau) = 0 \quad \tau > T.$



(a) (b) (c)

FIG. 1

At time $t = T$

$$x_T = \frac{F}{\omega} \int_0^T \sin \omega(T - \tau)d\tau \; ; \qquad v_T = F \int_0^T \cos \omega(T - \tau)d\tau.$$

Making the restriction that $\sin \omega(T - \tau) \doteq \omega(T - \tau)$,

$$\text{then} \quad \cos \omega(T - \tau) \doteq 1 - \tfrac{1}{2}\, \omega^2(T - \tau)^2$$

$$\therefore \qquad x_T = \frac{F}{\omega} \int_0^T \omega T - \omega\tau . d\tau = \tfrac{1}{2}FT^2$$

$$v_T = F \int_0^T 1 - \tfrac{1}{2}\omega^2(T^2 - 2T\tau + \tau^2) = FT(1 - \frac{1}{6}\,\omega^2 T^2).$$

Substituting x_T and v_T in (1) then gives:

$$x = \frac{FT}{\omega}\;(1 - \frac{1}{6}\,\omega^2 T^2)\,\sin \omega(t - T) + FT.\frac{T}{2}\cos \omega(t - \tau)$$

and after putting $\omega T = \lambda$, and $FT = A$, the area under the pulse, then

$$x = \frac{A}{\omega}\left((1 - 1/6\lambda^2)\sin\omega(t-T) + \tfrac{1}{2}\lambda\cos\omega(t-T)\right) \quad (4)$$

for $t > T$.

Triangular Pulse. $f(\tau) = \frac{t}{T} \cdot F \quad 0 < \tau < T; \quad f(\tau) = 0 \quad \tau > T.$

Making the same restrictions as in the previous case, then

$$x_T = \frac{F}{\omega T}\int_0^T T.\tau - \tau^2.d\tau = \frac{1}{6}FT^2$$

$$v_T = \frac{F}{T}\int_0^T \tau - \tfrac{1}{2}\omega^2(T^2\tau - 2T\tau^2 + \tau^3)d\tau = \tfrac{1}{2}FT(1 - \frac{1}{12}\omega^2T^2).$$

Again substituting for x_T and v_T in (1), now putting $A = \frac{1}{2}FT$ gives the result

$$x = \frac{A}{\omega}\left((1 - \frac{1}{12}\lambda^2)\sin\omega(t - T) + \frac{1}{3}\lambda\cos\omega(t - T)\right) \quad (5)$$

Triangular Pulse. $f(\tau) = F(1 - \tau/T) \quad 0 < \tau < T; \quad f(\tau) = 0 \quad \tau > T.$

Using a similar procedure to the previous cases, then

$$x = \frac{A}{\omega}\left((1 - \frac{5\lambda^2}{12})\sin\omega(t - T) + \frac{2}{3}\lambda\cos\omega(t - T)\right) \quad (6)$$

Equations (4), (5) and (6) may be arranged into the following form

$$x = \frac{A}{\omega}(1 - \frac{1}{12}\lambda^2 + \frac{1}{36}\lambda^4)\sin(\omega(t - T) + \phi_1); \quad \phi_1 = \tan^{-1}\frac{\lambda}{2(1 - \frac{1}{6}\lambda^2)} \quad (7)$$

$$x = \frac{A}{\omega}(1 - \frac{1}{18}\lambda^2 + \frac{1}{144}\lambda^4)\sin(\omega(t - T) + \phi_2); \quad \phi_2 = \tan^{-1}\frac{\lambda}{3(1 - \frac{1}{12}\lambda^2)} \quad (8)$$

$$x = \frac{A}{\omega}(1 - \frac{7}{18}\lambda^2 + \frac{25}{144}\lambda^4)\sin(\omega(t - T) + \phi_3); \quad \phi_3 = \tan^{-1}\frac{\lambda}{3(1 - \frac{5}{12}\lambda^2)} \quad (9)$$

These results show that, provided $\lambda < \frac{1}{2}$, the most important factor which influences the response is the area under the curve, i.e. the magnitude of the impulse; the shape of the curve is of secondary importance. Therefore, for all practical purposes any forcing function can be divided up into a finite number of rectangular pulses. Further, instead of using equation (4) for computing the response to each impulse the simpler form

$$x = \frac{A}{\omega} \sin \omega(t-T) \qquad t > T \tag{10}$$

may be used.

At the worst, for $\lambda = \frac{1}{2}$, the error in the amplitude is approximately 5% and this is when a pulse of the type shown in Fig. 1(c) is being approximated. Most of the time the pulses will be of a trapezoidal form and the error will be much smaller. In equation (10) the effect of the phase angle, ϕ, has been neglected. This is reasonable since ϕ is dependent only on the magnitude of the step size and not on the size of the impulse, and, provided that the step size is constant, the error in the time lag may be made up by simply moving the time origin along $\lambda/2$.

The Matrix Form for the Superposition Process

Once the forcing function has been idealised into a finite number of rectangular pulses, the total response may be found by superimposing, with the appropriate time lag, the response from each one. If the impulse response and forcing functions can be stored numerically the superposition process may be expressed as a matrix multiplication, and in this form is easily evaluated on a digital computer. The matrix form may be deduced as follows:

Let $\underset{\sim}{F_r}$ be an array of numbers, each number being the value of the input force at the time $r \times \Delta t$, where Δt is the time increment. Let $\underset{\sim}{h_s}$ be an array of numbers each being the value of the unit response function at the time $s \times \Delta t$, as may be calculated from equation (10). The time increment, Δt, needs to be the same in both the unit response and the forcing arrays, and providing it is sufficiently small the response, w, at the time $s \times \Delta t$ is given by:

$$w_s = \Delta t(F_1 h_s + F_2 h_{s-1} + F_3 h_{s-2} + F_4 h_{s-3} + \dots)$$

$$w_s = \Delta t \sum_{r=1}^{s} F_r h_{s-r+1}$$

This equation may in turn be expressed as a matrix multiplication. If $\underset{\sim}{w}$ is now an array containing the numerical values of the response history at the time intervals Δt, the relationship between $\underset{\sim}{w}$ and $\underset{\sim}{F}$ is

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \cdot \\ \cdot \\ w_s \end{bmatrix} = \Delta t \begin{bmatrix} h_1 & & & & & \\ h_2 & h_1 & & & & \\ h_3 & h_2 & h_1 & & & \\ \cdot & h_3 & h_2 & h_1 & & \\ \cdot & \cdot & \cdot & \cdot & h_1 & \\ h_s & h_{s-1} & h_{s-2} & h_{s-3} & \cdots\cdots & h_1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F_n \end{bmatrix}$$

$$\{w\} = \Delta t \; [H] \; \{F\} \qquad (11)$$

H, is a square lower triangular matrix of order s x s; the non-zero elements of the r th column containing the first $(s - r + 1)$ terms of the unit response $\underset{\sim}{h}$. It should be noted that, h, may be for any response parameter, i.e. displacement, velocity, acceleration or stress; and, although for purposes of illustration it is convenient to consider H as a full square matrix, in the computation this would involve an excessive amount of storage space; and in fact only the unit response, $\underset{\sim}{h}$, needs to be stored.

If a normal mode approach is to be used the above may be extended to a system with a finite number of degrees of freedom: the displacements then become the generalised displacements and the forces the generalised forces for each mode. For this case equation (11) becomes

$$\{q\} = \Delta t \; [R] \; \{Q\} \qquad (12)$$

The response at point (x, y) on the system is then

$$\{w\} = \Delta t(\psi_1,(x,y) \; [R_1] \; \{Q_1\} + \psi_2 \; [R_2]\{Q_2\} \; \ldots) \qquad (13)$$

R is an integrating matrix similar to H, containing the unit impulse response array for the generalised co-ordinates, q. Q is the array for the generalised forces and $\underset{\sim}{\psi}$ the mode shape. The number of modes to use and therefore the number of times that equation (12) needs to be applied will depend upon the accuracy required in the solution.

The equations (11) and (12) represent the Duhammel Integral:

$$w = \int_0^t F(\tau) \quad h(t - \tau)d\tau,$$

in an approximate matrix form. There are, however, several advantages to be gained from this formulation: (i) once the impulse response and forcing is defined the solution is obtained simply by a matrix multiplication no matter what degree of complexity the forcing function may be; this could be an array of random numbers, (ii) the presence of damping does not increase the labour in obtaining a solution as it does in the analytical methods.

TRANSIENT RESPONSE OF A SIMPLY SUPPORTED PLATE

The results of the previous sections are now applied to the response of a simply-supported plate to Sonic booms. In obtaining a solution a normal mode approach is used and any damping present in the system is assumed not to couple these modes. Also, it will be assumed that the plate is vibrating in a vacuo so that there is no acoustic back pressure acting. Three response parameters are to be computed: the displacement, stress and acceleration in order to compare the different effects of the higher modes.

General Theory

The equation of motion of a uniform plate in forced vibration is

$$\frac{\partial^2 w}{\partial t^2} + \frac{Eh^2}{12\rho(1 - \mu^2)} \left(\frac{\partial^4 w}{dx^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = \frac{P(x,y,t)}{\rho h} \quad (14)$$

To obtain a solution of this equation the right hand side is first put equal to zero and in this form it is satisfied by a single mode, ψ_r, with time dependence $\sin \omega_r t$ giving:

$$\frac{Eh^2}{12\rho(1 - \mu^2)} \left(\frac{\partial^4 \psi_r}{\partial x^4} + 2\frac{\partial^4 \psi_r}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi_r}{\partial y^4} \right) = \omega_r^2 \psi_r \qquad (15)$$

A solution of equation (14) may be obtained in a series form by making the substitutions

$$w = \sum_{r=1}^{\infty} q_r(t)\ \psi(x,y) \qquad p = \sum p_r(t)\ \psi(x,y)$$

giving

$$\sum_{r=1}^{\infty} \ddot{q}\psi_r + \sum_{r=1}^{\infty} \frac{Eh^2}{12\rho(1-\mu^2)} \left(\frac{\partial^4\psi_r}{\partial x^4} + 2\frac{\partial^4\psi_r}{\partial x^2\partial y^2} + \frac{\partial^4\psi_r}{\partial y^4} \right) = \frac{1}{\rho h}\sum p_r\psi_r$$

and after applying equation (15), this equation reduces to

$$\sum(\ddot{q}_r + \omega_r{}^2 q_r)\psi_r = \frac{1}{\rho h} \sum p_r(t)\ \psi_r(x,y)$$

introducing a viscous damping coefficient, $2\omega_r\nu\dot{q}_r$, into each term of the left hand side, then equating coefficients of ψ_r gives

$$\ddot{q}_r + 2\omega_r\nu\dot{q}_r + q_r = \frac{1}{\rho h}\ p_r(t) \tag{16}$$

$$P_r(t) = \int_0^b\int_0^a p(x,y,t)\ \psi_r(x,y)\ dxdy \ \div \int_0^b\int_0^a \Psi_r{}^2(x,y)dxdy$$

For a simply-supported plate $\psi_{mn}(x,y) = \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}$

and $$\int_0^b\int_0^a \psi_{mn}^2(x,y)dxdy = \int_0^b\int_0^a \sin^2\frac{m\pi x}{a} \cdot \sin^2\frac{n\pi y}{b}\ dx.dy = \frac{a.b}{4}$$

therefore $$p_{m,n}(t) = \frac{4}{a.b}\int_0^b\int_0^a p(x,y,t)\psi_{m,n}\ dx.dy$$

Response to Plane Pressure Pulse at Normal Incidence

When a plane pressure pulse is normally incident on a flat plate the forcing pressure $p(x, y, t)$ can be expressed in the separable form

$$p(x, y, t) = \hat{p} \cdot f_1(\tau) \cdot f_2(x, y)$$

$\hat{p}$ is the maximum pressure; $\hat{f}_1(\tau) = 1$ and $f_1(x,y) = 1$

then $$p_{m,n} = \frac{4\hat{p}}{a.b} \int_0^b \int_0^a \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} \, dx.dy$$

$$= 16 \frac{\hat{p}}{\pi^2} \cdot c_{mn} \quad \text{where} \quad c_{m,n} = 1/m.n$$

It is useful to work in terms of dimensionless parameters and this may be achieved by expressing the responses in the form of a factor x the maximum stiffness response in the first mode, which may be obtained by neglecting the inertia term; the response would then follow the form of the forcing function. Since, under these conditions, the response is dominated by the fundamental mode the factor for the displacements represents, approximately, the dynamic amplification.

$$\hat{q}_{11\,\text{stiff}} = \frac{1}{\rho h} \frac{\hat{p}_{11}}{\omega_{11}^2} = \frac{16\hat{p}}{\rho h \pi^2 \omega_{11}^2}$$

Now if $q_{mn} = d_{mn} \times \hat{q}_{11\,\text{stiff}}$, then substituting this in equation (16) gives:

$$\ddot{d}_{mn} + 2\omega_{mn} \nu \dot{d}_{mn} + \omega^2_{mn} d_{mn} = c_{mn} \omega^2_{11} f_1(\tau)$$

and using the Duhammel Integral gives

$$d_{mn} = \omega_{11}^2 c_{mn} \int_0^t \frac{f_1(\tau) e^{-\omega_{mn}\nu(t-\tau)}}{\omega_{mn}\sqrt{1-\nu^2}} \sin \omega_{mn} \sqrt{1-\nu^2}(t-\tau).d\tau$$

Now if $R_{mn} = \omega_{mn}/\omega_{11}$; $t = \dfrac{r\lambda}{\omega_{11}}$; $\tau = \dfrac{s\lambda}{\omega_{11}}$

where λ is a dimensionless time factor, some fraction of the lowest period of the system, $2\pi/\omega_{11}$, then in the notation of the previous section:

$$d_{mn}(s\lambda) = \lambda\, c_{mn} \sum_{r=1}^{s} \frac{f(r\lambda)e^{-R_{mn}\nu(r-s+1)\lambda}}{R_{mn}\sqrt{1-\nu^2}} \sin(R_{mn}\sqrt{1-\nu^2}(r-s+1)\lambda)$$

$$\{d_{mn}\} = \lambda\, c_{mn} \left[R\right]\{f\}$$

The dimensionless displacement, $\bar{w}(x,y)$, can now be written in terms of the normal modes:

$$\{\bar{w}\} = \lambda(\psi_1 c_1 \left[R\right]_1 + \psi_2 c_2 \left[R\right]_2 + \psi_3 c_3 \left[R\right]_3 + \ldots)\ \{f\}$$

The suffix notation used now refers to the order in which each generalised co-ordinate occurs in the frequency scale.

The Acceleration and Stress Responses

It has already been mentioned that the unit impulse response could be computed for any parameter. The impulse acceleration response is

$$\ddot{w} = -\omega_1 e^{-\nu\omega_1 t}\ (2\nu \cos \sqrt{1-\nu^2}\ t + \frac{(1-2\nu^2)}{\sqrt{1-\nu^2}} \sin \omega t)$$

The computed value in the dimensionless form is:

$$\bar{\ddot{w}}(x,y) = \ddot{w}(x, y)/\ \omega_{11}^2\ \hat{q}_{11\ stiff}\ \psi_{11}(x,y) \qquad (17)$$

The stresses in the x and y directions are computed from the two equations

$$\sigma_x = \frac{-Eh}{(1-\mu^2)} \left\{ \frac{\partial^2\omega}{\partial x^2} + \mu \frac{\partial^2\omega}{\partial y^2} \right\}$$

$$\sigma_y = \frac{-Eh}{(1-\mu^2)} \left\{ \frac{\partial^2\omega}{\partial y^2} + \mu \frac{\partial^2\omega}{\partial x^2} \right\}$$

After substituting for ω in terms of the normal modes, that is $\bar{\omega} = \sum d_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$, in these equations the stresses may be expressed in the dimensionless form

$$\bar{\sigma}_x(x,y) = \sigma_x / \frac{16 E\hat{p}}{(1 - \mu^2)ab\omega^2{}_{11}} = \sum_m \sum_n (m^2 + \frac{\mu n^2}{\beta^2}) d_{mn}\psi_{mn}(x,y)$$

$$\bar{\sigma}_y(x,y) = \sigma_y / \frac{16E\hat{p}}{(1 - \mu^2)ab\omega^2{}_{11}} = \sum_m \sum_n (\frac{n^2}{\beta^2} + \mu m^2) d_{mn} \psi_{mn}(x, y) \qquad (18)$$

The Range for the Experimental Loading Parameters

The range of data used in the computation which is considered to be of practical interest is given below. The upper limit for the period ratio could have been much higher than 3, but beyond this value the response is easily predictable as the displacement response follows the time history of the forcing function.

Period Ratio	τ/T	$\frac{1}{2}$ to 3
Rise Time		T/20 to T/4

For the isochronous case, $\tau/T = 1$, both a single and double 'N' and two damping coefficients, V = .002 and .02, were considered.

Aspect Ratio	b/a	=	1.5

COMPUTATION PROCEDURE

A FORTRAN programme was written, based on the theory given in section 5, to evaluate the response of the plate under the various loading conditions. Since the computation time was important when using the programmes a step size had to be found that was compatible with the nature of the problem. For good resolution the step size needed to be a fraction of the lowest period of the idealised system and it was necessary, therefore, to have an estimate of the number of modes that were effectively contributing to the response under the different loadings. This was found by a trial and error procedure: first computing the response the response with a large number of modes, and then comparing this with the response that was obtained when the number of modes and step size were halved.

When the pulse is normally incident on the plate only the odd modes are excited, and for the period ratios considered it was necessary to use the first three modes in the set, i.e. $m = 1$, $n = 1, 3, 5$, to give good accuracy. The effect of the higher modes $n = 3, 5$ were more significant for the accelerations than for any other parameters. Generally, for displacements and stresses, a step size, $\lambda = \pi/20$ was found to be sufficient though for the accelerations it was necessary to have $\lambda = \pi/40$.

RESULTS

Some typical response curves are shown in Figs. 2 to 13, these are included in order to illustrate the salient features about the nature of the response. In order to compress the amount of data it is necessary to choose only the important quantities: the maximum values of the displacements, stresses and accelerations for the different loading variables are shown in Tables 1 and 2. Table 1 shows the values for the forced motion and Table 3 the values for the ensuing free vibration. Any marked contribution from the higher modes increases the number of peaks in a given time; the effect is accounted for by the quantity, N_p, the number of positive peaks in one cycle of the fundamental mode; these values are also included in the tables.

Figs. 2 to 9 show the time history of the stresses and displacements for the different 'n' waves. With the particular plate considered there is very little difference in the form of the stress and displacement responses, so that it is not necessary to show them both. A brief survey of these curves shows that the effects of the loading may be different depending upon whether the system is being forced or whether it is in the subsequent state of free vibration; it is necessary, therefore, to specify the location in time.

During the period over which the pulse is acting, when the initial rise in pressure is taking place, a decrease in rise time causes an increase in the magnitude of the response. Figs. 2 to 9. This is because decreasing the rise time effectively increases the net impulse and hence the displacement. It might be considered that decreasing the rise time increases the content of the higher modes, but for the displacements and stresses the effects must be small as the response appears to be dominated by the fundamental mode in all the cases studied.

As might be expected, from simple frequency response considerations, the overall forced motion is governed by the particular value of the period ratio. When τ/T is small as in Figs. 8 and 9, both $\bar{\omega}$ and $\bar{\sigma}$ are small; when τ/T is large ($1\frac{1}{2}$ to 3) as in Figs. 2 to 5, the magnitude is greater. The maximum values seem to lie between 1.5 and 2.0; their position and magnitudes being strongly influenced by the rise time. Under these conditions the form of the forced motion may be clearly distinguished with the harmonic motion superimposed.

When the plate is vibrating freely at the time, t, greater than τ it is the overall shape of the pulse and its duration which is important. Changes in the load parameters bring variations in the form of the response, but not always in the same direction. For instance, if Figs. 2 and 3 are compared, it may be seen in the former that decreasing the rise time increased the maximum response; but, in the latter the exact opposite occurs. The fact that the response in a particular mode may be small at zero in the free vibration, even though there is substantial movement during the application of the pulse is interesting. It is shown in the Appendix, for an undamped mode, that this occurs when the Fourier Integral of the pulse is zero at the natural frequency ω_n. If this condition is almost satisfied substantial changes in the response may result by only a minor change in either the forcing function or system characteristics. When the effect is to decrease the response it gives rise to an apparent increase in the damping.

The largest dynamic amplification occurs when the period ratio is unity, the isochronous condition, where the push-pull effect of the boom matches up with the natural frequency of the fundamental mode. Fig. 6 shows the variation in the resonant response with the rise time; the maximum value, about 2.6, occurs when the rise time is one quarter of the fundamental period; the lowest, about 2.1, obtained by extrapolating the illustrated results, occurs when the rise time is zero.

It is possible to have a twin boom in the wake of the main one due to ground reflection, and at a certain height the pressure loading will be that of a double 'N'. The effects of this for the isochronous condition is shown in Fig. 7, it may be noted that the dynamic amplification factors are now double those for the single boom, the maximum now being 5.2. When the damping factor was reduced from 0.02 to 0.002 the maximum value increases a further 10% to 5.7.

Figs. 10 to 13 show some of the acceleration curves. The most striking difference between these and the stresses is that the effect of the higher modes is more noticeable. When the rise time decreases the initial accelerations and the contributions of these modes also increases, as shown in Fig. 10. Because of its dependence on the higher modes the acceleration does not change with alteration of the period ratio in the manner that the other parameters do; the maximum values do not fall off for τ/T less than 1. This is shown clearly in Fig. 13 where for τ/T equal to $\frac{1}{2}$, the maximum value is 2.1 compared with the values in Fig. 10, τ/T equal to 3, of -1.5.

DISCUSSION

The approximate method used in this work gives results which are compatible to those obtained from analytical methods, as in (1). Though it has been applied here to evaluating the response to regular pulse shapes there is no added difficulty in obtaining the response to more complex wave forms.

Apart from the special isochronous case it appears from the results that there will in general be a dynamic amplification of about 2. Since the boom overpressure is not expected to be much above 3 lbs/ft.^{2} and the static breaking pressure of windows in the region of 80 lbs./ft.^{2} there is no reason to expect failure to occur under the load of a single boom.

The fact that the fundamental mode dominates the displacement and stress responses in all but a few cases is important as it means that the plate may be idealized as a single degree of freedom systems. However, it is difficult to make generalisations for other types of loading without considering the matter further. Under the conditions that the response of the plate has been studied, the number of modes which make a significant contribution is strongly dictated by the spacing of their natural frequencies relative to that of the fundamental mode. The frequency ratio, R_{mn}, for the first 9 modes are shown in Table 3 below; of these only one third are active under normal incidence, those marked with an asterisk. These are reasonably well separated.

Table 3

m, n	1, 1	1, 2	2, 1	1, 3	2, 2	2, 3	1, 4	3, 1	2, 4
R_{mn}	1.00*	1.90	3.07	3.70*	4.00	5.54	5.60	6.24*	7.76

As far as the (1, 3) and (3, 1) modes are concerned, for the range of period ratio considered, a crude approximation to the modal coefficients may be obtained by assuming the forcing function is a step input. The relative magnitude of the modal displacements in the series is $1/mn.R^2_{mn}$, so that the (1, 3) mode has only about 2.5% of the fundamental contribution and the (3, 1) mode less than 1%. The other modes are brought into play when the boom arrives obliquely at the plate surface. The significance of these are shown in part II of this paper (NASA CR-1176).

CONCLUSIONS

A method for determining the response of a structure to an arbitrary waveform has been presented. This has been applied to finding the response of a simply supported plate to 'n' waves at normal incidence. Both the period ratio, τ/T, and the rise time were varied. The following conclusions are drawn.

(i) Effect of Period Ratio τ/T: The maximum amplification factor occurs when τ/T is unity and in this condition the maximum value is sensitive to a variation in the rise time. The greatest magnification factor for displacement is 2.6.

(ii) Effect of Rise Time: In the duration of the pressure pulse, decreasing the rise time increases the magnitude of the response, and increases the contributions from the higher modes. When τ/T equals one, the maximum amplification factor occurs when the rise time is $\frac{1}{4}$ of the duration of the pulse. It is then about 2.6. For zero rise time the amplification factor is about 2.1. When a double 'N' excites the system under isochronous conditions these factors are doubled.

(iii) For the plate (aspect ratio 1.5) there is little difference between the displacement and stress time histories as these were almost completely dominated by the fundamental mode. However, the accelerations are affected more by the higher modes and, consequently, the response contains more peaks.

(iv) Under certain conditions the response for the time $t > \tau$, (i.e. when the system is left vibrating freely) is very small. This is dependent more on the overall shape of the pulse and its relation to the fundamental period of the system than to any other single parameter.

APPENDIX

The Condition for an Undamped Oscillator to have Zero Response after the Application of a Pulse

If the duration of the pulse is, T_p, then the displacement and velocity at the time $\tau = T$ is given by:

$$x(T) = \frac{1}{\omega_n} \int_0^T F(\tau) \sin \omega_n (T - \tau) d\tau$$

$$\dot{x}(T) = \int_0^T F(\tau) \cos \omega_n (T - \tau) d\tau.$$

Now, for zero response, $x(T) = \dot{x}(T) = 0.$ These conditions may be expressed by a single complex equation if use is made of the phase plane and plot $\dot{x}(T)/\omega_n$ as the abscissa and $x(T)$ as ordinate then:

$$W(T) = \frac{\dot{x}(T)}{\omega_n} + jx(T)$$

$$= \frac{1}{\omega_n} \int_0^{T_p} F(\tau)(\cos \omega_n (T - \tau) + j \sin \omega_n (T - \tau)) d\tau$$

$$= \frac{1}{\omega_n} \int_0^{T_p} F(\tau) e^{j\omega_n (T-\tau)} d\tau = \frac{e^{j\omega T}}{\omega_n} \int_0^{T_p} F(\tau) e^{-j\omega_n \tau} d\tau$$

$$= \frac{e^{j\omega_n T_p}}{\omega_n} \int_{-\infty}^{\infty} F(\tau) e^{-j\omega_n \tau} d\tau \qquad F(\tau) = 0 < \tau < T_p$$

The term inside the integral is the Fourier Transform of the input $F(\tau)$, and the condition for zero response after the application of a pulse is that the Fourier Transform should be zero at the frequency, ω_n.

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TABLE 1

FORCED MOTION

τ/T	R.T.	$\bar{w}$		$\bar{\sigma}x$		$\bar{\sigma}y$		$\ddot{\bar{w}}$	
		MAX	N_p	MAX	N_p	MAX	N_p	MAX	N_p
3	T/2	1.5	1	1.64	1	1.05	1		
	T/4	1.63	1	1.78	1	1.13	1	-1.37	1
	T/8	-1.64	1	-1.87	1	-1.23	1	-1.40	3
2	T/2	1.45	1	1.61	1	1.05	1		
	T/4	1.54	1	1.69	1	1.08	1	-1.31	3
	T/8	-1.83	1	-2.09	1	-1.36	1	1.83	3
1.5	3T/8	-2.18	1	-2.41	1	-1.53	1	1.90	
	T/4	-1.73	1	-1.87	1	-1.23	1	1.47	3
	T/8	-1.44	1	-1.67	1	-1.09	1	1.49	2
1	T/4	-2.4	1	-2.73	1	-1.79	1		
	T/10	-2.19	1	-2.54	1	-1.70	1	2.63	3
	T/20	-2.05	1	-2.31	1	-1.52	1	2.33	3
0.5	3T/20	0.70	1	0.86	4	0.68	4		
	T/10	0.73	1	0.92	4	0.71	4	-1.36	4
	T/20	0.74	1	0.94	4	0.74	4	-1.49	6

TABLE 2

FREE MOTION

τ/T	R.T.	$\bar{w}$		$\bar{\sigma}x$		$\bar{\sigma}y$		$\ddot{\bar{w}}$	
		MAX	N_p	MAX	N_p	MAX	N_p	MAX	N_p
3	T/2	0.195	1	0.223	1	0.149	1	-	-
	T/4	1.21	1	1.37	1	0.915	1	-0.97	1
	T/8	1.57	1	1.80	1	1.24	1	-1.08	1
2	T/2	0.13	1	0.153	1	0.12	1	-	-
	T/4	1.40	1	1.59	1	1.06	1	-1.08	1
	T/8	1.73	1	1.96	1	1.31	1	-1.58	1
1.5	3T/8	1.83	1	2.09	1	1.40	1	1.78	1
	T/4	1.12	1	1.28	1	0.82	1	-1.15	1
	T/8	.381	1	0.46	1	0.36	3	-0.76	6
1	T/4	2.24	1	2.53	1	1.66	1	-	-
	T/10	2.09	1	2.38	1	1.66	4	-2.36	3
	T/20	1.95	1	2.19	1	1.52	1	-2.10	1
0.5	3T/20	-0.93	1	-1.08	1	-0.79	1		
	T/10	-1.03	1	-1.15	1	-0.77	1	-1.11	4
	T/20	-1.13	1	-1.40	1	-0.94	1	1.92	6

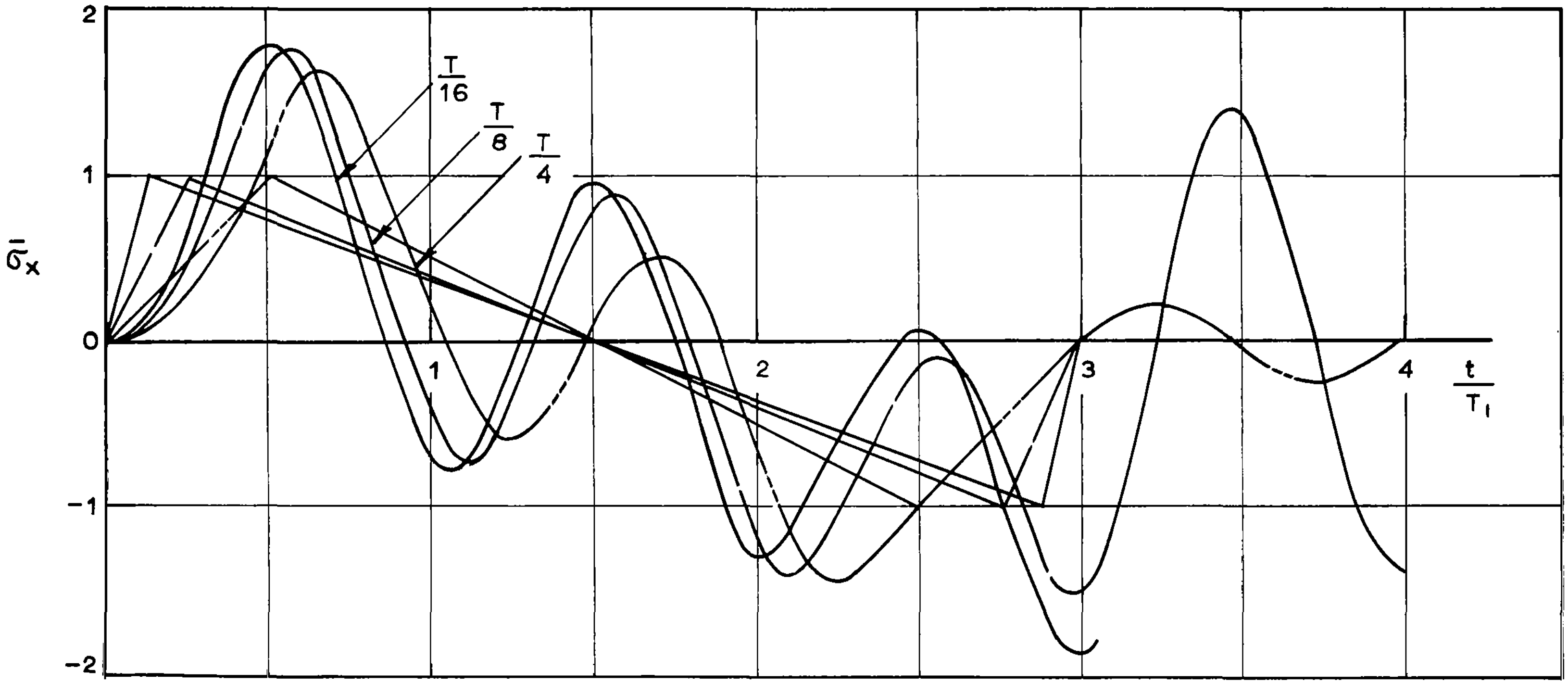

FIG. 2. $\bar{\sigma}x$, Effect of Rise Time, Period Ratio, 3.

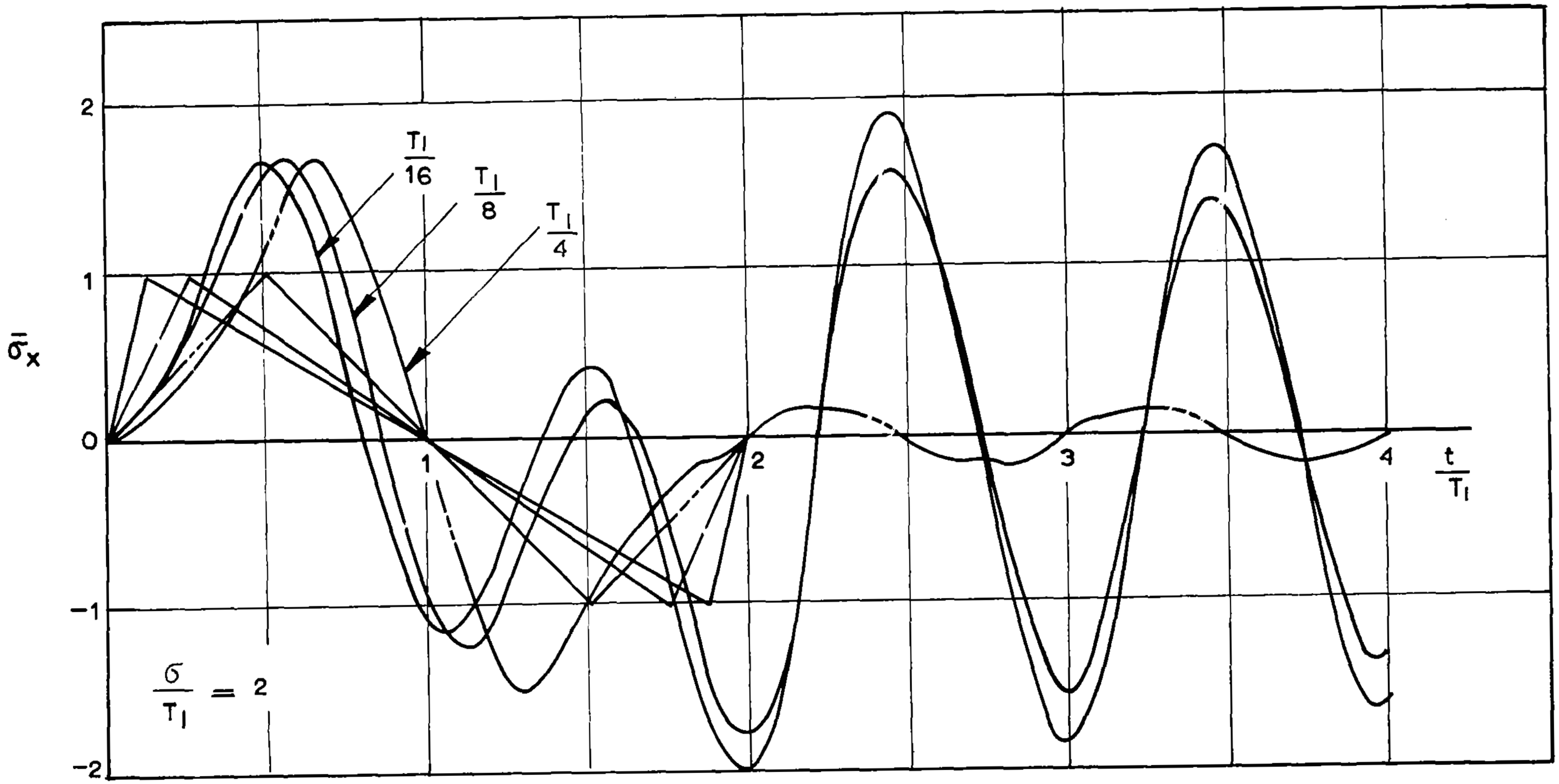

FIG. 3. $\bar{\sigma}x$, Effect of Rise Time, Period Ratio, 2.

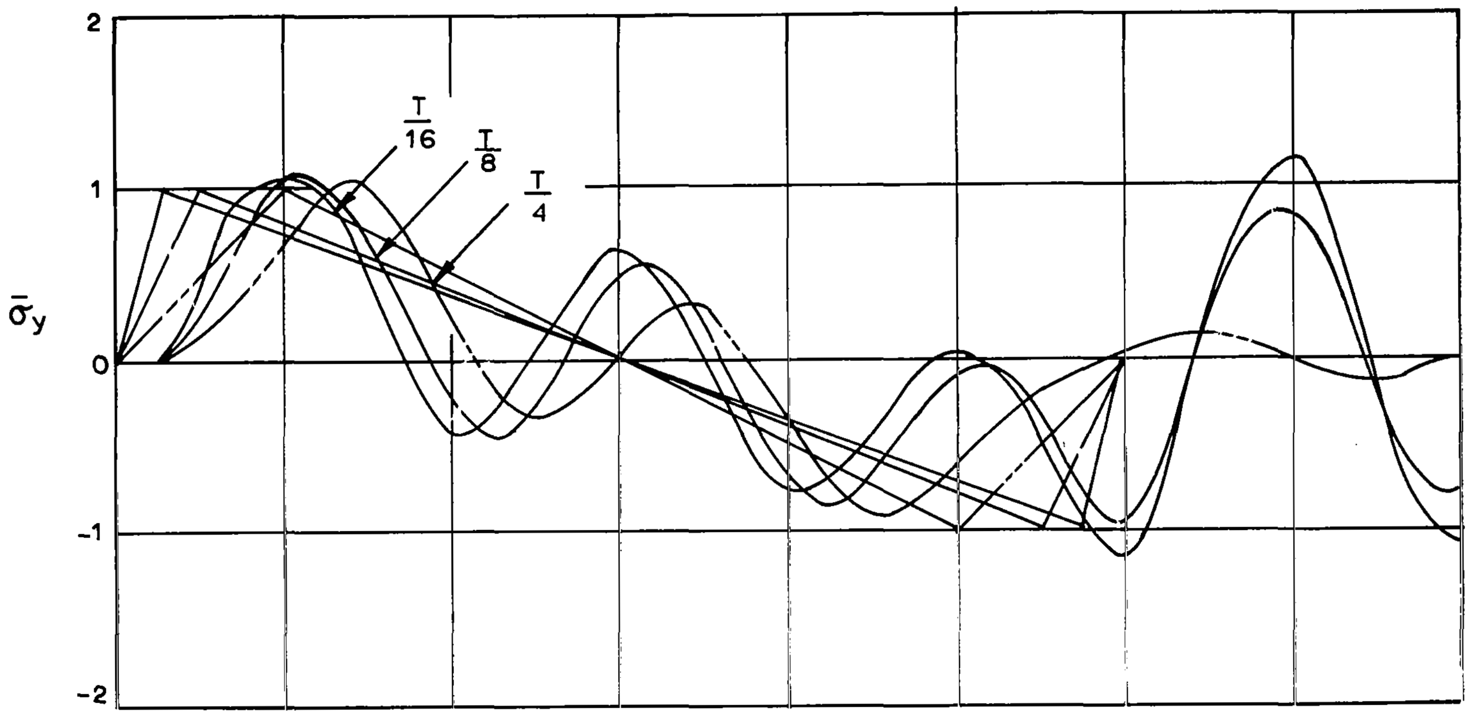

FIG. 4. $\bar{\sigma}$y, Effect of Rise Time, Period Ratio, 3.

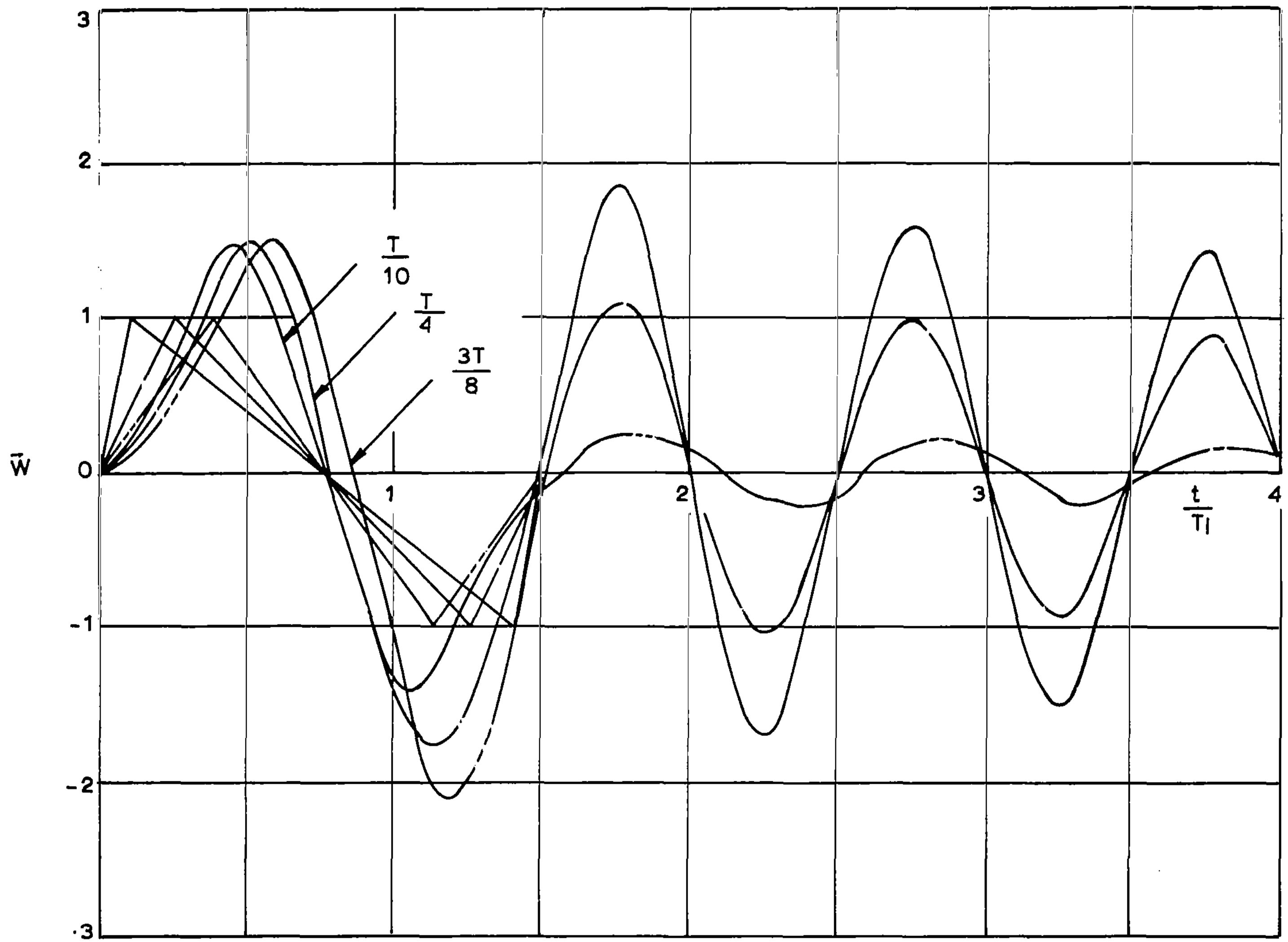

FIG. 5. $\bar{w}$, Effect of Rise Time, Period Ratio, $1\frac{1}{2}$.

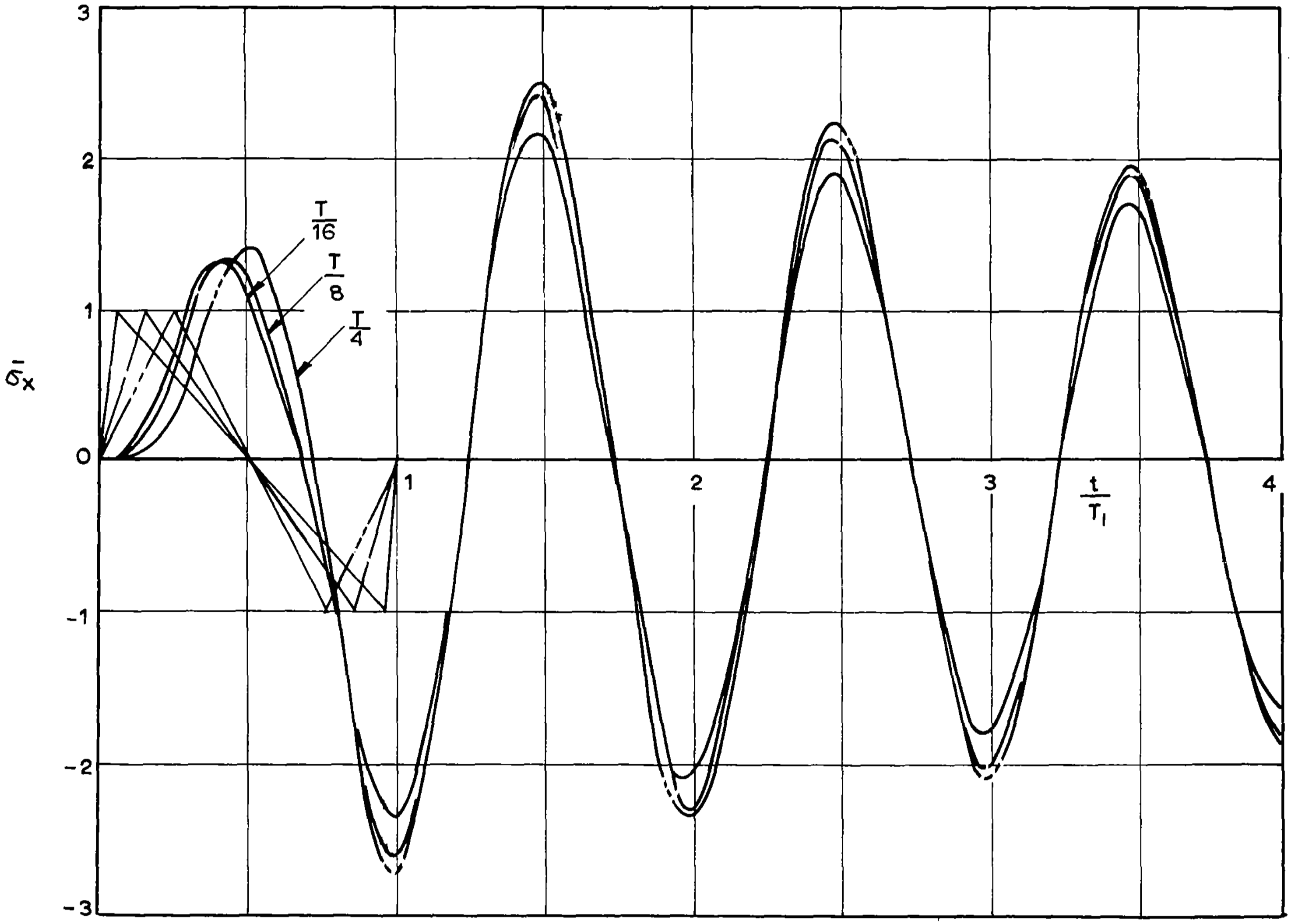

FIG. 6. $\bar{\sigma}x$, Effect of Rise Time, Period Ratio, 1 (Isochronous Case).

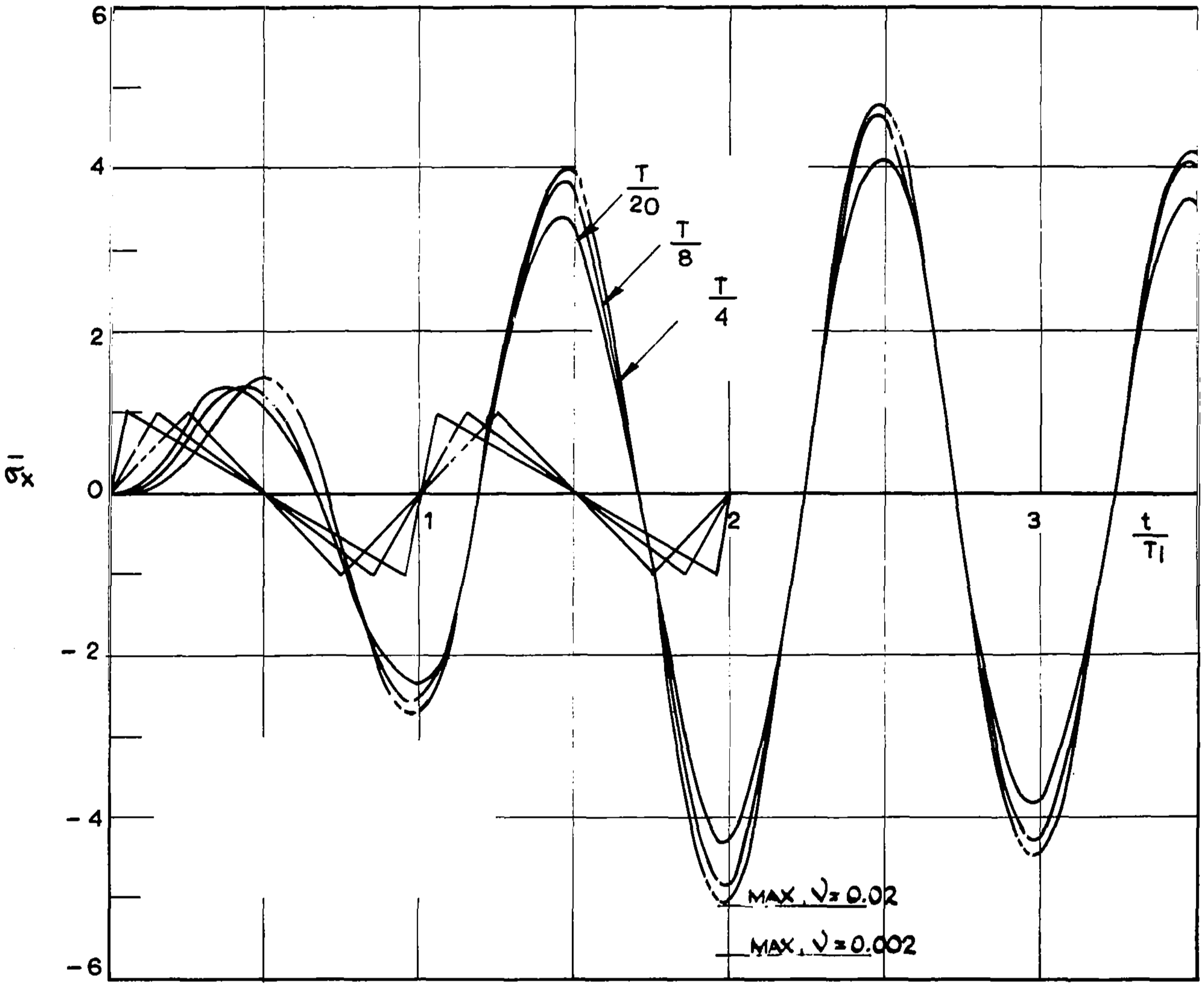

FIG. 7. $\bar{\sigma}x$, Effect of Rise Time, Double N, Isochronous Case.

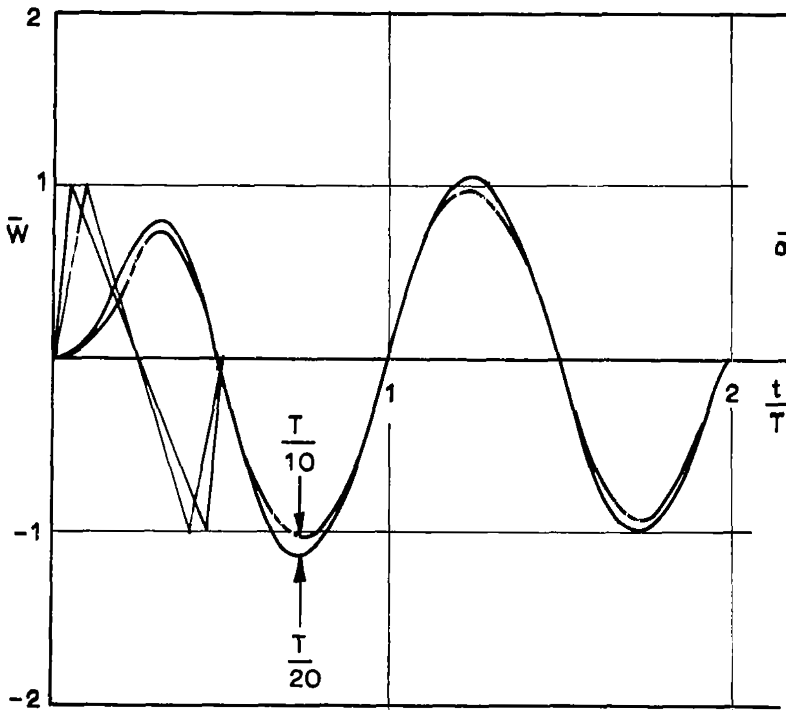

FIG. 8. $\bar{w}$, Period Ratio, ½.

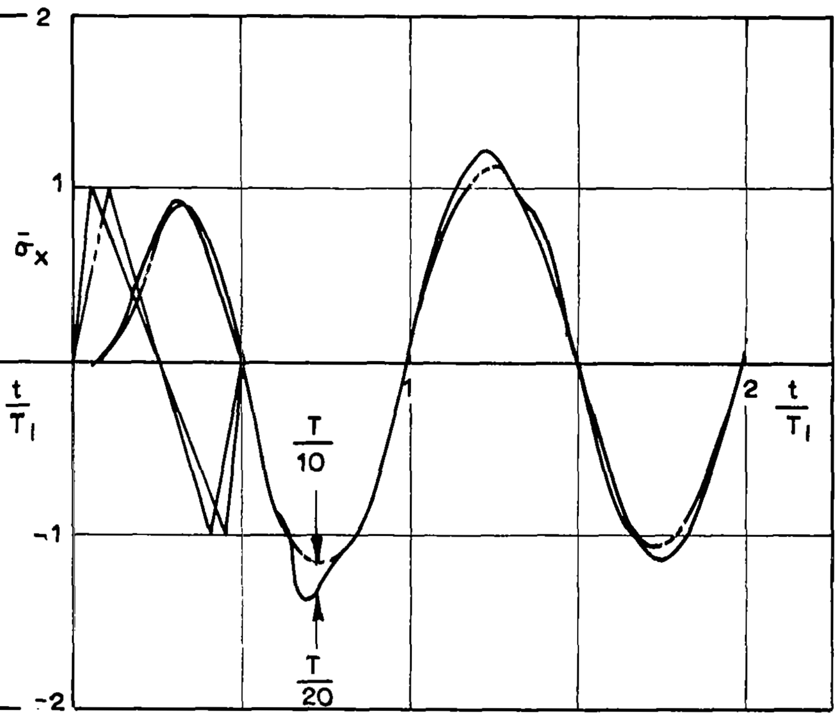

FIG. 9. $\bar{\sigma}x$, Period Ratio, ½.

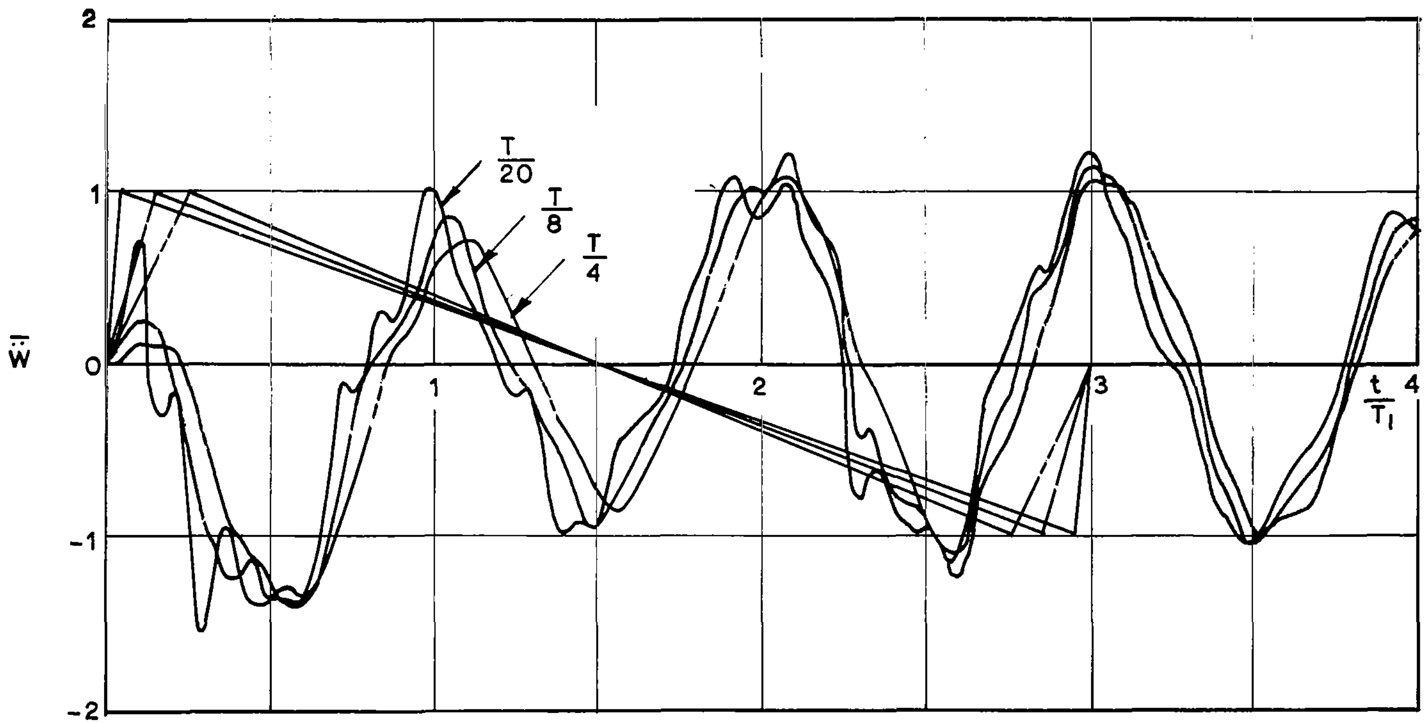

FIG. 10. $\ddot{\bar{w}}$, Effect of Rise Time, Period Ratio, 3.

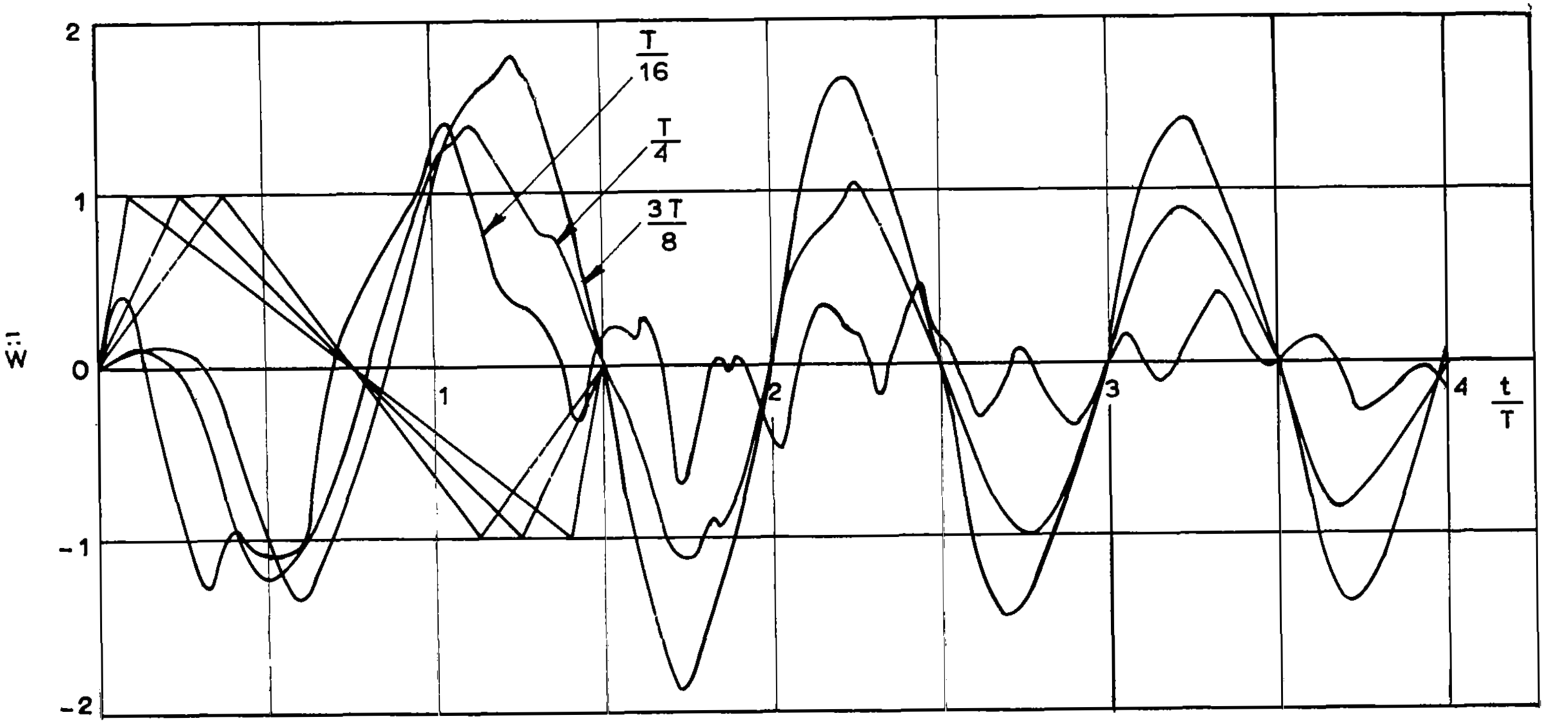

FIG. 11. $\bar{\ddot{w}}$, Effect of Rise Time, Period Ratio, $1\frac{1}{2}$.

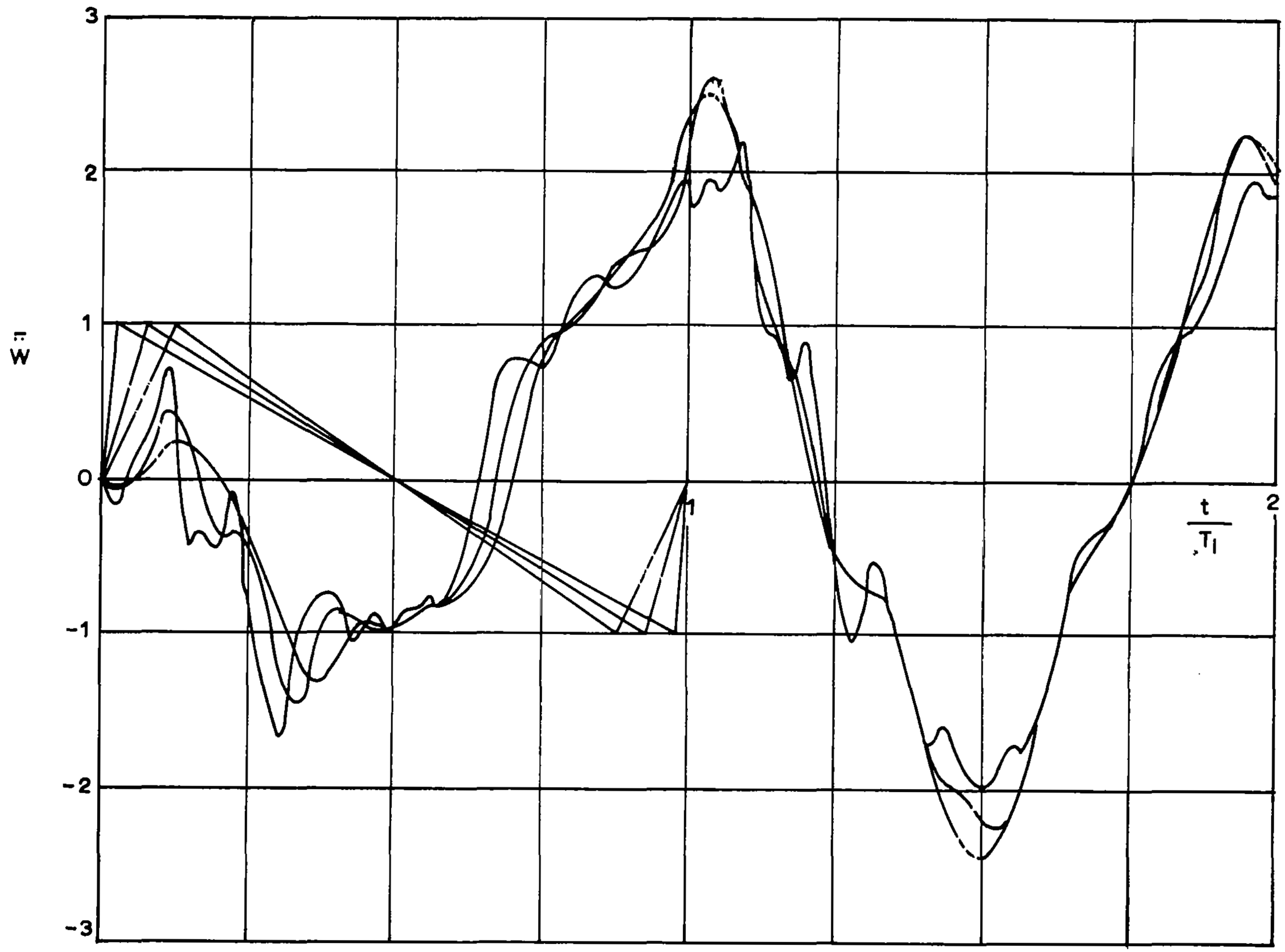

FIG. 12. $\ddot{\bar{w}}$, Effect of Rise Time, Period Ratio, 1 (Isochronous Case).

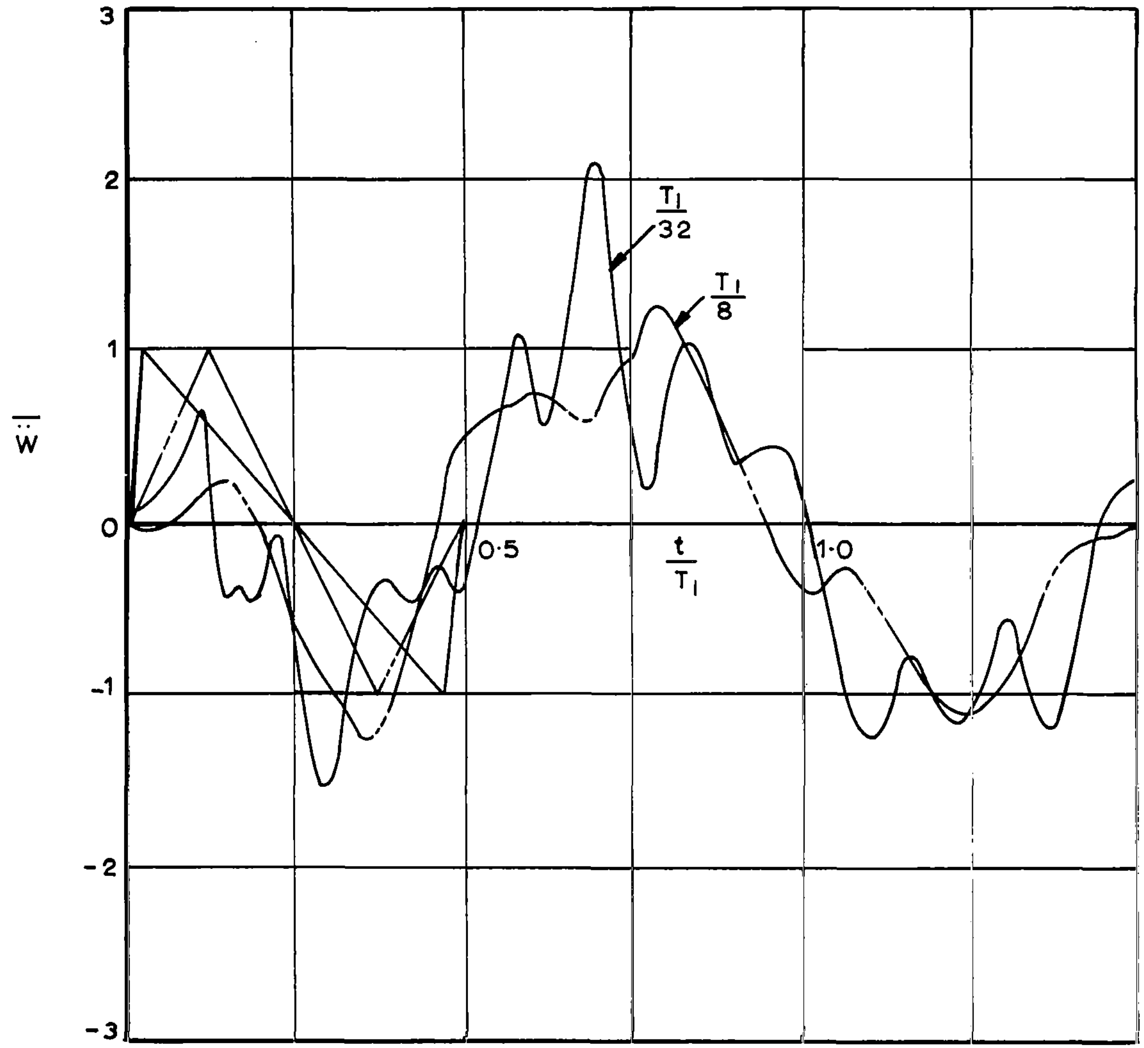

FIG. 13. $\ddot{\bar{w}}$, Effect of Rise Time, Period Ratio, $\frac{1}{2}$.

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