CASCADE THEORY OF PLASMA TURBULENCE

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Cascade Theory of Plasma Turbulence*

by

C.M. Tchen

Abstract

The Fourier analysis of the hydrodynamic and electrostatic equations has given a stochastic hierarchy of equations of correlations. This hierarchy, which is slowly convergent, is transformed into a more convergent cascade hierarchy, describing the mixing of a group of small eddies with a group of bigger ones which provide a gradient type of diffusive background flow. A quasilinear approximation is applied to the motion of the small eddies at the earliest closure. It determines the turbulent diffusion from the nonlinear effect of the small eddies on their background flow. The spectral equations for velocity and electrostatic fluctuations are derived and the spectral laws are obtained. In the inviscid ranges, the two power spectra follow the k^{-3} law. The spectra in the dissipative ranges, together with the mixed range (initial-diffusive range) are also derived. In the framework of hydrodynamic turbulence, the present scheme degenerates to the Heisenberg equation of spectrum, corrected for its divergence at large wave numbers.

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I. INTRODUCTION

The study of plasma turbulence has recently gained its importance in several directions. Turbulent fluctuations in electric and magnetic fields have been observed in solar photosphere [Vasilyeva, 1963, 1964], and in interplanetary space. Turbulent spectra have been determined in laboratory plasmas [D'Angelo et al., 1966; Chen, 1965] . The effects of turbulence on the scattering of an electromagnetic radiation from an ionosphere or a wake of a space vehicle is a recognized important problem. An incident radiation of frequency ω_0 and wave-number $\underset{\sim}{k}_0$, will be scattered by the plasma $(\omega , \underset{\sim}{k})$ at the frequency ω' and wave number $\underset{\sim}{k}'$:

$$\omega' = \omega_0 \pm \omega \quad ,$$

$$\underset{\sim}{k}' = \underset{\sim}{k}_0 \pm \underset{\sim}{k} \quad .$$

The scattered intensity is proportional to the cross section, or proportional to the spectrum of density n

$$< n(\omega,\underset{\sim}{k}) \overset{*}{n} (\omega, \underset{\sim}{k}) > \quad .$$

We can distinguish two cases:

A. If an oscillation is present in the plasma, the hydrodynamic theory predicts easily a spectrum in the form of a δ-function, peaked at the electron plasma frequency, and the kinetic theory predicts a broadening of the spectrum. A quasilinear theory exhibits the nonlinear interation between the wave and the plasma [Tchen, 1961] .

B. If the plasma medium is turbulent, the spectrum can be divided into an inertial range and a dissipative range. The spectral distribution in a turbulent plasma will be investigated in the present work.

The theories of plasma turbulence, connected with the backscatter of electromagnetic radiation, can be divided into three groups: One is based on the gradient type of mixing, the second on the modal transfer, and the third on the random mixing. The first group, as represented by the works of Gallet [1955], Villars and Weisskoff [1955], proposes that the r.m.s. of the difference of density fluctuations at two points

$$< (\Delta n)^2 >^{\frac{1}{2}} \equiv n_1$$

is proportional to the gradient of the mean density N in the following manner :

$$n_1 = -\ell \text{ grad } N \quad , \tag{1}$$

where ℓ is the separation length. This formula is known in fluid mechanics as the mixing length hypothesis, first proposed by Boussinesq 1897]. The second group is represented by the works of Krasilnikoff [1949], Silverman [1956], and more recently, by Salpeter and Treiman [1964]. They applied the Kolmogoroff [1941] theory of fluid turbulence to the radio scattering from charged particles. Since the Kolmogoroff theory predicts a velocity fluctuation according to the law

$$u_1^2 \sim \ell^{2/3} \quad , \tag{2a}$$

where

$$u_1^2 \equiv < (\Delta u)^2 > ,$$

and if the suspended particles, which serve as scatterers, follow the fluid motion, then it is legitimate to write accordingly

$$n'^2 \sim \ell^{2/3} , \tag{2b}$$

a formula independent of grad N . The third group introduces a random walk model, because of the stochastic nature of the diffusion of particles. This leads to a correlation between densities of the exponential type [Tchen, 1952]:

$$< n(0)\ n(\ell) > / < n^2 > = e^{-\kappa \ell} , \tag{3}$$

or, in terms of n_1 ,

$$\frac{1}{2} \frac{n_1^2}{< n^2 >} = 1 - \frac{< n(0)\ n(\ell) >}{< n^2 >} ,$$

$$= 1 - e^{-\kappa \ell} ,$$

where $1/\kappa$ is the length of correlation.

The above formulas of correlations can be converted into formulas of spectral functions by means of a Fourier transform:

$$F(k) = \int_0^\infty d\ell < u(0)\ u(\ell) > \cos k\ell ,$$

$$G(k) = \int_0^\infty d\ell < n(0)\ n(\ell) > \cos k\ell . \tag{4}$$

Then (1), (2b) and (3) are transposed respectively into the following:

$$\text{mixing length,} \quad G(k) \sim k^{-3} , \tag{5}$$

$$\text{Kolmogoroff,} \quad G(k) \sim k^{-5/3}, \tag{6}$$

Booker [1959] - Gordon [1952]:

$$G(k) \sim (\kappa^2 + k^2)^{-1} \tag{7}$$

Formula (7) is recognized as the Booker [1959] formula, widely applied to the radio scattering problem. Although sometimes convenient, the random walk model yielding (3) and (7) lacks the nonlinear dynamical basis. On the other hand, the mixing length formulas (1) and (5) break down in a turbulent medium which does not possess a mean density gradient. Finally the Kolmogoroff law (2a) is attractive, because of its implicit nonlinear dynamics of velocity fluctuations, but its extension to density fluctuations (2b) can only be valid under very restricted conditions. In order to generalize this result, and to explore the nonlinear interaction between density and velocity fluctuations, theories have been developed for the determination of the density spectrum of neutral particles under various turbulent conditions [Tchen, 1965]. However, as the particles are treated as neutral, the theories do not apply to plasmas.

For the treatment of plasma turbulence, we shall confine ourselves to stationary, isotropic and homogeneous electrostatic turbulence, without the presence of any external field, and shall attempt to determine the spectra of turbulence and electrostatic fluctuations. In order to illustrate the dominant gross physical features of plasma turbulence, we shall outline the problem first by similitude considerations (Section 2).

A mathematical program, based upon the nonlinear Navier-Stokes equation of motion and its Fourier decomposition would call for a stochastic hierarchy of equations of correlations, involving the problem of its closure. The sampling summation by the diagram technique in the perturbation theory helps in the solution of the hierarchy. Shut'ko [1965] has shown that, with a certain order of approximation, the method leads to Kraichnan's stochastic equations, and to a system of equations yielding the Kolmogoroff spectrum on a more complex approximation. It is by now generally agreed that the Kolmogoroff law is correct, but one wishes that it could be understood without requiring the above elaborate method, as it will be difficult to extend to plasma turbulence. On the other hand, the dimensional arguments of

Kolmogoroff, and the Heisenberg [1944] equation based upon the analogy between the dissipations by molecular and turbulent motions, should be clarified on a mathematical ground. The main difficulty of a mathematical theory of turbulence resides in the slow convergence of the stochastic hierarchy, and in the highly skew nature of the odd order correlations. Therefore we shall transform this stochastic hierarchy, governing the statistical behavior of individual eddies, into a cascade hierarchy governing two groups of eddies. The latter hierarchy becomes highly convergent as it embodies the cascade process. As a test of its effectiveness, the first approximation degenerates the hierarchy into the Heisenberg equation of spectrum. The velocity cascade is developed in Section 3, and the density cascade in Section 4. Perturbations of the cascades lead to spectral equations (Section 5), and their solutions determine the spectral laws (Section 6). Finally the effects of molecular motions on the turbulent diffusions are investigated in Section 7. The new spectra are convergent at large wave numbers and for high order moments.

2. SIMILITUDE CONSIDERATIONS

In a homogeneous system, we use the Navier Stokes equation for describing the turbulent fluctuations in velocity $\underset{\sim}{u}$ of the plasma

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{u} \frac{\partial}{\partial \underset{\sim}{x}}\right)\underset{\sim}{u} = \underset{\sim}{E} + \nu \frac{\partial^2 \underset{\sim}{u}}{\partial \underset{\sim}{x}^2} + \underset{\sim}{X} , \qquad (8)$$

where $\underset{\sim}{X}$ is a stochastic force (noise), representing fluctuations in pressure and brownian movements. The pressure is lumped in this way, because of its lesser importance in the study of energy spectrum. We assume

$$< \underset{\sim}{X}\underset{\sim}{u} > \cong 0 \quad .$$

Further ν is the kinematic viscosity, and $\underset{\sim}{E}$ is the electrostatic force per unit mass, which is the product of the electrostatic field by e/m, e is the charge and m is the mass. If n is the density fluctuation, then $\underset{\sim}{E}$ and n are related by the Poisson equation

$$\frac{\partial \underset{\sim}{E}}{\partial \underset{\sim}{x}} = \varpi^2 n \ , \tag{9}$$

with

$$\varpi^2 = 4\pi e^2/m \ .$$

A diffusion equation is used to describe the fluctuations in concentration, or number-density n, as follows:

$$\frac{\partial n}{\partial t} + \underset{\sim}{u} \frac{\partial n}{\partial \underset{\sim}{x}} = \lambda \frac{\partial^2 n}{\partial \underset{\sim}{x}^2} \ , \tag{10}$$

where λ is the molecular diffusion coefficient.

It is to be noted the effects of compressibility may manifest themselves through $\underset{\sim}{X}$ and the molecular transport coefficients ν and λ, and are considered not important in the inertial range of the spectra.

It may be worthwhile to recall the essential argument of similitude for arriving at the Kolmogoroff [1944] law of turbulent spectrum (2a). The inertial range of the turbulent spectrum in a neutral fluid is characterized by a constant flux of energy transfer between the modes (modal transfer) without viscous dissipations. As determined by the nonlinear term of (8), the energy transfer in a length ℓ is dimensionally

$$(\text{velocity})^3/\ell \quad ,$$

or more specifically,

$$u_1^3/\ell = \text{constant.} \tag{11a}$$

This leads immediately to the Kolmogoroff law (2a). This spectrum is expected to drop by the action of viscous dissipation. However, in a plasma the spectrum (2a) should first drop on account of its partition and randomization into electrostatic fields, in the form of a correlation

$$< \underset{\sim}{E}u > \quad ,$$

or

$$E_1^2 \tau \quad ,$$

where

$$E_1^2 \equiv <(\Delta E)^2> \quad ,$$

and

$$\tau = \ell/u_1 \tag{11b}$$

is the duration of correlation. Thus for a plasma, (11a) should include electrostatic fields and therefore be replaced by

$$\frac{u_1^3}{\ell} = \text{const} \; E_1^2 \; \frac{\ell}{u_1} \quad . \tag{11c}$$

Applying the same argument to the nonlinear term in (10), after being multiplied by the density, we can write

$$u_1 \frac{n_1^2}{\ell} = \text{constant} \; . \tag{12}$$

Finally (9) gives

$$\frac{E_1}{\ell} = \varpi^{-2} \; n_1 \quad . \tag{13}$$

The three equations (11c), (12), and (13) easily determine the three variables as follows:

$$u_1 = \text{const} \; \ell \; ,$$

$$E_1 = \text{const} \; \ell \; ,$$

$$n_1 = \text{const} \; ,$$

corresponding to the following spectra

$$F = \text{const} \; k^{-3} \; , \tag{14a}$$

$$G_E = \text{const} \; k^{-3} \; , \tag{14b}$$

$$G = \text{const} \; k^{-1} \; , \tag{14c}$$

for velocity, electrostatic field and density fluctuations respectively.

3. CASCADE PROCESS FOR THE MIXING OF EDDIES

If the velocity $\underset{\sim}{u}(t,\underset{\sim}{x})$ is decomposed into Fourier components:

$$\underset{\sim}{u}(\underset{\sim}{x}) = \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, \underset{\sim}{u}(\underset{\sim}{k}') \, e^{i\underset{\sim}{k}'\underset{\sim}{x}} \quad ,$$

covering wave numbers of all amplitudes from 0 to ∞, we can distinguish two parts: one part covering wave-numbers of amplitudes from 0 to k, as denoted by

$$\underset{\sim}{u}_o(\underset{\sim}{x}) = \int_0^k d\underset{\sim}{k}' \, \underset{\sim}{u}(\underset{\sim}{k}') \, e^{i\underset{\sim}{k}'\underset{\sim}{x}} \quad , \tag{15a}$$

and the other part covering wave-numbers of amplitudes from k to ∞, as denoted by

$$\underset{\sim}{u}'(\underset{\sim}{x}) = \int_k^{\infty} d\underset{\sim}{k}' \, \underset{\sim}{u}(\underset{\sim}{k}') \, e^{i\underset{\sim}{k}'\underset{\sim}{x}} \quad , \tag{15b}$$

with

$$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{u}_o(\underset{\sim}{x}) + \underset{\sim}{u}'(\underset{\sim}{x}) \quad . \tag{16}$$

Here

$$\int_0^k d\underset{\sim}{k}' \qquad \text{and} \qquad \int_k^{\infty} d\underset{\sim}{k}'$$

are volume integrals inside and outside of a sphere of radius k. As we consider a turbulent motion of velocity $\underset{\sim}{u}$ without the presence of a mean flow, the notations $\underset{\sim}{u}_o$ and $\underset{\sim}{u}'$ represent respectively the velocities of big and small eddies, or waves of wave-numbers smaller or greater than k.

The formulas (15) can be more conveniently written in the following form:

$$\underset{\sim}{u}_o(\underset{\sim}{x}) = \int_{-\infty}^{\infty} d\underset{\sim}{k}' \; \underset{\sim}{u}_o(\underset{\sim}{k}') \; e^{i\underset{\sim}{k}'\underset{\sim}{x}} , \tag{17a}$$

$$\underset{\sim}{u}'(\underset{\sim}{x}) = \int_{-\infty}^{\infty} d\underset{\sim}{k}' \; \underset{\sim}{u}'(\underset{\sim}{k}') \; e^{i\underset{\sim}{k}'\underset{\sim}{x}} , \tag{17b}$$

after the introduction of

$$\underset{\sim}{u}_o(\underset{\sim}{k}') = \begin{cases} \underset{\sim}{u}(\underset{\sim}{k}') , & \text{for } 0 \leq k' \leq k , \\ 0 , & \text{for } k < k' \leq \infty , \end{cases}$$

$$\underset{\sim}{u}'(\underset{\sim}{k}') = \begin{cases} 0 , & \text{for } 0 \leq k' < k , \\ \underset{\sim}{u}(\underset{\sim}{k}') , & \text{for } k \leq k' \leq \infty . \end{cases} \tag{18}$$

A distinction between the two motions $\underset{\sim}{u}'$ and $\underset{\sim}{u}_o$ are useful: The movements of the small eddies $\underset{\sim}{u}'$ exert on the large scale flow $\underset{\sim}{u}_o$ a stress related to diffusion, while the large scale flow configuration $\underset{\sim}{u}_o$ shapes a gradient type of background flow, in which medium the small eddies are to evolve and to draw momentum.

As a result of such an interplay a general flow of energy from $\underset{\sim}{u}_o$ into $\underset{\sim}{u}'$ in the form of a cascade process can be accounted. Since the small eddies do not embody the major energy, the equation governing the evolution of $\underset{\sim}{u}'$ can be linearized, while the equation for $\underset{\sim}{u}_o$ remains nonlinear.

The Fourier decomposition as presented in (17) and (18), means that the function $\underset{\sim}{u}'(\underset{\sim}{x})$ is truncated at the length $1/k$, so that an average

$$< \ldots >_k \tag{19}$$

over such a length will smooth out all configurations of smaller scales, i.e. $\underset{\sim}{u}'(\underset{\sim}{x})$, while keeping $\underset{\sim}{u}_o(\underset{\sim}{x})$ intact. Under this circumstance, an equation governing $\underset{\sim}{u}_o$ can be derived by substituting (16) into the equation of motion (8) for $\underset{\sim}{u}$, and by subsequently performing the average. We have

$$\begin{aligned} < \underset{\sim}{u}'(\underset{\sim}{x}) >_k &= 0 \quad , \\ < \underset{\sim}{u}_o(\underset{\sim}{x}) >_k &= \underset{\sim}{u}_o(\underset{\sim}{x}) \quad . \end{aligned} \tag{20}$$

Distinction should be made between the above short truncation, of length $1/k$, which smooths out the $\underset{\sim}{u}'$ motion by an average (19), and leave the large scale motion $\underset{\sim}{u}_o$ intact, compare(20), and a long truncation of length $\ell_o \to \infty$, which would average out the $\underset{\sim}{u}_o$ motion too

$$< \underset{\sim}{u}_o(\underset{\sim}{x})> = 0 \quad .$$

Such an average will be denoted by brackets without a subscript.

Thus by an average of the type (19) over (8), we obtain

$$\frac{\partial \underset{\sim}{u}_o}{\partial t} + \underset{\sim}{u}_o \frac{\partial \underset{\sim}{u}_o}{\partial \underset{\sim}{x}} = \underset{\sim}{E}_o + \nu \frac{\partial^2 \underset{\sim}{u}_o}{\partial \underset{\sim}{x}^2} - \langle \underset{\sim}{u} \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}} \rangle_k \quad . \tag{21a}$$

By substracting (21a) from (8), we further obtain

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{u}_o \frac{\partial}{\partial \underset{\sim}{x}} \right) \underset{\sim}{u}' + \alpha \underset{\sim}{u}' = - \underset{\sim}{u}' \frac{\partial \underset{\sim}{u}_o}{\partial \underset{\sim}{x}} + \underset{\sim}{E}' + \underset{\sim}{X} \tag{21b}$$

The linearized equation (21b) has omitted the terms

$$- \underset{\sim}{u}' \frac{\partial \underset{\sim}{u}'}{\partial \underset{\sim}{x}} + \langle \underset{\sim}{u}' \frac{\partial \underset{\sim}{u}'}{\partial \underset{\sim}{x}} \rangle_k \quad .$$

which are assumed negligible in the present quasilinear approximation. Like in (16), we have also written here

$$\underset{\sim}{E} = \underset{\sim}{E}_o + \underset{\sim}{E}' \quad ,$$

and $\alpha \underset{\sim}{u}'$ has been chosen, in the place of $\nu \, \partial^2 \underset{\sim}{u}'/\partial \underset{\sim}{x}^2$, to represent a damping. This means that

$$\alpha = \text{const} \, \frac{\nu}{\ell^2} \quad . \tag{22}$$

Integration of (21b) along the trajectory of the u' motion gives

$$\underset{\sim}{u}'(t,\underset{\sim}{x}) = - \frac{\partial \underset{\sim}{u}_o}{\partial \underset{\sim}{x}} \int_{t_o}^{t} dt' \, e^{-\alpha(t-t')} \, \underset{\sim}{u}'(t',\underset{\sim}{x}')$$

$$+ \int_{t_o}^{t} dt' \, e^{-\alpha(t-t')} \, [\underset{\sim}{E}'(t',\underset{\sim}{x}') + \underset{\sim}{X}(t',\underset{\sim}{x}')], \qquad (23)$$

where

$$\underset{\sim}{x}' = \underset{\sim}{x} - \underset{\sim}{u}_o(t-t') \, .$$

On the basis of (23), we can calculate

$$< u'_\ell \frac{\partial u'_i}{\partial x_\ell} >_k = - \frac{\partial^2 u_{oi}}{\partial x_j x_\ell} \int_{t_o}^{t} dt' \, e^{-\alpha(t-t')} <u'_j(t',x') \, u'_\ell(t,x)>_k ,$$

$$\cong - \nu_k \frac{\partial^2 u_{oi}}{\partial x_j^2} \quad , \qquad (24)$$

where

$$\nu_k = \int_{t_o}^{t} dt' \, e^{-\alpha(t-t')} < u'_\ell(t',\underset{\sim}{x}') \, u'_\ell(t,\underset{\sim}{x}) > \qquad (25)$$

is the turbulent coefficient of diffusion, as contributed by small eddies. Formula (24) was obtained in the assumption of locally isotropic and homogeneous turbulence in the $\underset{\sim}{u}'$-field. Thus the effects of the motion of the small eddies are to contribute to a diffusion in the large scale motion $\underset{\sim}{u}_o$, as was expected, and to modify the equation of motion (21a) into the following:

$$\frac{\partial \underset{\sim}{u}_o}{\partial t} + \underset{\sim}{u}_o \frac{\partial \underset{\sim}{u}_o}{\partial \underset{\sim}{x}} = \underset{\sim}{E}_o + (\nu + \nu_k) \frac{\partial^2 \underset{\sim}{u}_o}{\partial \underset{\sim}{x}^2} \tag{26}$$

Equations (21b) and (26) govern the basic process of eddy mixing in the form of a cascade.

We note that the last term on the right hand side of (26) gives already the ingredient for the structure of the turbulent dissipation, necessary in the derivation of the Heisenberg equation.

4. DENSITY CASCADE

The method used in Section 3 to develop the velocity cascade on the basis of the dynamical equation (8), can be repeated to derive the density cascade on the basis of the diffusion equation (10), by making the superposition

$$n = n_o + n' \quad ,$$

into big and small eddies like in (1b). The system of quasilinear equations governing n_o and n' follows from (10), and is of the form

$$\frac{\partial n_o}{\partial t} + \underset{\sim}{u}_o \frac{\partial n_o}{\partial \underset{\sim}{x}} = \lambda \frac{\partial^2 n_o}{\partial \underset{\sim}{x}^2} - < \underset{\sim}{u}' \frac{\partial n'}{\partial \underset{\sim}{x}} >_k \ , \tag{27a}$$

$$\frac{\partial n'}{\partial t} + \underset{\sim}{u}_o \frac{\partial n'}{\partial \underset{\sim}{x}} = - \beta n' - \underset{\sim}{u}' \frac{\partial n_o}{\partial \underset{\sim}{x}} \tag{27b}$$

where

$$-\beta n' \quad ,$$

in the place of

$$\lambda \frac{\partial^2 n'}{\partial \underset{\sim}{x}^2} \quad ,$$

represents a damping of the wave motion n'. After integration of (27b) and some simplification, we find

$$< \underset{\sim}{u}' \frac{\partial n'}{\partial \underset{\sim}{x}} >_k = - \lambda_k \frac{\partial^2 n_o}{\partial \underset{\sim}{x}^2} \quad , \tag{28}$$

with

$$\lambda_k = \int_{t_o}^{t} dt' \, e^{-\beta(t-t')} < \underset{\sim}{u}'(t',\underset{\sim}{x}') \cdot \underset{\sim}{u}'(t,\underset{\sim}{x}') >_k \tag{29}$$

being the coefficient of turbulent diffusion of particles. Upon substitution of (28), we rewrite (27a) as follows:

$$\frac{\partial n_o}{\partial t} + \underset{\sim}{u}_o \frac{\partial n_o}{\partial \underset{\sim}{x}} = (\lambda + \lambda_k) \frac{\partial^2 n_o}{\partial \underset{\sim}{x}^2} \quad . \tag{30}$$

Equations (27b) and (30) governs the density cascades.

5. SPECTRAL EQUATIONS

We shall study the velocity and the density spectra, and the correlation between the velocity and the density.

(a) Velocity spectrum

The function $\underset{\sim}{u}_o(\underset{\sim}{x})$, defined by (15a), is convenient to study the spectral function of velocity F . By raising it to the square, we can write

$$[u_o(x)]^2 = \int_0^k \int_0^k d\underset{\sim}{k}' \, d\underset{\sim}{k}'' \, \underset{\sim}{u}(\underset{\sim}{k}') \cdot \underset{\sim}{u}(\underset{\sim}{k}'') \, e^{i(\underset{\sim}{k}'+\underset{\sim}{k}'')\underset{\sim}{x}} ,$$

and its average over a long truncation length $\ell_o \to \infty$ is

$$\tfrac{1}{2} < [u_o(x)]^2 > = \tfrac{1}{2} \lim_{\ell_o \to \infty} \frac{1}{(2\ell_o)^3} \int_{-\ell_o}^{+\ell_o} d\underset{\sim}{x} \int_0^k \int_0^k d\underset{\sim}{k}' \, d\underset{\sim}{k}'' \, \underset{\sim}{u}(\underset{\sim}{k}') \cdot \underset{\sim}{u}(\underset{\sim}{k}'') \, e^{i(\underset{\sim}{k}'+\underset{\sim}{k}'')\underset{\sim}{x}} ,$$

$$= \tfrac{1}{2}\left(\frac{\pi}{\ell_o} \right)^3 \int_0^k d\underset{\sim}{k}' \, \underset{\sim}{u}(\underset{\sim}{k}') \cdot \underset{\sim}{u}(-\underset{\sim}{k}') ,$$

$$= \int_0^k dk \, F(k) , \qquad (31a)$$

as

$$\int_{-\infty}^{\infty} d\underset{\sim}{x} \, e^{i\underset{\sim}{k}\underset{\sim}{x}} = (2\pi)^3 \, \delta(k_1) \, \delta(k_2) \, \delta(k_3)$$

Further the derivatives can be calculated in the same way. For example we have from (15a)

$$\frac{\partial u_{oi}}{\partial x_j} = \int_0^k d\underset{\sim}{k}' \; ik_j' \; u_i(\underset{\sim}{k}') \, e^{i\underset{\sim}{k}' \underset{\sim}{x}} ,$$

and

$$< (\frac{\partial u_{oi}}{\partial x_j})^2 >= \lim_{\ell_o \to \infty} \frac{1}{(2\,\ell_o)^3} \int_{-\ell_o}^{+\ell_o} d\underset{\sim}{x} \int_0^k \int_0^k d\underset{\sim}{k}' \, d\underset{\sim}{k}'' \; ik_j' \; ik_j''$$

$$\times \; \underset{\sim}{u}(\underset{\sim}{k}') \cdot \underset{\sim}{u}(\underset{\sim}{k}'') \, e^{i(\underset{\sim}{k}' + \underset{\sim}{k}'')\underset{\sim}{x}} ,$$

$$= (\frac{\pi}{\ell_o})^3 \int_0^k dk' \; k'^2 \; \underset{\sim}{u}(\underset{\sim}{k}') \cdot \underset{\sim}{u}(-\underset{\sim}{k}') ,$$

$$= 2 \int_0^k dk' \; k'^2 \; F(k') \equiv R_o . \qquad (31b)$$

Obviously when we put $k \to \infty$ in (31), we reduce to

$$\tfrac{1}{2} < u^2 > = \int_0^\infty dk \; F(k) ,$$

and

$$< (\frac{\partial u_i}{\partial x_j})^2 > = 2 \int_0^\infty dk \; k^2 \; F(k) \equiv R .$$

(b) Density spectrum

It is to be noted that G and G_E are related through (9), as follows:

$$G_E = \varpi^4 k^{-2} G \quad . \tag{32}$$

In analogy with (31a), the density spectrum G and the electrostatic spectrum G_E assume the expressions:

(33)

$$\tfrac{1}{2} < [\underset{\sim}{n}_o(\underset{\sim}{x})]^2 > = \int_0^k dk' \, G(k') \ ,$$

$$\tfrac{1}{2} < [\underset{\sim}{E}_o(\underset{\sim}{x})]^2 > = \int_0^k dk' \, G_E(k') \ ,$$

and

$$< (\partial n_o/\partial x_j)^2 > = 2 \int_0^k dk' \, k'^2 \, G(k') \equiv J_o \ ,$$

which become

$$\tfrac{1}{2} < n^2 > = \int_0^\infty dk \, G(k) \ ,$$

$$\tfrac{1}{2} < E^2 > = \int_0^\infty dk \, G_E(k) \ ,$$

and

$$< (\partial n/\partial x_j)^2 > = 2 \int_0^\infty dk \, k^2 \, G(k) \equiv J \ ,$$

respectively, when $k \to \infty$.

Similarly,

$$< \left(\frac{\partial E_{oi}}{\partial x_j}\right)^2 > = 2 \int_0^k dk\, k^2 G_E = \varpi^4 \int_0^k dk\, G \equiv S_o \quad ,$$

$$< \left(\frac{\partial E_i}{\partial x_j}\right)^2 > = 2 \int_0^\infty dk\, k^2\, G_E \equiv S \quad .$$

(c) Correlation and Diffusion

We may remark that $<u'\ell>$, which is the basis of ν_k, is a diffusion in the configurational space. Similarly,

$$D_E = < \underset{\sim}{E}_o \cdot \underset{\sim}{u}_o >$$

can be regarded as a diffusion by electrostatic fluctuations, i.e., a diffusion in the velocity space. Its structure can be determined by the equations (9), (26) and (30) governing $\underset{\sim}{E}_o$ and $\underset{\sim}{u}_o$. By multiplying the equations for

$$\frac{\partial \underset{\sim}{u}_o}{\partial t} \quad \text{and} \quad \frac{\partial \underset{\sim}{E}_o}{\partial t}$$

by $\underset{\sim}{E}_o$ and $\underset{\sim}{u}_o$ respectively, we have

$$\frac{\partial D_E}{\partial t} + \frac{\partial}{\partial x_j} <u_{oj}\, u_{oi}\, E_{oi}> = <E_o{}^2> - (\tilde{\nu} + \tilde{\lambda}) < \frac{\partial E_{oi}}{\partial x_j} \frac{\partial u_{oi}}{\partial x_j} >$$

$$+ \frac{1}{2} (\tilde{\nu} + \tilde{\lambda}) \frac{\partial^2 D_E}{\partial x_j^2} \quad , \qquad (34)$$

which reduces to

$$\frac{\partial D_E}{\partial t} = \langle E_o{}^2 \rangle - (\tilde{\nu} + \tilde{\lambda}) \left\langle \frac{\partial E_{oi}}{\partial x_j} \frac{\partial u_{oi}}{\partial x_j} \right\rangle \quad ,$$

on account of homogeneous turbulence. Here

$$\tilde{\nu} = \nu + \nu_k \ ,$$

and

$$\tilde{\lambda} = \lambda + \lambda_k$$

are the total viscosities.

According to (11b), the time scale of the development of the diffusion D_E is

$$\tau = \left\langle \left(\frac{\partial u_{ii}}{\partial x_j} \right)^2 \right\rangle^{-1/2} .$$

Therefore, we can rewrite (34) in the form

$$\tau^{-1} D_E = \langle E_o{}^2 \rangle - (\tilde{\nu} + \tilde{\lambda}) \left\langle \frac{\partial E_{oi}}{\partial x_j} \frac{\partial u_{oi}}{\partial x_j} \right\rangle \quad ,$$

or

$$D_E = D_E^o - D_E' \quad ;$$

where

$$D_E^o = \int_{t_o \to -\infty}^{t} dt' \langle \underset{\sim}{E}_o(t', \underset{\sim}{x}') \cdot \underset{\sim}{E}_o(t, \underset{\sim}{x}) \rangle \quad ,$$

$$D_E' = (\tilde{\nu} + \tilde{\lambda}) \left\langle \left(\frac{\partial E_{oi}}{\partial x_j} \right)^2 \right\rangle^{1/2} C_E = C_E (\tilde{\nu} + \tilde{\lambda}) S_o{}^{1/2} \quad , \qquad (35a)$$

$$= C_E \, (\tilde{\nu} + \tilde{\lambda}) \, \varpi^2 \langle n_o{}^2 \rangle^{1/2} \quad ;$$

and

$$C_E = < \frac{\partial E_{oi}}{\partial x_j} \frac{\partial u_{oi}}{\partial x_j} > \Big/ < \left(\frac{\partial E_{oi}}{\partial x_j}\right)^2 >^{\frac{1}{2}} < \left(\frac{\partial u_{or}}{\partial x_s}\right)^2 >^{\frac{1}{2}}$$

is the correlation coefficient.

It is to be remarked that D_E^o governs the big eddies, and therefore is not relevant to the derivation of the universal spectrum (small eddies).

The functions ν_k , λ_k are written in terms of F and G as follows:

$$\begin{pmatrix} \nu_k \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \chi_u \\ \chi_n \end{pmatrix} \int_k^\infty dk \; (F/k^3)^{\frac{1}{2}} \quad . \tag{35b}$$

We have left out the damping effect in the eddy viscosities. For the sake of simplification of writing, we shall replace all the coefficients

$$\chi_u, \; \chi_n, \; C_E$$

by unity.

(d) Spectral equations

In order to derive the equations governing the temporal evolution of the spectral functions F and G, it suffices to multiply (26) and (27a) by u_o and n_o respectively, and to perform a spatial average over an infinite interval $\ell_o \to \infty$. Thus, we have

$$\begin{aligned} \frac{\partial}{\partial t} \langle u_o^2 \rangle &= - (\nu + \nu_k) R_o - D_E' \; , \\ \frac{\partial}{\partial t} \langle n_o^2 \rangle &= - (\lambda + \lambda_k) J_o \end{aligned} \tag{36}$$

Here we have left out the convection terms, as they should disappear in a homogeneous turbulence.

With the aid of (33), it is a simple matter to transform (36) into the following equations determing the spectra:

$$\frac{\partial}{\partial t} \int_0^k dk\ F(k) = -\ 2(\nu + \nu_k) \int_0^k dk\ k^2\ F(k) - D_E'(k) \quad ,$$
$$\frac{\partial}{\partial t} \int_0^k dk\ G(k) = -2(\lambda + \lambda_k) \int_0^k dk\ k^2\ G(k) \quad , \tag{37}$$

where ν_k, λ_k and $D_E'(k)$ have been defined by (35).

In a statistically stationary turbulence, the time derivatives vanish, and the spectral equations (37) reduce to

$$\tilde{\nu}\ R_o + D_E' = \epsilon_\nu \quad , \tag{38a}$$

$$\tilde{\lambda}\ J_o = \epsilon_\lambda \quad , \tag{38b}$$

$$D_E' = (\tilde{\nu} + \tilde{\lambda})\ S_o^{\frac{1}{2}} \quad .$$

Here,

$$\epsilon_\nu = \nu\ R \ ,$$

and

$$\epsilon_\lambda = \lambda\ J$$

are the dissipations.

The three equations (38a), (38b) and (32) determine the three spectral functions F, G and G_E.

It is interesting to note that in the framework of hydrodynamic turbulence, the above system (38) degenerates simply to

$$\tilde{\nu} R_o = \epsilon_\nu , \tag{39}$$

the fundamental equation of spectrum proposed by Heisenberg [see Heisenberg, 1948; Tchen, 1954], on the basis of an analogy between the dissipations by molecular and turbulent diffusions. Thus, it is shown that the Heisenberg equation (39) fits well into the quasi-linear approximation of the present cascade theory describing the interplay between the two motions $\underset{\sim}{u}_o$ and $\underset{\sim}{u}'$.

6. SPECTRAL LAWS

We shall solve the system of spectral equations for various ranges.

(1) Inviscid ranges

We shall investigate the spectral laws in the inviscid ranges, also called the inertial and convective ranges, where the inertia term of the Navier-Stokes equation and the convection term of the diffusion equation play the main role. We have here

$$\nu_k \gg \nu \quad ,$$

$$\nu_k \gg \lambda \quad ,$$

$$\nu \sim \lambda \quad ,$$

$$\epsilon_\nu \sim \epsilon_\lambda \; .$$

This requires a gas of high ionization and high Reynolds number. In these ranges, the system of equations (38) reduces to

$$\nu_k \; J_o = \epsilon_\lambda \quad , \tag{40a}$$

and

$$\nu_k(R_o + 2S_o^{\frac{1}{2}}) = \epsilon_\nu \quad . \tag{40b}$$

Unlike the case of hydrodynamic turbulence, where the inertial range was characterized by a constant modal transfer

$$\nu_k \; R_o \quad ,$$

the occurance of diffusion in the electrostatic turbulence should drain this transfer. For such a balance to exist, the spectra must have developed sufficiently toward the large wave number, so that

$$\begin{aligned} R_o &\cong R \quad , \\ J_o &\cong J \quad ; \end{aligned} \tag{41}$$

but the molecular actions are still ineffective.

Under such circumstances, we shall differentiate (40b) to get

$$-\left(\frac{F}{k^3}\right)^{\frac{1}{2}} (R + 2\,S^{\frac{1}{2}}) + 2(k^2 F + S^{-\frac{1}{2}} \varpi^4 G) \int_k^\infty dk \sqrt{\frac{F}{k^3}} = 0 \quad . \tag{42}$$

Here use has been made of (41). Solving (40a) and (42), we find

$$F = Ak^{-3} , \tag{43}$$

$$G = Bk^{-1} ; \tag{44a}$$

where A,B satisfy the relations:

$$A^{\frac{1}{2}} B = \epsilon_\lambda ,$$

$$A = (R + 2\omega'^2) - S^{-\frac{1}{2}} \varpi^4 B .$$

It can be verified that the last term

$$S^{-\frac{1}{2}} \varpi^4 B$$

has a negligible contribution as compared to the other terms, because the ratio

$$\frac{S^{-\frac{1}{2}} \varpi^4}{(R+2\omega'^2)^{\frac{3}{2}}} \quad BA^{\frac{1}{2}}$$

is of the order of

$$\frac{\lambda}{\nu_k} ,$$

which is negligible in the present case. We thus obtain

$$A = R + 2\omega'^2 ,$$

$$B = \epsilon_\lambda (R + 2\omega'^2)^{-\frac{1}{2}} ,$$

where

$$\omega'^2 = 4\pi \langle n^2 \rangle^{\frac{1}{2}} e^2/m \quad .$$

From (32) and (44a), we find the spectrum of electrostatic field

$$G_E = B\varpi^4 k^{-3} \quad . \tag{44b}$$

It follows from

$$E = - \nabla\varphi \quad ,$$

the spectrum of electrostatic potential

$$G_\varphi = B\varpi^4 k^{-5} \quad . \tag{44c}$$

It is to be noted that the turbulent spectrum F in (43) drops faster than the $k^{-5/3}$ law (Kolmogoroff law), because here the velocity cascade is drained by a diffusion from electrostatic fluctuations. The k^{-5} law (44c) for the electrostatic potential is in agreement with several experimental results [D'Angelo, et. al 1966; Chen, 1965]. The present method gives the spectral results (43), (44a) and (44b) identical to (14a), (14c) and (14b), obtained earlier by similitude considerations (Section 2).

(2) Dissipative ranges

In the dissipative ranges of the spectra, the molecular viscosity and the molecular diffusion are important. The ranges of wave numbers are such that

$$\nu_k \ll \nu \,, \quad \text{and} \quad \nu_k \ll \lambda \,.$$

By differentiating the spectral equations (38), we have

$$-\left(\frac{F}{k^3}\right)^{\frac{1}{2}} [R_o + 2S_o^{\frac{1}{2}}]_{k\to\infty} + \nu\,[2k^2F + 2\varpi^4 S_o^{-\frac{1}{2}} G]_{k\to\infty} = 0,$$

$$-\left(\frac{F}{k^3}\right)^{\frac{1}{2}} J_o + 2\lambda k^2 G = 0. \tag{45a}$$

By replacing

$$(R_o, J_o, S_o) \text{ by } (R, J, S) \,,$$

and neglecting

$$2\varpi^4 S_o^{-\frac{1}{2}} G \,,$$

for large k, we find

$$-\left(\frac{F}{k^3}\right)^{\frac{1}{2}} (R + 2S^{\frac{1}{2}}) + 2\nu k^2 F = 0 \,,$$

$$-\left(\frac{F}{k^3}\right)^{\frac{1}{2}} J + 2\lambda k^2 G = 0 \,.$$

The solutions are:

$$F = \left(\frac{\epsilon_\nu}{2\nu^2}\right)^2 \left(1 + 2\omega'^2\tau^2\right)^2 k^{-7} \,,$$

$$G = \frac{\epsilon_\nu}{2\nu^2}\,\frac{\epsilon_\lambda}{2\lambda^2}\left(1 + 2\omega'^2\tau^2\right) k^{-7} \;; \tag{45b}$$

where

$$\tau^{-2} = R.$$

When the density fluctuations are negligible

$$\omega' \cong 0 ,$$

(45) degenerates to the dissipative spectrum of hydrodynamic turbulence

$$F = \left(\frac{\epsilon_\nu}{2\nu^2}\right)^2 k^{-7} ,$$

which is the spectral law proposed by Heisenberg.

(3) Convection and diffusion of charged particles under strong inertial turbulent regime

When the charged particles are of low concentration, we can assume

$$S_o^{\frac{1}{2}} \ll R_o .$$

Moreover, when the strong turbulent field of a high Reynolds number is maintained in its inertial regime, under which the charged particles undergo a convection or a diffusion, we have

$$\nu \ll \lambda .$$

Under such circumstances, the turbulent spectrum F is not much modified by the weak electrostatic diffusion, but satisfies equation (38a) which is simplified to the form

$$\nu_k R_o = \epsilon_\nu \quad ,$$

and gives the Kolmogoroff law

$$F = A_1 \, k^{-5/3} \quad ,$$

with

$$A_1 = \left(\frac{8\epsilon_\nu}{9\chi}\right)^{2/3} .$$

On the other hand, the diffusion of the particles satisfies equation (38b), rewritten as

$$J_o = \frac{\epsilon_\lambda}{\lambda + \nu_k} \quad .$$

After a simple differentiation, we obtain

$$G = \frac{\epsilon_\lambda}{(\lambda + \nu_k)^2} \frac{1}{2k^2} \left(\frac{F}{k^3}\right)^{\frac{1}{2}} ,$$

$$= \frac{2}{3} \frac{\epsilon_\lambda}{\lambda} k_d^{-3} \frac{(k/k_d)^{-5/3}}{[1 + (k/k_d)^{4/3}]^2} \quad ; \qquad (47a)$$

$$= \begin{cases} \frac{2}{3} \frac{\epsilon_\lambda}{\lambda} k_d^{-4/3} k^{-5/3} , & k \ll k_D \; , \\ \frac{2}{3} \frac{\epsilon_\lambda}{\lambda} k_d^{4/3} k^{-13/3} , & k \gg k_D \; ; \end{cases} \qquad (47b)$$

where

$$k_d = \left(\frac{3}{8} \frac{\epsilon_\nu}{\chi\lambda^3}\right)^{1/4}$$

is a critical wave number, characteristic of the transition between the convection and the dissipation of the density spectrum. The -13/3 law (47b) has been considered in experiments [Gratstein et. al, 1966].

If the density fluctuations in the dissipative range and the turbulent field in the inertial range are not correlated, the following spectrum is obtained

$$G = A_1 \frac{\epsilon_\lambda}{3\lambda^3} k^{-17/3} , \tag{48}$$

instead of (47b). The spectral law (48) agrees with the one related to the diffusion of neutral particles in a turbulent fluid [Tchen, 1965].

7. EFFECTS OF DAMPING ON TURBULENT DIFFUSION

The Heisenberg law in (45) and (46) would give a divergent integral

$$< \left(\frac{\partial u_i}{\partial x_j}\right)^{2m} > = 2 \int_0^\infty dk\, k^{2m} F(k), \tag{49}$$

for $m \geq 3$. This difficulty of divergence has worried investigators of hydrodynamic turbulence in the past [Pao, 1965]. Here we shall show that it can be remediated by including the effects of damping in the turbulent diffusion as provided by the present theory. According to (22) and (25) the damping factor is

$$\exp(-\nu k^2 \tau)$$

where τ is the life time of the small eddies; as characterized by the

2 parameters ν and ϵ_ν, it is

$$\tau = \text{const } (\nu/\epsilon_\nu)^{1/2} \quad ,$$

yielding a damping factor

$$e^{-k^2/k_\nu^2} \quad ,$$

with

$$k_\nu = \left(\frac{3}{8}\chi^2\right)^{1/4} (\epsilon_\nu/\nu^3)^{1/4} \quad .$$

The numerical coefficient can be found from a determination of the viscous cutoff of the inertial spectrum [Tchen, 1965]. Thus, by attaching the damping factor to the diffusion formulas (25) and (35b), we find

$$\nu_k = \chi \int_k^\infty dk \, (F/k^3)^{1/2} \, e^{-(k/k_\nu)^2} \quad , \tag{50a}$$

instead of (35b). In the initial range of the spectrum where

$$k \ll k_\nu \quad ,$$

the damping is ineffective.

Similar arguments applied to the damping of the particle diffusion, with a factor

$$e^{-\lambda k^2 \tau}$$

yield

$$\lambda_k = \chi \int_k^\infty dk\ (F/k^3)^{1/2}\ e^{-(k/k_D)^2} \quad , \tag{50b}$$

where

$$k_D = \text{const}\ (\epsilon_\nu/\lambda^2\nu)^{1/4} \quad .$$

On the basis of the new diffusion coefficients (50a) and (50b) including the effects of damping, the spectra in the dissipative ranges take the new formulas,

$$F = \left(\frac{\chi\epsilon_\nu}{2\nu^2}\right)^2 k^{-7}\ e^{-2(k/k_\nu)^2} \quad , \tag{51a}$$

and

$$G = \frac{\chi\epsilon_\nu}{2\nu^2}\,\frac{\chi\epsilon_\lambda}{2\lambda^2}\ k^{-7}\ e^{-2(k/k_D)^2} \quad , \tag{51b}$$

instead of (45b). Further, the inertial-diffusion spectrum, as derived from

$$\lambda k^2 G - \chi\ (F/k^3)^{1/2}\ e^{-k^2/k_D^2} \int_0^{k\to\infty} dk\ k^2 G = 0 \quad ,$$

a modified form of (45a), becomes

$$G = A_1^{1/2}\ \frac{\epsilon_\lambda \chi}{2\lambda^2}\ k^{-13/3}\ e^{-k^2/k_D^2} \quad , \tag{51c}$$

instead of (47b).

Unlike the Heisenberg formula (45b), the formulas (51) are convergent at large wave numbers and for high order moments (49).

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