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A Method for Obtaining Desired Sensitivity Characteristics with Optimal Controls

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Abstract

An approach is offered to the problem of designing feedback optimal controls to minimize the effects of parameter variations on system performance. It is found that the optimal control can be expressed as a linear function of the state. The resulting design is superior, from the standpoint of trajectory sensitivity, to systems whose controls depend explicitly only upon the time. The solution of an example problem is included.

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1. Introduction

Consider a dynamical system described by the vector differential equation

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), \underline{u}(t), a, t] \quad ; \quad \underline{x}(t_0) = \underline{x}_0 \tag{1}$$

where $\underline{x}(t)$ is an n-dimensional state vector, $\underline{u}(t)$ is an m-dimensional vector of control variables, specified as a function of time, and "a" is a constant parameter. If $\underline{x}(a, t)$ is a solution of (1) for the assumed value of "a", then the perturbed trajectory, as a result of the parameter variation Δa, is described to first order by*

$$\underline{x}_p(a+\Delta a, t) = \underline{x}(a, t) + \frac{\partial \underline{x}(a, t)}{\partial a} \Delta a \tag{2}$$

That is, to first order, $\underline{x}_p(a+\Delta a, t)$ is the solution of (1) with "a" replaced by $a+\Delta a$. It should be emphasized that both "a" and Δa are constant in time. Furthermore, it can be shown[1] that the "sensitivity variables", defined as

$$\underline{x}_a(t) = \frac{\partial \underline{x}(a, t)}{\partial a}$$

satisfy the differential equations

$$\dot{\underline{x}}_a(t) = \frac{\partial \underline{f}}{\partial \underline{x}} \underline{x}_a(t) + \frac{\partial \underline{f}}{\partial a} \quad ; \quad \underline{x}_a(t_0) = \underline{0} \tag{3}$$

where it is understood that the partial derivatives of $\underline{f}$ are evaluated along $\underline{x}(a, t)$.

Several authors[1, 2, 3] have studied the system of equations (1, 3) with the purpose of determing an open-loop control $\underline{u}(t)$ to suppress the effects of a parameter variation Δa. That is, the control is designed so that $\underline{x}_a(t)$ is small, in some sense. The methods of optimal control theory are applied to the problem of obtaining a control $\underline{u}(t)$ to minimize a performance index involving $\underline{x}(t)$, $\underline{x}_a(t)$ and $\underline{u}(t)$. A salient feature of all these treatments is that the control is implemented only as an explicit function of the time; hence the control is open loop.

*All partial derivatives appearing in this report are assumed to exist.

The possibility of designing feedback controls $\underline{u}_1[\underline{x}(t), t]$ to obtain desired sensitivity characteristics, has also been considered. In this situation, equations (1, 3) are altered significantly because of the fact that the control is an explicit function of the state. The equations become

$$\dot{\underline{x}}_1(t) = \underline{f}\left\{\underline{x}_1(t), \underline{u}_1[\underline{x}_1(t), t], a, t\right\} \quad ; \quad \underline{x}_1(t_0) = \underline{x}_0 \tag{4}$$

$$\dot{\underline{x}}_{a_1}(t) = \left[\frac{\partial \underline{f}}{\partial \underline{x}_1} + \frac{\partial \underline{f}}{\partial \underline{u}_1}\frac{\partial \underline{u}_1}{\partial \underline{x}_1}\right]\underline{x}_{a_1}(t) + \frac{\partial \underline{f}}{\partial a} \quad ; \quad \underline{x}_{a_1}(t_0) = 0 \tag{5}$$

The presence of the partial derivative of the control with respect to the state in Eq. (5) has led previous authors[3, 4] to the conclusion that optimal control techniques cannot be used to determine $\underline{u}_1[\underline{x}_1(t), t]$ unless its functional dependence on $\underline{x}_1(t)$ is known or assumed a priori. Kriendler[3] treats this problem with respect to linear systems and proposes a method for approximating $\partial\underline{u}_1/\partial\underline{x}_1$. Dougherty, et al.,[4] consider nonlinear systems and suggest that one must impose a structural form on the control at the onset, so that $\partial\underline{u}_1/\partial\underline{x}_1$ can be expressed in terms of control variables and state variables. Neither approach yields truly optimal solutions. In the first case,[3] one must be content with a differential equation for $\underline{x}_a(t)$ which is approximate; in the second,[4] the control is optimal only with respect to the class of controls admitted by the imposed structure.

The purpose of this report is to present a method of obtaining an optimal control for Eqs. (4) and (5). It is shown that the feedback realization of the optimal control is nonunique and that a control which is linear in $\underline{x}_1(t)$ is one admissible realization of the feedback optimal control. To illustrate the benefits of feedback control, an example, providing a comparison with an open-loop design, is presented.

2. Definitions: Open-Loop, Feedback, and Closed-Loop Controls

The purpose of this section is to provide precise definitions for the familiar adjectives — open-loop, feedback, and closed-loop — which apply to this discussion. Although the material in this section is well known to many readers, it is included for completeness.

Suppose an open-loop control $\underline{u}(t)$, $t_0 \leqq t \leqq T$, which accomplishes some objective is known for any system of equations having the form of Eqs. (1). Assume that "a" is known perfectly. With the initial conditions $\underline{x}_0$ and t_0 regarded as parameters, $\underline{u}(t)$ can be written implicitly as

$$\underline{u}(t) = \underline{u}[\underline{x}_0, t_0, t] \tag{6}$$

The term, open-loop, implies that $\underline{u}(t)$ is suitable only for a particular set of initial conditions.

If the right-hand side of Eq. (6) is available in closed form, a closed-loop control is obtained by making the change of variables

$$\underline{x}_0 \longrightarrow \underline{x}(t)$$

$$t_0 \longrightarrow t$$

and is written as

$$\underline{u}_2[\underline{x}(t), t] = \underline{u}[\underline{x}(t), t, t] \tag{7}$$

A closed-loop control attains the desired objectives regardless of the initial conditions provided the actual state, $\underline{x}(t)$, is known perfectly. This self-correcting property of closed-loop controls is well known and is the motivation for implementing them whenever possible, assuming the parameter "a" in Eqs. (1) is perfectly known.

In many applications $\underline{u}[\underline{x}_0, t_0, t]$ in Eq. (6) is not available in closed form so that a closed-loop control cannot be derived. Such is often the case with solutions of optimal control problems. Now observe that given an open-loop control, a feedback control $\underline{u}_1[\underline{x}_0, t_0, \underline{x}(t), t]$ can be defined by

$$\underline{u}_1[\underline{x}_0, t_0, \underline{x}(t), t] = \underline{u}[\underline{x}_0, t_0, t] + \underline{g}[\underline{x}_0, t_0, \underline{x}(t), t] \tag{8}$$

where $\underline{g}[\ ]$ is any function such that

$$\underline{g}[\underline{x}_0, t_0, \underline{x}_0(t), t] \equiv \underline{0} \tag{9}$$

and $\underline{x}_0(t)$ is the solution to Eqs. (1) with $\underline{u}(t)$ defined by Eq. (6). Clearly, Eqs. (6) and (7) are special cases of Eq. (8). The feedback control as defined here in general combines the characteristics of open-loop and closed-loop controls. Thus it is also referred to as a "semi-closed-loop" control. It depends upon particular initial conditions but is also affected by the actual state, $\underline{x}(t)$. As an example, $\underline{g}[\ ]$ might be selected so as to maintain $\underline{x}(t) \cong \underline{x}_0(t)$.

With respect to these definitions, closed-loop control is usually most desirable for achieving an objective, provided the parameter in Eq. (1) is known perfectly. In the next section it is shown that feedback (semi-closed-loop) controls are most appropriate for systems in which sensitivity variables are included in the definition of the system state to account for possible uncertainty in constant parameters. In the sequel the explicit dependence of open-loop and feedback controls upon t_0 and $\underline{x}_0$ is understood and is omitted in the notation.

3. The Effect of Feedback Controls Upon Sensitivity State Variables

To understand the effect of feedback control upon sensitivity state variables, $\underline{x}_a(t)$, consider an open-loop optimal control problem associated with Eqs. (1) and (3).

Open-Loop Optimal Control Problem

Find the optimal open-loop control $\underline{u}(t) = \underline{u}^*(t)$ such that

$$
\begin{aligned}
J^*[\underline{u}^*(t)] &= \min_{\underline{u}} J[\underline{u}(t)] \\
J[\underline{u}(t)] &= \phi[\underline{x}(T), \underline{x}_a(T), T] + \int_{t_o}^{T} L[\underline{x}(t), \underline{x}_a(t), \underline{u}(t), t]\,dt
\end{aligned}
\tag{10}
$$

subject to the constraints

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), \underline{u}(t), a, t] \quad ; \quad \underline{x}(t_0) = \underline{x}_0 \tag{11}$$

$$\dot{\underline{x}}_a(t) = \frac{\partial \underline{f}}{\partial \underline{x}} \underline{x}_a(t) + \frac{\partial \underline{f}}{\partial a} \quad ; \quad \underline{x}_a(t_0) = \underline{0} \tag{12}$$

$$\underline{\psi}[\underline{x}(T), \underline{x}_a(T), T] = \underline{0} \tag{13}$$

Note that Eqs. (11) and (12), taken together, have the form of Eqs. (1). Hence, if $\underline{u}^*(t)$ can be found explicitly in the form of Eq. (6), one might formally implement the associated closed-loop optimal control

$$\underline{u}^*[\underline{x}(t),\ \underline{x}_a(t),\ t,\ t] \tag{14}$$

by making the identifications $\underline{x}_0 \longrightarrow \underline{x}(t)$, $\underline{x}_a(t_0) \longrightarrow \underline{x}_a(t)$, and $t_0 \longrightarrow t$. However, although this is the closed-loop solution to the mathematical optimization problem posed by Eqs. (10)-(13), the state variables $\underline{x}_a(t)$ defined by Eqs. (12) do not have the proper physical significance unless $\underline{u}(t)$ is open-loop. Remember that Eq. (12) is derived[1] assuming $\underline{u}(t)$ is unaffected by a perturbation in "a"; the fact that the closed-loop control depends upon $\underline{x}(t)$ and $\underline{x}_a(t)$, both of which change with such a perturbation,contradicts this assumption. Before the control can be permitted to depend explicitly upon the state, a consistent differential equation for $\underline{x}_a$ must be derived.

To demonstrate the effect of a feedback control upon $\underline{x}_a$, rewrite Eq. (1) as

$$\dot{\underline{x}}(t) = \underline{f}\left\{\underline{x}(t),\ \underline{u}[\underline{x}(t),\ \underline{x}_a(t),\ t],\ a,\ t\right\}\ ;\ \underline{x}(t_0) = \underline{x}_0 \tag{15}$$

where it is assumed a priori that $\underline{u}$ is explicitly a differentiable function of $\underline{x}(t)$ and $\underline{x}_a(t)$. The perturbed trajectory differential equations are, to first order,

$$\dot{\underline{x}}_p \cong \underline{f} + \left[\frac{\partial \underline{f}}{\partial \underline{x}} + \frac{\partial \underline{f}}{\partial \underline{u}}\frac{\partial \underline{u}}{\partial \underline{x}}\right](\underline{x}_p - \underline{x}) + \frac{\partial \underline{f}}{\partial \underline{u}}\frac{\partial \underline{u}}{\partial \underline{x}_a}(\underline{x}_{a_p} - \underline{x}_a) + \frac{\partial \underline{f}}{\partial a}\Delta a \tag{16}$$

where $\underline{x}_{a_p}(t)$ is the perturbed value of $\underline{x}_a(t)$. With the definitions*

$$\begin{aligned} \underline{x}_a(t) &= \lim_{\Delta a \to 0} \frac{\underline{x}_p(t) - \underline{x}(t)}{\Delta a} \equiv \frac{\partial \underline{x}(t)}{\partial a} \\ \underline{x}_{aa}(t) &= \lim_{\Delta a \to 0} \frac{\underline{x}_{a_p}(t) - \underline{x}_a(t)}{\Delta a} \equiv \frac{\partial \underline{x}_a(t)}{\partial a} \end{aligned} \tag{17}$$

it follows by subtraction of Eq. (15) from Eq. (16) that

$$\dot{\underline{x}}_a = \left[\frac{\partial \underline{f}}{\partial \underline{x}} + \frac{\partial \underline{f}}{\partial \underline{u}}\frac{\partial \underline{u}}{\partial \underline{x}}\right]\underline{x}_a + \frac{\partial \underline{f}}{\partial \underline{u}}\frac{\partial \underline{u}}{\partial \underline{x}_a}\underline{x}_{aa} + \frac{\partial \underline{f}}{\partial a}\ ;\ \underline{x}_a(t_0) = \underline{0} \tag{18}$$

*Observe that $\partial \underline{x}_a(t)/\partial a \neq \partial^2 \underline{x}(t)/\partial a^2$ in general because the partial derivatives in Eq. (16) are known only in terms of the nominal value of "a".

Now one can make two observations. First, Eq. (18) differs from Eq. (12) in the presence of two terms derived from the dependence of $\underline{u}$ on $\underline{x}$ and $\underline{x}_a$. Thus the closed-loop optimal control in Eq. (14) produces a change in the sensitivity dynamics, and the new system, obtained by replacing Eq. (12) with Eq. (18), is in general not optimized. This fact has been observed by Kriendler[3] and Dougherty, et al.[4]

In the second place, before one can solve an optimal control problem with the new sensitivity equations, $\underline{x}_{aa}(t)$ in Eq. (18) must be known. This term is caused by the dependence of $\underline{u}$ upon $\underline{x}_a$. Writing Eq. (18) symbolically as

$$\dot{\underline{x}}_a = \underline{f}_a[\underline{x}, \underline{x}_a, \underline{x}_{aa}, \underline{u}(\underline{x}, \underline{x}_a, t), t] \tag{19}$$

and repeating the limiting process, the differential equation for $\underline{x}_{aa}$, which includes the unknown term $(\partial^2 \underline{x}_a(t)/\partial a^2$ can be derived. If we attempt to describe this latter quantity, the third derivative of $\underline{x}_a$ with respect to "a" will appear, etc. Thus, one concludes that if $\underline{u}$ depends upon $\underline{x}_a$ explicitly, there is no finite dimensional dynamic system that defines the state $\underline{x}$ and its sensitivity to the parameter "a" unless $\partial^i \underline{x}_a(t)/\partial a^i \equiv \underline{0}$ for some $i \geqq 1$. To expedite the theoretical discussion this difficulty is avoided by requiring that $\underline{u}$ be expressed as a function only of $\underline{x}$ and t so the quantity $\partial \underline{u}/\partial \underline{x}_a$ in Eq. (18) is zero. This imposes a <u>semi-closed-loop</u> structure on the control.

With the objective that a feedback control of the type given by Eq. (8) is to be implemented, we now examine the problem of obtaining an <u>optimal</u> feedback control.

4. <u>The Feedback Optimal Control Problem</u>

The feedback optimal control problem is initially formulated as follows:

<u>Problem 1</u>

Find the optimal feedback control $\underline{u}[\underline{x}(t), t] = \underline{u}^*[\underline{x}(t), t]$ such that

$$J^*\left\{\underline{u}^*[\underline{x}^*(t), t]\right\} = \min_{\underline{u}[\underline{x}(t), t]} J[\underline{u}(t), \partial\underline{u}(t)/\partial\underline{x}(t)] \tag{20}$$

$$J\left[\underline{u}, \frac{\partial \underline{u}}{\partial \underline{x}}\right] = \phi[\underline{x}(T), \underline{x}_a(T), T] + \int_{t_0}^{T} L\left[\underline{x}(t), \underline{x}_a(t), \underline{u}(t), \frac{\partial \underline{u}(t)}{\partial \underline{x}(t)}, t\right] dt$$

subject to the constraints

$$\underline{\dot{x}}(t) = \underline{f}[\underline{x}(t),\ \underline{u}[\underline{x}(t), t], a,\ t] \quad ; \quad \underline{x}(t_0) = \underline{x}_0 \tag{21}$$

$$\underline{\dot{x}}_a(t) = \left[\frac{\partial \underline{f}}{\partial \underline{x}} + \frac{\partial \underline{f}}{\partial \underline{u}} \frac{\partial \underline{u}}{\partial \underline{x}} \right] \underline{x}_a(t) + \frac{\partial \underline{f}}{\partial a} \quad ; \quad \underline{x}(t_0) = 0 \tag{22}$$

$$\underline{\psi}[\underline{x}(T),\ \underline{x}_a(T),\ T] = \underline{0} \tag{23}$$

where $\underline{x}^*(t)$ is the solution to Eqs. (21) with $\underline{u}^*[\underline{x}^*(t), t]$ substituted for $\underline{u}[\underline{x}(t), t]$.

Observe that the quantity $\partial\underline{u}/\partial\underline{x}$ appears explicitly in the performance index, $J[\underline{u}, \partial\underline{u}/\partial\underline{x}]$. Problem 1 can be transformed into a "conventional" optimization problem if $\underline{u}$ and $\partial\underline{u}/\partial\underline{x}$ are regarded as separate sets of control variables, $\underline{u}(t)$ and $K(t)$ respectively, the latter being a matrix. Then, assuming optimal controls $\underline{u}^*(t)$ and $K^*(t)$ are determined, we seek a feedback realization $\underline{u}^*[\underline{x}(t), t]$ such that

$$\underline{u}^*[\underline{x}^*(t), t] = \underline{u}^*(t) \tag{24}$$

$$\left. \frac{\partial \underline{u}^*}{\partial \underline{x}} \right|_{\underline{x}^*(t)} = K^*(t) \tag{25}$$

Following this procedure define

<u>Problem 2</u>

Find the optimal controls $\underline{u}(t) = \underline{u}^*(t)$ and $K(t) = K^*(t)$ such that

$$J^*\left\{\underline{u}^*(t),\ K^*(t)\right\} \equiv \min_{\underline{u}(t),\, K(t)} J[\underline{u}(t),\ K(t)] \tag{26}$$

$$J[\underline{u}(t),\ K(t)] = \phi[\underline{x}(T),\ \underline{x}_a(T),\ T] + \int_{t_0}^{T} L[\underline{x}(t),\ \underline{x}_a(t),\ \underline{u}(t),\ K(t),\ t]\,dt$$

subject to the constraints

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), \underline{u}(t), a, t] \quad ; \quad \underline{x}(t_0) = \underline{x}_0 \tag{27}$$

$$\dot{\underline{x}}_a(t) = \left[\frac{\partial \underline{f}}{\partial \underline{x}} + \frac{\partial \underline{f}}{\partial \underline{u}} K(t)\right] \underline{x}_a(t) + \frac{\partial \underline{f}}{\partial a} \quad ; \quad \underline{x}_a(t_0) = 0 \tag{28}$$

$$\psi[\underline{x}(T), \underline{x}_a(T), T] = \underline{0} \tag{29}$$

Various necessary conditions for existence of a solution to Problem 2 are given in Reference 5. Assuming that the optimal controls exist and can be determined, it is readily seen from Eqs. (24) and (25) that an optimal feedback control law is given by any function

$$\underline{u}^*[\underline{x}(t), t] = \underline{u}^*(t) + \underline{g}[K^*(t), \underline{x}(t), t] \tag{30}$$

if

$$\underline{g}[K^*(t), \underline{x}^*(t), t] = 0 \tag{31}$$

$$\left.\frac{\partial \underline{g}}{\partial \underline{x}}\right|_{\underline{x}^*(t)} = K^*(t) \tag{32}$$

There may be many functions $\underline{g}$ that satisfy these conditions; hence the realization of $\underline{u}^*[\underline{x}(t), t]$ is not unique. However, all admissible realizations are equivalent from the standpoint of the solution to Problem 2. In particular, it is natural to choose a linear feedback control law of the form

$$\underline{u}^*[\underline{x}(t), t] = \underline{u}^*(t) + K^*(t)[\underline{x}(t) - \underline{x}^*(t)] \tag{33}$$

which satisfies Eqs. (30) through (32).

In the previous section it is argued that the feedback control is not to depend explicitly upon $\underline{x}_a(t)$, thereby giving it a semi-closed loop character. This property is accentuated by the fact that $\underline{u}^*(t)$ and $K^*(t)$ in general depend upon $\underline{x}_0$ and t_0. Because of the condition established by Eq. (25), it is an improper (and in general a nonoptiomal) procedure to "close-the-loop" with the identifications $\underline{x}_0 \longrightarrow \underline{x}(t)$ and $t_0 \longrightarrow t$. Semi-closed-loop controls have appeared in other contexts in the control systems literature.[6]

It is clear that the feedback control is superior to or at least as good as an open-loop control because $K^*(t) = 0$ is an admissible solution to Problem 2. In this event, Eqs. (26)-(29) describe an open-loop optimization problem. A comparison of these types of control is presented in the next section.

Finally, one should recognize that a special interpretation must be applied to Problem 2 and its solution. Because $\underline{u}^*[\underline{x}(t), t]$ is semi-closed-loop in nature, it is not the optimal control in the sense of Problem 2 if $\underline{x}(t)$ deviates from $\underline{x}^*(t)$. However, if this deviation is caused by a perturbation in "a", the "error"

$$\Delta\underline{x}(t) = \underline{x}(t) - \underline{x}^*(t)$$

is small to the extent that $\underline{x}_a^*(t)$ along $\underline{x}^*(t)$ is also small.

To summarize, the main contribution of this report is recognition that Optimal Control Problem 1 can be formulated as Problem 2 and the optimal control can be realized by Eq. (33) with the properties described above.

5. A Numerical Example

There does not appear to be a wide class of systems for which optimization Problem 2 admits a closed form solution for the control. The numerical solution given by Cassidy and Lee[7] does not appear to be applicable to this formulation. In this section we discuss a simple special case for which analytical results are obtainable.

Given a dynamical system

$$\dot{x}(t) = u(t) + a \quad ; \quad x(0) = x_0 \tag{34}$$

where "a" is a parameter whose "nominal" value is taken to be zero, optimal sensitivity designs of both open-loop and closed-loop types are compared.

A. Open-loop Control Problem

The dynamical system for the case where u is not explicitly a function of the state is

$$\dot{x}(t) = u(t) \quad ; \quad x(0) = x_0 \tag{35}$$

$$\dot{x}_a(t) = 1 \quad ; \quad x_a(0) = 0 \tag{36}$$

Find $u(t) = u^*(t)$ such that

$$J_1[u(t)] = \frac{1}{2} \left\{ x^2(T) + \eta x_a^2(T) + R \int_0^T u^2(t) dt \right\} \tag{37}$$

is minimized where η and R are positive weighting constants and T is specified.

We observe that $x_a(t)$ is uncontrollable; that is, it is independent of u(t). Thus, only Eq. (35) and the terms in Eq. (37) dependent upon x(T) and u(t) need be considered in determining $u^*(t)$. Applying the necessary conditions for an optimal trajectory,[5] we find that

$$u^*(t) = -\frac{x_0}{R + T} \tag{38}$$

$$x^*(t) = x_0 \left[1 - \frac{t}{R + T} \right] \tag{39}$$

$$x^*_a(t) = t \tag{40}$$

$$J^*[u^*(t)] = \frac{1}{2} \left[\eta T^2 + \frac{x_0^2 R}{R + T} \right] \tag{41}$$

Equations (38)-(41) provide a comparison with the closed-loop optimal control derived below. Because the portion of the problem depending upon x(t) is the linear regulator problem, this solution is truly optimal.[5]

B. Feedback Optimal Control Problem

For a feedback control the dynamical system is represented by

$$\dot{x}(t) = u(t) \qquad ; \qquad x(0) = x_0 \tag{42}$$

$$\dot{x}_a(t) = k(t)x_a(t) + 1 \quad ; \quad x_a(0) = 0 \tag{43}$$

We wish to find $u(t) = u^*(t)$ and $k(t) = k^*(t)$ to minimize

$$J[u(t), k(t)] = \frac{1}{2}\left\{x^2(T) + \eta x_a^2(T) + R \int_0^T [u^2(t) + \beta k^2(t)]dt\right\} \tag{44}$$

where η, R, and β are positive weighting constants, T is specified, and no explicit terminal constraints are imposed.

One finds that the necessary conditions for an optimal trajectory are uniquely satisfied by

$$u^*(t) = -\frac{x_0}{R + T} \tag{45}$$

$$x^*(t) = x_0\left(1 - \frac{t}{R + T}\right) \tag{46}$$

$$k^*(t) = -2\gamma \tan \gamma t \tag{47}$$

$$x_a^*(t) = \frac{1}{\gamma} \cos \gamma t \sin \gamma t \tag{48}$$

$$J_2^*[u^*(t), k^*(t)] = \frac{1}{2}\left\{\frac{x_0^2 R}{R + T} + \frac{\eta}{\gamma^2} \cos^2 \gamma T \sin^2 \gamma T + 4R\beta\gamma(\tan \gamma T - \gamma T)\right\} \tag{49}$$

where γ is the unique solution to

$$2\gamma^3 - \frac{\eta}{R\beta} \cos^3 \gamma T \sin \gamma T = 0 \tag{50}$$

that lies in the range

$$0 < \gamma T < \frac{\pi}{2} \tag{51}$$

Using the linear representation of the feedback control, given by Eq. (19), we have

$$u^*[x(t), t] = -(2\gamma \tan \gamma t)\left[x(t) - x_0\left(1 - \frac{t}{R+T}\right)\right] - \frac{x_0}{R+T} \tag{52}$$

It can be verified by examining the Hamiltonian of this problem[5] that this solution yields a local minimum of J_2; global optimality has not been proved.

Upon comparing Eqs. (38)-(41) with (45)-(51), we notice the equations for $u^*(t)$ and $x^*(t)$ are identical in both cases. This is a result of the separable nature of the equations of motion, J_1 and J_2. An analytical comparison of J_1^* and J_2^* is not available; however, it has been argued that $J_2^* \leqq J_1^*$ if Eqs. (45)-(48) are globally optimal. Numerical evidence that $J_2^* \leqq J_1^*$, for particular parameter values at least, is given in the following table:

T	0.5	1.0	1.5	3.0	6.0
J_1^*	0.458	0.75	1.33	4.63	18.1
J_2^*	0.453	0.648	0.88	1.36	1.71

Comparison of J_1^* with J_2^* for various values of T with $\eta = R = \beta = x_0 = 1$.

Graphs of $x_a^*(t)$ for both cases (Eqs. (40) and (48)) are shown in Fig. 1. The objective of optimal sensitivity design for this example is to achieve a

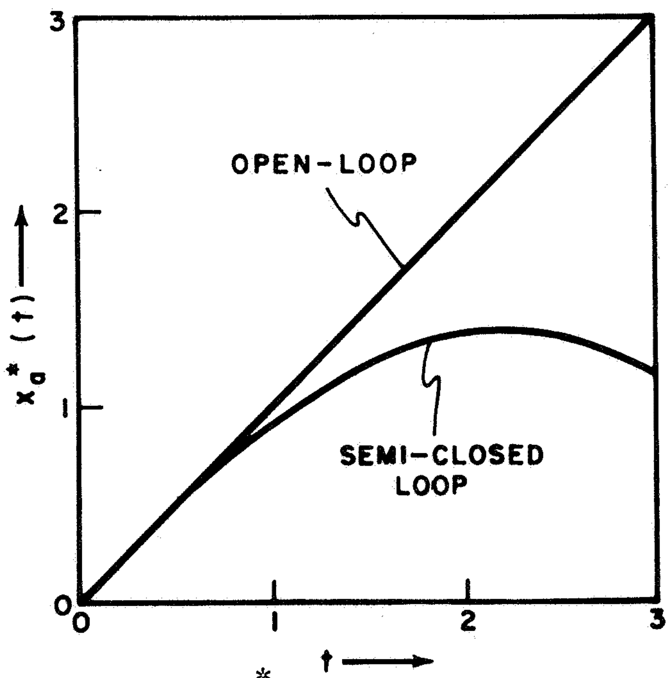

Fig. 1. Comparison of $x_a^*(t)$ for both open-loop and semi-closed loop optimal controls for T = 3, $\eta = R = \beta = x_0 = 1$.

low value of $x_a(T)$. It is evident that the semi-closed-loop control configuration is superior from this point of view.

Summary and Conclusions

In this report an approach to the problem of designing feedback control systems with trajectory sensitivity characteristics that are optimal, in some sense, is presented. The results are more general than those previously available in that a truly optimal design, based upon minimization of a performance index, can, in principle, be obtained, assuming one exists. Certain aspects of this technique remain to be studied. Whether such a method is competitive with statistical techniques for dealing with parameter uncertainty remains to be determined. The proper choice of performance index to provide desirable trajectory sensitivity characteristics should be investigated.

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