NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1332

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Approved by:

D. R. Bartz, Manager
Research and Advanced Concepts Section

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Abstract

A complete, written set of the conservation equations of a viscous, heat-conducting fluid is given in curvilinear orthogonal coordinates. Scale factors for a number of coordinate systems are tabulated for convenience in expressing the equations in various coordinates.

Conservation Equations of a Viscous, Heat-Conducting Fluid in Curvilinear Orthogonal Coordinates

I. Introduction

Although formulation of the conservation equations of a viscous, heat-conducting fluid in curvilinear orthogonal coordinates is well known through vector and tensor analysis (Refs. 1 and 2), a complete, written-out set of equations, including the energy equation, is not readily available in any given source. The momentum equation was given by Goldstein (Ref. 3) in curvilinear orthogonal coordinates for an incompressible, constant-property fluid, and by Tsien (Ref. 4) for a compressible, variable-property fluid. Only the commonly used special cases of the set of equations in rectangular, cylindrical, and spherical coordinates appear in the literature (e.g., Ref. 5). The purpose of this paper is to briefly present the complete set of equations in stationary, curvilinear orthogonal coordinates. For convenience in expressing the equations in various coordinates, scale factors for eleven coordinate systems are tabulated.

II. Conservation Equations

Three forms of the energy equation are considered, one form of which may be best suited for a particular application. These relations involve the total enthalpy H_t,

$$\rho \frac{\partial H_t}{\partial t} + \rho (\mathbf{V} \cdot \nabla) H_t = \frac{\partial p}{\partial t} - \nabla \cdot \mathbf{q} + \nabla \cdot (\tau \cdot \mathbf{V}) + \mathbf{F} \cdot \mathbf{V} + W$$

the internal energy E,

$$\rho \frac{\partial E}{\partial t} + \rho (\mathbf{V} \cdot \nabla) E + p \nabla \cdot \mathbf{V} = - \nabla \cdot \mathbf{q} + \tau : (\nabla \mathbf{V}) + W$$

and the enthalpy H,

$$\rho \frac{\partial H}{\partial t} + \rho (\mathbf{V} \cdot \nabla) H - \left[\frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla) p \right] = - \nabla \cdot \mathbf{q} + \tau : (\nabla \mathbf{V}) + W$$

To complete the set of conservation equations, the continuity and momentum equations are, respectively,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = - \frac{1}{\rho} \nabla p + \frac{\mathbf{F}}{\rho} + \frac{1}{\rho} \nabla \cdot \tau$$

The quantities that appear in these equations are identified in the nomenclature.

By use of the same notation as used by Goldstein (Ref. 3), the orthogonal coordinates are taken as α, β, and γ such that the elements of length at α, β, and γ in the directions of increasing α, β, and γ are $h_1 d\alpha$, $h_2 d\beta$, and $h_3 d\gamma$, respectively. The differential arc length ds is, then,

$$(ds)^2 = h_1^2 (d\alpha)^2 + h_2^2 (d\beta)^2 + h_3^2 (d\gamma)^2$$

If u, v, and w are components of the velocity vector $\mathbf{V}$ in direction of increasing α, β, and γ, the continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{1}{h_1 h_2 h_3} \times \left[\frac{\partial}{\partial \alpha}(h_2 h_3 \rho u) + \frac{\partial}{\partial \beta}(h_1 h_3 \rho v) + \frac{\partial}{\partial \gamma}(h_1 h_2 \rho w) \right] = 0$$

The momentum equation written in the α, β, and γ directions is

$$\alpha: \frac{\partial u}{\partial t} + \frac{u}{h_1}\frac{\partial u}{\partial \alpha} + \frac{v}{h_2}\frac{\partial u}{\partial \beta} + \frac{w}{h_3}\frac{\partial u}{\partial \gamma} + \frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \beta} + \frac{uw}{h_1 h_3}\frac{\partial h_1}{\partial \gamma} - \frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \alpha} - \frac{w^2}{h_1 h_3}\frac{\partial h_3}{\partial \alpha} = -\frac{1}{\rho}\frac{1}{h_1}\frac{\partial p}{\partial \alpha} + \frac{F_\alpha}{\rho} + \frac{1}{\rho}(\nabla \cdot \tau)_\alpha$$

$$\beta: \frac{\partial v}{\partial t} + \frac{u}{h_1}\frac{\partial v}{\partial \alpha} + \frac{v}{h_2}\frac{\partial v}{\partial \beta} + \frac{w}{h_3}\frac{\partial v}{\partial \gamma} + \frac{vu}{h_1 h_2}\frac{\partial h_2}{\partial \alpha} + \frac{vw}{h_2 h_3}\frac{\partial h_2}{\partial \gamma} - \frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \beta} - \frac{w^2}{h_2 h_3}\frac{\partial h_3}{\partial \beta} = -\frac{1}{\rho}\frac{1}{h_2}\frac{\partial p}{\partial \beta} + \frac{F_\beta}{\rho} + \frac{1}{\rho}(\nabla \cdot \tau)_\beta$$

$$\gamma: \frac{\partial w}{\partial t} + \frac{u}{h_1}\frac{\partial w}{\partial \alpha} + \frac{v}{h_2}\frac{\partial w}{\partial \beta} + \frac{w}{h_3}\frac{\partial w}{\partial \gamma} + \frac{wu}{h_1 h_3}\frac{\partial h_3}{\partial \alpha} + \frac{wv}{h_2 h_3}\frac{\partial h_3}{\partial \beta} - \frac{u^2}{h_1 h_3}\frac{\partial h_1}{\partial \gamma} - \frac{v^2}{h_2 h_3}\frac{\partial h_2}{\partial \gamma} = -\frac{1}{\rho}\frac{1}{h_3}\frac{\partial p}{\partial \gamma} + \frac{F_\gamma}{\rho} + \frac{1}{\rho}(\nabla \cdot \tau)_\gamma$$

The components of the divergence of the symmetric viscous stress tensor τ in the α, β, and γ direction (Ref. 6)[1] are:

$$(\nabla \cdot \tau)_\alpha = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \alpha}(h_2 h_3 \tau_{\alpha\alpha}) + \frac{\partial}{\partial \beta}(h_1 h_3 \tau_{\alpha\beta}) + \frac{\partial}{\partial \gamma}(h_1 h_2 \tau_{\gamma\alpha}) \right] + \tau_{\alpha\beta}\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \beta} + \tau_{\gamma\alpha}\frac{1}{h_1 h_3}\frac{\partial h_1}{\partial \gamma} - \tau_{\beta\beta}\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \alpha} - \tau_{\gamma\gamma}\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial \alpha}$$

$$(\nabla \cdot \tau)_\beta = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \alpha}(h_2 h_3 \tau_{\alpha\beta}) + \frac{\partial}{\partial \beta}(h_1 h_3 \tau_{\beta\beta}) + \frac{\partial}{\partial \gamma}(h_1 h_2 \tau_{\beta\gamma}) \right] + \tau_{\alpha\beta}\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \alpha} + \tau_{\beta\gamma}\frac{1}{h_2 h_3}\frac{\partial h_2}{\partial \gamma} - \tau_{\alpha\alpha}\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \beta} - \tau_{\gamma\gamma}\frac{1}{h_2 h_3}\frac{\partial h_3}{\partial \beta}$$

$$(\nabla \cdot \tau)_\gamma = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \alpha}(h_2 h_3 \tau_{\gamma\alpha}) + \frac{\partial}{\partial \beta}(h_1 h_3 \tau_{\beta\gamma}) + \frac{\partial}{\partial \gamma}(h_1 h_2 \tau_{\gamma\gamma}) \right] + \tau_{\gamma\alpha}\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial \alpha} + \tau_{\beta\gamma}\frac{1}{h_2 h_3}\frac{\partial h_3}{\partial \beta} - \tau_{\alpha\alpha}\frac{1}{h_1 h_3}\frac{\partial h_1}{\partial \gamma} - \tau_{\beta\beta}\frac{1}{h_2 h_3}\frac{\partial h_2}{\partial \gamma}$$

The components of the viscous stress tensor for a Stokes' fluid are related to the components of the rate of strain tensor by

$$\tau_{\alpha\alpha} = \lambda \nabla \cdot \mathbf{V} + \mu e_{\alpha\alpha}$$
$$\tau_{\beta\beta} = \lambda \nabla \cdot \mathbf{V} + \mu e_{\beta\beta}$$
$$\tau_{\gamma\gamma} = \lambda \nabla \cdot \mathbf{V} + \mu e_{\gamma\gamma}$$
$$\tau_{\alpha\beta} = \tau_{\beta\alpha} = \mu e_{\alpha\beta}$$
$$\tau_{\alpha\gamma} = \tau_{\gamma\alpha} = \mu e_{\alpha\gamma}$$
$$\tau_{\beta\gamma} = \tau_{\gamma\beta} = \mu e_{\beta\gamma}$$

where the divergence of the velocity vector is

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \alpha}(h_2 h_3 u) + \frac{\partial}{\partial \beta}(h_1 h_3 v) + \frac{\partial}{\partial \gamma}(h_1 h_2 w) \right]$$

[1]The h_1, h_2, and h_3 used by Love are the reciprocals of those used herein.

and the components of the rate of strain tensor are (Ref. 3):

$$\frac{1}{2}e_{\alpha\alpha} = \frac{1}{h_1}\frac{\partial u}{\partial \alpha} + \frac{v}{h_1h_2}\frac{\partial h_1}{\partial \beta} + \frac{w}{h_3h_1}\frac{\partial h_1}{\partial \gamma}$$

$$\frac{1}{2}e_{\beta\beta} = \frac{1}{h_2}\frac{\partial v}{\partial \beta} + \frac{w}{h_2h_3}\frac{\partial h_2}{\partial \gamma} + \frac{u}{h_1h_2}\frac{\partial h_2}{\partial \alpha}$$

$$\frac{1}{2}e_{\gamma\gamma} = \frac{1}{h_3}\frac{\partial w}{\partial \gamma} + \frac{u}{h_1h_3}\frac{\partial h_3}{\partial \alpha} + \frac{v}{h_2h_3}\frac{\partial h_3}{\partial \beta}$$

$$e_{\alpha\beta} = \frac{h_2}{h_1}\frac{\partial}{\partial \alpha}\left(\frac{v}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial \beta}\left(\frac{u}{h_1}\right)$$

$$e_{\alpha\gamma} = \frac{h_1}{h_3}\frac{\partial}{\partial \gamma}\left(\frac{u}{h_1}\right) + \frac{h_3}{h_1}\frac{\partial}{\partial \alpha}\left(\frac{w}{h_3}\right)$$

$$e_{\beta\gamma} = \frac{h_3}{h_2}\frac{\partial}{\partial \beta}\left(\frac{w}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial \gamma}\left(\frac{v}{h_2}\right)$$

The second viscosity coefficient λ is related to the shear viscosity μ (first viscosity coefficient) by $\lambda = -2/3\,\mu$ if the bulk viscosity coefficient defined by $\kappa = \lambda + 2/3\,\mu$ is *zero.* Otherwise, λ is given by

$$\lambda = \kappa - \frac{2}{3}\mu$$

In the various forms of the energy equations, the operator $(\mathbf{V}\cdot\nabla)$ applied to a scalar f, such as H_t, E, p, or H, gives the convection of that quantity by the flow,

$$(\mathbf{V}\cdot\nabla)f = u\frac{1}{h_1}\frac{\partial f}{\partial \alpha} + v\frac{1}{h_2}\frac{\partial f}{\partial \beta} + w\frac{1}{h_3}\frac{\partial f}{\partial \gamma}$$

The divergence of the heat flux vector $\mathbf{q}$ is

$$\nabla\cdot\mathbf{q} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial \alpha}(h_2h_3q_\alpha) + \frac{\partial}{\partial \beta}(h_1h_3q_\beta) + \frac{\partial}{\partial \gamma}(h_1h_2q_\gamma)\right]$$

In particular, if the heat flux vector is given by Fourier's heat-conduction law, $\mathbf{q} = -k\nabla T$, then the components are

$$q_\alpha = -k\frac{1}{h_1}\frac{\partial T}{\partial \alpha}, \qquad q_\beta = -k\frac{1}{h_2}\frac{\partial T}{\partial \beta},$$

$$q_\gamma = -k\frac{1}{h_3}\frac{\partial T}{\partial \gamma}$$

The rate at which work is done by body forces is, simply,

$$\mathbf{F}\cdot\mathbf{V} = F_\alpha u + F_\beta v + F_\gamma w$$

The rate at which work is done by the viscous stresses is given by

$$\nabla\cdot(\tau\cdot\mathbf{V}) = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial \alpha}\left[h_2h_3(\tau_{\alpha\alpha}u + \tau_{\beta\alpha}v + \tau_{\gamma\alpha}w)\right] + \frac{\partial}{\partial \beta}\left[h_1h_3(\tau_{\alpha\beta}u + \tau_{\beta\beta}v + \tau_{\gamma\beta}w)\right] + \frac{\partial}{\partial \gamma}\left[h_1h_2(\tau_{\alpha\gamma}u + \tau_{\beta\gamma}v + \tau_{\gamma\gamma}w)\right]\right\}$$

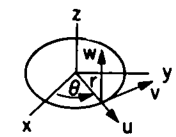
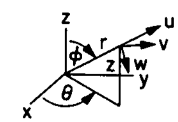
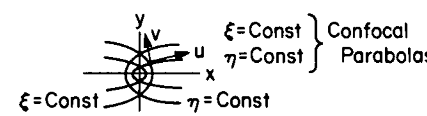
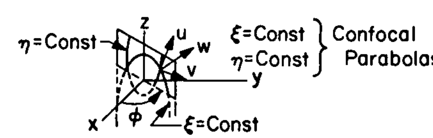
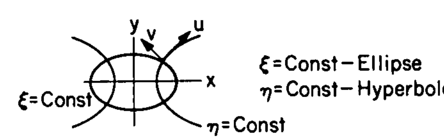
Lastly, the rate of dissipation of energy takes the form

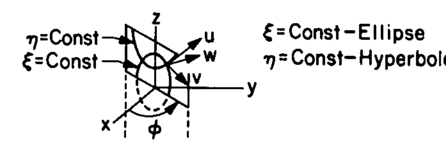
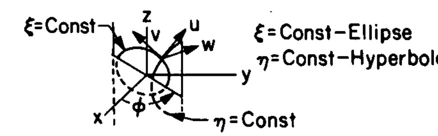
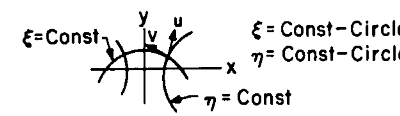
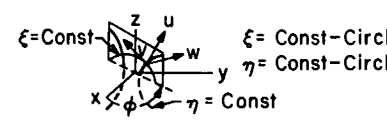
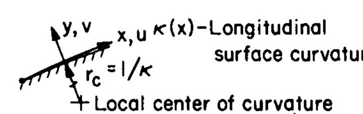
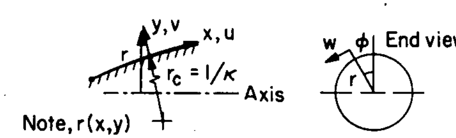
$$\begin{aligned}\tau:(\nabla\mathbf{V}) = {} & \tau_{\alpha\alpha}\left(\frac{1}{h_1}\frac{\partial u}{\partial \alpha} + \frac{v}{h_1h_2}\frac{\partial h_1}{\partial \beta} + \frac{w}{h_1h_3}\frac{\partial h_1}{\partial \gamma}\right)\\ & + \tau_{\beta\beta}\left(\frac{1}{h_2}\frac{\partial v}{\partial \beta} + \frac{u}{h_1h_2}\frac{\partial h_2}{\partial \alpha} + \frac{w}{h_2h_3}\frac{\partial h_2}{\partial \gamma}\right)\\ & + \tau_{\gamma\gamma}\left(\frac{1}{h_3}\frac{\partial w}{\partial \gamma} + \frac{u}{h_1h_3}\frac{\partial h_3}{\partial \alpha} + \frac{v}{h_2h_3}\frac{\partial h_3}{\partial \beta}\right)\\ & + \tau_{\alpha\beta}\left(\frac{1}{h_2}\frac{\partial u}{\partial \beta} + \frac{1}{h_1}\frac{\partial v}{\partial \alpha} - \frac{v}{h_1h_2}\frac{\partial h_2}{\partial \alpha} - \frac{u}{h_1h_2}\frac{\partial h_1}{\partial \beta}\right)\\ & + \tau_{\alpha\gamma}\left(\frac{1}{h_3}\frac{\partial u}{\partial \gamma} + \frac{1}{h_1}\frac{\partial w}{\partial \alpha} - \frac{w}{h_1h_3}\frac{\partial h_3}{\partial \alpha} - \frac{u}{h_1h_3}\frac{\partial h_1}{\partial \gamma}\right)\\ & + \tau_{\beta\gamma}\left(\frac{1}{h_3}\frac{\partial v}{\partial \gamma} + \frac{1}{h_2}\frac{\partial w}{\partial \beta} - \frac{w}{h_2h_3}\frac{\partial h_3}{\partial \beta} - \frac{v}{h_2h_3}\frac{\partial h_2}{\partial \gamma}\right)\end{aligned}$$

This rate of dissipation of energy term usually appears in the literature as Φ.

Table 1 presents descriptive information on a number of orthogonal coordinate systems for which the conservation equations can be readily written by use of the foregoing relations. The last two entries in Table 1, in which the coordinates are taken along and normal to the surface, are useful in analyzing internal and external boundary-layer flows. For many flow problems in these coordinates, the dominant viscous stress is the shear stress that lies in the plane of β = const ($\tau_{\alpha\beta}$ for a two-dimensional flow and $\tau_{\alpha\beta}$, $\tau_{\gamma\beta}$ for a three-dimensional flow), and the important heat-flux component is normal to the surface, q_β.

Table 1. Coordinate systems and scale factors

1. Orthogonal coordinate system, and 2. orthogonal coordinates α, β, γ	Rectangular coordinates			Scale factors h_1 h_2 h_3			Coordinate configuration
	x	y	z	h_1	h_2	h_3	
Cylindrical r, θ, z	$r \cos \theta$	$r \sin \theta$	z	1	r	1	
Spherical r, ϕ, θ	$r \cos \theta \sin \phi$	$r \sin \theta \sin \phi$	$r \cos \phi$	1	r	$r \sin \phi$	
Parabolic cylindrical ξ, η, z	$\frac{1}{2}(\xi^2 - \eta^2)$	$\xi\eta$	z	$\sqrt{\xi^2 + \eta^2}$	$\sqrt{\xi^2 + \eta^2}$	1	
Paraboloidal ξ, η, ϕ	$\xi\eta \cos \phi$	$\xi\eta \sin \phi$	$\frac{1}{2}(\xi^2 - \eta^2)$	$\sqrt{\xi^2 + \eta^2}$	$\sqrt{\xi^2 + \eta^2}$	$\xi\eta$	
Elliptic cylindrical ξ, η, z	$a \cosh \xi \cos \eta$ $a = \text{const}$	$a \sinh \xi \sin \eta$	z	$a\sqrt{\sinh^2 \xi + \sin^2 \eta}$	$a\sqrt{\sinh^2 \xi + \sin^2 \eta}$	1	

Prolate spheroidal ξ, η, ϕ	$a \sinh \xi \sin \eta \cos \phi$ $a = \text{const}$	$a \sinh \xi \sin \eta \sin \phi$	$a \cosh \xi \cos \eta$	$a \sqrt{\sinh^2 \xi + \sin^2 \eta}$	$a \sqrt{\sinh^2 \xi + \sin^2 \eta}$	$a \sinh \xi \sin \eta$	
Oblate spheroidal ξ, η, ϕ	$a \cosh \xi \cos \eta \cos \phi$ $a = \text{const}$	$a \cosh \xi \cos \eta \sin \phi$	$a \sinh \xi \sin \eta$	$a \sqrt{\sinh^2 \xi + \sin^2 \eta}$	$a \sqrt{\sinh^2 \xi + \sin^2 \eta}$	$a \cosh \xi \cos \eta$	
Bipolar ξ, η, z	$\dfrac{a \sinh \eta}{\cosh \eta - \cos \xi}$ $a = \text{const}$	$\dfrac{a \sin \xi}{\cosh \eta - \cos \xi}$	z	$\dfrac{a}{\cosh \eta - \cos \xi}$	$\dfrac{a}{\cosh \eta - \cos \xi}$	1	
Toroidal ξ, η, ϕ	$\dfrac{a \sinh \eta \cos \phi}{\cosh \eta - \cos \xi}$ $a = \text{const}$	$\dfrac{a \sinh \eta \sin \phi}{\cosh \eta - \cos \xi}$	$\dfrac{a \sin \xi}{\cosh \eta - \cos \xi}$	$\dfrac{a}{\cosh \eta - \cos \xi}$	$\dfrac{a}{\cosh \eta - \cos \xi}$	$\dfrac{a \sinh \eta}{\cosh \eta - \cos \xi}$	
Local coordinates along surface (Ref. 3) x, y, z	—	—	—	$1 + \kappa y$	1	1	
Local coordinates along surface (Ref. 3) Symmetric about axis x, y, ϕ	—	—	—	$1 + \kappa y$	1	r	

Nomenclature

e_{ij} = components of rate of strain tensor

E = internal energy per unit mass

f = scalar

$\mathbf{F}$ = body force per unit volume

$h_1 h_2 h_3$ = scale factors

H = static enthalpy per unit mass

H_t = total enthalpy, $H_t = H + V^2/2$

k = thermal conductivity

p = static pressure

$\mathbf{q}$ = heat-flux vector

q_β = heat-flux component normal to the surface

t = time

T = temperature

u = velocity component in α direction

v = velocity component in β direction

$\mathbf{V}$ = velocity vector

w = velocity component in γ direction

W = heat generation per unit volume

α, β, γ = orthogonal coordinates

κ = bulk viscosity

λ = second viscosity coefficient

μ = shear viscosity

ρ = density

τ = viscous stress tensor

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