Houston, Texas, April 17-19, 1968.

# A DESCRIBING FUNCTION ANALYSIS IN CLOSED LOOP SYSTEMS WITH PULSE FREQUENCY MODULATORS*

R. V. Monopoli
University of Massachusetts
Amherst, Massachusetts

J. F. Fitzgerald
Bell Telephone Laboratories, Inc.
Whippany, New Jersey



Abstract - Equivalent circuits are presented for integral and sigma pulse frequency modulators (IPFM and ΣPFM). Both contain linear elements and a nonlinearity with step discontinuities and hysteresis. A describing function is developed for this nonlinearity, and applied in the usual manner to the stability analysis of a unity feedback closed loop system with the modulator in the forward path. The transfer function for the linear part of the system is assumed to have a single pole at the origin. Analog computer results indicate that this analysis gives better predicted values for limit cycle periods than the quasi-describing function analysis previously reported.

## I. INTRODUCTION

The application of pulse frequency modulation in feedback systems is of interest in present day technology [1-5]. In this paper, an equivalent circuit and a describing function analysis based on this equivalent is presented for a certain class of these systems.

Pulse frequency modulation (PFM) is the coding of information into the interval between pulses. A common scheme for implementing the coding process is to integrate the input signal and decide on the emission of a pulse whenever the integral reaches a certain level. This is known as integral pulse frequency modulation (IPFM) and has been the subject of considerable study [3-5]. A more general PFM scheme was introduced in [1] and is known as sigma pulse

*Research reported herein was supported in part by NASA under grant NGR-22-010-018.

frequency modulation (ΣPFM). ΣPFM feedback systems have two basic advantages over IPFM feedback systems, improved stability and ease of physical implementation.

A frequency dependent sinusoidal describing function was proposed by Li[3] for IPFM. However, it cannot be applied to ΣPFM feedback systems [5]. Pavlidis and Jury presented a quasi-describing function, based on a square wave input, for IPFM and the subclass of ΣPFM known as Neural pulse frequency modulation (NPFM). In this paper an IPFM equivalent circuit proposed by Meyer [4] is used as the basis to develop an equivalent circuit for NPFM. This equivalent contains linear elements and a nonlinearity with step discontinuities and hysteresis. Use of this equivalent makes conventional sinusoidal describing function techniques convenient to apply to both IPFM and NPFM feedback systems. The transfer function for the linear part of the system is assumed to have one pole at the origin.

Analog computer results indicate that the approach presented here gives better predicted values for limit cycle periods than does the quasi-describing function.

## II. DEFINITIONS AND EQUIVALENT MODELS FOR IPFM AND ΣPFM

If the on-time of a pulse is very short compared with the response time of the system, the pulses can be assumed to be impulses, and IPFM can be described by the following equations [1]

$$\frac{dp}{dt} = x - ry \tag{1}$$

$$y = \mathrm{sgn}(p)\ \delta(|p| - r) \tag{2}$$

where x denotes the input, y the output, p the value of the integrator output, and r the pulse triggering and integrator reset level. Note that if an impulse occurs at $t = t_o$, then $p(t_o-) = r$ and $p(t_o+) = 0$.

If a continuous function of p, g(p) that is odd and monotonically increasing is added to the left hand side of (1) it leads to the equations describing ΣPFM [1], i.e.

$$\frac{dp}{dt} + g(p) = x - ry \tag{3}$$

$$y = \text{sgn}\,(p)\delta(|p| - r) \tag{4}$$

If $g(p) = cp$ where c is a positive constant equations (3) and (4) define NPFM.

An alternate way of describing these two PFM processes is developed in [6]. Based on the equivalent circuit for IPFM proposed by Myer [4] it is shown that the input output relationship of equations (1) and (2) can be represented in block diagram form as shown in Fig. 1(a) with N as shown in Fig. 1(b). Furthermore the block diagram in Fig. 1(a) may be manipulated to produce the ΣPFM equivalent and the NPFM equivalent shown in Figs. 2(a) and 2(b) respectively [6]. These equivalents are significant for two reasons. First, for large q(t), N may be reasonably approximated as a linear element with unity gain. This large signal linear approximation lead to good predicted results for the transient response to large step inputs [6]. Furthermore, the nonlinear element N admits a sinusoidal describing function which can be readily derived as opposed to a describing function which assumes that x(t) is sinusoidal [3].

## III. DERIVATION OF THE DESCRIBING FUNCTION

In the describing function method the sinusoidal input to the nonlinear element is usually assumed to have zero average value. This assumption is not very useful for NPFM systems, however, since the output of the nonlinear element can exhibit only an even number of pulses per half cycle if driven by such an input. This eliminates from consideration the very important case of one pulse per half cycle and all other cases where the observed limit cycle contains an odd number of pulses per half cycle. For this reason the describing function is developed with q(t) assumed to be of the form

$$q(t) = A \sin \omega t + b \qquad (5)$$

where $A > 0$ and $0 \leq b \leq 1/2$. For this choice of q(t), the following properties of f(q) are evident by inspection of figure 1(b):

$$f(q + M) = f(q) + M; \text{ for } M \text{ an integer} \qquad (6)$$

$$-f(-q) = f(q) \qquad (7)$$

Because of these properties, all useful first harmonics of f(q) may be computed with A and b restricted as in (5).

For $b = 0$ only an even number of pulses per half cycle can occur, and for $b = 1/2$, only an odd number can occur. In the former case 2n pulses are produced if $n \leq A < n + 1$ (for n = 0, 1, 2,...) and in the latter $2n + 1$ pulses are produced for $n + 1/2 \leq A < n + 3/2$ (for n = 0, 1, 2, 3,...). The number of pulses per half cycle which occur for various combinations of A and b may be odd or even. If $n \leq A + b < n + 1$ (for n = 1, 2, 3,...) then $2n - 1$ pulses occurs for $A - b < n$ and 2n pulses occur for $A - b \geq n$. These results are indicated graphically in Fig. 4 which shows the A-b plane separated into triangular regions by straight lines. For a given pair (A,b), the number of pulses per half cycle is uniquely specified. The boundaries for these triangular regions will prove useful in the graphical representation of the negative inverse describing function.

In computing the describing function, let $\omega t = \Theta$. Since $f(A \sin \Theta + b)$ changes only when $A \sin (\Theta +) + b = m$ and $A \sin (\Theta -) + b \neq m$ for $m = 0, \pm 1, \pm 2,$ ..., it may be represented by the following series of unit step functions:

$$f(A \sin \Theta + b) = \sum_{m=1}^{n} [u(\Theta - \sin^{-1}((m-b)/A)) - u(\Theta - \Pi + \sin^{-1}((m-1-b)/A))]$$

$$- \sum_{m=1}^{j} [u(\Theta - \Pi - \sin^{-1}((m+b)/A)) - u(\Theta - 2\Pi + \sin^{-1}((m-1+b)/A))] \quad (8)$$

where j equals the greatest integer in A - b

The first harmonic of f (A sin Θ + b) will be denoted by $f_1$ (A sin Θ + b) and in phasor form by $\gamma_1 = \alpha_1 + j\beta_1$ where

$$\alpha_1 = \frac{1}{\Pi} \int_o^{2\Pi} f(A \sin \Theta + b) \cos \Theta d\Theta \quad (9)$$

$$\beta_1 = \frac{1}{\Pi} \int_o^{2\Pi} f(A \sin \Theta + b) \sin \Theta d\Theta \quad (10)$$

Performing the integration on (9) and (10) yields

$$\alpha_1 = -\frac{1}{\Pi} ((2n-1)/A + \{\frac{1}{A}\}) \quad (11)$$

The bracketed term, { }, does not appear if j = n-1.

$$\beta_1 = \left[\frac{2}{\Pi} \sum_{m=-n+2}^{n-1} \sqrt{1-((m-b)/A)^2}\right](1 - \delta_{n1}) + \frac{1}{\Pi}\sqrt{1 - ((n-b)/A)^2}$$

$$+ \frac{1}{\Pi}\sqrt{1 - ((n-1+b)/A)^2} + \frac{1}{\Pi} \{\sqrt{1 - ((n-1+b)/A)^2} + \sqrt{1 - ((n-b)/A)^2}\} \quad (12)$$

where $\delta_{n1} = 1$ if n = 1 and $\delta_{n1} = 0$ if n > 1

In order to obtain the describing function, one need only normalize $\gamma_1$ with respect to A; thus,

$$N = \gamma_1/A \quad (13)$$

## IV. APPLICATION OF THE DESCRIBING FUNCTION TO NPFM FEEDBACK SYSTEMS

An NPFM feedback system is shown in Fig. 3(a). A modified form of this system's block diagram shown in Fig. 3(b) is seen to be suitable for describing function analysis. The loop transfer function, H(s), is

$$H(s) = \frac{(1/rc)sP(s) - 1}{s/c + 1} \tag{14}$$

where P(s), the plant transfer function, is assumed to have one pole at the origin. Limit cycle amplitudes and frequencies are determined by solving the equation

$$H(j\omega) = -1/N \tag{15}$$

graphically using either a polar plot or a magnitude-phase plot. The negative inverse describing function, -1/N, is plotted in Fig. 5 for n = 1 and 2, i.e. for one and two pulses per half cycle. Since no stable limit cycles with more than two pulses per half cycle were observed experimentally, no other characteristics are shown.

It is to be noted that the -1/N plot is not a single curve. Instead, each triangular region in Fig. 4 leads to a unique bounded region in Fig. 5. Thus, for n = 1 and n = 2, -1/N may be anywhere in the regions indicated in Fig. 5. Thus, a range of predicted values of limit cycle frequencies is determined by the two intersections where the H (jω) plot enters and leaves each region. This is indicated in the results included in Tables I and II below.

| | | Fitzgerald | | Pavlidis | |
|---|---|---|---|---|---|
| Example | n | $T_p$ | $T_e$ | $T_p$ | $T_e$ |
| 1 | 1 | 6.2 - 7.8 | 6.4 | 5.4 | 3.5 |
| 2 | 1 | 4.2 - 5.1 | 4.4 | 5.4 | 3.6 |
| 2 | 2 | 3.7 - 3.9 | 3.9 | 3.5 | 2.0 |

$T_p$ = predicted period, $T_e$ = experimental period

Table I

| Example | n | $T_p$ | $T_e$ | $A_p$ | $A_e$ |
|---|---|---|---|---|---|
| 3 | 1 | 2.35 - 2.85 | 2.6 | 1.0 | 1.1 |
| 3 | 2 | 2.1 - 2.3 | 2.2 | 1.4 | 1.6 |
| 4 | 1 | 3.15 - 3.5 | 3.1 | 0.6 | 1.0 |

$A_p$ = predicted amplitude, $A_e$ = experimental amplitude

Table II

In Fig. 5, (15) is solved graphically for four examples. The curves labeled I, II, III, and IV are those of $H(j\omega)$ for examples 1, 2, 3, and 4 respectively. Examples 1 and 2 are virtually the same examples used in [1] with the exception that K and r have been scaled to suit a TR-10 analog computer.

Relevant data for these examples is as follows:

Example 1: $P(s) = \frac{K}{s(s+1)}$; $H(s) = \frac{3-s}{(2s+1)(s+1)}$

$K = 2,\ r = 1,\ c = 0.5$ $\qquad$ ($K = 10,\ r = 5,\ c = 0.5$ in [1])

Example 2: Same as Example 1 except $K = 4$

($K = 20$ in [1]). $H(s) = \frac{7-s}{(2s+1)(s+1)}$

Example 3: $P(s) = \frac{K(5s+1)}{s(s^2+s+1)}$; $H(s) = \frac{-s^2+9s+1}{(s+1)(s^2+s+1)}$

$K = 2,\ r - c - 1.$

Example 4: $P(s) = \frac{K(5s+1)}{s(s+1)^2}$; $H(s) = (1/3)\ \frac{-3s^2+14s+1}{(s+1)^3}$

$K = 4/3,\ r = c = 1$

As seen from Tables I and II, experimental periods fall within the range of predicted periods for all examples except 4. Predicted and experimental amplitudes were in better or worse agreement as q(t) was more or less sinusoidal. Examples 1 and 2 show that predicted periods are better than those obtained using the quasi describing function. No comparison of predicted amplitude is made with results in [1] since different signals are involved. No explanation is offered for discrepancies in experimental periods presented here and in [1] since none is known to the authors.

## V Conclusions

An equivalent circuit has been developed for ΣPFM and NPFM which allows convenient application of conventional describing function techniques to a class of closed loop systems containing these types of modulators. Predicted results for limit cycle characteristics are in quite good agreement with experimental for the examples presented. If adequate low pass filtering is not provided by H(s), then predicted and experimental results are not always in such good agreement. A comparison of this technique with the quasi describing function method shows this approach better for the two examples considered.

## REFERENCES


1. Pavlidis, T. and E. I. Jury, "Analysis of a New Class of Pulse Frequency Modulated Feedback Systems", I.E.E.E. Trans. on Automatic Control, V., AC-10, Jan. 1965, pp. 35-43.

2. Pavlidis, T., Pulse Frequency Modulated Feedback Systems, Ph.D. Dissertation, University of California, Berkeley, June 1964.

3. Li, C. C., Integral Pulse Frequency Modulated Feedback Systems, Ph. D. Dissertation, Northwestern University, June 1961.

4. Meyer, A. U., Pulse Frequency Modulation and its Effects in Feedback Control Systems, Ph.D. Dissertation, Northwestern University, Aug. 1961.

5. Li. C. C. and R. W. Jones, "Integral Pulse Frequency Modulated Feedback Systems", 2nd I.F.A.C. Congress, Basile, Switzerland, 1963, pp. 217/1 - 417/8.

6. Fitzgerald, J. F., Frequency Domain Techniques for Pulse Frequency Modulated Feedback Systems, M. S. Thesis, University of Massachusetts, July, 1967.

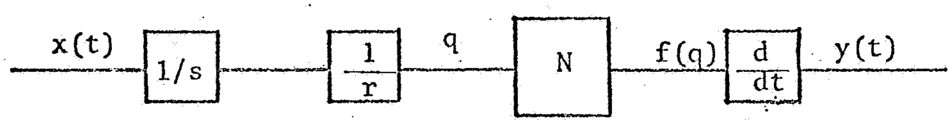



Fig. 1(a)

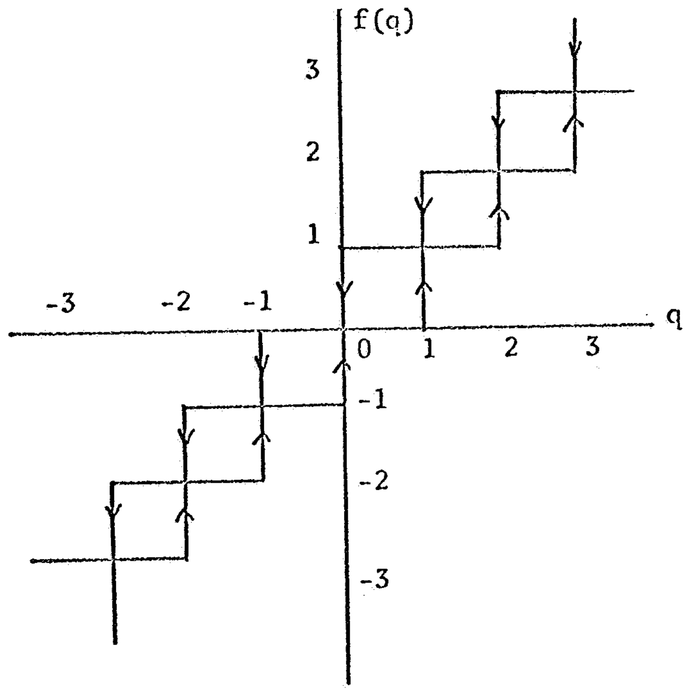



Fig. 1(b)

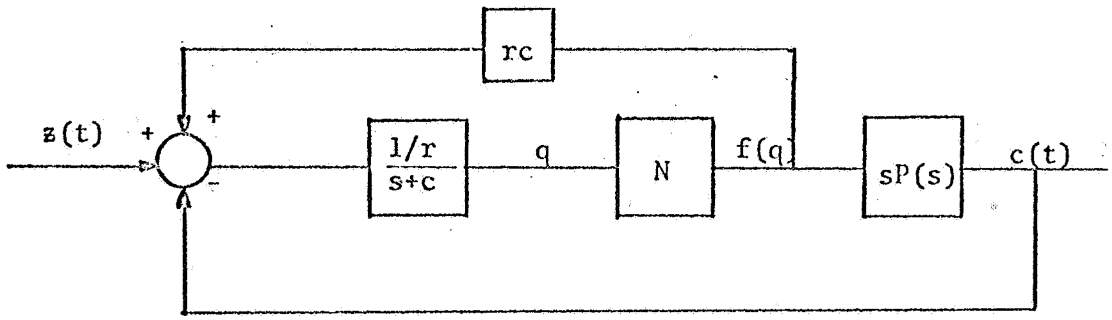



Fig. 3(a)

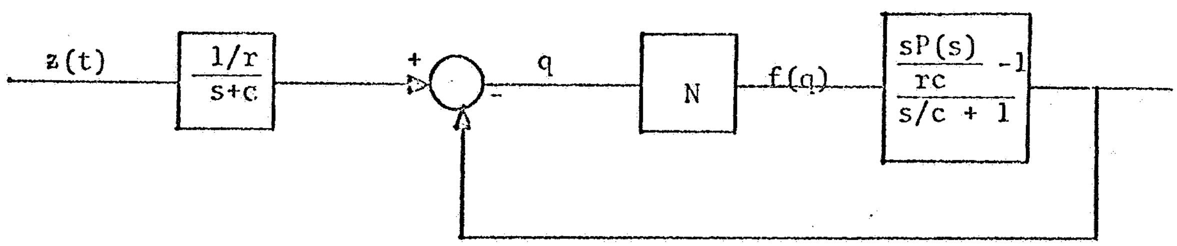



Fig. 3(b)

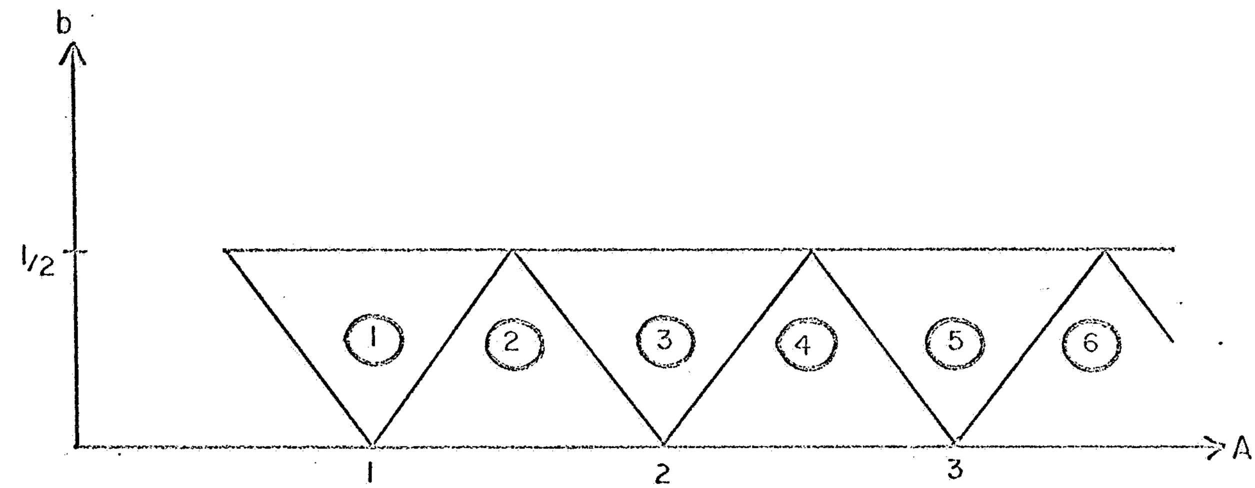
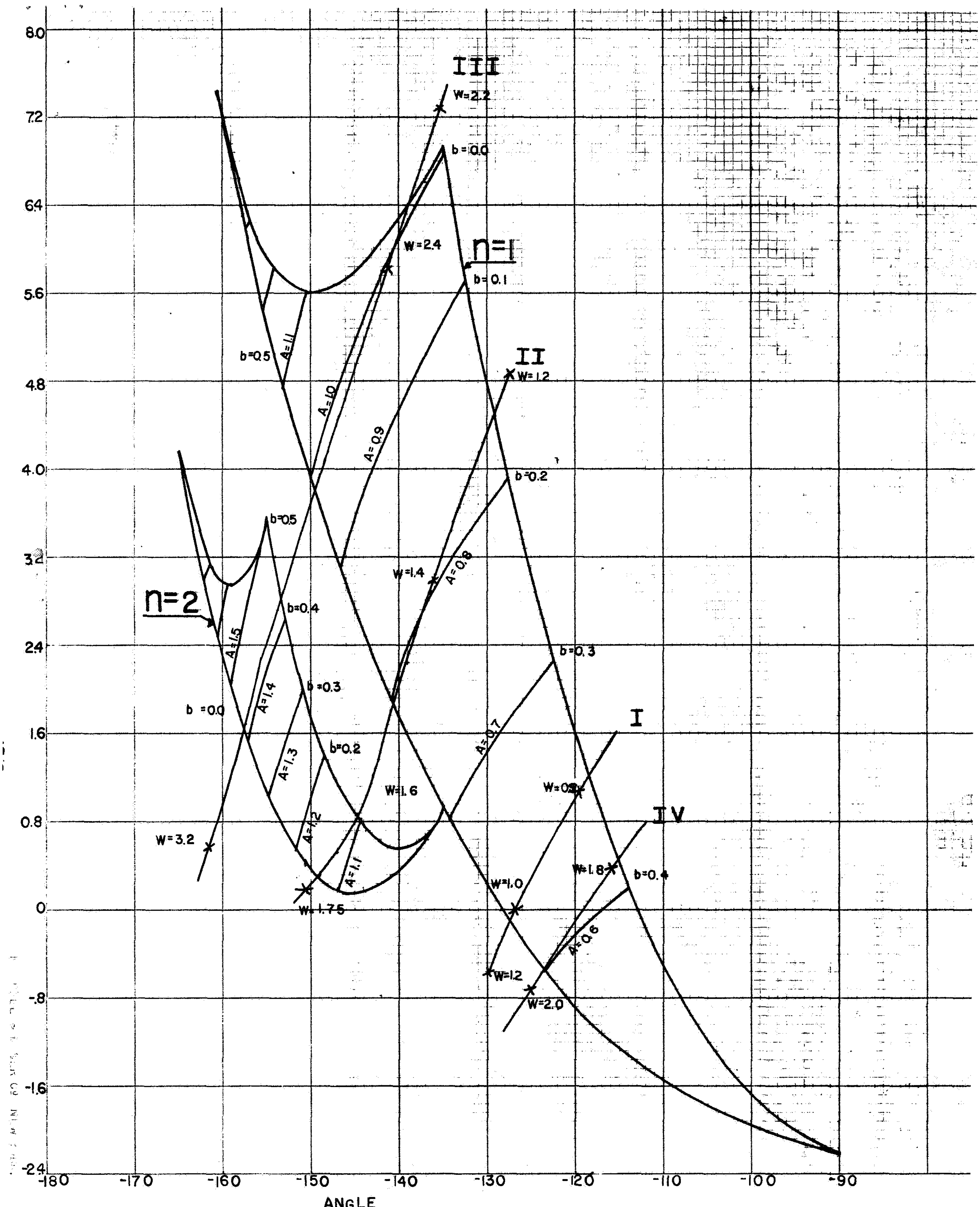



Fig. 4



III
W=2.2
b=0.0
W=2.4
n=1
b=0.1
II
W=1.2
b=0.5
A=1.1
A=1.0
A=0.9
b=0.2
b=0.5
W=1.4
A=0.8
n=2
b=0.4
A=1.5
b=0.3
b=0.3
A=1.4
b=0.0
A=0.7
I
A=1.3
b=0.2
W=1.6
IV
W=3.2
A=1.2
W=1.8
b=0.4
A=1.1
W=1.0
W=1.75
A=0.6
W=1.2
W=2.0
8.0
7.2
6.4
5.6
4.8
4.0
3.2
2.4
1.6
0.8
0
-.8
-1.6
-2.4
-180
-170
-160
-150
-140
-130
-120
-110
-100
-90
ANGLE


Fig. 5

# LIST OF FIGURES