FLUID, THERMAL AND AEROSPACE SCIENCES

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[redacted]

ERRATA

LAMINAR FLOW ANALYSIS OF PLANE DISK SLINGER SEAL

by Robert L. Rosenthal and Eli Reshotko

p. 12: eq. (25), the last terms should read:

$$\frac{EcS^2}{Re}\left[\left(\frac{\partial v_o}{\partial z}\right)^2 + \left(\frac{\partial u_o}{\partial z}\right)^2\right]$$

p. 14: eqs. (31) and (32) should be:

$$\frac{S^2}{Re} F''' - 2FF'' + F'^2 - G^2 = - \frac{E}{r} \frac{dp_o}{dr} \qquad (31)$$

$$\frac{S^2}{Re} G'' - 2G'F + 2GF' = 0 \qquad (32)$$

p. 15 l. 19: change $\frac{Re}{S}$ to $\frac{Re}{S^2}$

p. 16: eq. (36) and (38) should be

$$F = \frac{Re}{S^2}\left[F_o + \left(\frac{Re}{S^2}\right) F_1 + \left(\frac{Re}{S^2}\right)^2 F_2 + \ldots.\right] \qquad (36)$$

$$B = B_o + \left(\frac{Re}{S^2}\right) B_1 + \left(\frac{Re}{S^2}\right)^2 B_2 + \ldots \qquad (38)$$

p. 19 l. 12: ... and boundary conditions (34), (35))

p. 20 l. 4: change $\frac{Re}{S}$ to $\frac{Re}{S^2}$

p. 20, lines (11) and (12): This text should read:... under the conditions that the two functions and their first three derivatives match at $z = 1/2$ and also that $G'(0) = G'(1) = 0$.

p. 21 l. 10: ... at $\frac{Re}{S^2} = 25$ ...

p. 23 l. 17: change Karmn to Karman

p. 24 l. 19: change $\frac{Re}{S}$ to $\frac{Re}{S^2}$

p. 25 The numerator of equation (57) should read

$$|g'_D(0)| - |g'_H(0)|$$

p. 49 Reference 4: Change volume <u>30</u> to volume <u>49</u>

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ABSTRACT

The flow and temperature fields are considered in the region between a rotating and stationary disk. The governing equations are changed by a radial transformation into ordinary differential equations. Distinctions between regimes of large and small Reynolds numbers are clarified and the equations are solved accordingly. The trend toward boundary layer formation is confirmed by these solutions.

The thermal problem is treated in a similar manner with Prandtl number also being of importance in this case.

Finally, quantities which are of design importance are discussed based on the dynamic and thermal solutions.

ACKNOWLEDGEMENTS

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LIST OF SYMBOLS

a	gap width between disk and housing
$\bar{a}$, $\bar{b}$	constants in asymptotic solution
B	radial pressure gradient strength
c	specific heat
c_1, c_2	constants in thermal boundary layer solution
C_g, C_h	constants in asymptotic solution
E	Euler No. $(= P/\rho\omega^2 R^2)$
Ec	Eckert No. $(= R^2\omega^2/cT_R)$
F	radially transformed axial velocity function
f	axial velocity boundary layer function
$\bar{f}$	asymptotic axial velocity function
G	radially transformed azimuthal velocity function
g	azimuthal velocity boundary layer function
$\bar{g}$	asymptotic azimuthal velocity function
Ge	dimensionless core rate of rotation in the boundary layer limit
$\bar{h}$	derivative of $\bar{f}$
i	$\sqrt{-1}$
I_o	zeroth order energy integral
I	radially transformed energy integral
$\bar{j}$	derivative of $\bar{h}$
K	asymptotic value of f
K_D	asymptotic value of disk boundary layer temperature

K_H	asymptotic value of housing boundary layer temperature
$\bar{k}$	derivative of $\bar{g}$
k	thermal conductivity
$\mathcal{O}(\ )$	order of magnitude
p	dimensionless pressure
$\bar{p}$	dimensional pressure
P	characteristic pressure
Pr	Prandtl No. (= $\mu c/k$)
r	dimensionless radial co-ordinate
$\bar{r}$	dimensional radial co-ordinate
R	radius of disk
$\bar{r}_1$, $\bar{r}_2$	radial distances of the two liquid-gas interfaces from the axis of rotation
R_1, R_2	surfaces of constant radius
Re	Reynolds No. (= $\rho\ \omega R^2/\mu$)
S	geometry factor (= R/a)
$\bar{S}$	derivative of $\bar{t}$
T	dimensionless temperature
$\bar{T}$	dimensional temperature
T_R	characteristic temperature
T_w	dimensionless housing wall temperature
t	thermal boundary layer temperature function
$\bar{t}$	asymptotic temperature function
t_1, t_2	homogeneous solutions to the thermal boundary layer equation

t_p	a particular solution to the thermal boundary layer equation
u	dimensionless radial velocity component
$\bar{u}$	dimensional radial velocity component
v	dimensionless azimuthal velocity component
$\bar{v}$	dimensional azimuthal velocity component
Ve	dimensionless core azimuthal velocity component
w	dimensionless axial velocity component
$\bar{w}$	dimensional axial velocity component
z	dimensionless axial co-ordinate
$\bar{z}$	dimensional axial co-ordinate
ξ	axial boundary layer co-ordinate
θ	radially transformed temperature function
λ	exponential constant in asymptotic solution
μ	viscosity
ρ	density
ϕ	azimuthal co-ordinate
ω	angular velocity of disk

SUBSCRIPTS

0,1,2, ...	ordered components
B	pertaining to Bodewadt's solution
i	index for variable numerical subscripts
D	reference to disk boundary layer problem
H	reference to housing boundary layer problem

SUBSCRIPTS CONTINUED

w	housing wall

primes denote differentiation

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CHAPTER I

INTRODUCTION

1. Description of Device

With the development of space power systems has arisen the need for a seal demanding little power for operation and possessing very high sealing abilities across pressure differences of approximately one atmosphere. The slinger seal has been proposed to meet these requirements.

The slinger seal is a fluid, shaft sealing device. The configuration is presented in Figure 1. Fastened to, and spinning with, the shaft is a thin disk. This assembly is enclosed in a cylindrical, stationary housing. Liquid occupies the narrow gap between the disk and housing faces. Through the action of viscosity, the angular motion of the disk is imparted to the liquid slinging it outward to the housing periphery. The liquid acts as the sealing medium, and the pressure difference that can be supported is related to the gap distance and the difference of radial distances, from the axis of rotation, of the two interfaces.

Important secondary flows arise. There is a radially outward secondary flow along the rotating disk and a radially inward flow along the housing wall. In order to prevent leakage of the fluid, the integrity of the interface must be maintained. This requires adequate surface tension at the interface to arrest the radial inflow and redirect it axially so that it may flow outward along the rotating disk. This report does not investigate the interface dynamics, but it is recognized that such an analysis would strongly depend on the dynamics in the main body of the fluid away from the neighborhood of the interface.

The flow can be either laminar or turbulent. The subject of this report is the analysis of the fluid field for laminar flow. Specifically, the region of interest is that which comprises the bulk of the fluid exclusive of the end effects of the interface and the radial end wall.

2. Previous Investigations of Related Problems

The qualitative aspects of a fluid between two infinite, rotating disks were discussed by Batchelor [1]. For the special case of one of the disks held stationary, Batchelor's problem becomes the model adopted for this analysis.

Within the laminar flow regime, two limiting cases can be identified: 1) fully viscous behavior, and 2) boundary layer behavior. In the fully viscous limit, the effects of viscosity dominate everywhere across the gap. This is analogous to a fully developed Couette flow. By contrast, in the boundary layer limit, the effects of viscosity are contained in thin layers adjacent to the disk and housing wall.

Core rotation is assumed between the two boundary layers when solving for that limiting case. Each boundary layer is treated separately with an appropriate matching condition invoked to determine the angular velocity of the core. Karman [7] considered the problem of a rotating disk in an infinite extent of fluid at rest; and Bodewadt [8] a stationary disk in an infinite extent of rotating fluid. A study of both fluid and disk rotation was made by Rogers and Lance [9], [15].

This report calculates the core rate of rotation by two methods. The first embodies continuity and the second is based on an integral momentum balance. Results of these two methods disagree slightly, and reasons are given for accepting the momentum result. Schultz-Grunow [5] examined the two disk problem and invoked an integral momentum balance, assuming parabolic velocity profiles, to determine the rate of core rotation.

A clear and complete picture of the transition from the viscous limit to the boundary layer limit is difficult to achieve. As Re/S^2, Reynolds number based on gap width, increases, the present numerical solutions become divergent as do series expanded in terms of Re/S^2. However, numerical solutions have been carried far enough by Grohne [2], [16] and by Lance and Rogers [15] to establish the type of departure from fully viscous behavior. The present paper also determines the core angular velocity in the boundary layer limit.

The associated heat transfer problem may also be identified with two types of limiting behavior which, as in the dynamic problem, determine the method of analysis. Depending on the absolute and relative magnitudes of Re/S^2 and Pr, the temperature distribution will range from the parabolic conduction type to a thermal boundary layer at each disk.

There has been very little previous work done in connection with the thermal aspects of the problem. Millsaps and Pohlhausen [6] have solved the case of heat transfer for the Karman problem, and the compressibility aspects are discussed by Ostrach and Thornton [17].

3. Objectives of Analysis

The objectives of this laminar flow analysis are:

1) For finite Re/S^2, to resolve the discrepancies in the mode of departure from the viscous limit.
2) To review matching criteria in the boundary layer limit and provide appropriate solutions.
3) To solve the related heat transfer problem.
4) To formulate design criteria for the use of slinger seals.

CHAPTER II

GOVERNING EQUATIONS

1. Assumptions

The fluid model chosen is an incompressible, constant property, Newtonian fluid. Body forces and effects of radiation are neglected. The bounding walls of the fluid are impermeable, and the fluid on the surface of those walls moves with the velocity of the walls. The flow is steady and symmetric about the axis of rotation of the disk.

The assumptions and the geometrical confines of the fluid are similar to those of the fluid in a vortex magnetohydrodynamic generator analyzed by Loper [14]. Much of the development of the basic equations parallels Loper's development.

2. Non-Dimensionalization of Equations

The problem is most easily treated in cylindrical coordinates. Steady flow and rotational symmetry allow time and azimuthal derivatives to be neglected.

The continuity, Navier-Stokes, and energy equations, under the assumptions specified, are as follows:

$$\frac{1}{\bar{r}} \frac{\partial(\bar{u}\,\bar{r})}{\partial \bar{r}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0 \qquad (1)$$

$$\rho\left(\bar{u}\,\frac{\partial\bar{u}}{\partial\bar{r}} - \frac{\bar{v}^2}{\bar{r}} + \bar{w}\,\frac{\partial\bar{u}}{\partial\bar{z}}\right) = -\frac{\partial\bar{p}}{\partial\bar{r}} + \mu\left(\frac{\partial^2\bar{u}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{u}}{\partial\bar{r}} - \frac{\bar{u}}{\bar{r}^2} + \frac{\partial^2\bar{u}}{\partial\bar{z}^2}\right) \tag{2}$$

$$\rho\left(\bar{u}\,\frac{\partial\bar{v}}{\partial\bar{r}} + \frac{\bar{u}\bar{v}}{\bar{r}} + \bar{w}\,\frac{\partial\bar{v}}{\partial\bar{z}}\right) = \mu\left(\frac{\partial^2\bar{v}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{v}}{\partial\bar{r}} - \frac{\bar{v}}{\bar{r}^2} + \frac{\partial^2\bar{v}}{\partial\bar{z}^2}\right) \tag{3}$$

$$\rho\left(\bar{u}\,\frac{\partial\bar{w}}{\partial\bar{r}} + \bar{w}\,\frac{\partial\bar{w}}{\partial\bar{z}}\right) = \frac{\partial\bar{p}}{\partial\bar{z}} + \mu\left(\frac{\partial^2\bar{w}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{w}}{\partial\bar{r}} + \frac{\partial^2\bar{w}}{\partial\bar{z}^2}\right) \tag{4}$$

$$\rho c\left(\bar{u}\,\frac{\partial\bar{T}}{\partial\bar{r}} + \bar{w}\,\frac{\partial\bar{T}}{\partial\bar{z}}\right) = k\left(\frac{\partial^2\bar{T}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{T}}{\partial\bar{r}} + \frac{\partial^2\bar{T}}{\partial\bar{z}^2}\right) + \mu\left[2\left(\frac{\partial\bar{u}}{\partial\bar{r}}\right)^2 + 2\left(\frac{\bar{u}}{\bar{r}}\right)^2 + 2\left(\frac{\partial\bar{w}}{\partial\bar{z}}\right)^2 + \left(\frac{\partial\bar{v}}{\partial\bar{z}}\right)^2 + \left(\frac{\partial\bar{u}}{\partial\bar{z}} + \frac{\partial\bar{w}}{\partial\bar{r}}\right)^2 + \left(\frac{\partial\bar{v}}{\partial\bar{r}} - \frac{\bar{v}}{\bar{r}}\right)^2\right] \tag{5}$$

In an effort to simplify these equations by comparing the orders-of-magnitude of the various terms, the variables will be non-dimensionalized as follows.

$$r = \frac{\bar{r}}{R}, \quad z = \frac{\bar{z}}{a}, \quad p = \frac{\bar{p}}{P}, \quad u,v,w = \frac{\bar{u},\bar{v},\bar{w}}{R\omega}, \quad T = \frac{\bar{T}}{T_R} \tag{6}$$

Substitution of equations (6) into equations (1)-(5) yields the following dimensionless set of equations:

$$\frac{1}{r}\frac{\partial(ur)}{\partial r} + S\frac{\partial w}{\partial z} = 0 \tag{7}$$

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} + Sw\frac{\partial u}{\partial z} = -E\frac{\partial p}{\partial r} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + S^2\frac{\partial^2 u}{\partial z^2}\right) \tag{8}$$

$$u\frac{\partial v}{\partial r} + \frac{uv}{r} + Sw\frac{\partial v}{\partial z} = \frac{1}{Re}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + S^2\frac{\partial^2 v}{\partial z^2}\right) \tag{9}$$

$$u\frac{\partial w}{\partial r} + Sw\frac{\partial w}{\partial z} = -SE\frac{\partial p}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + S^2\frac{\partial^2 w}{\partial z^2}\right) \tag{10}$$

$$\begin{aligned} u\frac{\partial T}{\partial r} + Sw\frac{\partial T}{\partial z} = {} & \frac{1}{RePr}\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + S^2\frac{\partial^2 T}{\partial z^2}\right] \\ & + \frac{Ec}{Re}\left[2\left(\frac{\partial u}{\partial r}\right)^2 + 2\frac{u^2}{r^2} + 2S^2\left(\frac{\partial w}{\partial z}\right)^2 + S^2\left(\frac{\partial v}{\partial z}\right)^2\right. \\ & \left. + S^2\left(\frac{\partial u}{\partial z}\right)^2 + 2S\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + \left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 - 2\frac{v}{r}\frac{\partial v}{\partial r} + \frac{v^2}{r^2}\right] \end{aligned} \tag{11}$$

Where: Re = Reynolds No. = $\frac{\rho\omega R^2}{\mu}$

E = Euler No. = $\frac{P}{\rho\omega^2 R^2}$

S = Geometry Factor = $\frac{R}{a}$

Ec = Eckert No. = $\frac{R^2\omega^2}{cT_R}$

Pr = Prandtl No. = $\frac{\mu c}{k}$

3. Boundary Conditions

A. Velocity Boundary Conditions

Let $\bar{z} = 0$ be the face of the rotating disk. The non-slip and impermeability conditions on the disk and on the face of the stationary housing $(\bar{z} = a)$ are:

$$\begin{aligned} \bar{u}(\bar{r},0) &= 0 & \bar{u}(\bar{r},a) &= 0 \\ \bar{v}(\bar{r},0) &= \omega\bar{r} & \bar{v}(\bar{r},a) &= 0 \\ \bar{w}(\bar{r},0) &= 0 & \bar{w}(\bar{r},a) &= 0 \end{aligned} \tag{12}$$

which in non-dimensional form are:

$$\text{at } z = 0: \quad u = 0, \quad v = r, \quad w = 0 \tag{13}$$

$$\text{at } z = 1: \quad u = 0, \quad v = 0, \quad w = 0 \tag{14}$$

An additional, integral, condition on the radial velocity, u , is obtained by noting that there can be no net radial flow across the gap. This condition is expressed by:

$$\int_0^1 u\,dz = 0 \tag{15}$$

B. Thermal Boundary Conditions

If the disk is not cooled, then maintenance of steady state conditions requires that all heat generated by viscous dissipation be removed at the housing wall; therefore, heat transfer at the disk is zero

$$\text{at} \quad z = 0, \qquad \frac{\partial T}{\partial z} = 0 \tag{16}$$

The ability to cool the housing wall suggests that at $z = 1$, the temperature could be maintained at any arbitrary level T_w, where T_w is usually constant.

$$\text{at} \quad z = 1, \qquad T = T_w \tag{17}$$

The disk temperature is left unspecified and will result from the solution of the energy equation.

4. Series Expansion of Dependent Variables

The non-dimensionalization has served to consolidate the many quantities of the problem into a few familiar, dimensionless, parameters such as Re, Pr, etc.

Each term in the differential equations consists of a variable portion of nominally unit order modified by a dimensionless coefficient. The salient features of the problem can be isolated by solving equations consisting of the largest terms of the complete equations based on their indicated magnitudes. This problem leads also to mathematical simplification as well.

To this end, the dependent variables will be expanded in power series of some dimensionless parameter. Three such parameters, or their reciprocals are available; S, E, and Re . Of these three, S is the only one which is consistently large, as the geometry will always embody a large aspect ratio.

The expansions of the independent variables will be done in powers of $1/S$.

$$
\begin{aligned}
u &= u_o + \frac{1}{S} u_1 + \frac{1}{S^2} u_2 + \dots \dots \dots \\
v &= v_o + \frac{1}{S} v_1 + \dots \dots \dots \dots \\
w &= w_o + \frac{1}{S} w_1 + \dots \dots \dots \dots \\
p &= p_o + \frac{1}{S} p_1 + \dots \dots \dots \dots \\
T &= T_o + \frac{1}{S} T_1 + \dots \dots \dots \dots
\end{aligned}
\qquad (18)
$$

The functions u_o, u_1, u_2, . . . v_o, v_1, . . . etc. are nominally of unit order of magnitude.

In the geometrically limiting case of infinite disks, for which equations (18) must remain valid, the zeroth order terms will be the only term in each series. It is these terms, therefore, that must satisfy the boundary conditions. For finite $1/S$, the higher order terms must then assume zero values at the boundaries.

Equations (7) to (11) are to be expanded with the series (18) and the ordering is accomplished by collecting the terms multiplied by a given power of S and setting each collection equal to zero.

To leading order, the continuity equation yields

$$\frac{\partial w_o}{\partial z} = 0 \tag{19}$$

Boundary conditions on w determine that

$$w_o = 0 \tag{20}$$

Ordering the "z" momentum equation (10) and taking into account that $w_o = 0$, yields

$$\frac{\partial p_o}{\partial z} = 0 \tag{21}$$

Since $\frac{\partial w_o}{\partial z} = 0$, the next order terms of the continuity equation are:

$$\frac{1}{r}\frac{\partial (ru_o)}{\partial r} + \frac{\partial w_1}{\partial z} = 0 \tag{22}$$

The leading significant terms of the "r" and "ϕ" momentum equations (8) and (9) are:

$$u_o \frac{\partial u_o}{\partial r} + w_1 \frac{\partial u_o}{\partial z} - \frac{v_o^{\,2}}{r} = -E\frac{\partial p_o}{\partial r} + \frac{S^2}{Re}\frac{\partial^2 u_o}{\partial z^2} \tag{23}$$

$$u_o \frac{\partial v_o}{\partial r} + w_1 \frac{\partial v_o}{\partial z} + \frac{u_o v_o}{r} = \frac{S^2}{Re}\frac{\partial^2 v_o}{\partial z^2} \tag{24}$$

To leading significant order, the energy equation is:

$$u_o \frac{\partial T_o}{\partial r} + w_1 \frac{\partial T_o}{\partial z} = \frac{S^2}{RePr} \frac{\partial^2 T_o}{\partial z^2} + \frac{EcS^2}{Re} \left(\frac{\partial v_o}{\partial z}\right)^2 + \left(\frac{\partial u_o}{\partial z}\right)^2 \tag{25}$$

$$z = 0, \qquad \frac{\partial T_o}{\partial z} = 0 \tag{26}$$

$$z = 1, \qquad T_o = T_w \tag{27}$$

Equations (22) - (25) constitute a mathematical statement of the hydrodynamic flow in and thermal behavior of the fluid. The incompressibility assumption results in the uncoupling of the momentum equation from the energy equation allowing the temperature distribution to be calculated after the flow field has been determined. The solution of the flow field is described in the next chapter and the solution of the energy equation is treated separately in Chapter IV.

CHAPTER III

SOLUTION OF THE FLOW FIELD

The distributions of radial, azimuthal and axial velocity result from simultaneous solution of the continuity and three momentum equations.

1. Karman Similarity

The equations (9) and (10) are identically the equations governing the flow over a disk rotating in an infinite extent of fluid. The difference between that problem and the slinger seal lies in the boundary conditions.

Karman [7] noted that these partial differential equations may be reduced to ordinary equations in z by the following transformations:

Let:

$$w_1 = 2F(z) \tag{28}$$

In order that continuity, equation (7), be satisfied, u_o must have the following form:

$$u_o = -rF'(z) \tag{29}$$

Also, let

$$v_o = rG(z) \tag{30}$$

$F(z)$ and $G(z)$ are functions of z only. Primes denote differentiation with respect to z. The argument, z, of $F(z)$

and G(z) will no longer be carried along as it is understood.

The use of this transformation in the present problem depends on its compatibility with the boundary conditions (13)-(15). It can be shown in fact that the use of this transformation automatically requires that there be no net radial flow in this problem (equation 15). If there were to be a net radial through-flow then one could not have Karman similarity.

Substitution of the transformation, equations (28), (29), and (30) into the "r" and "ϕ" momentum equations (8) and (9), yields

$$\frac{S^2}{Re} F''' - 2F'' + F'^2 - G^2 = -\frac{E}{r}\frac{dp_o}{dr} \tag{31}$$

$$\frac{S^2}{Re} G'' - 2G'F + 2gf' = 0 \tag{32}$$

Recalling that $\frac{\partial p_o}{\partial z} = 0$, the right-hand side of equation (31) is a function of r, only, while the left-hand side is a function of z, only. Therefore, each side of that equation can at most be equal to a mutual constant, defined as B.

$$\frac{dp_o}{dr} = -\frac{B}{E} r \tag{33}$$

One integration of equation (33) reveals that p_o varies with r^2. The strength of this variation depends on B.

The boundary conditions in terms of F and G are:

$$z = 0: \quad F' = 0, \quad G = 1, \quad F = 0 \tag{34}$$

$$z = 1: \quad F' = 0, \quad G = 0, \quad F = 0 \tag{35}$$

Equations (31), (32) and (34), (35) are the transformed governing equations and boundary conditions on the velocities.

Note that the equations are ordinary, non-linear, and coupled in the two unknowns, F and G. Also note that there are two derivatives on G, three derivatives on F, two boundary conditions on G, but <u>four</u> boundary conditions on F. The extra condition to be met is associated with the proper determination of B. The correct B will be that for which all the boundary conditions are satisfied.

2. Range of Re/S^2 and Types of Flow

Equations (31) and (32) are seen to depend on the one parameter Re/S^2. The magnitude of this parameter will determine the types of flow and the approach to solving the equations.

For small values of Re/S^2, the viscous forces dominate. For the limiting viscous case of $Re/S^2 \to 0$, there will be no axial nor radial velocities and the azimuthal velocity decreases linearly.

Large values of Re/S will result in boundary layer flow; presumably separate boundary layers at the disk and at the housing. The boundary conditions for this type of flow are determined from the assumption that there exists a potential core of fluid rotating as a solid body in the central region between the two boundary layers.

Intermediate values of Re/S^2 represent transitional behavior between the two extremes just discussed. Investigation of this range of Re/S^2 will shed light on the nature of the trend toward core rotation as Re/S^2 increases.

3. Viscous Limit and Its Environs

The methods of solution for small Re/S^2 will now be discussed in detail.

A. Series Solution

For small Re/S^2, the functions $F(z)$, $G(z)$, and the constant B have been expanded in the series:

$$F = \frac{Re}{S^2} F_o + \left(\frac{Re}{S}\right) F_1 + \left(\frac{Re}{S^2}\right)^2 F_2 + \ldots \tag{36}$$

$$G = G_o + \frac{Re}{S^2} G_1 + \left(\frac{Re}{S^2}\right)^2 G_2 + \ldots\ldots \tag{37}$$

$$B = B_o + \frac{Re}{S^2} B_1 + \left(\frac{Re}{S}\right)^2 B_2 + \ldots\ldots \tag{38}$$

where F_o and G_o satisfy the boundary conditions equations (34) and (35), and the higher order functions F , F' , and G satisfy homogeneous boundary conditions at $z = 0$ and $z = 1$.

$$z = 0, \quad F_i = F'_i = G_i = 0$$

$$\text{for } i = 1,2,3, \ldots$$

$$z = 1, \quad F_i = F'_i = G_i = 0$$

(39a,b)

Substitution of the series into equations (31) and (32) and collecting coefficients of like powers of Re/S^2 gives the following sequence of equations.

$$F_o''' - G_o^2 = B_o$$
$$G_o'' = 0 \qquad (40a,b)$$

$$F_1''' - 2G_oG_1 = B_1$$
$$G_1'' = 0 \qquad (41a,b)$$

$$F_2''' - 2F_oF_o'' + F_o'^2 - 2G_oG_2 - G_1^2 = B_2$$
$$G_2'' + 2F_o'G_o - 2F_oG_o' = 0 \qquad (42a,b)$$

$$F_3''' - 2(F_oF_1'' + F\ F_o'') + 2F_o'F_1' - 2(G_oG_3 + G_1G_2) = B_3$$
$$G_3'' + 2(F_o'G_1 + F\ 'G_o) - 2(F_oG_1' + F_1G_o') = 0 \qquad (43a,b)$$

$$F_4''' - 2(F_oF_2'' + F_1F_1'' + F_o''F_2) + 2F_o'F_2' + F_1'^2$$
$$- (2G_oG_4 + 2G_1G_3 + G_2^2) = B_4$$
$$(44a,b)$$
$$G_4'' + 2(F_o'G_2 + F_1'G_1 + F_2'G_o) - 2(F_oG_2' + F_1G_1'$$
$$+ F_2G_o') = 0$$

Inspection of these equations and the associated boundary conditions reveals that the odd-subscripted functions F_i, G_i, B_i are identically zero. This fact somewhat simplifies the remaining equations when solving for the even-subscripted F_i, G_i, and B_i.

Equations (40a,b) may be solved by twice integrating $G_o'' = 0$, substituting into equation (40a) and performing three integrations on F_o''' . The five integration constants and B_o are determined to satisfy the six appropriate boundary conditions. The resulting F_o and G_o are substituted into equations (42a,b) to obtain G_2 and the F_2 . This sequence is carried out for each successive even-subscripted set of equations.

In this procedure, each F_i and G_i evolves as a polynomial and, although the principle is simple, the polynomials become lengthy and the process is very tedious. Furthermore, convergence of the series (36) to (38) cannot be guaranteed particularly for $Re/S^2 > 1$.

The series solutions, as far as they were carried out, are

$$G_o = 1 - z \tag{45a}$$

$$F_o = -\frac{1}{20} z^2 + \frac{7}{60} z^3 - \frac{1}{12} z^4 + \frac{1}{60} z^5 \tag{45b}$$

$$B_o = -\frac{3}{10} \tag{45c}$$

$$G_2 = \frac{1}{30}\left(\frac{2}{21}z^7 - \frac{2}{3}z^6 + \frac{17}{10}z^5 - 2z^4 + z^3 - \frac{9}{70}z\right) \qquad (46a)$$

$$\begin{aligned} F_2 = \; & 0.00011586z^3 - 0.000357z^4 + 0.000143z^5 + \\ & 0.000361z^6 - 0.0005198z^7 + 0.000327z^8 - \\ & 0.000111z^9 + 0.0000243z^{10} + 0.00000261z^{11} + \\ & 0.0000294z^{12} \end{aligned} \qquad (46b)$$

$$B_2 = 0.00069516 \qquad (46c)$$

The zeroth terms F_o, F_o', and G_o are presented in Figure 2. Deviation of $G(z)$ from the linear G_o will indicate if and how core rotation will form for larger Re/S^2.

B. Numerical Integration

Numerically integrating the two point boundary value problem (equations (31), (32) and boundary conditions (34), (34)) could be done only up to $Re/S^2 = 5.70$. Details of the numerical procedure and its limitation up to $Re/S^2 = 5.70$ are discussed in Appendix A.

The series solutions using only the zeroth and second terms are compared to the numerical solutions by looking at G' at $z = 0$ and $z = 1$ and the constant B for various Re/S^2 up to 5.70 (See Table 1).

Good agreement appears between the two methods for values of Re/S^2 less than 1, as might be expected, while discrepancies are held to be about 1% anywhere in the range $0 < Re/S^2 < 5.70$.

Grohne [2] and Soo [3] have also looked at this problem for small but finite Re/S . Grohne had his higher order G_i satisfy the same boundary condition as G_o so that the overall series for $G = G_o + \left(\frac{Re}{S}\right)^2 G_2 + \ldots$ could not satisfy the proper boundary condition $G(0) = 1$ except for the case $Re/S^2 = 0$.

Soo's zeroth order solution is in agreement with equations (45a, b, c). Rather than taking higher order corrections to be the higher order terms in series such as equation (36)-(38), Soo preferred to calculate the corrections by two separate series expansions; one from $z = 0$ forward to $z = \frac{1}{2}$, and another from $z = 1$ backward to $z = \frac{1}{2}$, under the conditions that the two also that $G'(0) = G'(1)$. This last condition was argued from the fact that across any constant radius surface, the flux of angular momentum is small, therefore the torques on the disk and housing must be equal. The numerical results show this to be not quite true. For example, for $Re/S^2 = 1.0$, calculations show (Refer to Table 1) that $G'(0) = - 1.0042$ while $G'(1) = 0$ 0.99872. Although the difference is small, important qualitative differences arise between Soo's method and the numerical solution for small but finite Re/S^2 .

Referring to Figure 3, Soo's solution of $G(z)$ deviates from $G_o(z)$ in an "S" shape indicating that for finite Re/S^2 , boundary layers are beginning to form at both walls. The numerical solution deviates from $G_o(z)$ by falling below $G_o(z)$, indicating a trend toward boundary layer formation adjacent to the disk only.

Before concluding on the finality of this trend, it was decided that if at all possible, the solutions should be taken to larger Re/S^2 .

C. Analog Solution

The equations (31) and (32) were programmed on an analog computer. Solutions to the two point boundary value problem were obtained up to $Re/S^2 = 50$. Figure 4 presents the analog solutions of $G(z)$ for $Re/S^2 = 0, 10, 25, 50$. Although boundary layer formation is more pronounced at the disk side, it is clearly evident that at $Re/S^2 - 25$ an inflection point appears indicating boundary layer behavior at the housing side. The deviation from the linear case is still totally on one side of G_o . However, as the boundary layer thickness decreases, it is reasonable to suggest that $G(z)$ near the housing will cross over to the top side of G_o. Notice that $G(z)$ for $Re/S^2 = 50$ has already crossed to the top side of $G(z)$ for $Re/S^2 = 25$. The analog solutions have been the strongest evidence yet of a trend toward the formation of a rotating core at large values of Re/S^2 .

4. Boundary Layer Limit

A. Boundary Layer Equations

Having established the likelihood of boundary layer formation at both walls and core rotation between them, we are now in a position to look at each boundary layer individually.

For large Re/S^2, define the boundary layer transformations

$$\xi = (Re/S^2)^{1/2} z \tag{47}$$

$$f(\xi) = (Re/S^2)^{1/2} F(z) \qquad g(\xi) = G(z) \tag{48a,b}$$

The radial pressure gradient existing in the core will be the gradient within both boundary layers. Viscous effects are absent in the core; its angular velocity is constant but there is also axial flow through the core. The Bernoulli equation for the core is:

$$E \frac{dp}{dr} = \frac{Ve^2}{r} \tag{49}$$

where Ve is the azimuthal velocity, linear in r . Substituting into equation (30) yields

$$\frac{E}{r} \frac{dp_o}{dr} = Ge^2 \tag{50}$$

Ge is an unknown constant whose physical importance is that the pressure gradient at large Re/S^2 is related to it. Determination of Ge will be by one of several "matching conditions" which will be discussed later.

Substitution of equations (47), (48a,b) and (50) into equations (31) and (32) give the following

$$f''' - 2f''f + f'^2 - (g^2 - Ge^2) = 0 \tag{51}$$

$$g'' - 2g'f + 2gf' = 0 \tag{52}$$

B. Boundary Conditions for Disk Problem and for Housing Problem

Equations (51) and (52) are valid for either the disk or housing boundary layer problem. However, the two problems are distinguished by their boundary conditions.

For the disk, $\xi = 0$ on the disk surface and increases positively into the fluid. For the housing $\xi = 0$ on the housing surface and increases positively into the fluid.

Disk problem:

$$\begin{aligned} &\xi = 0: \quad && f = 0, \quad g = 1, \quad f' = 0 \\ &\xi \to \infty: \quad && f' \to 0, \quad g \to Ge \end{aligned} \tag{53}$$

Housing problem:

$$\begin{aligned} &\xi = 0: \quad && f = 0, \quad g = 0, \quad f' = 0 \\ &\xi \to \infty: \quad && f' \to 0, \quad g \to Ge \end{aligned} \tag{54}$$

Notice that for the special case of $Ge = 0$, the disk problem reduces to that of a spinning disk in an infinite body of fluid at rest. Karmn [7] has solved this problem. Unfortunately, no scaling exists such that f and g for $Ge \neq 0$ may be obtained using Karman's solution for $Ge = 0$.

For the special case of $Ge = 1$, the housing problem reduces to that of a stationary disk in a infinite fluid rotating with non-dimensional velocity of 1. This problem has been analyzed by Bodewadt [8]. It is possible to scale Bodewadt's solution to arbitrary Ge by the transformation:

$$f = \frac{1}{2}Ge^{1/2}f_B; \quad g = Ge g_B; \quad \xi = Ge^{-1/2}\xi_B \qquad (55a,b,c)$$

where $f_B(\xi_B)$ and $g_B(\xi_B)$ are solutions to Bodewadt's problem.

C. Matching Conditions

It should be noted that for both the disk and housing problems, a family of an infinite number of solutions may be generated by allowing Ge to vary between 0 and 1 . Batchelor [1] and Stewartson [4] propose that the correct Ge , and hence the correct solutions in the two boundary layers, be determined by matching $f(\infty)$ for the disk and housing problems. This is basically a statement of continuity that the axial outflow from the housing must equal the axial inflow towards the disk.

Schultz-Grunow [5] attacked the problem in the following way. Assuming core rotation, he integrated the equations of motion over both boundary layer thicknesses assuming parabolic profiles for the radial and azimuthal velocities within the boundary layers and neglecting the axial component. Then, matching the frictional shears at the disk and housing walls, the value of $Ge = 0.54$ resulted. It has been pointed out previously in this report that for finite Re/S the shears at the two bounding walls are not equal. This suggests another condition for the determination of Ge, viz. an angular momentum balance on a toroidal control volume with a rectangular cross-section whose control surfaces are composed of the disk, the housing, and two cylindrical surfaces each of a different constant radius. This angular momentum integral condition is derived in Appendix B and takes into account the transport of angular momentum across the surfaces of constant radius.

For several solutions to the disk and housing problem that have been computed for a common value of Ge , it has been observed that the axial velocities of the two boundary layers are not necessarily equal. If the momentum integral condition is satisfied by a value of Ge for which the axial velocities do not match, then the continuity condition must be reckoned with by altering the picture of the core velocities to include radial flow. This is discussed in detail, along with the derivation of the momentum integral condition, in Appendix B.

The axial velocity match is:

$$f_D(\infty) = f_H(\infty) \tag{56}$$

The angular momentum matching condition is:

$$\frac{g'_D(\;) - g'_H(0)}{\left|\int_{D^o}^{\infty} f'g\,d\xi\right| - \left|\int_{H^o}^{\infty} f'g\,d\xi\right| + \left(\left|f_H(\infty)\right| - \left|f_D(\infty)\right|\right)Ge} = 4 \tag{57}$$

The boundary layer equations (51) and (52) have been solved numerically (see Appendix A) for the cases of Ge = 0.29, 0.31, and 0.33. Refer to Appendix C and Figures 5 and 6 for the calculation of the matching conditions.

Graphically, the axial velocity match determines Ge to be 0.308 while the momentum integral condition determines Ge to be 0.313 .

For Ge = 0.313 the difference in the axial velocities is 0.005. This means that the core radial flow (outward, because $f_H(\infty) > f_D(\infty)$) is about 0.006/0.380 = 1.5% of the radial flow in the disk boundary. The disk and housing boundary layer functions appear in Tables 2 and 3 and Figures 7 and 8.

CHAPTER IV

SOLUTION OF ENERGY EQUATION

1. Radial Transformation and Boundary Conditions

Having obtained solutions for the velocities, the energy equation may now be solved. The energy equation (25) is

$$\frac{S^2}{RePr}\frac{\partial^2 T_o}{\partial z^2} - w_1\frac{\partial T_o}{\partial z} - u_o\frac{\partial T_o}{\partial r} = -\frac{EcS^2}{Re}\left[\left(\frac{\partial v_o}{\partial z}\right)^2 + \left(\frac{\partial u_o}{\partial z}\right)^2\right] \tag{25}$$

and has the boundary conditions

$$z = 0: \quad \frac{\partial T_o}{\partial z} = 0 \qquad\qquad z = 1: \quad T_o = T_w(r) \qquad (26,27)$$

Recalling equations (28) - (30),

$$w_1 = 2F(z), \qquad u_o = -\,rF'(z), \qquad v_o = rG(z)$$

a similarity transformation exists for the temperature T_o of the form:

$$T_o = Ec(r^2\theta(z) + \text{const.}) \tag{58}$$

where const. may assume any value because only derivatives of T_o appear in equation (25).

Substitution of the velocity and temperature similarity transformations into equation (26) yields:

$$\frac{S^2}{RePr}\theta'' - 2F\theta' + 2F'\theta = -\frac{S^2}{Re}\left[G'^2 + F''^2\right] \tag{59}$$

The boundary conditions are:

$$z = 0: \qquad \theta'(0) = 0 \tag{60a}$$

If a constant housing wall temperature is assumed, $T_w(r) = T_w$, then const. in equation (58) is:

$$\text{const.} = T_w/Ec$$

So that

$$\text{at} \quad z = 1, \quad \theta(1) = 0 \tag{60b}$$

Equation (59) with boundary conditions (60a, b) is sufficient to describe the temperature distribution in the fluid field.

Equation (59) is an ordinary, second order, non-homogeneous, linear, differential equation for the variable $\theta(z)$ with non-constant coefficients. There are two physical parameters in this problem, Re/S^2 and Pr , as opposed to only one parameter, Re/S^2, in the momentum equations.

2. Range of Parameters and Types of Temperature Distribution

The general types of temperature profiles may be deduced from the relative and absolute magnitudes of the dimensionless parameters Re/S^2 and Pr .

I. Limiting case of $Re/S^2 \to 0$.

The solution of θ for $Re/S^2 \to 0$ is rather easily obtained and will be solved for, summarily. For any finite Pr , no matter how small or large, the coefficient of θ'' tends to infinity. A conduction profile may be expected throughout the region. Since the viscous limiting case considers dissipation, one might expect θ to have a parabolic variation.

II. Finite Re/S^2 .

For finite Re/S^2 and Pr , the thermal behavior ranges from fully viscous, as in Case I, to the development of thermal boundary layers as the number $S^2/(RePr)$ ranges from large to small.

III. Re/S^2 very large.

For $Pr \ll 1$, the intermediate type behavior as discussed in Case II will often prevail. Therefore, to focus on the limiting regimes, the thermal conditions will be studied for $Pr \sim O(1)$ and $Pr \gg 1$ so that both Re/S^2 and $RePr/S^2$ are very large. One immediate qualitative comparison can be drawn. The order of the thermal boundary layer thickness is less than or equal to the order of the velocity boundary layer thickness depending on, respectively, whether Pr is greater than or equal to unit order of magnitude.

Solutions will be investigated in detail only for the limiting cases:

1) Re/S^2 very small
2) Re/S^2 and $RePr/S^2$ very large $(Pr \geq O(1))$

3. Series Solutions

For small values of Re/S^2 , a series expansion for θ in powers of Re/S^2 is assumed.

$$\theta = \theta_o + \left(\frac{Re}{S^2}\right)\theta_1 + \left(\frac{Re}{S^2}\right)^2 \theta_2 + \ldots \tag{61}$$

The boundary conditions being

$$\theta_i'(0) = \theta_i(1) = 0, \quad \text{for all } i \tag{62}$$

Substitution of equation (61) and the series expansions of the velocities, equations (36), (37), into the energy equation (59) and collecting terms of like powers of Re/S^2 results in the following sequence of differential equations.

$$\frac{1}{Pr}\theta_o'' = -G_o'^2 \tag{63}$$

$$\frac{1}{Pr}\theta_1'' = -2G_o'G_1' \tag{64}$$

$$\frac{1}{Pr}\theta_2'' - 2F_o\theta_o' + 2F_o'\theta_o = -(2G_o'G_2' + G_1'^2 + F_o''^2) \tag{65}$$

$$\frac{1}{Pr}\theta_3'' - 2(F_o\theta_1' + F_1\theta_o') + 2(\theta_o F_1' + F_o'\theta_1) =$$

$$-2(G_o'G_3' + G_1'G_2' + F_o''F_1'') \tag{66}$$

Recalling that odd-subscripted F_i and G_i vanish identically, and taking into account the boundary conditions equation (62), inspection reveals that odd-subscripted θ_i vanish, also. The solutions to the equations for θ_i are obtained in a manner similar to the series solutions for the velocities. Tediousness increases rapidly with higher order terms. The solutions to θ_o and θ_2 are

$$\theta_o = \frac{Pr}{2}(1 - z^2) \tag{67}$$

where equation (67) also represents the temperature distribution for the limiting case of $Re/S^2 \to 0$, and

$$\begin{aligned}\theta_2 = {} & Pr(-0.00278z^8 + 0.02222z^7 - 0.06778z^6 + \\ & 0.10000z^5 - 0.07417z^4 + 0.02333z^3 - \\ & 0.00071z^2 - 0.00012) + Pr^2(0.00089z^8 - \\ & 0.00397z^7 + 0.00111z^6 + 0.01667z^5 - \\ & 0.02917z^4 + 0.01667z^3 - 0.00220)\end{aligned} \tag{68}$$

4. Thermal Boundary Layer Limit and Boundary Conditions

For large values of Re/S^2 , the velocity coefficients of the temperature terms and the viscous dissipation terms are given as the velocity boundary layer functions, $f(\xi)$ and $g(\xi)$, and the boundary layer co-ordinate, ξ . The temperature, therefore, will also be solved for in the ξ co-ordinate system.

Allow

$$\theta(z) = t(\xi) \tag{70}$$

Substituting equation (70) and equations (47) and (48a,b) for the co-ordinate and velocities into the energy equation (59), the result is

$$\frac{1}{Pr} t'' - 2ft' + 2f't = -[g'^2 + f''^2] \tag{71}$$

Equation (71) is the thermal boundary layer equation valid at both the disk and housing. For each of these two separate applications, the co-ordinate ξ is considered to increase positively into the fluid. Since this was the case for the velocity boundary layer problems, the functions f and g can be taken from the solution for the dynamic boundary layers.

Different sets of thermal boundary conditions will distinguish the thermal boundary layer problems at the disk and housing as described by equation (71). It has already been shown that for $Pr \geq \mathcal{O}(1)$ thermal boundary layers will exist. The physical picture then is the existence of steep temperature gradients near the disk and housing and slopes of much lesser magnitude in the rotating core where there is no dissipation. The asymptotic boundary conditions for each problem will be that t becomes constant at large ξ . It should be kept in mind that the slopes of the temperature profiles as ξ approaches infinity are not really zero but are much smaller than slopes within the boundary layers. For boundary conditions:

Disk:

$$\xi = 0: \quad t' = 0; \qquad \xi \to \infty: \quad t = K_D \tag{72a,b}$$

Housing:

$$\xi = 0: \quad t = 0; \qquad \xi \to \infty: \quad t = K_H \tag{73a,b}$$

One should also remember that K_D does not have to equal K_H . There being no heat transfer at the disk, the heat generated in the disk boundary layer must somehow be brought to the housing wall. To this end, conduction is assumed across the rotating core.

The proper difference, $K_D - K_H$, will later be reconciled with the thermal boundary layers in an overall integral balance.

A. Solution To The Disk Thermal Problem

The solution to equation (71) with boundary conditions (72a, b) should not be difficult owing to the linearity of the energy equation. The solution may be constructed according to the following theorem:

Given a linear, ordinary, second order differential equation of the type (71), (i.e. it may have non-constant coefficients, be non-homogeneous, and of the two-point boundary value type), the solution is given by

$$t = c_1 t_1 + c_2 t_2 + t_p \tag{74}$$

where t_1 and t_2 are any two linearly independent solutions to the homogeneous form of equation (71), t_p is any particular (non-homogeneous) solution, and c_1 and c_2 are chosen to satisfy the boundary conditions equations (72a, b). The functions t_p , t_1 , t_2 and their first two derivatives have been computed for Pr = 10 and 50 . Tables 4 through 9 list those functions. For each Pr , recall that it is required that the functions t_1 and t_2 are linearly independent. Linear independence of the functions may be observed by a casual inspection of the functions. More rigorously, linear independence is proven in Appendix D.

Returning now to the determination of c_1 and c_2 through satisfaction of the boundary conditions, we require the values of derivatives at $\xi = 0$ and the functions themselves as $\xi \to \infty$.

For Pr = 10:

$$t_p'(0) = 4.0000, \quad t_1'(0) = 4.0000, \quad t_2'(0) = 0 \tag{75}$$

$$t_p(\infty) = 15.679, \quad t_1(\infty) = 21.753, \quad t_2(\infty) = 5.941 \tag{76}$$

For Pr = 50:

$$t_p'(0) = 4.0000, \quad t_1'(0) = 4.0000, \quad t_2'(0) = 0 \tag{77}$$

$$t_p(\infty) = 6.760, \quad t_1(\infty) = 33.627, \quad t_2(\infty) = 13.290 \tag{78}$$

Invoking the boundary conditions (72a, b) and substituting the above values into equation (74) gives the following values for c_1 and c_2 .

For Pr = 10:

$$c_1 = -1.0 \qquad c_2 = \frac{K_D + 6.074}{5.941} \tag{79}$$

For Pr = 50:

$$c_1 = -1.0 \qquad c_2 = \frac{K_D + 26.867}{13.290} \tag{80}$$

Remaining is the task of constructing a thermal solution to the housing boundary layer for Pr = 10 and 50 for which the disk thermal solution is now known.

B. Solution To The Housing Thermal Problem

It had been expected that the relative ease with which the disk thermal solution had been obtained would be indicative of the effort required to obtain the housing thermal solution. Unfortunately, all attempts to date have failed to converge for any value of Pr . The various numerical methods employed are discussed in Appendix E.

5. Energy Matching Conditions

The solution of the housing thermal boundary layer equation will depend on the housing asymptotic temperature, K_H , in a way similar to that in which the disk solution depends on K_D through the constants c_1 and c_2 . The problem now faced is analogous to the determination of Ge in the velocity solutions, only now there are two asymptotic constants to be calculated, K_D and K_H . Two matching conditions are required. They are derived in Appendix F and are conservation of energy applied to two control volumes, one having the same surface as the volume used in the angular momentum matching condition and the other being similar in shape but whose bounding constant ξ planes are the disk and a plane sufficiently far from the disk to include only the disk velocity and thermal boundary layers.

The two energy conditions are

$$\left|g(0)g'(0)\right|_D - 4\left[\left|\int_{D\,o}^{\infty} f'td\xi\right| - \left|\int_{H\,o}^{\infty} f'td\xi\right| + \left(|f_H(\infty)| - |f_D(\infty)|\right)\frac{K_H + K_D}{2}\right] - 4\left[\left|\int_{D\,o}^{\infty}\frac{1}{2}(f'^2 + g^2)f'd\xi\right| - \left|\int_{H\,o}^{\infty}\frac{1}{2}(f'^2 + g^2)f'd\xi\right| + \frac{1}{2}\left(\left(|f_h(\infty)| - |f_D(\infty)|\right)^2 + Ge^2\right)\cdot\left(|f_H(\infty)| - |f_D(\infty)|\right)\right] - \frac{1}{Pr}\left|t_H'(0)\right| = 0 \qquad (81)$$

$$\left|g(0)g'(0)\right|_D - 4\left|\int_{D\,o}^{\infty} f'td\xi\right| - 4\left|\int_{D\,o}^{\infty}\frac{1}{2}(f'^2 + g^2)f'd\xi\right| - \left(\frac{Re}{S^2}\right)^{-1/2}\frac{1}{Pr}(K_D - K_H) = 0 \qquad (82)$$

where $\left|g(0)g'(0)\right|_D = 0.52423.$

For a given Pr , $\int_{D\,o}^{\infty} f'td\xi$ will be a linear function of K_D or K_H depending whether the integration is performed over the disk or housing boundary layer. Equations (81) and (82) then represent two linear algebraic equations for the unknowns K_D and K_H . These matching conditions are not universal in the sense that they depend on Re/S^2 and Pr .

6. Energy Solution for $Pr = 1$

A particularly simple method of solution to the temperature problem, the Reynolds analogy, exists for the special case of $Pr = 1$. Consider the momentum equations (23) and (24) multiplied by u_o and v_o, respectively, resulting in

$$u_o \frac{\partial(\frac{1}{2} u_o^2)}{\partial r} + w_1 \frac{\partial(\frac{1}{2} u_o^2)}{\partial z} - \frac{u_o v_o^2}{r} = - Eu_o \frac{\partial p_o}{\partial r} + \left(\frac{S^2}{Re}\right) u_o \frac{\partial^2 u_o}{\partial z^2} \qquad (83)$$

and

$$u_o \frac{\partial(\frac{1}{2} v_o^2)}{\partial r} + w_1 \frac{\partial(\frac{1}{2} v_o^2)}{\partial z} + \frac{u_o v_o^2}{r} = \left(\frac{S^2}{Re}\right) v_o \frac{\partial^2 v_o}{\partial z^2} \qquad (84)$$

The fluid is now treated as if it were a compressible, perfect gas with constant properties and where the specific heat at constant pressure, c_p, is approximated by the specific heat of the fluid, c. The following modified form of the energy equation (5) results.

$$u_o \frac{\partial T_o}{\partial r} + w_1 \frac{\partial T_o}{\partial z} = \frac{S^2}{RePr} \frac{\partial^2 T_o}{\partial z^2} + \frac{EcS^2}{Re}\left[\left(\frac{\partial v_o}{\partial z}\right)^2 + \left(\frac{\partial u_o}{\partial z}\right)^2\right] + Eu_o \frac{\partial p_o}{\partial r} \qquad (85)$$

Adding equations (83) - (85) and setting $Pr = 1$ gives the following differential equation.

$$u_o \frac{\partial I_o}{\partial r} + w_1 \frac{\partial I_o}{\partial z} = \frac{S^2}{Re} \frac{\partial^2 I_o}{\partial z^2} \tag{86}$$

where

$$I_o = \frac{1}{Ec} T_o + \frac{1}{2} (u_o^{\,2} + v_o^{\,2}) \tag{87}$$

Equation (86) may be transformed into an ordinary differential equation by defining

$$I_o = r^2 I(z) + \text{const.} \tag{88}$$

Because only derivatives appear in equation (86), const. may assume any value. It is assigned the value of the housing wall temperature, $\frac{1}{Ec} T_w$. Recalling the similarity transformations for u_o and w_1 , equation (86) becomes

$$\frac{S^2}{Re} I'' - 2FI' + 2F'I = 0 \tag{89}$$

Equation (89) for $I(z)$ is identical to equation (32) for $G(z)$. To compare boundary conditions between $I(z)$ and $G(z)$ proceed as follows:

$$\text{at} \quad z = 1: \qquad I(1) = \frac{1}{r^2} \left(I_o(1) - \frac{T_w}{Ec} \right) \tag{90}$$

Substituting equation (87) into equation (90) and noting that

$T_o(1) = T_w$ and $u_o(r,1) = v_o(r,1) = 0$, we have

$$I(1) = 0 = G(1) \tag{91}$$

$$\text{at } z = 0: \qquad I'(0) = \frac{1}{r^2} I_o'(0) \tag{92}$$

Substituting equation (87) into equation (92) and noting that $T_o'(0) = 0$, $u_o(r,0) = 0$, and $v_o v_o' = G(0)G'(0)/r^2 = G'(0)/r^2$, we have

$$I'(0) = G'(0) \tag{93}$$

Having shown that $I(z)$ and $G(z)$ satisfy the same differential equation and boundary conditions, they are identically the same function.

$$I(z) = G(z) \quad , \qquad \text{for } Pr = 1 \tag{94}$$

An easy check can be carried out for the limiting case of $Re/S^2 \to 0$. From equation (58),

$$T_o(r,z) = (r^2\theta(z) + T_w)Ec$$

Therefore,

$$\theta = I - \frac{G^2}{2} = G - \frac{G^2}{2} \tag{95}$$

For $Re/S^2 = 0$, $G = (1 - z)$, and

$$\theta = \frac{1}{2}(1 - z^2) \tag{96}$$

which checks with equation (67) when $Pr = 1$ and $\theta = \theta_o$.

For large values of Re/S^2 (and $Pr = 1$), equation (89) remains a valid equation. The solution $I(z)$ therefore depends only on the availability of $G(z)$ which is known only up to $Re/S^2 = 5.70$. Equation (48b), however, gives $G(z) = g(\xi)$, so that in boundary layer co-ordinates

$$I(\xi) = g(\xi), \qquad Re/S^2 >> 1, \qquad Pr = 1 \qquad (97)$$

7. Asymptotic Solutions

Difficulties in obtaining a solution to the housing thermal boundary layer problem prompted an investigation of the asymptotic behavior of the boundary layer functions. Rogers and Lance [9] have derived the asymptotic solution for the velocities. Appendix G extends the analysis to the energy asymptotic solution. Equation (G12) gives the form of $g(\xi)$ for large ξ . The other boundary layer function has this same structure. The distinction between the housing and disk boundary layer problem lies in the sign of K. For the disk, the axial velocity is toward the disk, in the direction of decreasing ξ , hence K is negative. Conversely, for the housing, K is positive. The effect then is that, while the solutions are of a damped oscillatory nature for both disk and housing, the disk's oscillations have a longer period and are more heavily damped. This comparative behavior is verified by the solutions presented in Figures 7 and 8. Notice that for the Karman problem (Ge = 0) the asymptotic forms reduce to exponential decay and the oscillations are lost.

CHAPTER V

DESIGN CONSIDERATIONS

Design considerations for the slinger seal are of both a dynamic and a thermal nature. Four items have been identified as being of basic importance to the design, and whose dependence may be quantitatively specified as functions of the two parameters Re/S^2 and Pr. Design evaluations must take into account the presence of fluid on both sides of the disk. This is possible since the foregoing analysis applies to the fluid on either side. The Prandtl number Pr, depending only on fluid properties, is the same throughout, while Re/S^2, depending on geometry as well, may in general be different for the two sides.

1. Torque Requirements

The torque acting on one side of the disk is given by:

$$\text{torque} = \int_{r_o}^{R} \mu \frac{\partial \bar{v}}{\partial \bar{z}} \, 2\pi \, \bar{r}^2 \, d\bar{r}$$

where $\bar{r}_o$ is the radial distance from the axis of rotation of the interface and R is the disk radius. In non-dimensional variables:

$$\text{torque} = \frac{\pi \mu \omega}{2a} G'(0)(R^4 - \bar{r}_o^4) \tag{98}$$

where

$$G'(0) = -1 - 0.0042(Re/S^2)^2, \quad \text{for } 0 < \frac{Re}{S^2} < 5.7 \tag{99}$$

and

$$G'(0) = g_o'(0)\ (Re/S^2)^{1/2} = -0.52423\ (Re/S^2)^{1/2}, \text{ for large } Re/S^2 \tag{100}$$

Equation (99) is obtained from the series solution of G(z) and equation (100) is the boundary layer limit for large Re/S^2. For moderate values of Re/S^2 exact solutions are not known, however, data from the analog computer solution (see Table 10) has been used to fill in this range. The analog data seems to fit in well with the series and boundary layer solutions (see Figure 11), and G'(0) vs. Re/S^2 appears to be monotonically increasing.

2. Pressure Capabilities

The radial pressure gradient will determine the pressure difference that the seal can maintain. It has been shown that the radial pressure gradient is the dominating one for both boundary layer and viscous limiting flows and varies with r^2 according to equation (33).

Integrating equation (33) from any radial point to the outer edge of the disk yields, in dimensional form,

$$\bar{p} - \bar{p}_R = -\frac{\rho\omega^2}{2} B(\bar{r}^2 - R^2) \tag{101}$$

A similar formula is obtained from either side of the disk. Evaluating each of them at their respective interfaces and subtracting one from the other to eliminate $\bar{p}_R$, gives

$$\bar{p}_a - \bar{p}_b = -\frac{\rho\omega^2}{2}\left[B_a\,(\bar{r}_a^2 - R^2) - B_b(\bar{r}_b - R^2)\right] \tag{102}$$

Subscripts a and b denote opposite sides of the disk. Notice that B, being a function of Re/S^2, may differ on either side of the disk as Re/S^2 may differ, which has already been shown.

In practice, when large pressure differences occur, it would be an attractive feature to be able to control the interface radial distance difference $\bar{r}_a - \bar{r}_b$ without having to adjust the rotational speed of the disk. Assume that for a given $\bar{p}_a - \bar{p}_b$ it is desired to keep $\bar{r}_a = \bar{r}_b$. From equation (102) then,

$$B_a - B_b = \frac{\bar{p}_a - \bar{p}_b}{\frac{\rho\omega^2}{2}(R^2 - \bar{r}_a^2)} \tag{103}$$

For large Re/S^2 the interface position control is lost because both B_a and B_b go to the same values, namely the boundary layer asymptote where $B = Ge^2 = 0.098$. From the point of view of interface positioning, Re/S^2 should be small, but it should not be very close to zero because there the slope of B vs. Re/S^2 (see Figure 12) is small and the controlling of B is not sensitive to changes in Re/S^2.

In Figure 12 the intermediate ranges of Re/S^2 have been provided by the analog solutions. B is a monotonically decreasing function of Re/S^2. For greater pressure capabilities small Re/S^2 is recommended which will give the largest B.

Torque and pressure considerations are consistent in that both recommend operating at as low a Re/S^2 as possible. If interface position control is desired, low to moderate ranges of Re/S^2 are suggested.

3. Disk Temperature

The disk temperature is $\bar{T}(\bar{r},o)$ which in non-dimensional variables is given by

$$\bar{T}(\bar{r},o) - T_w = \frac{\omega^2}{c} \bar{r}^2 \theta(0) \tag{104}$$

For small Re/S^2 the series solution is valid and is

$$\theta(0) = Pr\left[\frac{1}{2} - \left(\frac{Re}{S^2}\right)^2 (0.00012 + 0.00220\ Pr)\right]$$

For large Re/S^2 equation (70) is

$$\theta(0) = t_D(0)$$

Unfortunately, the thermal boundary layer is not determinable because of failure to get a solution at the housing wall. For the cases of Pr = 10,50, however, for which families of solutions at the disk were obtained depending on the value of K_D, a range of K_D may be established. K_D must be positive which places a lower limit on it. An upper limit is derived by stipulating that K_D be less than the disk temperature. Consider the case of Pr = 10. From equations (74) and (79)

$$t_D(0) = - t_1(0) + \frac{K_D + 6.074}{5.941} t_2(0) + t_p(0), \quad Pr = 10 \qquad (106)$$

where $t_1(0) = 1 \qquad t_2(0) = 1 \qquad t_p(0) = 1$

For the upper limit of $K_D < t_D(0)$, equation (106) may be solved for K_D giving

$$0 < K_D < 1.23 \qquad Pr = 10 \qquad (107)$$

A similar calculation for Pr = 50 yields

$$t_D(0) = - t_1(0) + \frac{K_D + 26.867}{13.290} t_2(0) + t_p(0) \qquad (108)$$

substituting $K_D < t_D(0)$ results in

$$0 < K_D < 2.18 \qquad Pr = 50 \qquad (109)$$

For the ranges of K_D established in equations (107) and (109) the corresponding ranges of $t_D(0)$ may be easily calculated.

$$1.02 < t_D(0) < 1.23 \qquad Pr = 10,\ \text{large } Re/S^2 \qquad (110)$$

$$2.02 < t_D(0) < 2.18 \qquad Pr = 50,\ \text{large } Re/S^2 \qquad (111)$$

The trend here is of rising disk temperature with an increase in Pr.

4. Heat Transfer at Housing Wall

Heat transfer at the housing wall is given by

$$\text{heat transfer} = \int_{\bar{r}_o}^{R} k \frac{\partial \bar{T}(\bar{r},1)}{\partial \bar{z}} \, 2\pi \, \bar{r} d\bar{r}$$

which in non-dimensional variables assumes the form

$$\text{heat transfer} = \frac{\pi k \, \omega^2}{2ac} (R^4 - \bar{r}_o^4)\theta'(1) \tag{112}$$

For small values of Re/S^2 the series solution for $\theta'(1)$ is valid:

$$\theta'(1) = - Pr \left[1 - \left(\frac{Re}{S^2}\right)^2 (- 0.00148 + 0.00267 \, Pr) \right]$$

When $Re/S^2 = 0$, heat transfer varies with the first power of Pr, as is the case in Couette flow.

Practical considerations would suggest limiting the disk temperature and heat transfer demands, both of which may be accomplished by selecting a fluid with small Pr.

For the case of Pr = 1 the heat transfer is also known. Begininning with equation (90), substituting backward, differentiating with respect to z and evaluating at z = 1 where u = v = 0, we have

$$\theta'(1) = I'(1) \tag{114}$$

From the result of the Reynolds' analogy, equation (94):

$$\theta'(1) = G'(1) \tag{115}$$

For small Re/S^2, the series solution for $G(z)$ gives

$$\theta'(1) = G'(1) = -1 + (Re/S^2)^2 (2/1575) \tag{116}$$

and for large Re/S^2 the boundary layer velocity transformation gives

$$\theta'(1) = G'(1) = -g'_H(0) = -0.13567 \tag{117}$$

Substitution of equation (116) or (117) into equation (112) gives the complete heat transfer picture for $Pr = 1$.

5. Remarks

Although thermal considerations are unanimous in specifying a fluid with small Pr, and dynamically a small or moderately small Re/S^2 is recommended, fluids with small Pr, such as liquid metals, often have an extremely small viscosity. It is this property which presents difficulty in maintaining a low value of Re/S^2 even though the gap width, a, may be small. At this point, the practical importance of dynamic and thermal effects must be weighed against each other.

REFERENCES

1. Batchelor, G. K., "Note on a Class of Solutions of the Navier Stokes Equations Representing Steady Rotationally-Symmetric Flow," Quarterly Journal of Mechanics and Applied Mathematics, Vol. 4, 1951.

2. Grohne, D., "Uber die laminare Stromung in einer Kreiszylindrischen Dose mit rotierendem Deckel," Nachr. Akad. Wiss. Gottingen, math.-phys. Kl., IIa, See also Dorfman, L. A., "Hydrodynamic Resistance and the Heat Loss of Rotating Solids", Oliver and Boyd, publisher, 1963.

3. Soo, S. L., "Laminar Flow Over an Enclosed Rotating Disk", Trans. American Society Mechanical Engineers, Vol. 80.

4. Stewartson, K., "On the Flow Between Two Rotating Coaxial Disks," Proc. Cambridge Phil. Soc., Vol. 30, 1953.

5. Schultz-Grunow, F., "Der Reibungswiderstand rotierender Scheiben in Gehausen," Zeitschrift angew. Math. Mech., Vol. 15, 1935.

6. Millsaps, K. and K. Pohlhausen, "Heat Transfer by Laminar Flow from a Rotating Plate," Journal Aeronautical Sciences, Vol. 19, 1952.

7. Karman, Th. von, "Uber laminare und turbulente Reibung," Zeitschrift angew. Math. Mech., Vol. 1, 1921.

8. Bodewadt, U. T., "Die Drehstromung uber festem Grunde," Zeitschrift angew. Math. Mech., Vol. 20, 1940.

9. Rogers, M. H. and Lance, G. N., "The Rotationally Symmetric Flow of a Viscous Fluid in the Presence of an Infinite Rotating Disk," Journal of Fluid Mechanics, Vol. 7, April, 1960.

10. Fox, L., "Numerical Solution of Ordinary and Partial Differential Equations," Oxford, New York, Pergamon Press, 1962.

11. Nachtsheim, P. R. and P. Swigert, "Satisfaction of Asymptotic Boundary Conditions in Numerical Solutions of Systems of Non-Linear Equations of Boundary Layer Type," National Aeronautics and Space Administration, TN D-3004.

REFERENCES (Cont.)

12. Varga, R. S., Matrix Iterative Analysis, Englewood Cliffs, New Jersey, Prentice-Hall, 1962.

13. Schlichting, H., Boundary Layer Theory, 4th Edition, New York, McGraw-Hill, 1960.

14. Loper, D. E., "An Analysis of Confined Magnetohydrodynamic Vortex Flows," Case Institute of Technology, Plasma Research Technical Report No. A-37, Cleveland, 1965.

15. Lance G. N. and Rogers, M. H., The axially symmetric flow of a viscous fluid between two infinite rotating disks. Proc. Roy. Soc., London, A 266, 1962, 109-121

16. Grohne, D., "Uber die laminare Stromung in einer Kreiszylindrichen Dose mit rotierendem Deckel. Nachr. Akad. Wiss. Gottingen 12, 265, 1955.

17. Ostrach, S. and Thornton D. R., Commpressible Laminar Flow and Heat Transfer about a Rotating Isothermal Disk. NACA Tn 4320, August, 1958.

APPENDIX A

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

This appendix discusses the methods of numerical solutions to the 2 -point boundary value problem, equations (31), (32) and associated boundary conditions, and the method of solution to equations (51) and (52) where the associated boundary conditions are asymptotically specified. The 2 -point method will be discussed first, as the asymptotic method is an extension of the 2-point technique.

The 2 -point problem was solved by forward integration using the Runge-Kutta formula. Propagation of the solution is begun by specifying additional conditions at $z = 0$ equal to the number of boundary conditions at $z = 1$. These are in addition to the given initial conditions at $z = 0$. They must be guessed and are known to be correct when the solutions obtained satisfy the proper boundary conditions at $z = 1$. In general, it will be difficult to guess the proper initial condition and extremely difficult if several initial conditions are required. Fox [10] discusses a scheme for correcting the guessed initial conditions by the Newton-Raphson method, which is briefly reviewed here.

The values of the solutions at $z = 1$ obtained by the numerical integration formulas are envisioned as being functions of the required guessed initial conditions. In terms of equations (31) and (32) and their boundary conditions:

$$F'(1) = \text{fcn.}(F''(0),\ G'(0),\ B)$$

$$G(1) = \text{fcn.}(F''(0),\ G'(0),\ B)$$

$$F(1) = \text{fcn.}(F''(0),\ G'(0),\ B)$$

Deviation of F"(0), G'(0), and B from their correct values will result, when using the Runge-Kutta formula, in deviations of F'(1), G(1), F(1) from their correct values. Neglecting higher order terms:

$$\underline{F}'(1) - F'(1) = \frac{\partial F'(1)}{\partial F''(0)}(\underline{F}''(0) - F''(0)) + \frac{\partial F'(1)}{\partial G'(0)}(\underline{G}'(0) - G'(0)) + \frac{\partial F'(1)}{\partial B}(\underline{B} - B)$$

$$\underline{G}(1) - G(1) = \frac{\partial G(1)}{\partial F''(0)}(\underline{F}''(0) - F''(0)) + \ldots$$

$$\underline{F}(1) - F(1) = \frac{\partial F(1)}{\partial F''(0)}(\underline{F}''(0) - F''(0)) + \ldots$$

The underscored variables represent incorrect quantities. If values for the partials can be obtained, solutions to the above linear, algebraic equations give recommended changes for the guessed initial values, F"(0), G'(0), and B. The partials are obtained by differentiating the differential equations with respect to each initial condition, solving by forward integration, and evaluating at $z = 1$. There will be as many additional sets of differential equations for the partials as there are initial conditions to be adjusted.

The numerical solution to the boundary layer equations (51) and (52) was applied according to the method of Nachtsheim and Swigert [11]. This method tends to reduce the problem with asymptotic boundary conditions to a two-point boundary value problem. In addition to satisfying the end-point conditions (the end point selected at first at random) the condition that the slopes of the functions vanish at the end point is also imposed. The condition on the slopes presents more equations for adjusting initial guesses than there are initial guesses. A least square satisfaction of all the conditions (slopes and endpoints) is thus required. When initial values have changed an arbitrarily set amount from the starting guesses, assuming convergence, the end-point is moved outward. The process is repeated until the boundary conditions at the furthest calculated end-point agree with the specified asymptotic boundary conditions to within an arbitrarily set error of acceptance.

Neglecting higher order terms in the correction scheme equations is essentially a linearization. This is a good approximation only if the first guessed initial values are sufficiently close to their correct values. For the 2 -point boundary value problem, this requirement became exceedingly stringent as Re/S^2 increased, until for $Re/S^2 = 5.7$, convergence required starting guesses correct to six significant figures. Considering that the digital computer on which calculations were performed provided only for six significant places, the initial value correction scheme became useless for larger Re/S^2.

APPENDIX B

ANGULAR MOMENTUM INTEGRAL CONDITION

Consider the toroidal control volume as described in the text of the report. Let R_1 and R_2 be, respectively, the outer and inner radii of the cylindrical surfaces of the control volume

Apply the angular momentum theorem to the control volume; namely, the sum of torques on the control surfaces equals the time rate of change of angular momentum throughout the control volume

Surface torques will be considered first. The shear forces on the surfaces $\bar{r} = R_1$ and $\bar{r} = R_2$ are neglected in comparison to the shear forces on the disk and housing forces.

The total angular torque on the housing face is given by

$$M_H = \int_{\bar{r} = R_2}^{\bar{r} = R_1} 2 \pi \mu \frac{\partial \bar{v}}{\partial \bar{z}} r^{-2} \, d\bar{r}$$, evaluated at the housing.

Successive algebraic substitutions give

$$M_H = \frac{2 \pi \mu R^4 \omega G'(1)}{a} \int_{R_2/R}^{R_1/R} r^3 \, dr = \frac{\pi \mu \omega G'(1)}{2a} (R_1^{\,4} - R_2^{\,4}) \qquad \text{(B-1)}$$

Similarly, for the disk surface:

$$M_D = \frac{\pi \mu \omega G'(0)}{2a} (R_1^{\,4} - R_2^{\,4}) \qquad \text{(B-2)}$$

There can be no momentum flux across the control surfaces adjacent to the disk or housing.

The problem is steady state, and the rate of change of momentum throughout the control volume is given by the momentum flux across the surfaces of constant radius. Caution should be exercised to account for all angular momentum transported across these surfaces. For all housing and disk boundary layer solutions with a common value of Ge, it has been observed that the values of $f_D(\infty)$ and $f_H(\infty)$ generally do not match. From a continuity viewpoint, this means that there must be some radial flow in the core. This presents a question of the validity of the boundary condition in the housing and disk boundary layer problems, viz., as $\xi \to \infty$ $f'(\infty) = 0$, whereas it has just been shown that there may be a non-zero core radial velocity. The axial velocities which give rise to the core radial velocity are of the order of magnitude of S^2/Re. The resulting core radial velocity is then justifiably neglected in comparison to the radial velocity in the boundary layer which is nominally of unit order.

When considering momentum flux, however, the integrated value of the core radial velocity across the core width (approximately the gap width, a) may contribute significantly and must be retained in the momentum match.

Momentum flux across the surface $\bar{r} = R_2$ is given by

$$M_{R_2} = \int_o^a \rho\, \bar{\mu}\, 2\pi\, R^2{}_2\, \bar{v}\, d\bar{z}$$

$$= \rho\, \omega^2\, 2\, \pi a\, R_2{}^4 \int_o^1 F'(z)\, G(z) dz \qquad \text{(B-3)}$$

Similarly, for the surface $\bar{r} = R_1$:

$$M_{R_1} = \rho\omega^2\, 2\, \pi a\, R_1{}^4 \int_o^1 F'(z)\, G(z) dz \qquad \text{(B-4)}$$

Substitution of these quantities into the angular momentum law, cancelling $R_1{}^4 - R_2{}^4$, and recalling that $Re/S^2 = \rho\omega a^2/\mu$:

$$G'(1) - G'(0) = -4\, \frac{Re}{S^2} \int_o^1 F'(z)\, G(z) dz \qquad \text{(B-5)}$$

Change now to boundary layer co-ordinates by means of equations (47-48 a,b). The change of co-ordinates changes the integration by

$$\int_o^1 dz = \underbrace{\int_o^\infty d\xi}_{\text{disk}} + \underbrace{\int_\infty^o d\xi}_{\text{housing}} + \underbrace{\int d\xi}_{\text{core width}}$$

where

$$\underbrace{\int f' g d\xi}_{\text{core width}} = Ge \underbrace{\int f' d\xi}_{\text{core width}} = \left(f_H(\infty) - f_D(\infty)\right) Ge$$

Equation (C-5) in boundary layer co-ordinates is:

$$g'_D(0) - g'_H(0) = 4\left[\int_D^{\infty} f'g\, d\xi - \int_H^{\infty} f'g d\xi + (f_H(\infty) - f_D(\infty))\ Ge\right] \tag{B-6}$$

The physical significance of each term in equation (C-6) will be utilized as a check against sign errors. $g_D'(0)$ and $g_H'(0)$ represent torques acting on the control volumes two parallel, flat, ends. Since the senses of these torques are opposite one another, the left hand side of (B-6) should be the difference of these two terms if the absolute value for each is assumed.

In the integration terms, f' represents the radial velocity component, which is directly proportional to r (see equation (29). Near the disk, the radial velocity is entirely upwards while near the housing, the bulk of the radial velocity is downward. The two terms represent opposite effects of transferring momentum out and into the control volume, respectively. Thus, on the right hand side of equation (C-6) should appear the difference of these terms if the absolute value of each is assumed.

Since the core radial velocity represents a net outflow of angular momentum if $|f_H(\infty)|$ is greater than $|f_D(\infty)|$, and a net gain if the inequality is reversed, the right hand side of equation (B-6) will consistently represent a momentum loss if equation (B-6) is written

$$|g'_D(0)| - |g'_H(0)| = 4\left[\left|\int_D^\infty f'g\ d\xi\right| - \left|\int_H^\infty f'g\ d\xi\right| + \left(|f_H(\infty)| - |f_D(\infty)|\right) Ge\right] \qquad \text{(B-7)}$$

APPENDIX C

CALCULATION OF THE AXIAL VELOCITY MATCH AND MOMENTUM INTEGRAL CONDITION

A. Axial Velocity Match

Ge	$f_D(\infty)$	$f_H(\infty)$
0.29	.39210	.37157
0.31	.38213	.38419
0.33	.37174	.39641

B. Momentum Integral Condition

Ge	0.29	0.31	0.33
$g'_D(0)$	.53529	.52591	.51612
$g'_H(0)$	.12070	.13341	.14653
$\int_0^\infty f'\,g\,d\xi$ (D)	.19112	.19120	.19089
$\int_0^\infty f'g\,d\xi$ (H)	.08524	.09422	.10430
$f_D(\infty)$	.39210	.38213	.37174
$f_H(\infty)$	.37157	.38419	.39641
$\left[f_H(\infty) - f_D(\infty)\right]Ge$	-.00595	.00064	.00814
equation (57)	4.1488	4.0201	3.8680

APPENDIX D

LINEAR INDEPENDENCE OF THE HOMOGENEOUS TEMPERATURE FUNCTIONS

From the theory of linear differential equations, it is a well known condition that two homogeneous solutions of a second order equation are linearly independent if the following determinant (known as the Wronskian) is non-zero.

$$\begin{vmatrix} t_1(0) & t_2(0) \\ t_1'(0) & t_2'(0) \end{vmatrix} \neq 0 \qquad \text{(D-1)}$$

For the disk thermal boundary layer solutions for both Pr = 10 and 50, the initial conditions chosen for the homogeneous solutions were

$$t_1(0) = 1.0 \qquad t_2(0) = 1.0$$

$$t_1'(0) = 4.0 \qquad t_2'(0) = 0.0$$

The condition on the Wronskian is obviously fulfilled and the homogeneous solutions are therefore linearly independent.

APPENDIX E

UNSUCCESSFUL ATTEMPTS AT SOLVING THE HOUSING BOUNDARY LAYER ENERGY EQUATION

A review is given here of the numerical procedures (all of which have failed to converge) employed in the solution of the energy equation at the housing. The five methods attempted may be classified into one of two groups: forward integration methods and finite difference methods.

Characteristic of the forward integration methods is that the solution is propagated from the beginning of the interval. Using the superposition of homogeneous and particular solutions because the energy equation is linear, it is unnecessary to use any correction technique on the initial slopes as described in Appendix A. Runge-Kutta and Milne (Predictor-Corrector) schemes produced diverging solutions in spite of the asymptotic solutions (Appendix G) which indicate damped oscillations about a constant for large ξ. Imposing the requirement that $t'(\infty) \to 0$, and correcting initial values accordingly was not sufficient to cause convergence. The effect was to indicate virtually no change in initial slopes even though the condition $t'(\infty) = 0$ was far from being satisfied.

The finite difference formulation of the problem represents derivatives by central difference formulas. To within a truncation error of d^2, where d is the step length of the divisions of the range of ξ, equation (71) becomes

$$\left(\frac{1}{Prd^2} + \frac{f_n}{d}\right) t_{n-1} + \left(- \frac{2}{Prd^2} + 2\ f'_n\right) t_n + \left(\frac{1}{Prd^2} + \frac{f_n}{d}\right) t_{n-1} =$$

$$- {g'_n}^2 - {f''_n}^2 \qquad \text{(E-1)}$$

where the index n represents any subdivision in the interval.

Two alternative procedures may be applied to equation (E-1).

1) Iteration of the function $t(\xi)$: A reasonable form for $t(\xi)$ is assumed. t_n in equation (E-1) is solved for using the last known values of t_{n-1} and t_{n+1}. This procedure is repeatedly carried out over the range of ξ until it is clear that $t(\xi)$ converges to some function. In practice, this iterative procedure did not converge.

2) Linear System of equations: Equation (E-1) is applied to every internal step point of the interval. Resulting is a linear, algebraic set of equations for the unknown vector t_n. The known constant vector is the dissipation term and the matrix of coefficients evolves from the coefficients of t_{n-1}, t_n, and t_{n+1}. Although this system of equations may be of fairly large order, an exact solution is available (Varga [12]) due to the tri-diagonal nature of the matrix of coefficients. This methods converges if the matrix is diagonally dominant. For finite values of d, (which must be) the matrix is not diagonally dominant. This fact does not rule out the possibility of the solution, however, the method

did not converge.

APPENDIX F

STEADY STATE ENERGY INTEGRAL CONDITIONS FOR CONSTANT DENSITY FLUID

The two thermal matching conditions are derived herein.

Consider first the toroidal control volume defined for the angular momentum integral condition derivation. In steady state, the conservation of energy theorem states that the work rate done on the fluid boundaries by environmental forces equals the sum of internal and kinetic energy transported out of the control volume by fluid motion and the thermal energy conducted out of the control volume.

No normal forces contribute power as no boundary deflects normal to itself. Only the shear at the moving disk boundary contributes and is given by

$$\text{Work rate} = \int_{\bar{r}=R_2}^{\bar{r}=R_1} \mu \frac{\partial \bar{v}(0)}{\partial \bar{z}} \bar{v}(0)\, 2\pi \bar{r} d\bar{r} \qquad \text{(F-1)}$$

where $\bar{v}(0)$ is the disk velocity.

Successive substitutions into the dimensionless form result in

$$\text{Work rate} = \frac{2\pi \mu \omega^2}{4a} G(0)G'(0) \; (R_1^{\,4} - R_2^{\,4}) \qquad \text{(F-2)}$$

Consider now internal energy transported across control surfaces. This occurs only at the surface of constant radius as there are no normal velocity components at the solid boundaries.

$$\text{Internal energy transport} = 2\pi\rho c \left[R_2 \int_{\bar{z}=0}^{\bar{z}=a} \bar{u}\,\bar{T}\,d\bar{z}\Big|_{\bar{r}=R_2} \right.$$

$$\left. - R_1 \int_{\bar{z}=0}^{\bar{z}=a} \bar{u}\,\bar{T}\,d\bar{z}\Big|_{\bar{r}=R_1} \right] \qquad \text{(F-3)}$$

Successive substitutions into dimensionless form result in

$$\text{Internal energy transport} = 2\pi\rho a\,\omega^3 \int_{z=0}^{z=1} F'\,\theta\,dz\,(R_1^{\,4} - R_2^{\,4})$$

$$\text{(F-4)}$$

The dimensional form of the kinetic energy transport is

$$\text{K.E. transport} = \frac{1}{2}\rho\,2\pi \left[R_2 \int_{\bar{z}=0}^{\bar{z}=a} (\bar{u}^2 + \bar{v}^2 + \bar{w}^2)\bar{u}\,d\bar{z}\Big|_{R_2} \right.$$

$$\left. - R_1 \int_{\bar{z}=0}^{\bar{z}=a} (\bar{u}^2 + \bar{v}^2 + \bar{w}^2)\bar{u}\,dz\Big|_{R_1} \right] \qquad \text{(F-5)}$$

Successive substitution into dimensionless variables, and neglecting the axial velocity component for large S, yields

$$\text{K.E. transport} = \rho\omega^3\pi\, a(R_2^{\,4} - R_1^{\,4})\int_0^1 (F'^2 + G^2)F'\, dz$$

(F-6)

Thermal conduction at the disk is zero. Temperature gradients at the cylindrical surfaces of constant radius are neglected in comparison to gradients in the thermal boundary layer. The only conduction, therefore, is at the housing wall.

$$\text{Conduction} = \int_{\bar{r}=R_2}^{\bar{r}=R_1} k\, \frac{\partial\bar{T}(\bar{r},1)}{\partial\bar{z}}\, 2\,\pi\,\bar{r}\, d\bar{r} \qquad \text{(F-7)}$$

which, when put into dimensionless form becomes

$$\text{Conduction} = \frac{2\pi k}{4ac}\,\theta'(1)\,(R_1^{\,4} - R_2^{\,4}) \qquad \text{(F-8)}$$

Substitution of the work rate, energy transport, kinetic energy transport and conduction expressions into the conservation of energy theorem gives

$$G(0)G'(0) + 4\,\frac{Re}{S^2}\int_0^1 F'\,\theta\, dz + 4\,\frac{Re}{S^2}\int_0^1 \frac{1}{2}\,(F'^2 + G^2)F'\, dz$$

$$- \frac{1}{Pr}\,\theta'(1) = 0$$

Changing to boundary layer co-ordinates and recalling that

$$\int_0^1 dz = \int_{D0}^{\infty} d\xi + \int_{H^{\infty}}^{0} d\xi + \int_{\substack{\text{core}\\ \text{width}}} d\xi$$

the energy balance is:

$$|g(0)g'(0)|_D - 4\left[\left|\int_{D^0}^{\infty} f'td\xi\right| - \left|\int_{H^0}^{\infty} f'td\xi\right|\right.$$

$$\left. + \quad |f_H(\infty)| - |f_D(\infty)| \quad \frac{K_H + K_D}{2}\right] - 4\left[\left|\int_{D^0}^{\infty} \frac{1}{2}(f'^2 + g^2)f'd\xi\right.\right.$$

$$- \left|\int_{H^0}^{\infty} \frac{1}{2}(f'^2 + g^2)f'\ d\xi\right|$$

$$\left. + \frac{1}{2}\left\{\left(|f_H(\infty)| - |f_D(\infty)|\right)^2 + Ge^2\right\}\left(|f_H(\infty)| - |f_D(\infty)|\right)\right]$$

$$- \frac{1}{Pr}\,|t'_H(0)| = 0 \qquad \text{(F-10)}$$

where in the core the temperature has assumed an average value based on K_H and K_D and a mean radial velocity based on the difference of the axial velocities.

Absolute value signs have been inserted so that the algebraic sign preceding each term, can directly be associated with the term's physical significance. The convention adopted is that energies contributing to the control volume are positive.

The second thermal matching condition is derived by considering a control volume whose "z" width includes only the disk's velocity and thermal boundary layer thicknesses.

The shear work input at the disk boundary is the same as in the previous energy balance. Similarly, the internal and kinetic energy transported across the constant radius surfaces is the same as previously except the range of integration includes only the disk boundary layer thicknesses.

Energy conducted out of the constant z surface may be inferred from the constant conduction model of the core.

In boundary layer co-ordinates the second thermal matching condition is

$$|g(0)g'(0)|_D - 4\left|\int_{D^o}^{\infty} f'td\xi\right| - 4\left|\int_{D^o}^{\infty} \frac{1}{2}(f'^2 + g^2)f'd\xi\right| - \left(\frac{Re}{S^2}\right)^{-\frac{1}{2}} \frac{1}{Pr}(K_D - K_H) = 0 \qquad \text{(F-11)}$$

APPENDIX G

ASYMPTOTIC SOLUTIONS OF THE BOUNDARY LAYER EQUATIONS

The difficulty in obtaining a solution to equation (71) prompted an investigation of the asymptotic solutions (large ξ) of the momentum and energy equations. The uncoupling of these two sets of equations allows a solution of the momentum equations independent of the energy relation. Required is the solution to

$$f''' - 2f''f + f'^2 - (g^2 - Ge^2) = 0$$

$$g'' - 2g'f + 2gf' = 0 \tag{G-1}$$

for large ξ .

Let

$$f = K + \bar{f}$$

$$g = Ge + \bar{g} \tag{G-2}$$

where $k >> \bar{f}$ and $Ge >> \bar{g}$

K and Ge are the asymptotic values of f' and g respectively. It is required that as $\xi \to \infty$, $\bar{f}$ and $\bar{g}$ vanish.

Substitution of equations (G-2) into equations (G-1) and neglecting higher order terms results in the linear equations

$$\bar{h}'' - 2K\bar{h}' - 2\,Ge\,\bar{g} = 0$$
$$\bar{g}'' - 2K\bar{g}' + 2\,Ge\,\bar{h} = 0 \qquad \text{(G-3)}$$

where $\bar{f}'$ has been replaced by $\bar{h}$ because only derivatives of $\bar{f}$ appear.

The approach to the solution of equations (G-3) is facilitated by decomposing the system into four first order equations. Define additional variables $\bar{g}' = \bar{k}$ and $\bar{h}' = \bar{j}$. The barred notation is hereafter dropped, and equations (G-3) are written as

$$\begin{aligned} g' &= & & & & k \\ h' &= & & & j & \\ j' &= 2Geg & & & +2Kj & \\ k' &= & -2Geh & & & +2Kk \end{aligned}$$

Solutions are sought in the form of

$$g = C_g e^{\lambda\xi} \quad h = C_h e^{\lambda\xi} \quad j = C_j e^{\lambda\xi} \quad k = C_k e^{\lambda\xi}$$

Where the C's and λ's are constants. Substitution of these assumed forms of the solution into equations (G-4), and cancelling the exponentials, results in a linear set of homogeneous equations for the constants C_g, C_h, C_j, C_k.

In order for the solutions of the C's to be non-trivial, the determinant of the matrix of coefficients must vanish. This condition results in an algebraic, fourth degree equation for λ.

The four possibilities for λ indicates that for g, h, j, and k, there are four solutions for each. The linearity of the equations allows that the sum of these four solutions is also a solution. The fourth order equation is

$$[\lambda(2K - \lambda)]^2 = - 4Ge^2 \tag{G-5}$$

The form of equation (G-5) must admit complex roots for λ. Each of the four λ's will involve the positive square root of a complex number, either

$$\pm\sqrt{4K^2 - i8Ge} \qquad \text{or} \qquad \pm\sqrt{4K^2 + i8Ge}$$

where $i = \sqrt{-1}$.

Since there are two positive square roots of a complex number, there will result eight λ's; but only four of them are distinct. The four values may be compactly represented by

$$\lambda_{1,2,3,4} = K \pm \frac{1}{\sqrt{2}}\left[\sqrt{(K^4 + 4Ge^2)^{1/2} + K^2} \pm i\sqrt{(K^4 + 4Ge^2)^{1/2} - K^2}\right] \tag{G-6}$$

The function g is the sum of the four functions

$$g = C_{g1}e^{\lambda_1\xi} + C_{g2}e^{\lambda_2\xi} + C_{g3}e^{\lambda_3\xi} + C_{g4}e^{\lambda_4\xi} \tag{G-7}$$

It is required that as $\xi \to \infty$, $g \to 0$. This can only be satisfied if the λ's in the exponentials real parts are negative. For the moment, the negative restriction will be imposed on <u>both</u> the real and imaginary parts of λ.

Of the four λ's, the only one satisfying this condition is

$$\lambda = K - \frac{1}{\sqrt{2}}\left[\sqrt{(K^4 + 4Ge^2)^{1/2} + K^2} + i\sqrt{(K^4 + 4Ge^2)^{1/2} - K^2}\right] \tag{G-8}$$

If this is designated λ_1, then C_{g2}, C_{g3}, and C_{g4} must be chosen to be zero. In the homogeneous set of linear equations for the constants, C_h may be solved for in terms of C_g.

$$C_h = \frac{-2Ge}{(2K-\lambda)\lambda} C_g \tag{G-9}$$

Correspondingly, C_{h2}, C_{h3}, and C_{h4} are zero. If λ_1 is substituted into equation (G-9), the result is

$$C_{h1} = - iC_{g1} \tag{G-10}$$

Then

$$h = -ig = - iC_{g1}e^{\lambda_1 \xi} \tag{G-11}$$

Because of the factor i, the imaginary part of g is the real part of h. Since the imaginary part of λ_1 has already been shown to be negative, the requirement that $h \to 0$ as $\xi \to \infty$ is already satisfied. Taking C_{g1} to be the complex number a + ib, a and b are constants, the real parts of g and h as found above are taken as the solutions to equations (G-3). (Recall that these equations are linear, so that the real part alone is a solution.)

$$\bar{g} = \left\{ \exp\left[K - \frac{1}{\sqrt{2}} \sqrt{(K^4 + 4Ge^2)^{1/2} + K^2} \right] \right\} \left\{ a \cos\left[\frac{\xi}{\sqrt{2}} \sqrt{(K^4 + 4Ge^2)^{1/2} - K^2} \right] \right.$$

$$\left. + b \sin\left[\frac{\xi}{\sqrt{2}} \sqrt{(K^4 + 4Ge^2)^{1/2} - K^2} \right] \right\} \qquad \text{(G-12)}$$

and similarly

$$\bar{h} = \left\{ \exp\,[\ldots.] \right\} \left\{ b \cos\,[\ldots.] - a \sin\,[\ldots.] \right\} \qquad \text{(G-13)}$$

where the barred notation has been re-adopted. The asymptotic solution to the energy equation is to be investigated now.

$$\frac{1}{Pr}\, t'' - 2\, ft' + 2\, f't = -\, (g'^2 + f''^2) \qquad (71)$$

For large ξ, allow

$$t = K_H + \bar{t}, \quad \bar{t} << K_H \qquad \text{(G-14)}$$

Substituting equations (G-14) and (G-2) into equation (71), dropping higher order terms, define $\bar{t}' = \bar{S}$.

$$\frac{1}{Pr} \bar{S}' - 2K\bar{S} = - 2\bar{h}K_H \tag{G-15}$$

which is a first order, non-homogeneous, linear, equation. The solution is the sum of the particular and a complementary solution. Referring to equation (G-13) for $\bar{h}$, substitution will verify that a particular solution to equation (G-15) is

$$\bar{S}_{par} = \left\{ \exp [\ldots] \right\} \left\{ \alpha_1 b \cos [\ldots] + \alpha_2 a \sin [\ldots] \right\} \tag{G-16}$$

where α_1 and α_2 are real constants, and the square brackets are identified with those in equation (G-12). The complementary, (homogeneous), solution satisfies

$$\bar{S}'_{com} - \frac{2K}{Pr} \bar{S}_{com} = 0 \tag{G-17}$$

Integrating once:

$$S_{com} = \alpha_3 \exp \left(\frac{2K}{Pr} \xi \right) \tag{G-18}$$

where α_3 is an integration constant. The solution to equation (G-15) is the sum of S_{com} and S_{par},

$$S = \alpha_3 \exp \left(\frac{2K}{Pr} \xi \right) + \left\{ \exp [\ldots] \right\} \left\{ \alpha_1 b \cos [\ldots] + \alpha_2 a \sin [\ldots] \right\} \tag{G-19}$$

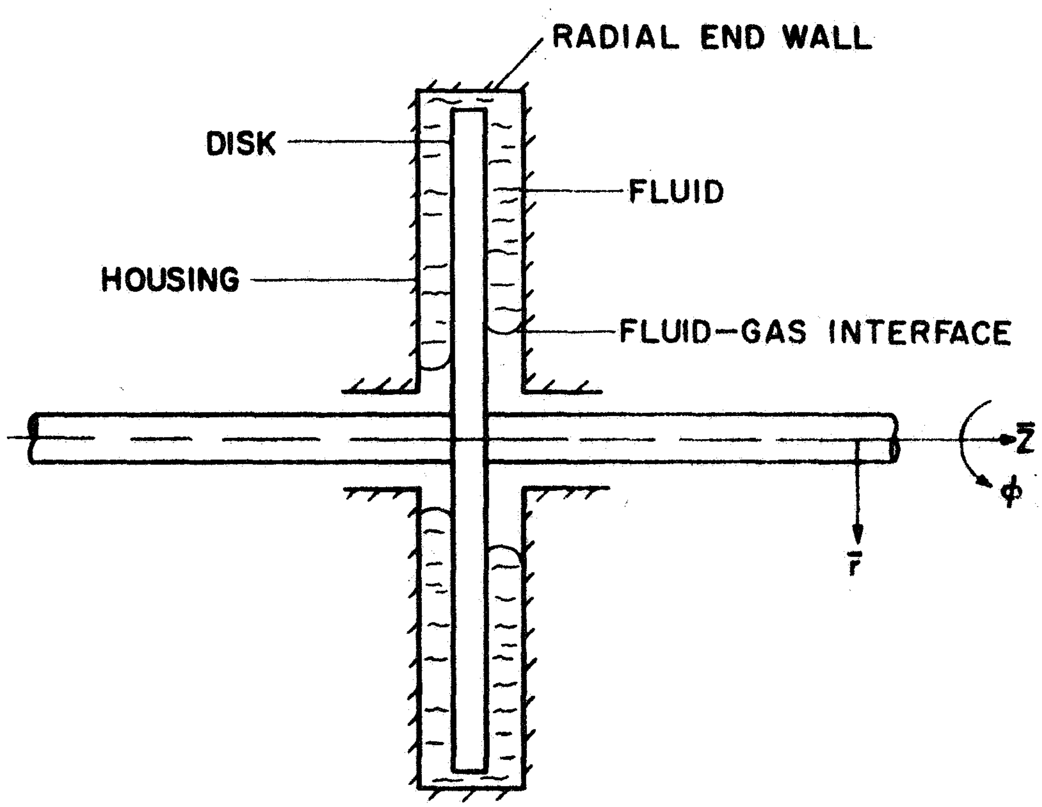
Since both K and Pr are positive, in order to satisfy $S \to 0$ as $\xi \to \infty$, α_3 must be chosen to be zero. Another way to see that $\alpha_3 = 0$ is to consider the special case of Pr = 1 and K_H = Ge. Under these conditions, equation (G-15) written in terms of $\bar{t}$ is

$$\bar{t}'' - 2K\bar{t}' + 2K_H\bar{h} = 0 \tag{G-20}$$

which is identical to equation (G-3) for $\bar{g}$. Both equations also require that as $\xi \to \infty$, $\bar{t}$, $\bar{g} \to 0$. Under these special circumstances, the solutions for $\bar{t}$ and $\bar{g}$ should be identical to within the constants multiplying them. This can only be if $\alpha_3 = 0$.

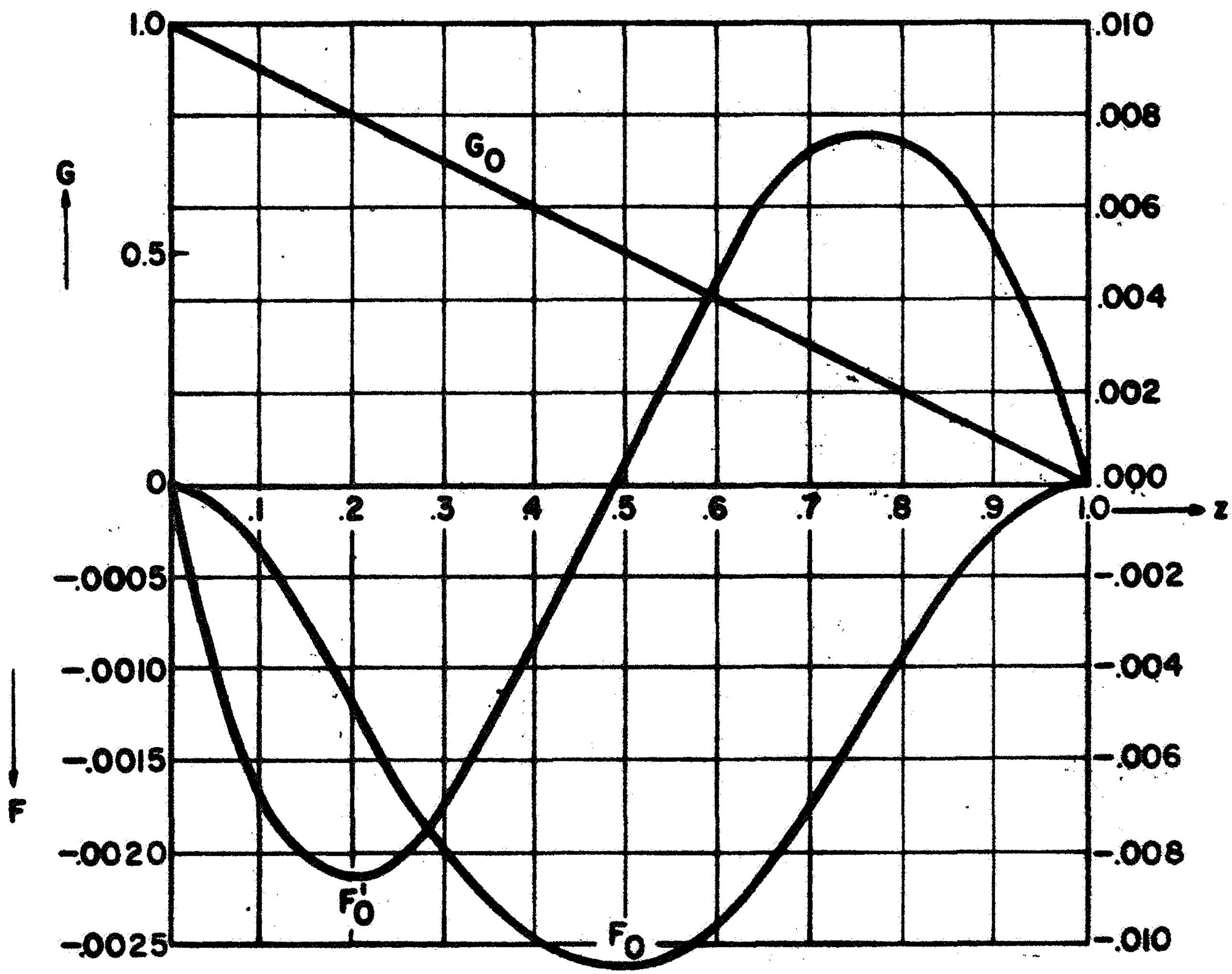
Therefore: $\bar{S} = \bar{t}' = \left\{\exp\,[\ldots]\right\} \left\{\alpha_1 b \cos\,[\ldots] + \alpha_2 a \sin\,[\ldots]\right\}$

The conclusion of this Appendix is that for the housing boundary layer problem, the asymptotic behavior of the velocities and temperature distribution is that of damped oscillations.



RADIAL END WALL
DISK
FLUID
HOUSING
FLUID–GAS INTERFACE
$\bar{Z}$
ϕ
$\bar{r}$

FIGURE 1



1.0
0.5
0
-.0005
-.0010
-.0015
-.0020
-.0025
G
F
.010
.008
.006
.004
.002
.000
-.002
-.004
-.006
-.008
-.010
F'
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
z
G_0
F_0'
F_0

FIGURE 2

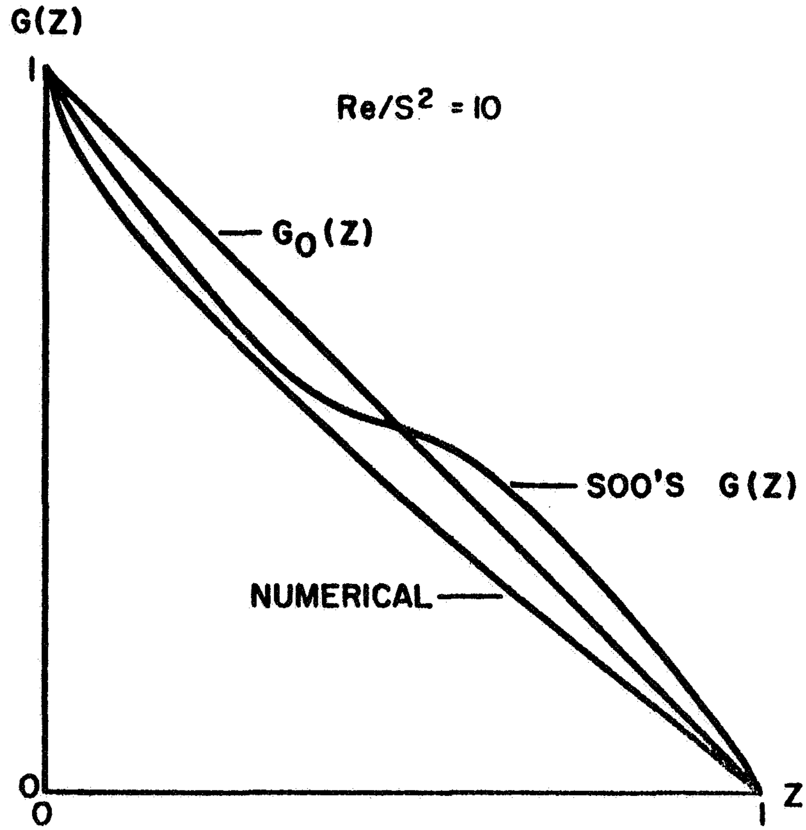


FIGURE 3

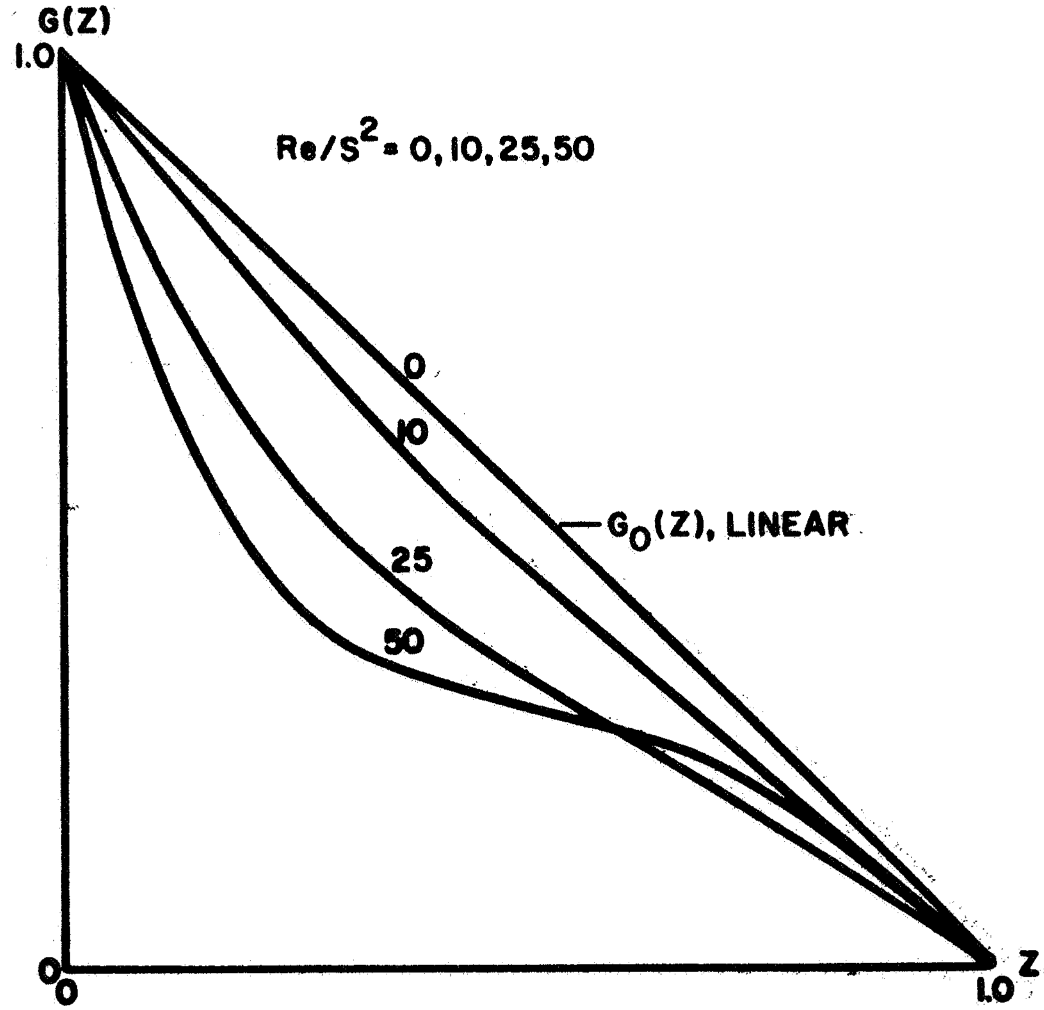


FIGURE 4

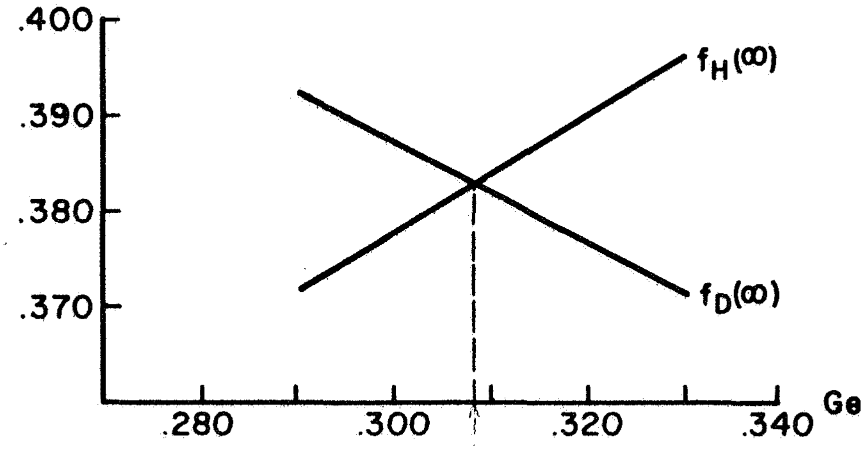
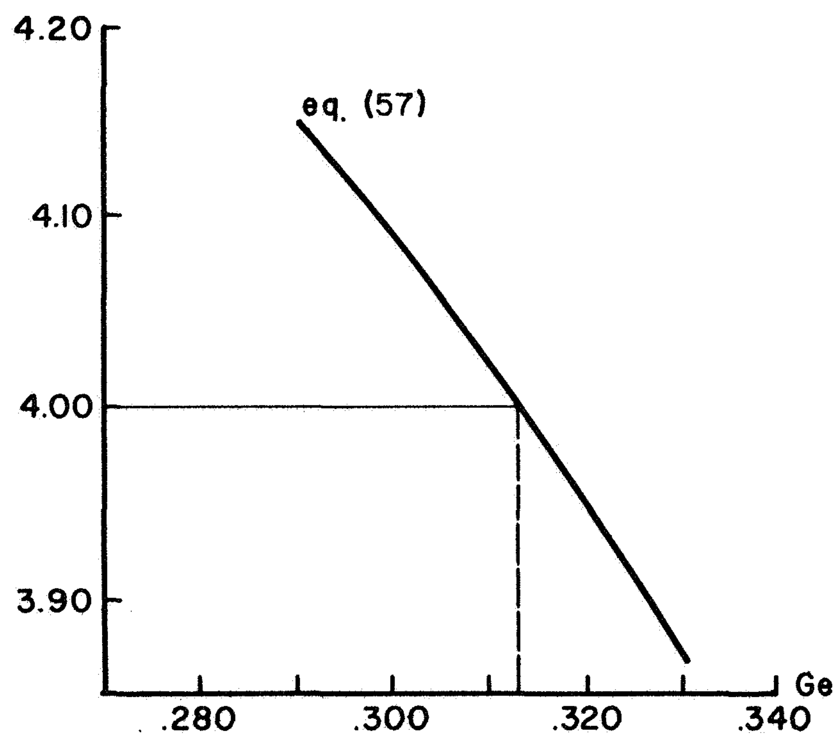


FIGURE 5



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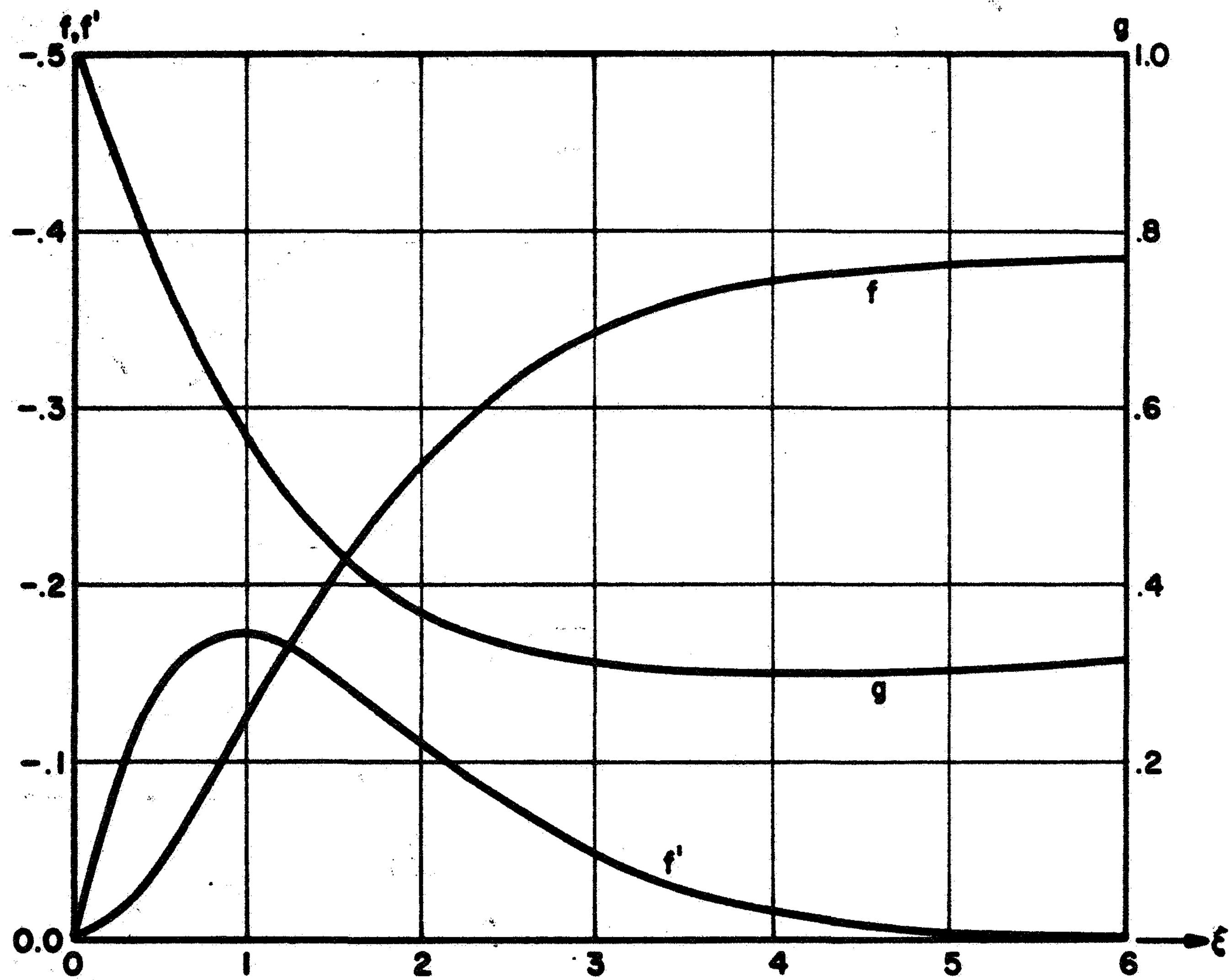

FIGURE 7

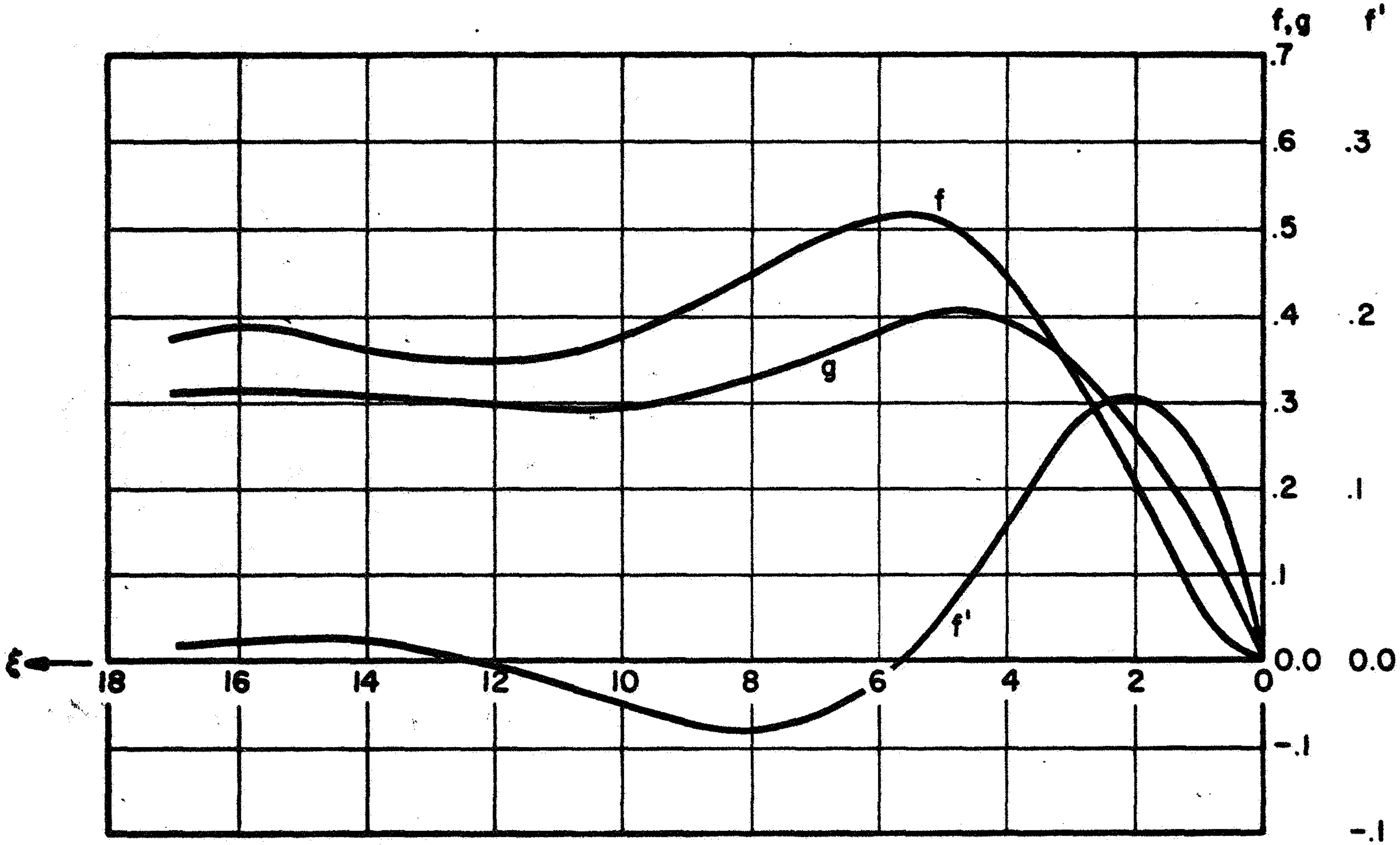

FIGURE 8

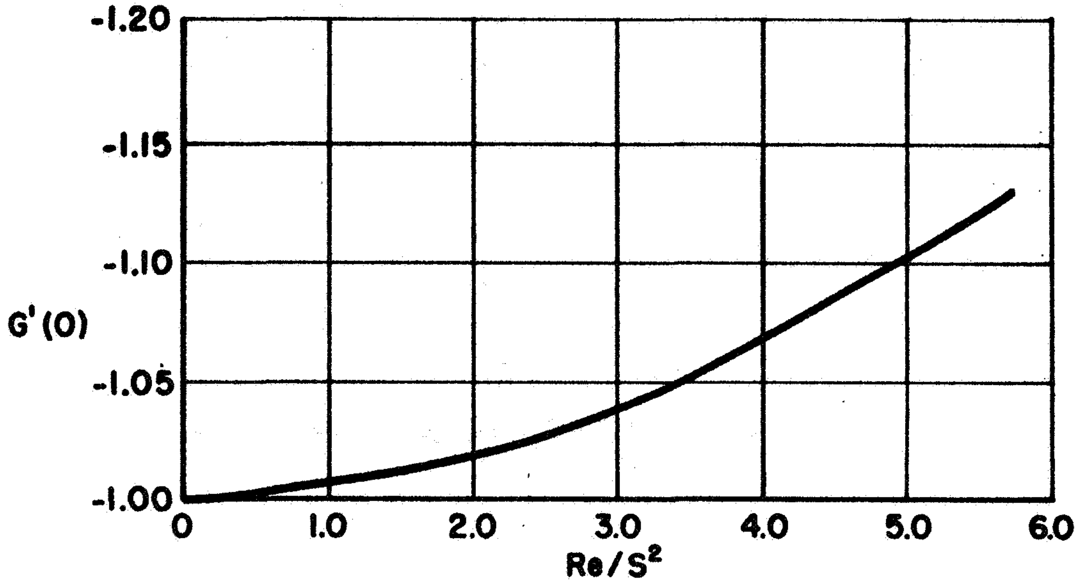

FIGURE 9

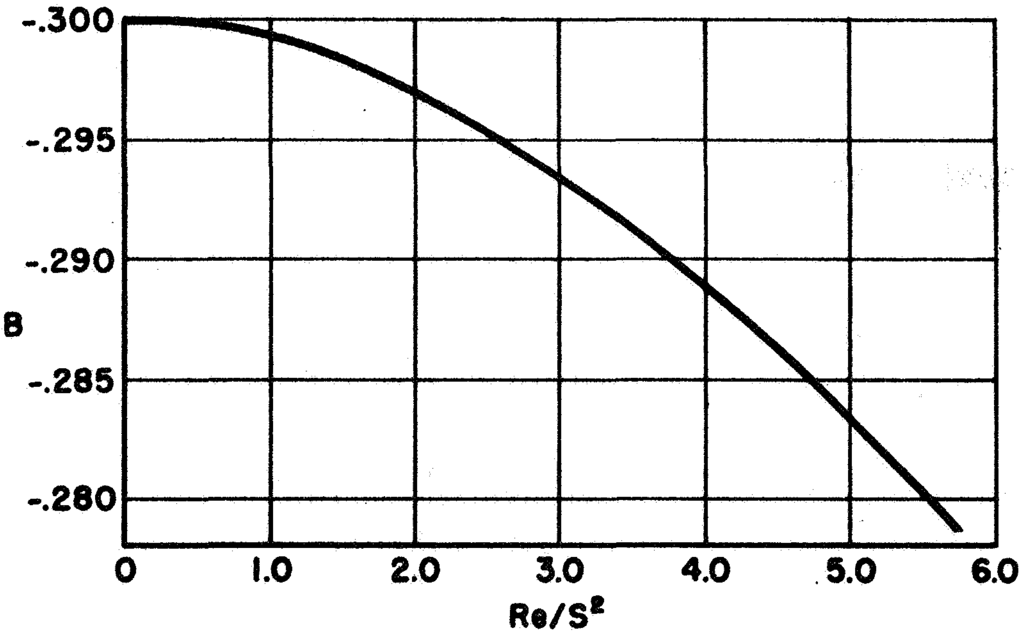

FIGURE 10

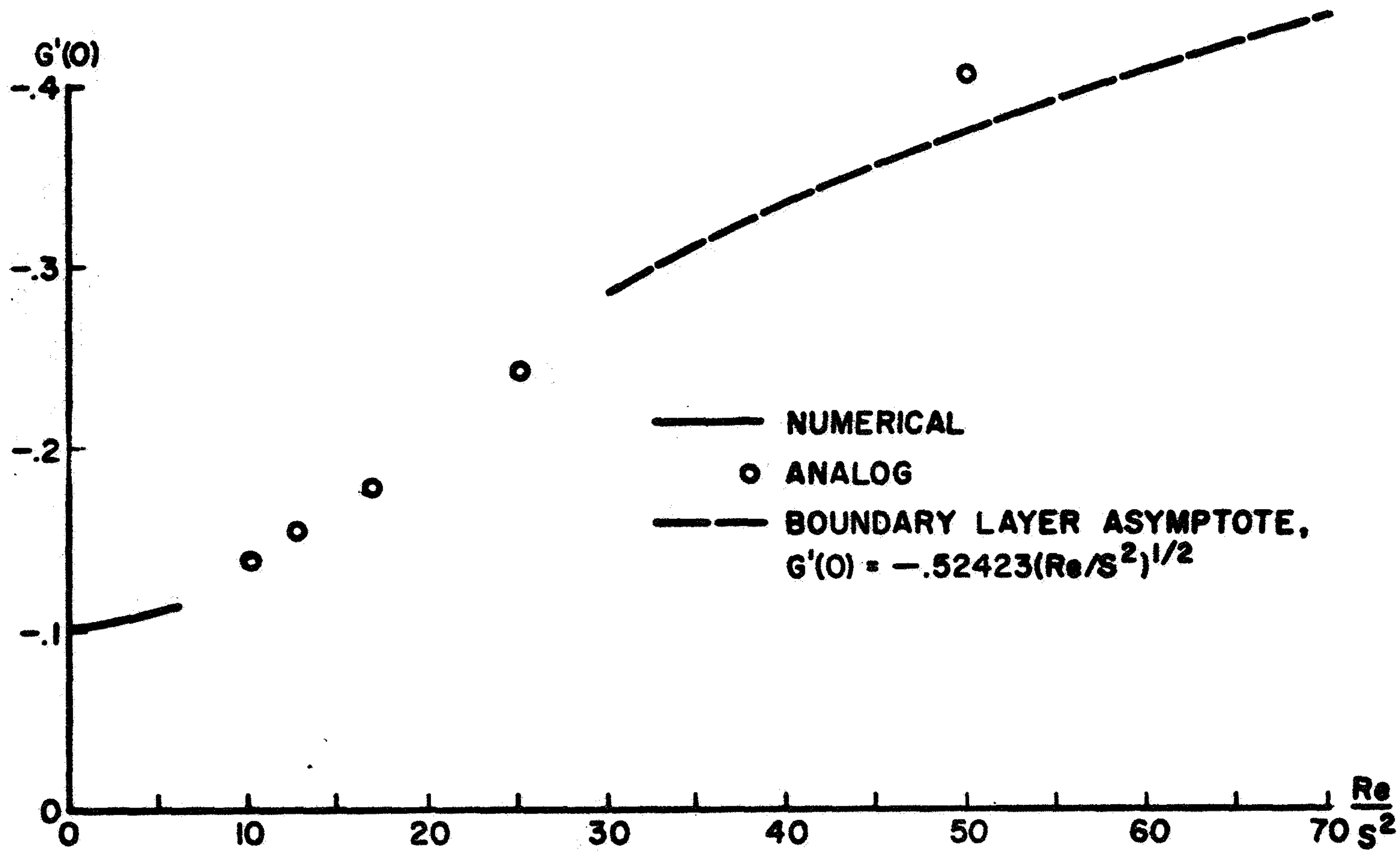
G'(O)
-.4
-.3
-.2
-.1
0
0
10
20
30
40
50
60
70
$\frac{Re}{S^2}$
NUMERICAL
ANALOG
BOUNDARY LAYER ASYMPTOTE,
$G'(0) = -.52423(Re/S^2)^{1/2}$

FIGURE 11

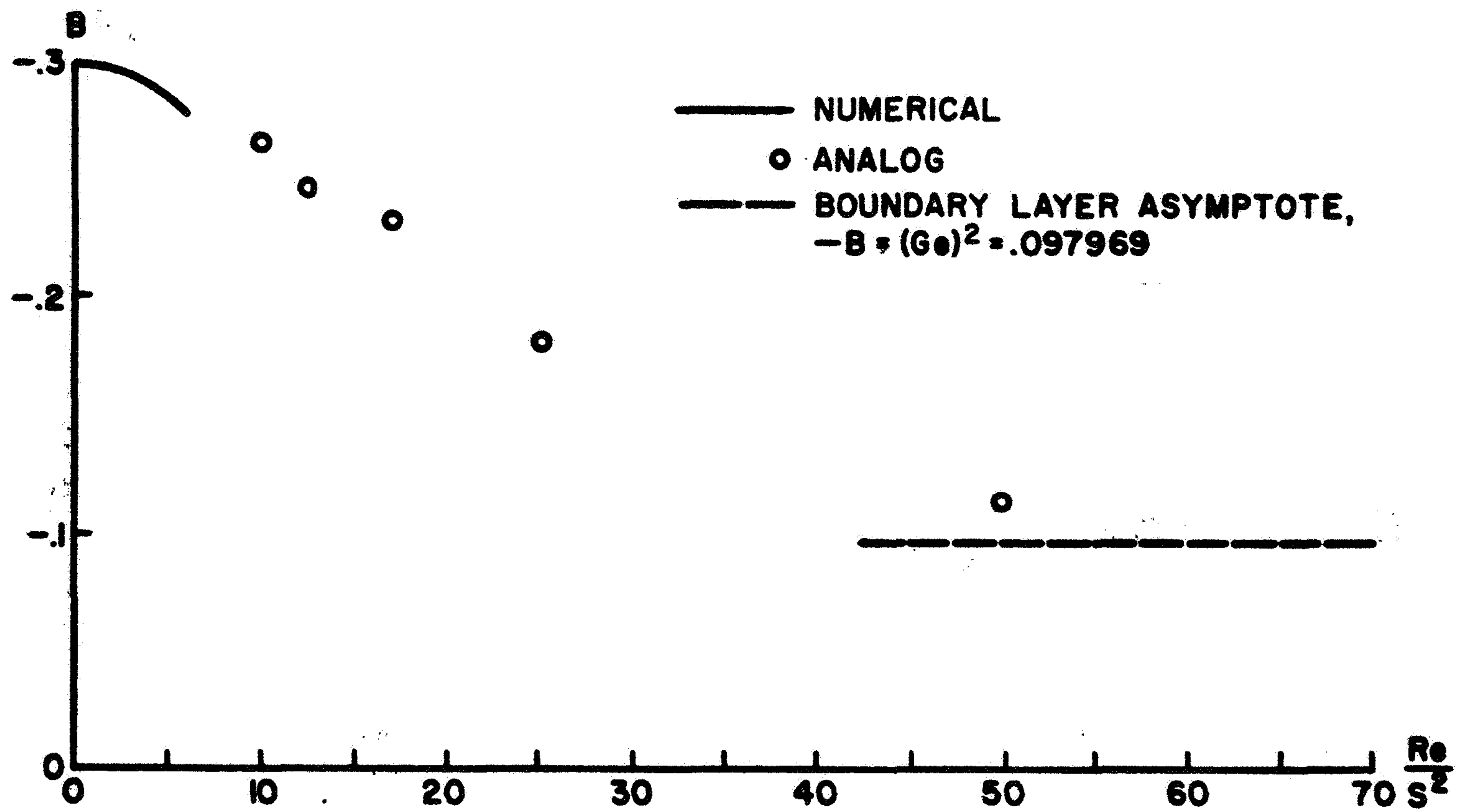
B
-.3
-.2
-.1
0
0
10
20
30
40
50
60
70
Re/S²
NUMERICAL
ANALOG
BOUNDARY LAYER ASYMPTOTE,
−B = (Ge)² = .097969

FIGURE 12

Table 1

Comparison of Series Solutions for Small Re/S^2,

using second order terms, with the numerical solutions

Re/S^2	G'(0) numer.	G'(0) series	G'(1) numer.	G'(1) series	B numer.	B series
0.01	-1.0000	-1.00000	-0.99999	-0.99999	-0.30000	-0.30000
0.10	-1.0000	-1.00003	-0.99998	-0.99998	-0.30000	-0.29993
1.00	-1.0042	-1.00428	-0.99872	-0.99873	-0.29923	-0.29930
1.25	-1.0066	-1.00667	-0.99801	-0.99808	-0.29881	-0.29891
2.00	-1.0170	-1.01712	-0.99497	-0.99492	-0.29697	-0.29722
2.50	-1.0265	-1.02675	-0.99223	-0.99206	-0.29529	-0.29566
3.33	-1.0465	-1.0476	-0.98624	-0.98589	-0.29178	-0.29228
3.64	-1.0555	-1.0567	-0.98395	-0.98317	-0.29020	-0.29079
4.00	-1.0662	-1.0685	-0.98036	-0.97968	-0.28836	-0.28888
4.44	-1.0818	-1.0844	-0.97631	-0.97497	-0.28567	-0.28630
5.00	-1.0999	-1.1070	-0.96769	-0.96825	-0.28326	-0.28262
5.70	-1.1266	-1.1391	-0.95805	-0.95860	-0.27909	-0.27741

Table 2

The disk boundary layer functions for Ge = 0.3135

ξ	f	g	f'	f"	g'
.0000	0.0000,+00	1.0000,+00	0.0000,+00	-4.7266,-01	-5.2423,-01
.0625	-8.6714,-04	9.6727,-01	-2.7822,-02	-4.1832,-01	-5.2247,-01
.1250	-3.4096,-03	9.3475,-01	-5.2373,-02	-3.6794,-01	-5.1755,-01
.1875	-7.3707,-03	9.0263,-01	-7.3895,-02	-3.2136,-01	-5.0994,-01
.2500	-1.2588,-02	8.7106,-01	-9.2620,-02	-2.7842,-01	-5.0007,-01
.3125	-1.8894,-02	8.4016,-01	-1.0877,-01	-2.3694,-01	-4.8852,-01
.3750	-2.6135,-02	8.1005,-01	-1.2255,-01	-2.0276,-01	-4.7502,-01
.4375	-3.4169,-02	7.8081,-01	-1.3418,-01	-1.6970,-01	-4.6049,-01
.5000	-4.2867,-02	7.5251,-01	-1.4363,-01	-1.3958,-01	-4.4498,-01
.5625	-5.2110,-02	7.2520,-01	-1.5168,-01	-1.1223,-01	-4.2873,-01
.6250	-6.1794,-02	6.9892,-01	-1.5791,-01	-8.7490,-02	-4.1196,-01
.6875	-7.1819,-02	6.7371,-01	-1.6267,-01	-6.5172,-02	-3.9463,-01
.7500	-8.2100,-02	6.4957,-01	-1.6611,-01	-4.5123,-02	-3.7752,-01
.8125	-9.2558,-02	6.2652,-01	-1.6835,-01	-2.7185,-02	-3.6015,-01
.8750	-1.0312,-01	6.0455,-01	-1.6954,-01	-1.1206,-02	-3.4285,-01
.9375	-1.1373,-01	5.8366,-01	-1.6979,-01	2.9584,-03	-3.2572,-01
1.0000	-1.2433,-01	5.6383,-01	-1.6921,-01	1.5448,-02	-3.0884,-01
1.0625	-1.3486,-01	5.4505,-01	-1.6789,-01	2.6395,-02	-2.9229,-01
1.1250	-1.4530,-01	5.2729,-01	-1.6594,-01	3.5925,-02	-2.7612,-01
1.1875	-1.5560,-01	5.1052,-01	-1.6343,-01	4.4156,-02	-2.6040,-01
1.2500	-1.6572,-01	4.9473,-01	-1.6044,-01	5.1199,-02	-2.4515,-01
1.3125	-1.7564,-01	4.7987,-01	-1.5705,-01	5.7160,-02	-2.3040,-01
1.3750	-1.8534,-01	4.6591,-01	-1.5332,-01	6.2137,-02	-2.1619,-01
1.4375	-1.9480,-01	4.5283,-01	-1.4930,-01	6.6221,-02	-2.0253,-01
1.5000	-2.0400,-01	4.4059,-01	-1.4506,-01	6.9500,-02	-1.8943,-01

Table 2 (continued)

ξ	f	g	f'	f"	g'
1.5625	-2.1293,-01	4.2914,-01	-1.4063,-01	7.2052,-02	-1.7669,-01
1.6250	-2.2158,-01	4.1846,-01	-1.3607,-01	7.3951,-02	-1.6492,-01
1.6875	-2.2994,-01	4.0651,-01	-1.3140,-01	7.5266,-02	-1.5352,-01
1.7500	-2.3800,-01	3.9926,-01	-1.2667,-01	7.6061,-02	-1.4268,-01
1.8125	-2.4577,-01	3.9067,-01	-1.2190,-01	7.6395,-02	-1.3238,-01
1.8750	-2.5324,-01	3.8270,-01	-1.1713,-01	7.6320,-02	-1.2263,-01
1.9375	-2.6042,-01	3.7533,-01	-1.1237,-01	7.5886,-02	-1.1341,-01
2.0000	-2.6729,-01	3.6651,-01	-1.0765,-01	7.5139,-02	-1.0470,-01
2.0625	-2.7387,-01	3.6223,-01	-1.0298,-01	7.4120,-02	-9.6492,-02
2.1250	-2.8017,-01	3.5644,-01	-9.8393,-02	7.2866,-02	-8.8764,-02
2.1875	-2.8617,-01	3.5112,-01	-9.3863,-02	7.1413,-02	-8.1501,-02
2.2500	-2.9190,-01	3.4624,-01	-8.9470,-02	6.9791,-02	-7.4666,-02
2.3125	-2.9736,-01	3.4178,-01	-8.5162,-02	6.8029,-02	-6.8300,-02
2.3750	-3.0255,-01	3.3770,-01	-8.0968,-02	6.6152,-02	-6.2325,-02
2.4375	-3.0748,-01	3.3398,-01	-7.6895,-02	6.4183,-02	-5.6744,-02
2.5000	-3.1217,-01	3.3060,-01	-7.2947,-02	6.2143,-02	-5.1537,-02
2.5625	-3.1660,-01	3.2753,-01	-6.9128,-02	6.0051,-02	-4.6686,-02
2.6250	-3.2081,-01	3.2475,-01	-6.5441,-02	5.7922,-02	-4.2175,-02
2.6875	-3.2479,-01	3.2225,-01	-6.1868,-02	5.5773,-02	-3.7985,-02
2.7500	-3.2855,-01	3.2000,-01	-5.8470,-02	5.3615,-02	-3.4099,-02
2.8125	-3.3210,-01	3.1798,-01	-5.5166,-02	5.1460,-02	-3.0501,-02
2.8750	-3.3545,-01	3.1618,-01	-5.2037,-02	4.9319,-02	-2.7174,-02
2.9375	-3.3861,-01	3.1458,-01	-4.9021,-02	4.7200,-02	-2.4103,-02
3.0000	-3.4158,-01	3.1316,-01	-4.6136,-02	4.5110,-02	-2.1273,-02
3.0625	-3.4438,-01	3.1192,-01	-4.3361,-02	4.3057,-02	-1.8670,-02
3.1250	-3.4700,-01	3.1082,-01	-4.0753,-02	4.1046,-02	-1.6276,-02
3.1875	-3.4947,-01	3.0988,-01	-3.8250,-02	3.9062,-02	-1.4086,-02
3.2500	-3.5179,-01	3.0906,-01	-3.5867,-02	3.7168,-02	-1.2080,-02
3.3125	-3.5396,-01	3.0836,-01	-3.3602,-02	3.5308,-02	-1.0248,-02
3.3750	-3.5599,-01	3.0778,-01	-3.1452,-02	3.3504,-02	-8.5795,-03
3.4375	-3.5789,-01	3.0729,-01	-2.9413,-02	3.1759,-02	-7.0616,-03
3.5000	-3.5967,-01	3.0689,-01	-2.7481,-02	3.0073,-02	-5.6847,-03

Table 2 (continued)

ξ	f	g	f'	f"	g'
3.5625	-3.6133,-01	3.0657,-01	-2.5653,-02	2.8468,-02	-4.4368,-03
3.6250	-3.6288,-01	3.0633,-01	-2.3924,-02	2.6885,-02	-3.3145,-03
3.6875	-3.6432,-01	3.0616,-01	-2.2291,-02	2.5383,-02	-2.3030,-03
3.7500	-3.6567,-01	3.0604,-01	-2.0750,-02	2.3942,-02	-1.3956,-03
3.8125	-3.6692,-01	3.0598,-01	-1.9297,-02	2.2563,-02	-5.8522,-04
3.8750	-3.6808,-01	3.0597,-01	-1.7928,-02	2.1244,-02	1.3637,-04
3.9375	-3.6916,-01	3.0600,-01	-1.6640,-02	1.9984,-02	7.7588,-04
4.0000	-3.7016,-01	3.0606,-01	-1.5429,-02	1.8783,-02	1.3398,-03
4.0625	-3.7109,-01	3.0616,-01	-1.4291,-02	1.7640,-02	1.8344,-03
4.1250	-3.7195,-01	3.0629,-01	-1.3222,-02	1.6553,-02	2.2653,-03
4.1875	-3.7275,-01	3.0644,-01	-1.2220,-02	1.5520,-02	2.6380,-03
4.2500	-3.7348,-01	3.0662,-01	-1.1281,-02	1.4541,-02	2.9574,-03
4.3125	-3.7416,-01	3.0681,-01	-1.0402,-02	1.3613,-02	3.2282,-03
4.3750	-3.7478,-01	3.0702,-01	-9.5789,-03	1.2735,-02	3.4547,-03
4.4375	-3.7536,-01	3.0724,-01	-8.8091,-03	1.1906,-02	3.6410,-03
4.5000	-3.7588,-01	3.0748,-01	-8.0896,-03	1.1123,-02	3.7909,-03
4.5625	-3.7637,-01	3.0772,-01	-7.4177,-03	1.0385,-02	3.9078,-03
4.6250	-3.7681,-01	3.0796,-01	-6.7906,-03	9.6899,-03	3.9949,-03
4.6875	-3.7722,-01	3.0822,-01	-6.2056,-03	9.0361,-03	4.0552,-03
4.7500	-3.7759,-01	3.0847,-01	-5.6603,-03	8.4220,-03	4.0915,-03
4.8125	-3.7793,-01	3.0873,-01	-5.1521,-03	7.8459,-03	4.1063,-03
4.8750	-3.7823,-01	3.0898,-01	-4.6788,-03	7.3060,-03	4.1019,-03
4.9375	-3.7851,-01	3.0924,-01	-4.2381,-03	6.8007,-03	4.0805,-03
5.0000	-3.7876,-01	3.0949,-01	-3.8260,-03	6.3284,-03	4.0441,-03
5.0625	-3.7899,-01	3.0974,-01	-3.4464,-03	5.8874,-03	3.9943,-03
5.1250	-3.7919,-01	3.0999,-01	-3.0914,-03	5.4763,-03	3.9330,-03
5.1875	-3.7938,-01	3.1024,-01	-2.7613,-03	5.0936,-03	3.8615,-03
5.2500	-3.7954,-01	3.1047,-01	-2.4542,-03	4.7377,-03	3.7813,-03
5.3125	-3.7968,-01	3.1071,-01	-2.1665,-03	4.4072,-03	3.6936,-03
5.3750	-3.7981,-01	3.1094,-01	-1.9027,-03	4.1008,-03	3.5995,-03
5.4375	-3.7992,-01	3.1116,-01	-1.6554,-03	3.8172,-03	3.5001,-03
5.5000	-3.8002,-01	3.1137,-01	-1.4251,-03	3.5550,-03	3.3962,-03

Table 2 (continued)

ξ	f	g	f'	f"	g'
5.5625	-3.8010,-01	3.1158,-01	-1.2106,-03	3.3131,-03	3.2886,-03
5.6250	-3.8017,-01	3.1178,-01	-1.0106,-03	3.0902,-03	3.1782,-03
5.6875	-3.8023,-01	3.1198,-01	-8.2401,-04	2.8852,-03	3.0655,-03
5.7500	-3.8027,-01	3.1217,-01	-6.4964,-04	2.6970,-03	2.9512,-03
5.8125	-3.8031,-01	3.1235,-01	-4.8654,-04	2.5246,-03	2.8358,-03
5.8750	-3.8033,-01	3.1252,-01	-3.3376,-04	2.3669,-03	2.7197,-03
5.9375	-3.8035,-01	3.1269,-01	-1.9038,-04	2.2231,-03	2.6034,-03
6.0000	-3.8036,-01	3.1285,-01	-5.5599,-05	2.0921,-03	2.4872,-03

Table 3

The Housing Boundary Layer Functions for Ge = 0.313

ξ	f	g	f'	f''	g'
.00000	0.00000	0.00000	0.00000	0.16534	0.13567
.70315	0.03523	0.09501	0.09230	0.09807	0.13359
1.40630	0.11932	0.18576	0.14005	0.03977	0.12278
2.10945	0.22364	0.26538	0.15144	-0.00495	0.10231
2.81260	0.32614	0.32816	0.13662	-0.03473	0.07556
3.51575	0.41206	0.37123	0.10591	-0.05040	0.04701
4.21890	0.47354	0.39477	0.06851	-0.05427	0.02059
4.92205	0.50855	0.40133	0.03164	-0.04944	-0.00096
5.62520	0.51934	0.39490	0.00026	-0.03917	-0.01625
6.32835	0.51085	0.38000	-0.02289	-0.02649	-0.02509
7.03150	0.48927	0.36096	-0.03701	-0.01382	-0.02818
7.73465	0.46079	0.34142	-0.04273	-0.00285	-0.02677
8.43780	0.43076	0.32402	-0.04165	0.00543	-0.02235
9.84410	0.38106	0.30110	-0.02727	0.01314	-0.01000
10.54725	0.36518	0.29615	-0.01788	0.01322	-0.00423
11.25040	0.35577	0.29488	-0.00907	0.01162	0.00040
11.95355	0.35207	0.29639	-0.00177	0.00903	0.00365
12.65670	0.35282	0.29969	0.00355	0.00608	0.00553
13.35985	0.35659	0.30390	0.00683	0.00330	0.00624
14.06300	0.36201	0.30826	0.00833	0.00111	0.00605
14.76615	0.36802	0.31226	0.00861	0.00007	0.00524
15.46930	0.37401	0.31552	0.00844	-0.00010	0.00390
16.17245	0.38002	0.31753	0.00884	0.00155	0.00161
16.87560	0.38688	0.31725	0.01105	0.00501	-0.00296

Table 4

A particular solution to the disk thermal boundary layer equation for Pr = 10

ξ	t	t'	t"
.0000	1.0000,+00	4.0000,+00	-4.9822,+00
.0625	1.2402,+00	3.6686,+00	-3.8550,+00
.1250	1.4633,+00	3.4517,+00	-2.7350,+00
.1875	1.6739,+00	3.2680,+00	-1.6440,+00
.2500	1.8764,+00	3.1951,+00	-6.0452,-01
.3125	2.0751,+00	3.1691,+00	3.6119,-01
.3750	2.2742,+00	3.2048,+00	1.2315,+00
.4375	2.4772,+00	3.2958,+00	1.9870,+00
.5000	2.6873,+00	3.4347,+00	2.6109,+00
.5625	2.9074,+00	3.6128,+00	3.0910,+00
.6250	3.1396,+00	3.8212,+00	3.4196,+00
.6875	3.3854,+00	4.0503,+00	3.5952,+00
.7500	3.6459,+00	4.2907,+00	3.6215,+00
.8125	3.9215,+00	4.5332,+00	3.5081,+00
.8750	4.2120,+00	4.7695,+00	3.2693,+00
.9375	4.5169,+00	4.9919,+00	2.9232,+00
1.0000	4.8350,+00	5.1942,+00	2.4907,+00
1.0625	5.1650,+00	5.3714,+00	1.9938,+00
1.1250	5.5051,+00	5.5197,+00	1.4545,+00
1.1875	5.8534,+00	5.6367,+00	8.9396,-01
1.2500	6.2081,+00	5.7214,+00	3.3088,-01
1.3125	6.5670,+00	5.7737,+00	-2.1841,-01
1.3750	6.9282,+00	5.7944,+00	-7.4074,-01
1.4375	7.2897,+00	5.7852,+00	-1.2260,+00
1.5000	7.6497,+00	5.7483,+00	-1.6673,+00
1.5625	8.0066,+00	5.6661,+00	-2.0603,+00
1.6250	8.3589,+00	5.6016,+00	-2.4032,+00
1.6875	8.7053,+00	5.4974,+00	-2.6959,+00
1.7500	9.0446,+00	5.3764,+00	-2.9401,+00
1.8125	9.3759,+00	5.2414,+00	-3.1384,+00
1.8750	9.6984,+00	5.0949,+00	-3.2942,+00
1.9375	1.0011,+01	4.9392,+00	-3.4111,+00
2.0000	1.0314,+01	4.7766,+00	-3.4932,+00
2.0625	1.0607,+01	4.6088,+00	-3.5444,+00
2.1250	1.0889,+01	4.4376,+00	-3.5684,+00
2.1875	1.1161,+01	4.2645,+00	-3.5688,+00
2.2500	1.1421,+01	4.0907,+00	-3.5489,+00
2.3125	1.1671,+01	3.9174,+00	-3.5117,+00
2.3750	1.1910,+01	3.7455,+00	-3.4598,+00
2.4375	1.2138,+01	3.5758,+00	-3.3956,+00
2.5000	1.2356,+01	3.4090,+00	-3.3213,+00

Table 4 (continued)

ξ	t	t'	t"
2.5625	1.2564,+01	3.2456,+00	-3.2387,+00
2.6250	1.2761,+01	3.0860,+00	-3.1495,+00
2.6875	1.2949,+01	2.9307,+00	-3.0549,+00
2.7500	1.3127,+01	2.7799,+00	-2.9565,+00
2.8125	1.3296,+01	2.6339,+00	-2.8551,+00
2.8750	1.3455,+01	2.4927,+00	-2.7518,+00
2.9375	1.3607,+01	2.3566,+00	-2.6473,+00
3.0000	1.3749,+01	2.2256,+00	-2.5425,+00
3.0625	1.3884,+01	2.0998,+00	-2.4378,+00
3.1250	1.4011,+01	1.9790,+00	-2.3339,+00
3.1875	1.4131,+01	1.8634,+00	-2.2312,+00
3.2500	1.4244,+01	1.7528,+00	-2.1301,+00
3.3125	1.4350,+01	1.6472,+00	-2.0308,+00
3.3750	1.4449,+01	1.5465,+00	-1.9337,+00
3.4375	1.4543,+01	1.4506,+00	-1.8391,+00
3.5000	1.4630,+01	1.3594,+00	-1.7470,+00
3.5625	1.4713,+01	1.2727,+00	-1.6576,+00
3.6250	1.4789,+01	1.1905,+00	-1.5711,+00
3.6875	1.4861,+01	1.1125,+00	-1.4876,+00
3.7500	1.4928,+01	1.0387,+00	-1.4070,+00
3.8125	1.4991,+01	9.6888,-01	-1.3295,+00
3.8750	1.5049,+01	9.0288,-01	-1.2550,+00
3.9375	1.5103,+01	8.4057,-01	-1.1835,+00
4.0000	1.5154,+01	7.8180,-01	-1.1151,+00
4.0625	1.5201,+01	7.2642,-01	-1.0497,+00
4.1250	1.5244,+01	6.7429,-01	-9.8731,-01
4.1875	1.5285,+01	6.2526,-01	-9.2779,-01
4.2500	1.5322,+01	5.7918,-01	-8.7113,-01
4.3125	1.5357,+01	5.3591,-01	-8.1727,-01
4.3750	1.5389,+01	4.9531,-01	-7.6612,-01
4.4375	1.5419,+01	4.5725,-01	-7.1763,-01
4.5000	1.5446,+01	4.2160,-01	-6.7171,-01
4.5625	1.5472,+01	3.8823,-01	-6.2828,-01
4.6250	1.5495,+01	3.5702,-01	-5.8725,-01
4.6875	1.5516,+01	3.2784,-01	-5.4855,-01
4.7500	1.5536,+01	3.0058,-01	-5.1207,-01
4.8125	1.5554,+01	2.7514,-01	-4.7774,-01
4.8750	1.5570,+01	2.5140,-01	-4.4547,-01
4.9375	1.5585,+01	2.2926,-01	-4.1517,-01
5.0000	1.5599,+01	2.0863,-01	-3.8676,-01
5.0625	1.5611,+01	1.8941,-01	-3.6016,-01
5.1250	1.5622,+01	1.7151,-01	-3.3527,-01
5.1875	1.5632,+01	1.5485,-01	-3.1202,-01
5.2500	1.5642,+01	1.3934,-01	-2.9033,-01

Table 4 (continued)

ξ	t	t'	t"
5.3125	1.5650,+01	1.2491,-01	-2.7012,-01
5.3750	1.5657,+01	1.1148,-01	-2.5132,-01
5.4375	1.5664,+01	9.8994,-02	-2.3385,-01
5.5000	1.5669,+01	8.7370,-02	-2.1764,-01
5.5625	1.5675,+01	7.6551,-02	-2.0263,-01
5.6250	1.5679,+01	6.6479,-02	-1.8874,-01
5.6875	1.5683,+01	5.7096,-02	-1.7591,-01
5.7500	1.5686,+01	4.8351,-02	-1.6408,-01
5.8125	1.5689,+01	4.0194,-02	-1.5320,-01
5.8750	1.5691,+01	3.2578,-02	-1.4320,-01
5.9375	1.5693,+01	2.5458,-02	-1.3402,-01
6.0000	1.5694,+01	1.8795,-02	-1.4298,-01
6.0625	1.5695,+01	1.1686,-02	-8.8904,-02
6.1250	1.5696,+01	7.2668,-03	-5.5280,-02
6.1875	1.5696,+01	4.5184,-03	-3.4372,-02
6.2500	1.5696,+01	2.8095,-03	-2.1372,-02
6.3125	1.5697,+01	1.7469,-03	-1.3289,-02
6.3750	1.5697,+01	1.0862,-03	-8.2632,-03
6.4375	1.5697,+01	6.7541,-04	-5.1380,-03
6.5000	1.5697,+01	4.1997,-04	-3.1948,-03
6.5625	1.5697,+01	2.6113,-04	-1.9865,-03
6.6250	1.5697,+01	1.6237,-04	-1.2351,-03
6.6875	1.5697,+01	1.0096,-04	-7.6803,-04
6.7500	1.5697,+01	6.2777,-05	-4.7755,-04
6.8125	1.5697,+01	3.9034,-05	-2.9694,-04
6.8750	1.5697,+01	2.4271,-05	-1.8463,-04
6.9375	1.5697,+01	1.5091,-05	-1.1480,-04
7.0000	1.5697,+01	9.3839,-06	-7.1385,-05
7.0625	1.5697,+01	5.8348,-06	-4.4386,-05
7.1250	1.5697,+01	3.6280,-06	-2.7599,-05
7.1875	1.5697,+01	2.2559,-06	-1.7161,-05
7.2500	1.5697,+01	1.4027,-06	-1.0670,-05
7.3125	1.5697,+01	8.7219,-07	-6.6349,-06
7.3750	1.5697,+01	5.4232,-07	-4.1255,-06
7.4375	1.5697,+01	3.3721,-07	-2.5652,-06
7.5000	1.5697,+01	2.0967,-07	-1.5950,-06
7.5625	1.5697,+01	1.3037,-07	-9.9178,-07
7.6250	1.5697,+01	8.1066,-08	-6.1668,-07
7.6875	1.5697,+01	5.0406,-08	-3.8345,-07
7.7500	1.5697,+01	3.1342,-08	-2.3842,-07
7.8125	1.5697,+01	1.9488,-08	-1.4825,-07
7.8750	1.5697,+01	1.2117,-08	-9.2182,-08
7.9375	1.5697,+01	7.5347,-09	-5.7318,-08
8.0000	1.5697,+01	4.6850,-09	-3.5640,-08

Table 5

A homogeneous solution to the disk thermal boundary layer equation for Pr = 10

ξ	t	t'	t"
.0000	1.0000,+00	4.0000,+00	-0.0000,+00
.0625	1.2500,+00	4.0000,+00	6.2459,-01
.1250	1.5013,+00	4.0433,+00	1.2968,+00
.1875	1.7567,+00	4.1325,+00	1.9870,+00
.2500	2.0191,+00	4.2682,+00	2.6656,+00
.3125	2.2913,+00	4.4490,+00	3.3034,+00
.3750	2.5762,+00	4.6721,+00	3.8726,+00
.4375	2.8762,+00	4.9328,+00	4.3476,+00
.5000	3.1954,+00	5.2248,+00	4.7069,+00
.5625	3.5296,+00	5.5406,+00	4.9334,+00
.6250	3.8860,+00	5.8719,+00	5.0163,+00
.6875	4.2633,+00	6.2096,+00	4.9512,+00
.7500	4.6616,+00	6.5445,+00	4.7406,+00
.8125	5.0804,+00	6.8678,+00	4.3933,+00
.8750	5.5189,+00	7.1712,+00	3.9240,+00
.9375	5.9754,+00	7.4474,+00	3.3519,+00
1.0000	6.4481,+00	7.6903,+00	2.6993,+00
1.0625	6.9348,+00	7.8952,+00	1.9903,+00
1.1250	7.4329,+00	8.0590,+00	1.2487,+00
1.1875	7.9400,+00	8.1798,+00	4.9770,-01
1.2500	8.4532,+00	8.2572,+00	-2.4231,-01
1.3125	8.9698,+00	8.2919,+00	-9.5365,-01
1.3750	9.4873,+00	8.2857,+00	-1.6221,+00
1.4375	1.0003,+01	8.2411,+00	-2.2371,+00
1.5000	1.0515,+01	8.1612,+00	-2.7916,+00
1.5625	1.1021,+01	8.0495,+00	-3.2813,+00
1.6250	1.1519,+01	7.9097,+00	-3.7050,+00
1.6875	1.2007,+01	7.7456,+00	-4.0635,+00
1.7500	1.2485,+01	7.5608,+00	-4.3593,+00
1.8125	1.2950,+01	7.3587,+00	-4.5962,+00
1.8750	1.3403,+01	7.1428,+00	-4.7785,+00
1.9375	1.3842,+01	6.9160,+00	-4.9114,+00
2.0000	1.4266,+01	6.6810,+00	-4.9998,+00
2.0625	1.4675,+01	6.4402,+00	-5.0488,+00
2.1250	1.5069,+01	6.1959,+00	-5.0633,+00
2.1875	1.5448,+01	5.9499,+00	-5.0477,+00
2.2500	1.5812,+01	5.7039,+00	-5.0063,+00
2.3125	1.6160,+01	5.4592,+00	-4.9428,+00
2.3750	1.6493,+01	5.2171,+00	-4.8608,+00
2.4375	1.6811,+01	4.9786,+00	-4.7631,+00
2.5000	1.7114,+01	4.7444,+00	-4.6527,+00

Table 5 (continued)

ξ	t	t'	t''
2.5625	1.7403,+01	4.5154,+00	-4.5317,+00
2.6250	1.7677,+01	4.2921,+00	-4.4024,+00
2.6875	1.7938,+01	4.0750,+00	-4.2667,+00
2.7500	1.8186,+01	3.8644,+00	-4.1260,+00
2.8125	1.8420,+01	3.6605,+00	-3.9819,+00
2.8750	1.8643,+01	3.4637,+00	-3.8356,+00
2.9375	1.8853,+01	3.2739,+00	-3.6881,+00
3.0000	1.9051,+01	3.0914,+00	-3.5404,+00
3.0625	1.9238,+01	2.9161,+00	-3.3933,+00
3.1250	1.9415,+01	2.7481,+00	-3.2475,+00
3.1875	1.9581,+01	2.5872,+00	-3.1036,+00
3.2500	1.9738,+01	2.4334,+00	-2.9620,+00
3.3125	1.9885,+01	2.2865,+00	-2.8232,+00
3.3750	2.0023,+01	2.1465,+00	-2.6876,+00
3.4375	2.0153,+01	2.0132,+00	-2.5555,+00
3.5000	2.0274,+01	1.8865,+00	-2.4271,+00
3.5625	2.0388,+01	1.7661,+00	-2.3025,+00
3.6250	2.0495,+01	1.6518,+00	-2.1820,+00
3.6875	2.0594,+01	1.5435,+00	-2.0656,+00
3.7500	2.0687,+01	1.4410,+00	-1.9535,+00
3.8125	2.0774,+01	1.3440,+00	-1.8456,+00
3.8750	2.0855,+01	1.2524,+00	-1.7420,+00
3.9375	2.0930,+01	1.1659,+00	-1.6426,+00
4.0000	2.1000,+01	1.0843,+00	-1.5475,+00
4.0625	2.1065,+01	1.0075,+00	-1.4566,+00
4.1250	2.1126,+01	9.3519,-01	-1.3699,+00
4.1875	2.1182,+01	8.6715,-01	-1.2872,+00
4.2500	2.1234,+01	8.0321,-01	-1.2085,+00
4.3125	2.1282,+01	7.4318,-01	-1.1337,+00
4.3750	2.1327,+01	6.8687,-01	-1.0627,+00
4.4375	2.1368,+01	6.3408,-01	-9.9542,-01
4.5000	2.1406,+01	5.8462,-01	-9.3167,-01
4.5625	2.1441,+01	5.3834,-01	-8.7139,-01
4.6250	2.1473,+01	4.9504,-01	-8.1445,-01
4.6875	2.1503,+01	4.5458,-01	-7.6074,-01
4.7500	2.1530,+01	4.1678,-01	-7.1013,-01
4.8125	2.1555,+01	3.8149,-01	-6.6250,-01
4.8750	2.1577,+01	3.4857,-01	-6.1773,-01
4.9375	2.1598,+01	3.1788,-01	-5.7570,-01
5.0000	2.1617,+01	2.8927,-01	-5.3629,-01
5.0625	2.1634,+01	2.6262,-01	-4.9938,-01
5.1250	2.1650,+01	2.3780,-01	-4.6486,-01
5.1875	2.1664,+01	2.1469,-01	-4.3262,-01
5.2500	2.1677,+01	1.9319,-01	-4.0254,-01

Table 5 (continued)

ξ	t	t'	t"
5.3125	2.1688,+01	1.7319,-01	-3.7451,-01
5.3750	2.1698,+01	1.5457,-01	-3.4844,-01
5.4375	2.1707,+01	1.3725,-01	-3.2421,-01
5.5000	2.1715,+01	1.2114,-01	-3.0173,-01
5.5625	2.1722,+01	1.0614,-01	-2.8091,-01
5.6250	2.1729,+01	9.2176,-02	-2.6165,-01
5.6875	2.1734,+01	7.9169,-02	-2.4387,-01
5.7500	2.1738,+01	6.7046,-02	-2.2747,-01
5.8125	2.1742,+01	5.5737,-02	-2.1238,-01
5.8750	2.1745,+01	4.5179,-02	-1.9851,-01
5.9375	2.1748,+01	3.5310,-02	-1.8579,-01
6.0000	2.1750,+01	2.6073,-02	-1.9834,-01
6.0625	2.1751,+01	1.6212,-02	-1.2332,-01
6.1250	2.1752,+01	1.0080,-02	-7.6685,-02
6.1875	2.1752,+01	6.2680,-03	-4.7682,-02
6.2500	2.1753,+01	3.8974,-03	-2.9648,-02
6.3125	2.1753,+01	2.4233,-03	-1.8435,-02
6.3750	2.1753,+01	1.5068,-03	-1.1462,-02
6.4375	2.1753,+01	9.3694,-04	-7.1275,-03
6.5000	2.1753,+01	5.8258,-04	-4.4318,-03
6.5625	2.1753,+01	3.6224,-04	-2.7556,-03
6.6250	2.1753,+01	2.2524,-04	-1.7134,-03
6.6875	2.1753,+01	1.4005,-04	-1.0654,-03
6.7500	2.1753,+01	8.7084,-05	-6.6247,-04
6.8125	2.1753,+01	5.4148,-05	-4.1192,-04
6.8750	2.1753,+01	3.3669,-05	-2.5612,-04
6.9375	2.1753,+01	2.0935,-05	-1.5925,-04
7.0000	2.1753,+01	1.3017,-05	-9.9026,-05
7.0625	2.1753,+01	8.0941,-06	-6.1573,-05
7.1250	2.1753,+01	5.0328,-06	-3.8286,-05
7.1875	2.1753,+01	3.1294,-06	-2.3806,-05
7.2500	2.1753,+01	1.9458,-06	-1.4802,-05
7.3125	2.1753,+01	1.2099,-06	-9.2040,-06
7.3750	2.1753,+01	7.5231,-07	-5.7230,-06
7.4375	2.1753,+01	4.6778,-07	-3.5585,-06
7.5000	2.1753,+01	2.9086,-07	-2.2126,-06
7.5625	2.1753,+01	1.8085,-07	-1.3758,-06
7.6250	2.1753,+01	1.1245,-07	-8.5547,-07
7.6875	2.1753,+01	6.9924,-08	-5.3192,-07
7.7500	2.1753,+01	4.3478,-08	-3.3074,-07
7.8125	2.1753,+01	2.7034,-08	-2.0565,-07
7.8750	2.1753,+01	1.6809,-08	-1.2787,-07
7.9375	2.1753,+01	1.0452,-08	-7.9512,-08
8.0000	2.1753,+01	6.4991,-09	-4.9440,-08

Table 6

A second homogeneous solution to the disk thermal boundary layer equation for Pr=10

ξ	t	t'	t"
.0000	1.0000,+00	0.0000,+00	-0.0000,+00
.0625	1.0000,+00	0.0000,+00	5.5644,-01
.1250	1.0010,+00	3.4771,-02	1.0462,+00
.1875	1.0053,+00	1.0013,-01	1.4709,+00
.2500	1.0144,+00	1.9202,-01	1.8307,+00
.3125	1.0300,+00	3.0638,-01	2.1249,+00
.3750	1.0533,+00	4.3911,-01	2.3522,+00
.4375	1.0853,+00	5.8606,-01	2.5121,+00
.5000	1.1268,+00	7.4307,-01	2.6045,+00
.5625	1.1784,+00	9.0597,-01	2.6307,+00
.6250	1.2401,+00	1.0707,+00	2.5935,+00
.6875	1.3121,+00	1.2334,+00	2.4973,+00
.7500	1.3941,+00	1.3907,+00	2.3481,+00
.8125	1.4857,+00	1.5392,+00	2.1532,+00
.8750	1.5861,+00	1.6763,+00	1.9212,+00
.9375	1.6947,+00	1.7998,+00	1.6611,+00
1.0000	1.8105,+00	1.9082,+00	1.3823,+00
1.0625	1.9326,+00	2.0004,+00	1.0937,+00
1.1250	2.0599,+00	2.0759,+00	8.0392,-01
1.1875	2.1914,+00	2.1347,+00	5.2010,-01
1.2500	2.3261,+00	2.1771,+00	2.4853,-01
1.3125	2.4629,+00	2.2039,+00	-5.9119,-03
1.3750	2.6009,+00	2.2162,+00	-2.3969,-01
1.4375	2.7392,+00	2.2151,+00	-4.5052,-01
1.5000	2.8771,+00	2.2020,+00	-6.3724,-01
1.5625	3.0138,+00	2.1783,+00	-7.9963,-01
1.6250	3.1488,+00	2.1453,+00	-9.3825,-01
1.6875	3.2814,+00	2.1044,+00	-1.0542,+00
1.7500	3.4113,+00	2.0569,+00	-1.1490,+00
1.8125	3.5380,+00	2.0039,+00	-1.2245,+00
1.8750	3.6612,+00	1.9466,+00	-1.2825,+00
1.9375	3.7808,+00	1.8858,+00	-1.3250,+00
2.0000	3.8965,+00	1.8225,+00	-1.3537,+00
2.0625	4.0082,+00	1.7574,+00	-1.3704,+00
2.1250	4.1158,+00	1.6911,+00	-1.3768,+00
2.1875	4.2192,+00	1.6242,+00	-1.3744,+00
2.2500	4.3184,+00	1.5573,+00	-1.3643,+00
2.3125	4.4135,+00	1.4906,+00	-1.3479,+00
2.3750	4.5044,+00	1.4246,+00	-1.3261,+00
2.4375	4.5912,+00	1.3595,+00	-1.2999,+00
2.5000	4.6740,+00	1.2956,+00	-1.2700,+00

Table 6 (continued)

ξ	t	t'	t"
2.5625	4.7529,+00	1.2331,+00	-1.2372,+00
2.6250	4.8279,+00	1.1721,+00	-1.2020,+00
2.6875	4.8991,+00	1.1128,+00	-1.1650,+00
2.7500	4.9668,+00	1.0553,+00	-1.1267,+00
2.8125	5.0308,+00	9.9971,-01	-1.0874,+00
2.8750	5.0915,+00	9.4595,-01	-1.0474,+00
2.9375	5.1489,+00	8.9414,-01	-1.0072,+00
3.0000	5.2031,+00	8.4430,-01	-9.6690,-01
3.0625	5.2542,+00	7.9643,-01	-9.2673,-01
3.1250	5.3024,+00	7.5053,-01	-8.8691,-01
3.1875	5.3478,+00	7.0658,-01	-8.4761,-01
3.2500	5.3906,+00	6.6458,-01	-8.0895,-01
3.3125	5.4308,+00	6.2448,-01	-7.7105,-01
3.3750	5.4685,+00	5.8624,-01	-7.3402,-01
3.4375	5.5039,+00	5.4984,-01	-6.9794,-01
3.5000	5.5371,+00	5.1522,-01	-6.6286,-01
3.5625	5.5682,+00	4.8233,-01	-6.2884,-01
3.6250	5.5973,+00	4.5113,-01	-5.9593,-01
3.6875	5.6245,+00	4.2155,-01	-5.6415,-01
3.7500	5.6499,+00	3.9356,-01	-5.3352,-01
3.8125	5.6736,+00	3.6707,-01	-5.0405,-01
3.8750	5.6957,+00	3.4204,-01	-4.7576,-01
3.9375	5.7163,+00	3.1642,-01	-4.4862,-01
4.0000	5.7354,+00	2.9615,-01	-4.2265,-01
4.0625	5.7532,+00	2.7516,-01	-3.9783,-01
4.1250	5.7697,+00	2.5540,-01	-3.7413,-01
4.1875	5.7851,+00	2.3682,-01	-3.5155,-01
4.2500	5.7993,+00	2.1936,-01	-3.3006,-01
4.3125	5.8124,+00	2.0297,-01	-3.0963,-01
4.3750	5.8246,+00	1.8758,-01	-2.9024,-01
4.4375	5.8358,+00	1.7317,-01	-2.7185,-01
4.5000	5.8462,+00	1.5966,-01	-2.5444,-01
4.5625	5.8558,+00	1.4702,-01	-2.3798,-01
4.6250	5.8645,+00	1.3520,-01	-2.2243,-01
4.6875	5.8726,+00	1.2414,-01	-2.0776,-01
4.7500	5.8800,+00	1.1382,-01	-1.9394,-01
4.8125	5.8868,+00	1.0418,-01	-1.8093,-01
4.8750	5.8930,+00	9.5199,-02	-1.6870,-01
4.9375	5.8987,+00	8.6815,-02	-1.5722,-01
5.0000	5.9039,+00	7.9002,-02	-1.4646,-01
5.0625	5.9085,+00	7.1723,-02	-1.3638,-01
5.1250	5.9128,+00	6.4945,-02	-1.2695,-01
5.1875	5.9166,+00	5.8635,-02	-1.1815,-01
5.2500	5.9201,+00	5.2763,-02	-1.0993,-01

Table 6 (continued)

ξ	t	t'	t"
5.3125	5.9232,+00	4.7299,-02	-1.0228,-01
5.3750	5.9260,+00	4.2215,-02	-9.5161,-02
5.4375	5.9285,+00	3.7485,-02	-8.8545,-02
5.5000	5.9307,+00	3.3084,-02	-8.2406,-02
5.5625	5.9326,+00	2.8987,-02	-7.6720,-02
5.6250	5.9343,+00	2.5174,-02	-7.1460,-02
5.6875	5.9357,+00	2.1621,-02	-6.6602,-02
5.7500	5.9370,+00	1.8310,-02	-6.2124,-02
5.8125	5.9380,+00	1.5222,-02	-5.8002,-02
5.8750	5.9389,+00	1.2338,-02	-5.4215,-02
5.9375	5.9396,+00	9.6435,-03	-5.0742,-02
6.0000	5.9401,+00	7.1207,-03	-5.4169,-02
6.0625	5.9404,+00	4.4276,-03	-3.3682,-02
6.1250	5.9407,+00	2.7530,-03	-2.0943,-02
6.1875	5.9408,+00	1.7118,-03	-1.3022,-02
6.2500	5.9409,+00	1.0644,-03	-8.0971,-03
6.3125	5.9409,+00	6.6184,-04	-5.0347,-03
6.3750	5.9410,+00	4.1152,-04	-3.1305,-03
6.4375	5.9410,+00	2.5588,-04	-1.9465,-03
6.5000	5.9410,+00	1.5910,-04	-1.2103,-03
6.5625	5.9410,+00	9.8932,-05	-7.5259,-04
6.6250	5.9410,+00	6.1515,-05	-4.6795,-04
6.6875	5.9410,+00	3.8249,-05	-2.9097,-04
6.7500	5.9410,+00	2.3783,-05	-1.8092,-04
6.8125	5.9410,+00	1.4788,-05	-1.1249,-04
6.8750	5.9410,+00	9.1953,-06	-6.9950,-05
6.9375	5.9410,+00	5.7175,-06	-4.3494,-05
7.0000	5.9410,+00	3.5551,-06	-2.7044,-05
7.0625	5.9410,+00	2.2105,-06	-1.6816,-05
7.1250	5.9410,+00	1.3745,-06	-1.0456,-05
7.1875	5.9410,+00	8.5466,-07	-6.5015,-06
7.2500	5.9410,+00	5.3142,-07	-4.0426,-06
7.3125	5.9410,+00	3.3043,-07	-2.5136,-06
7.3750	5.9410,+00	2.0546,-07	-1.5629,-06
7.4375	5.9410,+00	1.2775,-07	-9.7185,-07
7.5000	5.9410,+00	7.9436,-08	-6.0429,-07
7.5625	5.9410,+00	4.9393,-08	-3.7574,-07
7.6250	5.9410,+00	3.0712,-08	-2.3363,-07
7.6875	5.9410,+00	1.9096,-08	-1.4527,-07
7.7500	5.9410,+00	1.1874,-08	-9.0329,-08
7.8125	5.9410,+00	7.3833,-09	-5.6166,-08
7.8750	5.9410,+00	4.5908,-09	-3.4923,-08
7.9375	5.9410,+00	2.8545,-09	-2.1715,-08
8.0000	5.9410,+00	1.7749,-09	-1.3502,-08

Table 7

A particular solution to the disk thermal boundary layer equation for Pr = 50

ξ	t	t'	t"
.0000	1.0000,+00	4.0000,+00	-2.4911,+01
.0625	1.2013,+00	2.4430,+00	-1.9273,+01
.1250	1.3167,+00	1.2528,+00	-1.3693,+01
.1875	1.3686,+00	4.1594,-01	-8.3588,+00
.2500	1.3786,+00	-9.1162,-02	-3.4956,+00
.3125	1.3662,+00	-3.0414,-01	6.5791,-01
.3750	1.3483,+00	-2.7132,-01	3.8961,+00
.4375	1.3385,+00	-5.1063,-02	6.0923,+00
.5000	1.3464,+00	2.9350,-01	7.2334,+00
.5625	1.3779,+00	7.0134,-01	7.4267,+00
.6250	1.4353,+00	1.1201,+00	6.8763,+00
.6875	1.5179,+00	1.5108,+00	5.8350,+00
.7500	1.6231,+00	1.8494,+00	4.5510,+00
.8125	1.7474,+00	2.1254,+00	3.2252,+00
.8750	1.8868,+00	2.3386,+00	1.9910,+00
.9375	2.0375,+00	2.4950,+00	9.1584,-01
1.0000	2.1963,+00	2.6033,+00	1.6823,-02
1.0625	2.3605,+00	2.6726,+00	-7.1898,-01
1.1250	2.5279,+00	2.7108,+00	-1.3173,+00
1.1875	2.6968,+00	2.7244,+00	-1.8055,+00
1.2500	2.8657,+00	2.7185,+00	-2.2070,+00
1.3125	3.0337,+00	2.6968,+00	-2.5401,+00
1.3750	3.1999,+00	2.6625,+00	-2.8182,+00
1.4375	3.3634,+00	2.6179,+00	-3.0510,+00
1.5000	3.5239,+00	2.5650,+00	-3.2458,+00
1.5625	3.6807,+00	2.5053,+00	-3.4081,+00
1.6250	3.8335,+00	2.4402,+00	-3.5421,+00
1.6875	3.9820,+00	2.3708,+00	-3.6515,+00
1.7500	4.1261,+00	2.2981,+00	-3.7390,+00
1.8125	4.2654,+00	2.2230,+00	-3.8073,+00
1.8750	4.3999,+00	2.1462,+00	-3.8583,+00
1.9375	4.5295,+00	2.0683,+00	-3.8940,+00
2.0000	4.6541,+00	1.9898,+00	-3.9158,+00
2.0625	4.7738,+00	1.9114,+00	-3.9252,+00
2.1250	4.8885,+00	1.8333,+00	-3.9234,+00
2.1875	4.9984,+00	1.7559,+00	-3.9114,+00
2.2500	5.1033,+00	1.6795,+00	-3.8902,+00
2.3125	5.2035,+00	1.6044,+00	-3.8606,+00
2.3750	5.2990,+00	1.5308,+00	-3.8233,+00
2.4375	5.3900,+00	1.4588,+00	-3.7791,+00
2.5000	5.4764,+00	1.3687,+00	-3.7285,+00

Table 7 (continued)

ξ	t	t'	t"
2.5625	5.5586,+00	1.3205,+00	-3.6721,+00
2.6250	5.6365,+00	1.2543,+00	-3.6104,+00
2.6875	5.7104,+00	1.1902,+00	-3.5438,+00
2.7500	5.7803,+00	1.1282,+00	-3.4729,+00
2.8125	5.8464,+00	1.0684,+00	-3.3980,+00
2.8750	5.9088,+00	1.0108,+00	-3.3196,+00
2.9375	5.9678,+00	9.5544,-01	-3.2379,+00
3.0000	6.0233,+00	9.0223,-01	-3.1534,+00
3.0625	6.0757,+00	8.5120,-01	-3.0665,+00
3.1250	6.1249,+00	8.0232,-01	-2.9775,+00
3.1875	6.1712,+00	7.5557,-01	-2.8866,+00
3.2500	6.2147,+00	7.1089,-01	-2.7944,+00
3.3125	6.2556,+00	6.6826,-01	-2.7010,+00
3.3750	6.2938,+00	6.2762,-01	-2.6069,+00
3.4375	6.3297,+00	5.8892,-01	-2.5123,+00
3.5000	6.3632,+00	5.5210,-01	-2.4176,+00
3.5625	6.3946,+00	5.1712,-01	-2.3229,+00
3.6250	6.4238,+00	4.8392,-01	-2.2287,+00
3.6875	6.4512,+00	4.5243,-01	-2.1352,+00
3.7500	6.4767,+00	4.2259,-01	-2.0427,+00
3.8125	6.5004,+00	3.9435,-01	-1.9514,+00
3.8750	6.5225,+00	3.6765,-01	-1.8615,+00
3.9375	6.5430,+00	3.4242,-01	-1.7732,+00
4.0000	6.5621,+00	3.1860,-01	-1.6868,+00
4.0625	6.5798,+00	2.9615,-01	-1.6024,+00
4.1250	6.5962,+00	2.7498,-01	-1.5201,+00
4.1875	6.6114,+00	2.5506,-01	-1.4402,+00
4.2500	6.6254,+00	2.3632,-01	-1.3627,+00
4.3125	6.6384,+00	2.1871,-01	-1.2878,+00
4.3750	6.6504,+00	2.0217,-01	-1.2155,+00
4.4375	6.6614,+00	1.8665,-01	-1.1459,+00
4.5000	6.6716,+00	1.7210,-01	-1.0790,+00
4.5625	6.6809,+00	1.5847,-01	-1.0148,+00
4.6250	6.6895,+00	1.4571,-01	-9.5355,-01
4.6875	6.6973,+00	1.3377,-01	-8.9499,-01
4.7500	6.7045,+00	1.2261,-01	-8.3920,-01
4.8125	6.7110,+00	1.1218,-01	-7.8616,-01
4.8750	6.7170,+00	1.0245,-01	-7.3584,-01
4.9375	6.7224,+00	9.3367,-02	-6.8817,-01
5.0000	6.7274,+00	8.4895,-02	-6.4313,-01
5.0625	6.7318,+00	7.6998,-02	-6.0063,-01
5.1250	6.7358,+00	6.9639,-02	-5.6061,-01
5.1875	6.7394,+00	6.2784,-02	-5.2300,-01
5.2500	6.7427,+00	5.6401,-02	-4.8772,-01

Table 7 (continued)

ξ	t	t'	t"
5.3125	6.7456,+00	5.0458,-02	-4.5469,-01
5.3750	6.7481,+00	4.4926,-02	-4.2381,-01
5.4375	6.7504,+00	3.9775,-02	-3.9501,-01
5.5000	6.7523,+00	3.4980,-02	-3.6820,-01
5.5625	6.7540,+00	3.0514,-02	-3.4328,-01
5.6250	6.7555,+00	2.6354,-02	-3.2016,-01
5.6875	6.7567,+00	2.2477,-02	-2.9877,-01
5.7500	6.7577,+00	1.8860,-02	-2.7900,-01
5.8125	6.7585,+00	1.5484,-02	-2.6078,-01
5.8750	6.7591,+00	1.2330,-02	-2.4401,-01
5.9375	6.7596,+00	9.3790,-03	-2.2862,-01
6.0000	6.7599,+00	6.6143,-03	-2.5158,-01
6.0625	6.7599,+00	3.5719,-03	-1.3586,-01
6.1250	6.7600,+00	1.9289,-03	-7.3371,-02
6.1875	6.7600,+00	1.0417,-03	-3.9623,-02
6.2500	6.7600,+00	5.6256,-04	-2.1397,-02
6.3125	6.7600,+00	3.0380,-04	-1.1555,-02
6.3750	6.7600,+00	1.6406,-04	-6.2403,-03
6.4375	6.7600,+00	8.8600,-05	-3.3700,-03
6.5000	6.7600,+00	4.7647,-05	-1.8199,-03
6.5625	6.7600,+00	2.5639,-05	-9.8282,-04
6.6250	6.7600,+00	1.3954,-05	-5.3075,-04
6.6875	6.7600,+00	7.5357,-06	-2.8662,-04
6.7500	6.7600,+00	4.0695,-06	-1.5478,-04
6.8125	6.7600,+00	2.1976,-06	-8.3591,-05
6.8750	6.7600,+00	1.1868,-06	-4.5142,-05
6.9375	6.7600,+00	6.4092,-07	-2.4378,-05
7.0000	6.7600,+00	3.4612,-07	-1.3165,-05
7.0625	6.7600,+00	1.8691,-07	-7.1096,-06
7.1250	6.7600,+00	1.0094,-07	-3.8394,-06
7.1875	6.7600,+00	5.4512,-08	-2.0734,-06
7.2500	6.7600,+00	2.9438,-08	-1.1197,-06
7.3125	6.7600,+00	1.5897,-08	-6.0468,-07
7.3750	6.7600,+00	8.5853,-09	-3.2655,-07
7.4375	6.7600,+00	4.6363,-09	-1.7634,-07
7.5000	6.7600,+00	2.5038,-09	-9.5234,-08
7.5625	6.7600,+00	1.3521,-09	-5.1430,-08
7.6250	6.7600,+00	7.3020,-10	-2.7773,-08
7.6875	6.7600,+00	3.9433,-10	-1.4998,-08
7.7500	6.7600,+00	2.1295,-10	-8.0999,-09
7.8125	6.7600,+00	1.1500,-10	-4.3742,-09
7.8750	6.7600,+00	6.2105,-11	-2.3622,-09
7.9375	6.7600,+00	3.3538,-11	-1.2756,-09
8.0000	6.7600,+00	1.8112,-11	-6.8891,-10

Table 8

A homogeneous solution to the disk thermal
boundary layer equation for Pr = 50

ξ	t	t'	t''
.0000	1.0000,+00	4.0000,+00	-0.0000,+00
.0625	1.2500,+00	4.0000,+00	3.1229,+00
.1250	1.5065,+00	4.2167,+00	6.4525,+00
.1875	1.7835,+00	4.6599,+00	9.7446,+00
.2500	2.0949,+00	5.3244,+00	1.2700,+01
.3125	2.4539,+00	6.1865,+00	1.5002,+01
.3750	2.8715,+00	7.2040,+00	1.6364,+01
.4375	3.3556,+00	8.3193,+00	1.6600,+01
.5000	3.9102,+00	9.4659,+00	1.5665,+01
.5625	4.5352,+00	1.0577,+01	1.3673,+01
.6250	5.2265,+00	1.1597,+01	1.0868,+01
.6875	5.9770,+00	1.2484,+01	7.5661,+00
.7500	6.7775,+00	1.3215,+01	4.0813,+00
.8125	7.6181,+00	1.3784,+01	6.7397,-01
.8750	8.4898,+00	1.4197,+01	-2.4813,+00
.9375	9.3803,+00	1.4470,+01	-5.2961,+00
1.0000	1.0284,+01	1.4620,+01	-7.7484,+00
1.0625	1.1194,+01	1.4666,+01	-9.8561,+00
1.1250	1.2103,+01	1.4625,+01	-1.1663,+01
1.1875	1.3008,+01	1.4511,+01	-1.3207,+01
1.2500	1.3902,+01	1.4336,+01	-1.4526,+01
1.3125	1.4784,+01	1.4110,+01	-1.5653,+01
1.3750	1.5650,+01	1.3842,+01	-1.6613,+01
1.4375	1.6497,+01	1.3539,+01	-1.7426,+01
1.5000	1.7325,+01	1.3207,+01	-1.8108,+01
1.5625	1.8130,+01	1.2851,+01	-1.8675,+01
1.6250	1.8913,+01	1.2477,+01	-1.9138,+01
1.6875	1.9671,+01	1.2089,+01	-1.9506,+01
1.7500	2.0404,+01	1.1691,+01	-1.9791,+01
1.8125	2.1112,+01	1.1285,+01	-1.9999,+01
1.8750	2.1793,+01	1.0875,+01	-2.0138,+01
1.9375	2.2449,+01	1.0463,+01	-2.0215,+01
2.0000	2.3079,+01	1.0052,+01	-2.0235,+01
2.0625	2.3693,+01	9.6436,+00	-2.0204,+01
2.1250	2.4262,+01	9.2389,+00	-2.0125,+01
2.1875	2.4814,+01	8.8398,+00	-2.0004,+01
2.2500	2.5343,+01	8.4474,+00	-1.9843,+01
2.3125	2.5846,+01	8.0629,+00	-1.9647,+01
2.3750	2.6326,+01	7.6871,+00	-1.9418,+01
2.4375	2.6782,+01	7.3207,+00	-1.9158,+01
2.5000	2.7216,+01	6.9644,+00	-1.8871,+01

Table 8 (continued)

ξ	t	t'	t"
2.5625	2.7628,+01	6.6185,+00	-1.8559,+01
2.6250	2.8018,+01	6.2834,+00	-1.8223,+01
2.6875	2.8388,+01	5.9594,+00	-1.7867,+01
2.7500	2.8737,+01	5.6466,+00	-1.7490,+01
2.8125	2.9068,+01	5.3452,+00	-1.7097,+01
2.8750	2.9380,+01	5.0552,+00	-1.6688,+01
2.9375	2.9675,+01	4.7765,+00	-1.6264,+01
3.0000	2.9953,+01	4.5091,+00	-1.5828,+01
3.0625	3.0214,+01	4.2528,+00	-1.5382,+01
3.1250	3.0460,+01	4.0075,+00	-1.4926,+01
3.1875	3.0692,+01	3.7730,+00	-1.4463,+01
3.2500	3.0909,+01	3.5491,+00	-1.3994,+01
3.3125	3.1112,+01	3.3356,+00	-1.3520,+01
3.3750	3.1303,+01	3.1321,+00	-1.3043,+01
3.4375	3.1482,+01	2.9384,+00	-1.2565,+01
3.5000	3.1650,+01	2.7543,+00	-1.2087,+01
3.5625	3.1806,+01	2.5794,+00	-1.1610,+01
3.6250	3.1952,+01	2.4134,+00	-1.1136,+01
3.6875	3.2088,+01	2.2560,+00	-1.0665,+01
3.7500	3.2215,+01	2.1070,+00	-1.0200,+01
3.8125	3.2334,+01	1.9660,+00	-9.7424,+00
3.8750	3.2444,+01	1.8326,+00	-9.2914,+00
3.9375	3.2546,+01	1.7067,+00	-8.8490,+00
4.0000	3.2641,+01	1.5879,+00	-8.4161,+00
4.0625	3.2729,+01	1.4758,+00	-7.9935,+00
4.1250	3.2811,+01	1.3702,+00	-7.5821,+00
4.1875	3.2887,+01	1.2709,+00	-7.1824,+00
4.2500	3.2957,+01	1.1774,+00	-6.7951,+00
4.3125	3.3021,+01	1.0896,+00	-6.4205,+00
4.3750	3.3081,+01	1.0071,+00	-6.0593,+00
4.4375	3.3136,+01	9.2982,-01	-5.7115,+00
4.5000	3.3187,+01	8.5729,-01	-5.3775,+00
4.5625	3.3233,+01	7.8936,-01	-5.0575,+00
4.6250	3.3276,+01	7.2576,-01	-4.7513,+00
4.6875	3.3315,+01	6.6628,-01	-4.4592,+00
4.7500	3.3351,+01	6.1067,-01	-4.1809,+00
4.8125	3.3383,+01	5.5872,-01	-3.9163,+00
4.8750	3.3413,+01	5.1023,-01	-3.6654,+00
4.9375	3.3440,+01	4.6498,-01	-3.4278,+00
5.0000	3.3464,+01	4.2278,-01	-3.2032,+00
5.0625	3.3487,+01	3.8344,-01	-2.9913,+00
5.1250	3.3507,+01	3.4679,-01	-2.7919,+00
5.1875	3.3525,+01	3.1266,-01	-2.6045,+00
5.2500	3.3541,+01	2.8087,-01	-2.4287,+00

Table 8 (continued)

ξ	t	t'	t"
5.3125	3.3555,+01	2.5128,-01	-2.2641,+00
5.3750	3.3568,+01	2.2373,-01	-2.1103,+00
5.4375	3.3579,+01	1.9808,-01	-1.9668,+00
5.5000	3.3589,+01	1.7420,-01	-1.8332,+00
5.5625	3.3597,+01	1.5197,-01	-1.7091,+00
5.6250	3.3604,+01	1.3126,-01	-1.5939,+00
5.6875	3.3611,+01	1.1195,-01	-1.4874,+00
5.7500	3.3616,+01	9.3953,-02	-1.3889,+00
5.8125	3.3620,+01	7.7147,-02	-1.2982,+00
5.8750	3.3623,+01	6.1443,-02	-1.2147,+00
5.9375	3.3625,+01	4.6752,-02	-1.1380,+00
6.0000	3.3626,+01	3.2990,-02	-1.2548,+00
6.0625	3.3627,+01	1.7816,-02	-6.7765,-01
6.1250	3.3627,+01	9.6213,-03	-3.6595,-01
6.1875	3.3627,+01	5.1958,-03	-1.9762,-01
6.2500	3.3627,+01	2.8059,-03	-1.0672,-01
6.3125	3.3627,+01	1.5153,-03	-5.7636,-02
6.3750	3.3627,+01	8.1831,-04	-3.1125,-02
6.4375	3.3627,+01	4.4191,-04	-1.6808,-02
6.5000	3.3627,+01	2.3865,-04	-9.0773,-03
6.5625	3.3627,+01	1.2887,-04	-4.9020,-03
6.6250	3.3627,+01	6.9599,-05	-2.6472,-03
6.6875	3.3627,+01	3.7586,-05	-1.4296,-03
6.7500	3.3627,+01	2.0297,-05	-7.7204,-04
6.8125	3.3627,+01	1.0961,-05	-4.1693,-04
6.8750	3.3627,+01	5.9195,-06	-2.2515,-04
6.9375	3.3627,+01	3.1967,-06	-1.2159,-04
7.0000	3.3627,+01	1.7263,-06	-6.5664,-05
7.0625	3.3627,+01	9.3229,-07	-3.5460,-05
7.1250	3.3627,+01	5.0347,-07	-1.9150,-05
7.1875	3.3627,+01	2.7189,-07	-1.0341,-05
7.2500	3.3627,+01	1.4683,-07	-5.5848,-06
7.3125	3.3627,+01	7.9293,-08	-3.0160,-06
7.3750	3.3627,+01	4.2821,-08	-1.6287,-06
7.4375	3.3627,+01	2.3125,-08	-8.7958,-07
7.5000	3.3627,+01	1.2488,-08	-4.7500,-07
7.5625	3.3627,+01	6.7441,-09	-2.5651,-07
7.6250	3.3627,+01	3.6420,-09	-1.3852,-07
7.6875	3.3627,+01	1.9668,-09	-7.4810,-08
7.7500	3.3627,+01	1.0621,-09	-4.0400,-08
7.8125	3.3627,+01	5.7360,-10	-2.1817,-08
7.8750	3.3627,+01	3.0976,-10	-1.1782,-08
7.9375	3.3627,+01	1.6728,-10	-6.3627,-09
8.0000	3.3627,+01	9.0338,-11	-3.4361,-09

Table 9

A second homogeneous solution to the disk thermal boundary layer equation for Pr = 50

ξ	t	t'	t"
.0000	1.0000,+00	0.0000,+00	-0.0000,+00
.0625	1.0000,+00	0.0000,+00	2.7822,+00
.1250	1.0054,+00	1.7372,-01	5.2065,+00
.1875	1.0264,+00	4.9855,-01	7.2174,+00
.2500	1.0716,+00	9.4863,-01	8.7315,+00
.3125	1.1479,+00	1.4933,+00	9.6648,+00
.3750	1.2601,+00	2.0977,+00	9.9617,+00
.4375	1.4108,+00	2.7245,+00	9.6208,+00
.5000	1.6001,+00	3.3375,+00	8.7074,+00
.5625	1.8261,+00	3.9057,+00	7.3478,+00
.6250	2.0855,+00	4.4062,+00	5.7057,+00
.6875	2.3733,+00	4.8257,+00	3.9503,+00
.7500	2.6845,+00	5.1603,+00	2.2257,+00
.8125	3.0178,+00	5.4135,+00	6.3287,-01
.8750	3.3563,+00	5.5935,+00	-7.7535,-01
.9375	3.7079,+00	5.7103,+00	-1.9853,+00
1.0000	4.0649,+00	5.7744,+00	-3.0093,+00
1.0625	4.4243,+00	5.7949,+00	-3.8716,+00
1.1250	4.7837,+00	5.7797,+00	-4.5991,+00
1.1875	5.1410,+00	5.7351,+00	-5.2153,+00
1.2500	5.4946,+00	5.6660,+00	-5.7395,+00
1.3125	5.8431,+00	5.5768,+00	-6.1859,+00
1.3750	6.1853,+00	5.4708,+00	-6.5656,+00
1.4375	6.5203,+00	5.3509,+00	-6.8870,+00
1.5000	6.8473,+00	5.2197,+00	-7.1569,+00
1.5625	7.1656,+00	5.0792,+00	-7.3809,+00
1.6250	7.4748,+00	4.9314,+00	-7.5637,+00
1.6875	7.7744,+00	4.7780,+00	-7.7095,+00
1.7500	8.0642,+00	4.6205,+00	-7.8219,+00
1.8125	8.3439,+00	4.4602,+00	-7.9042,+00
1.8750	8.6134,+00	4.2981,+00	-7.9593,+00
1.9375	8.8726,+00	4.1354,+00	-7.9896,+00
2.0000	9.1215,+00	3.9729,+00	-7.9976,+00
2.0625	9.3602,+00	3.8113,+00	-7.9851,+00
2.1250	9.5888,+00	3.6514,+00	-7.9541,+00
2.1875	9.8074,+00	3.4936,+00	-7.9061,+00
2.2500	1.0016,+01	3.3386,+00	-7.8427,+00
2.3125	1.0215,+01	3.1866,+00	-7.7651,+00
2.3750	1.0404,+01	3.0381,+00	-7.6745,+00
2.4375	1.0585,+01	2.8933,+00	-7.5719,+00
2.5000	1.0756,+01	2.7524,+00	-7.4585,+00

Table 9 (continued)

ξ	t	t'	t"
2.5625	1.0919,+01	2.6157,+00	-7.3350,+00
2.6250	1.1073,+01	2.4833,+00	-7.2024,+00
2.6875	1.1219,+01	2.3552,+00	-7.0614,+00
2.7500	1.1357,+01	2.2316,+00	-6.9127,+00
2.8125	1.1488,+01	2.1125,+00	-6.7572,+00
2.8750	1.1611,+01	1.9979,+00	-6.5954,+00
2.9375	1.1728,+01	1.8877,+00	-6.4281,+00
3.0000	1.1838,+01	1.7820,+00	-6.2559,+00
3.0625	1.1941,+01	1.6808,+00	-6.0794,+00
3.1250	1.2038,+01	1.5838,+00	-5.8993,+00
3.1875	1.2130,+01	1.4912,+00	-5.7162,+00
3.2500	1.2215,+01	1.4027,+00	-5.5308,+00
3.3125	1.2296,+01	1.3183,+00	-5.3435,+00
3.3750	1.2371,+01	1.2378,+00	-5.1551,+00
3.4375	1.2442,+01	1.1613,+00	-4.9661,+00
3.5000	1.2508,+01	1.0885,+00	-4.7771,+00
3.5625	1.2570,+01	1.0194,+00	-4.5886,+00
3.6250	1.2628,+01	9.5384,-01	-4.4012,+00
3.6875	1.2682,+01	8.9165,-01	-4.2153,+00
3.7500	1.2732,+01	8.3275,-01	-4.0316,+00
3.8125	1.2779,+01	7.7701,-01	-3.8504,+00
3.8750	1.2822,+01	7.2432,-01	-3.6721,+00
3.9375	1.2863,+01	6.7454,-01	-3.4973,+00
4.0000	1.2900,+01	6.2757,-01	-3.3262,+00
4.0625	1.2935,+01	5.8328,-01	-3.1592,+00
4.1250	1.2967,+01	5.4156,-01	-2.9966,+00
4.1875	1.2997,+01	5.0229,-01	-2.8386,+00
4.2500	1.3025,+01	4.6535,-01	-2.6855,+00
4.3125	1.3050,+01	4.3064,-01	-2.5375,+00
4.3750	1.3074,+01	3.9806,-01	-2.3947,+00
4.4375	1.3096,+01	3.6748,-01	-2.2573,+00
4.5000	1.3116,+01	3.3882,-01	-2.1253,+00
4.5625	1.3134,+01	3.1197,-01	-1.9988,+00
4.6250	1.3151,+01	2.8683,-01	-1.8778,+00
4.6875	1.3166,+01	2.6332,-01	-1.7623,+00
4.7500	1.3181,+01	2.4135,-01	-1.6523,+00
4.8125	1.3193,+01	2.2082,-01	-1.5478,+00
4.8750	1.3205,+01	2.0165,-01	-1.4486,+00
4.9375	1.3216,+01	1.8377,-01	-1.3547,+00
5.0000	1.3226,+01	1.6709,-01	-1.2659,+00
5.0625	1.3234,+01	1.5154,-01	-1.1822,+00
5.1250	1.3242,+01	1.3706,-01	-1.1034,+00
5.1875	1.3249,+01	1.2357,-01	-1.0293,+00
5.2500	1.3256,+01	1.1100,-01	-9.5988,-01

Table 9 (continued)

ξ	t	t'	t"
5.3125	1.3261,+01	9.9311,-02	-8.9483,-01
5.3750	1.3266,+01	8.8423,-02	-8.3403,-01
5.4375	1.3271,+01	7.8287,-02	-7.7733,-01
5.5000	1.3275,+01	6.8851,-02	-7.2454,-01
5.5625	1.3278,+01	6.0063,-02	-6.7548,-01
5.6250	1.3281,+01	5.1877,-02	-6.2998,-01
5.6875	1.3283,+01	4.4248,-02	-5.8786,-01
5.7500	1.3285,+01	3.7132,-02	-5.4895,-01
5.8125	1.3287,+01	3.0490,-02	-5.1308,-01
5.8750	1.3288,+01	2.4283,-02	-4.8008,-01
5.9375	1.3289,+01	1.8477,-02	-4.4979,-01
6.0000	1.3290,+01	1.3038,-02	-4.9593,-01
6.0625	1.3290,+01	7.0413,-03	-2.6782,-01
6.1250	1.3290,+01	3.8025,-03	-1.4463,-01
6.1875	1.3290,+01	2.0535,-03	-7.8107,-02
6.2500	1.3290,+01	1.1089,-03	-4.2180,-02
6.3125	1.3290,+01	5.9887,-04	-2.2778,-02
6.3750	1.3290,+01	3.2341,-04	-1.2301,-02
6.4375	1.3290,+01	1.7465,-04	-6.6431,-03
6.5000	1.3290,+01	9.4319,-05	-3.5875,-03
6.5625	1.3290,+01	5.0935,-05	-1.9374,-03
6.6250	1.3290,+01	2.7507,-05	-1.0462,-03
6.6875	1.3290,+01	1.4854,-05	-5.6501,-04
6.7500	1.3290,+01	8.0221,-06	-3.0512,-04
6.8125	1.3290,+01	4.3322,-06	-1.6478,-04
6.8750	1.3290,+01	2.3395,-06	-8.8986,-05
6.9375	1.3290,+01	1.2634,-06	-4.8055,-05
7.0000	1.3290,+01	6.8229,-07	-2.5951,-05
7.0625	1.3290,+01	3.6846,-07	-1.4014,-05
7.1250	1.3290,+01	1.9898,-07	-7.5685,-06
7.1875	1.3290,+01	1.0745,-07	-4.0872,-06
7.2500	1.3290,+01	5.8030,-08	-2.2072,-06
7.3125	1.3290,+01	3.1338,-08	-1.1919,-06
7.3750	1.3290,+01	1.6923,-08	-6.4371,-07
7.4375	1.3290,+01	9.1395,-09	-3.4763,-07
7.5000	1.3290,+01	4.9356,-09	-1.8773,-07
7.5625	1.3290,+01	2.6654,-09	-1.0138,-07
7.6250	1.3290,+01	1.4394,-09	-5.4749,-08
7.6875	1.3290,+01	7.7733,-10	-2.9566,-08
7.7500	1.3290,+01	4.1978,-10	-1.5967,-08
7.8125	1.3290,+01	2.2669,-10	-8.6227,-09
7.8750	1.3290,+01	1.2242,-10	-4.6565,-09
7.9375	1.3290,+01	6.6113,-11	-2.5147,-09
8.0000	1.3290,+01	3.5703,-11	-1.3580,-09

TABLE 10

Initial Values to the Two-Point Boundary Value Problem from the Analog Solution

Re/S^2	B	G'(0)	F"(0)
10.0	.2674	-1.347	-.096
12.5	.2469	-1.520	-.119
16.6	.2337	-1.770	-.152
25.0	.1813	-2.420	-.314
50.0	.1138	-4.040	-.361