NSG-354 (SUPPL. 3)

CONSTRAINED MINIMIZATION UNDER VECTOR-VALUED CRITERIA IN LINEAR TOPOLOGICAL SPACES

N. O. Da Cunha
Technical Institute of Aeronautics
San Jose dos Campos
Sao Paulo, Brazil
and
E. Polak
Electronics Research Laboratory
Department of Electrical Engineering
and Computer Sciences
University of California, Berkeley

Introduction

Vector criterion optimization problems arise when several optimality criteria are relevant to a physical situation and their relative importance is not obvious. The first formulation of such a problem was given by the economist Pareto in 1896 (1), and since then discussions of vector valued optimization have kept reappearing in economics, operations research and, more recently, in control engineering. References (2), (3), (4), (5), (6), (7) form a representative sampling of the related literature in these fields.

In this paper we extend some of the necessary conditions and theorems on "scalarization" (i.e., the conversion of the problem into a family of optimization problems with a scalar criterion), which we gave in (7) for problems defined in R^n, to linear topological spaces. It will be observed that our derivation of necessary conditions follows the well trodden path established in (8), (9), (10).

The problem of scalarization is very important, since whenever scalarization is possible, standard non-linear programming algorithms become applicable. We concentrate on scalarization by weighting the components of the vector cost with strictly positive coefficients into a real valued function, since this represents the physically most meaningful case. Our major result in this area is theorem (4), for which we give a proof suggested by Prof. H. Halkin of University of California, San Diego, and which is shorter than our original proof in (7) and (11).

I. The Basic Problem and Necessary Conditions

Let E^s, where s is a positive integer, be the s-dimensional Euclidean space with the usual norm topology. Let $\mathcal{X}$ be a real, linear topological space; let $h : \mathcal{X} \to E^p$ and $r : \mathcal{X} \to E^m$ be continuous functions, and let Ω be a subset of $\mathcal{X}$. Furthermore, suppose that we are given a partial ordering in E^p with the property that for every y in E^p there exists an index set $J(y) \subset \{1, 2, \ldots, p\}$ and a ball $B(\varepsilon_0, y)$ with center y and radius $\varepsilon_0 > 0$ such that every $\tilde{y} \in B(\varepsilon_0, y)$ satisfies $\tilde{y}^i < y^i$ for all $i \in J(y)$, if and only if $\tilde{y} \prec y$ and $y \not\prec \tilde{y}$.

Definition 1: We shall call the index set $J(y)$ and the ball $B(\varepsilon_0, y)$, defined above, the critical index set and the critical neighborhood for the point y, respectively.

Examples: Suppose that $y_1 \prec y_2$ if and only if $y_1^i \leq y_2^i$ for $i = 1, 2, \ldots, p$, then we see that $J(y) = \{1, 2, \ldots, p\}$ for all $y \in E^p$. Again, suppose that $p > 1$ and $y_1 \prec y_2$ if and only if $\mathrm{Max}\{y_1^i \mid i = 1, 2, \ldots, p\} \leq \mathrm{Max}\{y_2^i = 1, 2, \ldots, p\}$. Now $J(y) \neq \{1, 2, \ldots, p\}$ and it is seen to change from point to point in E^p.

The problems we wish to consider can always be cast in the following standard form.

Basic Problem: Find a point $\hat{x}$ in $\mathcal{X}$, such that: (i) $\hat{x} \in \Omega$ and $r(\hat{x}) = 0$; (ii) for every x in Ω with $r(x) = 0$, the relation $h(x) \prec h(\hat{x})$ implies that $h(\hat{x}) \prec h(x)$.

As a first step in obtaining necessary conditions for a point $\hat{x}$ to be a solution to the Basic Problem, we introduce "linear" approximations to the set Ω and to the continuous functions h and r at $\hat{x}$.

Definition 2: We shall say that a convex cone $C(\hat{x}, \Omega)$ is a conical approximation to the constraint set Ω at the point $\hat{x} \in \Omega$ with respect to the functions h and r, if there exist continuous linear functions $h'(\hat{x}): \mathscr{X} \to E^p$ and $r'(\hat{x}): \mathscr{X} \to E^m$ such that for any finite collection $\{x_1, x_2, \ldots, x_k\}$ of linearly independent vectors in $C(\hat{x}, \Omega)$, there exists a continuous map ζ_ε from $\varepsilon S \triangleq \text{co}\{\varepsilon x_1, \varepsilon x_2, \ldots, \varepsilon x_k\}$, into $\Omega - \hat{x}$, for each ε, $0 \leq \varepsilon \leq 1$, and continuous functions $o_{h,\varepsilon}: \mathscr{X} \to E^p$ and $o_{r,\varepsilon}: \mathscr{X} \to E^m$, which satisfy (1), (2), (3) and (4) below:

$$\|o_{h,\varepsilon}(\varepsilon x)\|/\varepsilon \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly for } x \in S \tag{1}$$

$$\|o_{r,\varepsilon}(\varepsilon x)\|/\varepsilon \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly for } x \in S \tag{2}$$

$$h(\hat{x} + \zeta_\varepsilon(x)) = h(\hat{x}) + h'(\hat{x})(x) + o_{h,\varepsilon}(x), \quad \text{for all } x \in \varepsilon S,\ 0 \leq \varepsilon \leq 1 \tag{3}$$

$$r(\hat{x} + \zeta_\varepsilon(x)) = r(\hat{x}) + r'(\hat{x})(x) + o_{r,\varepsilon}(x), \quad \text{for all } x \in \varepsilon S,\ 0 \leq \varepsilon \leq 1 \tag{4}$$

Theorem 1: If $\hat{x}$ is a solution to the Basic Problem, if $C(\hat{x}, \Omega)$ is a conical approximation to Ω at $\hat{x}$, and if $J(h(\hat{x}))$ is the set of critical indices for $h(\hat{x})$, then there exist a vector μ in E^p and a vector η in E^m such that

(i) $\mu^i \leq 0$ for $i \in J(h(\hat{x}))$ and $\mu^i = 0$ for $i \in \bar{J}(h(\hat{x}))$;

(ii) $(\mu, \eta) \neq 0$;

(iii) $\langle \mu, h'(\hat{x})(x) \rangle + \langle \eta, r'(\hat{x})(x) \rangle \leq 0$ for all $x \in \overline{C(\hat{x}, \Omega)}$,

where $h'(\hat{x})$, $r'(\hat{x})$ are the linear continuous maps appearing in the definition of $C(\hat{x}, \Omega)$.

Proof: Let $\hat{x}$ be a solution to the Basic Problem. Let $J(h(\hat{x}))$ and $B(\varepsilon_0, h(\hat{x}))$ be, respectively, the critical index set and the critical neighborhood of $h(\hat{x})$ in E^p. Also, let q be the cardinality of $J(h(\hat{x}))$ and let f, $f'(\hat{x})$ be continuous functions from $\mathscr{X}$ into E^q defined by $f(x) = (f^1(x), \ldots, f^q(x))$, $f'(\hat{x})(x) = (f'^1(\hat{x})(x), \ldots, f'^q(\hat{x})(x))$, where $f^j = h^{i_j}$, $f'^j(\hat{x}) = h'^{i_j}(x)$, with $i_j \in J(h(\hat{x}))$ for $j = 1, 2, \ldots, q$ and $i_\alpha > i_\beta$ when $\alpha > \beta$. Now let

$$A(\hat{x}) = \{y \in E^q \mid y = f'(\hat{x})(x),\ x \in C(\hat{x}, \Omega)\}, \tag{5}$$

$$B(\hat{x}) = \{z \in E^m \mid z = r'(\hat{x})(x),\ x \in C(\hat{x}, \Omega)\}, \tag{6}$$

$$K(\hat{x}) = \{u \in E^q \times E^m \mid u = (f'(\hat{x})(x),\ r'(\hat{x})(x)),\ x \in C(\hat{x}, \Omega)\} \tag{7}$$

$$R = \{(y, 0) \in E^q \times E^m \mid y = (y^1, y^2, \ldots y^q),\ y^1 < 0,\ y^2 < 0, \ldots, y^q < 0,\ 0 \in E^m\}. \tag{8}$$

Examining (i), (ii), and (iii), we observe that if we define $\mu^i = 0$ for $i \in \bar{J}(h(\hat{x}))$, the complement of $J(h(\hat{x}))$ in $\{1, 2, \ldots, p\}$, then the claim of the theorem is that the convex sets $K(\hat{x})$ and R are separated in $E^q \times E^m$.

We now construct a proof by contradiction. Suppose that $K(\hat{x})$ and R are not separated in $E^q \times E^m$. Then,

(I) The convex sets $K(\hat{x})$ and R are not disjoint, i.e., $R \cap K(\hat{x}) \neq \phi$, the empty set.

(II) The convex cone $B(\hat{x})$ in E^m contains the origin as an interior point and hence $B(\hat{x}) = E^m$.

Statement (II) follows from the fact that if 0 is not an interior point of the convex set $B(\hat{x})$, then by the separation theorem (12), there exists a nonzero vector η_0 in E^m such that

$$\langle \eta_0, z \rangle \leq 0 \text{ for all } z \in B(\hat{x}). \tag{9}$$

Clearly, the vector $(0, \eta_0)$ in $E^q \times E^m$ separates R from $A(\hat{x}) \times B(\hat{x})$ and hence from $K(\hat{x})$, since $K(\hat{x}) \subset A(\hat{x}) \times B(\hat{x})$, contradicting our assumption that R and $K(\hat{x})$ are not separated.

Since the origin in E^m belongs to the non-void interior of $B(\hat{x})$ we can construct a simplex Σ in $B(\hat{x})$, with vertices $z_1, z_2, \ldots, z_{m+1}$, such that

(i) 0 is in the interior of Σ;

(ii) there exists a set of vectors $\{x_1, x_2, \ldots, x_{m+1}\}$ in $C(\hat{x}, \Omega)$ satisfying:

$$\text{(a)}\quad z_i = r'(\hat{x})(x_i) \text{ for } i = 1, 2, \ldots, m+1; \tag{10}$$

(b) $\zeta_1(x) \in ((\Omega - \hat{x}) \cap N)$ for all $x \in \text{co}\{x_1, x_2, \ldots, x_{m+1}\}$ (11)

where ζ_1 is the map entering the definition of a conical approximation and N is a neighborhood of 0 in $\mathcal{X}$ such that $h(\hat{x} + N) \subset B(\varepsilon_0, h(\hat{x}))$, where $B(\varepsilon_0, h(\hat{x}))$ is the critical neighborhood for $h(\hat{x})$. (Clearly, such an N exists since h is continuous).

(c) $y_i = f'(\hat{x})(x_i) < 0$ for $i = 1, 2, \ldots, m+1$. (12)

Let $\ell_1, \ell_2, \ldots, \ell_m$ be any basis in E^m, and let $Z: E^m \to E^m$, $X : E^m \to \mathcal{X}$ be linear operators defined by $Z\ell_i = (z_i - z_{m+1})$, $X\ell_i = (x_i - x_{m+1})$, respectively, with $i = 1, 2, \ldots, m$. Since $0 \in \text{int}\, \Sigma$, the vectors $(z_i - z_{m+1})$, $i = 1, 2, \ldots, m$, are linearly independent and hence the operator Z is nonsingular. Let Z^{-1} denote the inverse of Z. Clearly the map $z \to XZ^{-1}(z - z_{m+1}) + x_{m+1}$ from Σ into $\text{co}\{x_1, x_2, \ldots, x_{m+1}\}$ is continuous.

For $0 < \alpha \leq 1$, we now define a continuous map G_α from the simplex $\alpha\,\Sigma$ into E^m by

$$G_\alpha(\alpha z) = r(\hat{x} + \zeta_\alpha(\alpha\, XZ^{-1}(z - z_{m+1}) + \alpha x_{m+1})), \tag{13}$$

where ζ_α is the map specified by Definition 1.

Since $r(\hat{x}) = 0$, $r'(\hat{x})\, X = Z$, and $r'(\hat{x})(x_{m+1}) = z_{m+1}$, (13) becomes

$$G_\alpha(\alpha z) = \alpha z + o_r(\alpha\, XZ^{-1}(z - z_{m+1}) + \alpha x_{m+1}). \tag{14}$$

It now follows from (2) and Brouwer's fixed point theorem (13) that there exists an $\alpha_0 \in (0, 1]$ such that for every $\alpha \in (0, \alpha_0]$ we can find a $z_\alpha \in \Sigma$ satisfying $G_\alpha(\alpha z_\alpha) = 0$. Since by construction

$$f'^i(\hat{x})(XZ^{-1}(z - z_{m+1}) + x_{m+1}) < 0 \text{ for all } z \in \Sigma \text{ and } i = 1, 2, \ldots, q, \tag{15}$$

there exist by (1) $\alpha_1 > 0$, $\alpha_2 > 0, \ldots, \alpha_q > 0$ such that

$$f^i(\hat{x} + \zeta_\alpha(\alpha XZ^{-1}(x - z_{m+1}) + \alpha\, x_{m+1})) < f^i(\hat{x}) \text{ for all } z \in \Sigma,$$

$$\alpha \in (0, \alpha_i], \text{ and } i = 1, 2, \ldots, q, \qquad (16)$$

Let α^* be the minimum of $\{\alpha_0, \alpha, \ldots, \alpha_q\}$, and let $z^* \in \Sigma$ satisfy $G_{\alpha^*}(\alpha^* z^*) = 0$. Then, $x^* = \hat{x} + \xi_{\alpha^*}(\alpha^* XZ^{-1}(z^* - z_{m+1}) + \alpha^* x_{m+1})^{\alpha}$ is in Ω, $r(x^*) = 0$, $h(x^*) \prec h(\hat{x})$, and $h(\hat{x}) \not\prec h(x^*)$, which contradicts our assumption that $\hat{x}$, is a solution. Hence the theorem is true.

II. An Application to Optimal Control

We now illustrate the theory developed in the preceding section by obtaining a Pontryagin type maximum principle for an optimal control problem with a vector valued cost function.

The Optimal Control Problem: Let $z = (y, x)$, with $y \in E^p$, $x \in E^n$. Given a differential system

$$\frac{dz(t)}{dt} = F(x(t), u(t), \quad t \in [t_o, t_f], \qquad (17)$$

where $u(t) \in E^m$ is the control and $F = (c, f)$ is a map from $E^n \times E^m$ into E^{p+n} ($c(x, u) \in E^p$, $f(x, u) \in E^n$), continuous in u and continuously differentiable in x. Find a control $\hat{u}(t)$ and corresponding trajectory $\hat{z}(t)$ determined by (17), such that

(i) For $t \in [t_o, t_f]$, $\hat{u}$ is a measurable, essentially bounded function whose range is contained in an arbitrary but fixed subset U of E^m,

(ii) The following boundary conditions are satisfied:
(a) $\hat{z}(t_o) = \hat{z}_o = (0, \hat{x}_o)$, where $\hat{x}_o$ is a given vector in E^n, and (b) $g(\hat{x}(t_f)) = 0$, where $g : E^n \to E^\ell$ is a continuously differentiable map whose Jacobian $\frac{\partial g(x)}{\partial x}$ has maximum rank for all x satisfying $g(x) = 0$.

(iii) For every control $u(t)$ and corresponding trajectory $z(t)$, $t \in [t_o, t_f]$, satisfying (i) and (ii) above, the relation $y(t_f) \leq \hat{y}(t_f)$ implies that $y(t_f) = \hat{y}(t_f)$.

To transcribe the optimal control problem into the form of the Basic Problem, we take $\mathcal{H}$ to be the product space $\mathcal{S}_{p+n} = \mathcal{S} \times \mathcal{S} \times \ldots \times \mathcal{S}$, where $\mathcal{S}$ is the space of all real valued functions on $[t_o, t_f]$, which are either upper or lower semi-continuous at each point $t \in [t_o, t_f]$, with the pointwise topology. We define Ω to be the set of all absolutely continuous functions $z = (y, x)$ from $[t_o, t_f]$ into E^{p+n}, which for some u satisfying (i) above, satisfy the differential equation (17) for almost all $t \epsilon [t_o, t_f]$, with $z(t_o) = (0, x_0)$. Finally, we define $h(z) = y(t_f)$ and $r(z) = g(x(t_f))$. It is easy to show that both h and r are continuous.

Suppose the control $\hat{u}(t)$ and the corresponding trajectory $\hat{z}(t)$ solve the optimal control problem. To construct a conical approximation $C(\hat{z}, \Omega)$ to Ω, we follow L. W. Neustadt's derivation (8), which was based on a utilization of the cone of attainability given in (13) by Pontryagin et al. Let $I \subset [t_o, t_f]$ be the set of all points t at which $\hat{u}(t)$ is regular.

Let $\Phi(t, \tau)$ be the $(p+n) \times (p+n)$ matrix which satisfies the linear differential equation

$$\frac{d}{dt} \Phi(t, \tau) = \frac{\partial F}{\partial z} (\hat{x}(t), \hat{u}(t)) \Phi(t, \tau) \tag{18}$$

for almost all $t \epsilon [t_o, t_f]$ with $\Phi(\tau, \tau) = I_{p+n}$, the $(p+n)$ identity matrix.

For any $s \epsilon I$ and $v \epsilon U$ we define

$$\delta z_{s,v}(t) = \begin{cases} 0 \text{ for } t_o \leq t < s \\ \Phi(t, s) [F(\hat{x}(s), v) - F(\hat{x}(s), \hat{u}(s)] \\ \qquad \text{for } s \leq t \leq t_f \end{cases} \tag{19}$$

and

$$C(\hat{z}, \Omega) = \{\delta z \epsilon \mathcal{X} \mid \delta z(t) = \sum_{i=1}^{k} \alpha_i \delta z_{s_i, v_i}(t) ,$$

$$\{s_1, s_2, \ldots, s_k\} \subset I, \{v_1, v_2, \ldots, v_k\} \subset U, \text{ and}$$

$$\alpha_i \geq 0 \text{ for } i = 1, 2, \ldots, k, \text{ where } k \text{ is arbitrary finite}\} . \tag{20}$$

Finally, for every $\delta z = (\delta y, \delta x) \epsilon \mathcal{X}$, we define $h'(\hat{z})(\delta z) = \delta y(t_f)$ and $r'(\hat{z})(\delta z) = (g(\hat{x}(t_f)/\partial x) \delta x(t_f)$.

It now follows from theorem 1 that there exist a vector $\mu \leq 0$ in E^p and a vector $\eta \epsilon E^\ell$, $(\mu, \eta) \neq 0$, such that

$$\langle \mu, \delta y(t_f) \rangle + \langle \eta, \frac{\partial g(\hat{x}(t_f))}{\partial x} \delta x(t_f) \rangle \leq 0$$

$$\text{for all } \delta z \epsilon \overline{C(\hat{z}, \Omega)} \qquad (21)$$

Substituting for $\delta z(t_f)$ from (20) into (21) and making the usual identifications, we obtain the following maximum principle.

<u>Theorem 2</u>: If the control $\hat{u}(t)$ and the corresponding trajectory $\hat{z}(t) = (\hat{y}(t), \hat{x}(t))$ solve the optimal control problem, then there exist a vector $\psi_1 \leq 0$ in E^p and a vector valued function $\psi_2 : [t_o, t_f] \to E^n$, with $(\psi_1, \psi_2(t)) \not\equiv 0$ such that

(i) $$\frac{d\psi_2(t)}{dt} = - \left(\frac{\partial c(\hat{x}(t), \hat{u}(t))}{\partial x} \right)^T \psi_1 - \left(\frac{\partial f(\hat{x}(t), \hat{u}(t))}{\partial x} \right)^T \psi_2(t),$$

$$t \epsilon [t_o, t_f] \qquad (22)$$

(ii) $$\psi_2(t_f) = \left(\frac{\partial g(\hat{x}(t_f))}{\partial x} \right)^T \eta \text{ for some } \eta \epsilon E^\ell \qquad (23)$$

(iii) for every $v \epsilon U$ and almost all $t \epsilon [t_o, t_f]$,

$$\langle \psi_1, c(\hat{x}(f), \hat{u}(t)) \rangle + \langle \psi_2(t), f(\hat{x}(t), \hat{u}(t)) \rangle \geq$$

$$\langle \psi_1, c(\hat{x}(t), v) \rangle + \langle \psi_2(t), f(\hat{x}(t), v) \rangle \qquad (24)$$

III. <u>Reduction of a Vector-Valued Criterion to a Family of Scalar-Valued Criteria</u>

We now examine the possibility of solving the Basic Problem by weighting the components of the vector criterion function into a scalar criterion, thus reducing the problem to a family of scalar valued criterion problems. The weighting common in economics and in engineering is with strictly positive weights only. However, we shall also consider the degenerate case in which some of the weights can be zero.

In this section we restrict ourselves to the partial ordering defined as follows: Given y_1, y_2 in E^p, $y_1 \prec y_2$ if and only if $y_1^i \leq y_2^i$ for $i = 1, 2, \ldots, p$. With this ordering, for any vector y in E^p, the critical index set $J(y)$ is the set $\{1, 2, \ldots, p\}$.

In order to simplify our exposition, we combine the constraint set Ω with the set $\{x \in \mathcal{X} \mid r(x) = 0\}$ into a set $A = \Omega \quad \{x \in \mathcal{X} \mid r(x) = 0\}$.

Definition 3: We shall denote by P the problem of finding a point $\hat{x}$ in A such that for every x in A, the relation $h(x) \leq h(\hat{x})$ (componentwise) implies that $h(x) = h(\hat{x})$.

Definition 4: Given any vector λ in E^p, we shall denote by $P(\lambda)$ the problem of finding a point $\hat{x}$ in A such that $\langle \lambda, h(\hat{x}) \rangle \leq \langle \lambda, h(x) \rangle$ for all x in A.

Definition 5: Let Λ be the set of all vectors $\lambda = (\lambda^1, \ldots, \lambda^p)$ in E^p such that $\Sigma \lambda^i = 1$ and $\lambda^i > 0$ for $i = 1, 2, \ldots, p$; let $\overline{\Lambda}$ be the closure of Λ in E^p.

We shall consider the following subsets of $\mathcal{X}$:

$$L = \{x \in A \mid x \text{ solves } P\} \tag{25}$$

$$M = \{x \in A \mid x \text{ solves } P(\lambda) \text{ for some } \lambda \in \Lambda\} \tag{26}$$

$$N = \{x \in A \mid x \text{ solves } P(\lambda) \text{ for some } \lambda \in \overline{\Lambda}\}. \tag{27}$$

Remark: It is trivially verified that M is contained in L and in N. Furthermore, it is easy to show by example that if h is a continuous function, then the closure of the set M is contained in the set N and that this inclusion may be proper. (see (11). It can also be shown that (see(11)) if for each $\lambda \in \overline{\Lambda}$ either $P(\lambda)$ has a unique solution or else it has no solution, then the set L contains the set N.

Theorem 3: Suppose that h is a convex function (componentwise) and that A is a convex set. Then the set N contains the set L.

Proof: Let x be a point in L, i.e., x is a solution to the problem P. Let

$$\Delta = \{\alpha = (\alpha^1, \alpha^2, \ldots \alpha^p) \mid h^i(x) - h^i(\hat{x}) < \alpha^i, \; i = 1, 2, \ldots, p, \text{ for some } x \in A\}. \tag{28}$$

Since $\hat{x}$ is a solution to P, Δ does not contain the origin. Furthermore, since h is convex, Δ is a convex set in E^p. By the separation theorem (12) there exists a vector $\overline{\alpha}$ in E^p, $\overline{\alpha} \neq 0$ such that

$$\langle \overline{\alpha}, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta. \tag{29}$$

Since each α^i can be made as large as we wish, we must have $\overline{\alpha}^i \geq 0$ and hence $\overline{\alpha} \geq 0$. For any positive scalar $\varepsilon > 0$, let $\alpha = h(x) - h(\hat{x}) + \varepsilon e$ for some x in A and $e = (1, 1, \ldots, 1)$. The vector α is in Δ by definition, and hence, from (29)

$$\langle \overline{\alpha}, h(x) - h(\hat{x}) \rangle \geq -\varepsilon \langle \overline{\alpha}, e \rangle . \tag{30}$$

Relation (30) holds for every x in A, and since ε is arbitrary,

$$\langle \overline{\alpha}, h(x) - h(\hat{x}) \rangle \geq 0 \quad \text{for all } x \in A. \tag{31}$$

If we define $\overline{\lambda} = \overline{\alpha} / \sum_{i=1}^{P} \alpha^i$, then $\overline{\lambda} \in \overline{\Lambda}$ and,

$$\langle \overline{\lambda}, h(\hat{x}) \rangle \leq \langle \overline{\lambda}, h(x) \rangle \quad \text{for all } x \in A \tag{32}$$

But (32) implies that $\hat{x} \in N$.

Corollary: If A is convex and h is strictly convex (componentwise), then $L = N$.

Definition 6: We shall say that a solution $\hat{x}$ of the problem P is regular if the relation $h(\hat{x}) = h(y)$ implies that $\hat{x} = y$.

We shall say that the problem P is regular if every solution of P is a regular solution.

Remark: It is easy to verify that if h is convex and one of its components is strictly convex then P is regular.

Theorem 4: Suppose that the problem P is regular, that h is continuous and convex, and that the constraint set A is a closed convex subset of a Hausdorff, locally convex, linear topological space $\mathcal{X}$, with the property that for some closed convex neighborhood V of the origin, the set $A \cap (V+x)$

is compact for every x in A. Then the set L is contained in the closure of the set M.

Proof: Let $\hat{x}$ be any point in L, let $A' = A \cap (V + \hat{x})$, and let L', M' be defined by (25), (26), with A' taking the place of A. First we show that $h(L') \subset \overline{h(M')}$. Since A' is compact and h is continuous, $h(A')$ is closed, and for every $\varepsilon > 0$, $y - \varepsilon\overline{\Lambda}$ is closed by construction. Let $\hat{y}$ be any point in $h(L')$. Then, clearly, the closed, convex sets $h(A')$ and $(\hat{y} - \varepsilon\overline{\Lambda})$ are separated for every $\varepsilon > 0$. Consequently, for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) \in (0, \varepsilon]$ such that the closed convex sets $h(A')$ and $N(\hat{y} - \varepsilon\overline{\Lambda}, \delta(\varepsilon))$ are disjoint, where $N(\hat{y} - \varepsilon\overline{\Lambda}, \delta(\varepsilon)) = \{y = y' + y'' \mid y' \in (y - \varepsilon\overline{\Lambda}) \text{ and } \|y''\| \leq \delta(\varepsilon)\} \subset E^p$. Now, for every $\varepsilon > 0$, let $y_\varepsilon \in h(A')$ and $s_\varepsilon \in N(y - \varepsilon\overline{\Lambda}, \delta(\varepsilon))$ be such that $\|y_\varepsilon - s_\varepsilon\|$ is the minimum distance between these two closed and disjoint sets. Let λ be a normal to a separating hyperplane through y_ε, such that

$$\langle \lambda, y - y_\varepsilon \rangle \geq 0 \quad \text{for all } y \in h(A') \tag{33}$$

and

$$\langle \lambda, y - y_\varepsilon \rangle < 0 \quad \text{for all } y \in N(\hat{y} - \varepsilon\overline{\Lambda}, \delta(\varepsilon)) \tag{34}$$

then, from (33), $\langle \lambda, \hat{y} - y_\varepsilon \rangle \geq 0$, and hence for every $y \in \overline{\Lambda}$, $\varepsilon\langle \lambda, y \rangle > \langle \lambda, \hat{y} - y_\varepsilon \rangle > 0$. But this means that $\lambda = \alpha\lambda'$, with $\alpha > 0$ and $\lambda \in \Lambda$. Hence we may choose $\alpha = 1$, i.e., $\lambda = \lambda'$, thus proving that $y_\varepsilon \in h(M')$. We now show that $\lim_{\varepsilon \to 0+} y_\varepsilon = \hat{y}$, i.e., that $\hat{y} \in \overline{h(M')}$.

Indeed,

$$\|y_\varepsilon - \hat{y}\| \leq \|\hat{y} - s_\varepsilon\| + \|s_\varepsilon - y_\varepsilon\| \leq \|\hat{y} - s_\varepsilon\| + \|\hat{y} - s_\varepsilon\|$$

$$\leq 2(\varepsilon + \delta(\varepsilon)) \leq 4\varepsilon. \tag{35}$$

We now prove that $L \subset \overline{M}$. Again let $\hat{x}$ be any point in L, and let A', L', M', be defined as above. Then, from the above, $h(L') \subset \overline{h(M')}$ and, by inspection, $\hat{x} \in L'$. For $i = 1, 2, 3, \ldots$, let $y_i \in h(M')$ be such that $y_i \to h(\hat{x})$ and let $x_i \in A'$ be such that $h(x_i) = y_i$. Since A' is compact, $\{x_i\}$ contains a subsequence, $\{x_{i_j}\}$ which converges to a point $\overline{x} \in A'$. Since all subsequences of $\{y_i\}$ converge to $h(\hat{x})$, it follows from the continuity of h that $h(\overline{x}) = h(\hat{x})$, and from the regularity of the problem P that $\overline{x} = \hat{x}$. Now, since

$x_{i_j} \to \hat{x}$, there exists an integer n_0 such that $x_{i_j} \in \text{int}(V + \hat{x})$ for all $i_j \geq n_0$. It now follows from the convexity of h that for $i_j \geq n_0$ $x_{i_j} \in M$ and hence $L \subset \overline{M}$.

Conclusion

We should like to point out that our results can easily be extended to other types of ordering. For example, consider cone orderings on R^p of the following type: $y_1 \prec y_2$ if and only if $(y_1 - y_2) \in C$, where C is a given convex cone. When C has an interior, we let $q = p$ and modify (8) to read $R = \{(y, 0) \in E^p \times E^m | y \in \text{int } C, 0 \in E^m\}$. We then find that theorem 4 remains valid for cone orderings provided we replace the statement "$\mu \leq 0$" by "μ is in the cone polar to C." The scalarization theorems remain valid for cone orderings provided we replace the set Λ by the set $D = \{\lambda | \langle \lambda, y \rangle < 0$ for all $y \in C$ with $y \neq 0\}$, assuming, of course, that D is not the empty set.

Acknowledgment

We acknowledge with pleasure several interesting discussions with Professor H. Halkin of the University of California, San Diego.

The research reported herein was supported wholly by the National Aeronautics and Space Administration under Grant NsG-354, Supplement 3.

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