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AD 670 004

APPLICATION OF STATISTICAL SMOOTHING TECHNIQUES TO INERTIAL NAVIGATION

R. K. Mehra

Harvard University
Cambridge, Massachusetts

April 1968

GPO PRICE $ ______

CFSTI PRICE(S) $ ______

Hard copy (HC) 3.00

Microfiche (MF) ______

ff 653 July 65

FACILITY FORM 602

N68-36384 (ACCESSION NUMBER) (THRU)

20 (PAGES) 1 (CODE)

CR-97104 (NASA CR OR TMX OR AD NUMBER) 21 (CATEGORY)

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Office of Naval Research

Contract N00014-67-A-0298-0006

NR - 372 - 012

National Aeronautics and Space Administration

Grant NGR 22-007-068

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Technical Report No. 560

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Division of Engineering and Applied Physics

Harvard University Cambridge, Massachusetts

AD 670004

Office of Naval Research

Contract N00014-67-A-0298-0006 NR-372-012

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
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APPLICATION OF STATISTICAL SMOOTHING TECHNIQUES TO INERTIAL NAVIGATION

By

R. K. Mehra

Division of Engineering and Applied Physics

Harvard University Cambridge, Massachusetts

ABSTRACT

All inertial navigation systems are excited by random error inputs such as gyro drift rates, accelerometer errors etc. These errors make the position and velocity errors of the inertial navigator grow unbounded with time if no external measurements are made. Statistical smoothing techniques of Bryson-Frazier [13] and Rauch-Tung and Striebel [14] provide methods of estimating these random inputs from test runs of the system. (Smoothing involves estimating the state of a system at time t using measurements made both before and after time t.) In the present report, a ship's inertial navigation system is considered. It is shown that by using smoothing techniques, uncertainties in the knowledge of random error sources, particularily, gyro drift rates, can be significantly reduced.

1. Introduction

Kalman Filtering techniques have recently been applied to update Inertial Navigation Systems [1, 2, 3, 4]. The implementation of a Kalman Filter, however, assumes perfect knowledge of the system and noise parameters. In practice, these paramenters are either unknown or known only approximately and it becomes desirable to improve their estimates by making test runs on the system. The noisy data obtained during these test runs can be smoothed in the laboratory to obtain best estimates of the state history and of the noise waveforms which enter the system during these test runs. As statistical properties of the signals can be derived from their sample functions, the smoothed waveforms can be used to estimate system and noise parameters.

The smoothing equations derived in Ref. [13, 14] show that the covariance of the errors in the smoothed estimates can be precomputed. Therefore, a number of useful feasibility studies can be carried out and instrument specifications for test runs can be given before the actual tests are made. In this report, a feasibility study of this nature will be carried out for a Ship's Inertial Navigation System (SINS). The intent will be to demonstrate the usefulness of smoothing in recovering certain noise waveforms. In particular, the Gyro drift rate errors and ocean currents will be considered. It will be shown that the uncertainties in their estimates can be reduced significantly by use of smoothing. Results of Ref. [12] will be used to evaluate these results in view of the fact that the smoother is not optimal, since the system and noise parameters are not known exactly.

2. Formulation of the Problem

A Ship's Inertial Navigation System (Gimballed) typically consists of three gyros, two accelerometers, and a few integrators. The dynamic coupling among the three axes of the gyros is quite weak if the ship moves along a path of constant latitude and the time interval under consideration is snall compared to 24 hours.

Making use of these assumptions, only one axis of SINS will be considered here. It should be pointed out however, that the results obtained using the three axis model have been found to be the same as the results obtained using a single axis model. Therefore, it is sufficient to consider only the single-axis model which is easier to analyze.

Figure 1 shows a single-axis Schuler loop (or one axis of SINS). All the random forcing functions entering the system, (e_a, e_g, e_p, and e_v) are assumed to be stationary with exponential correlations. The mean square values of these functions are not known exactly, but the following a priori estimates are available:

1. Accelerometer error, e_a

   A priori RMS value = σ_a sec of arc

   Correlation time = $1/\beta_a$ hr

2. Gyro drift rate error, e_g

   A priori RMS value = σ_g sec of arc/hr

   Correlation time = $1/\beta_g$ hr

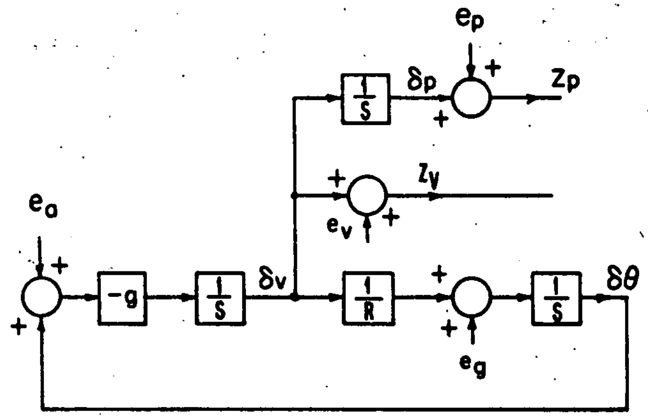

FIGURE 1 ERROR DIAGRAM OF A SHIP'S INERTIAL NAVIGATION SYSTEM

3. **Position Fix Error, e_p**

A priori RMS value = σ_p nm

Correlation time = $1/\beta_p$ hr

4. **Continuous Velocity Measurement Error (Ocean Currents), e_v**

A priori RMS value = σ_v kts

Correlation distance = d_v nm

Correlation time = $1/\beta_v = \frac{d_v}{V}$ hr

where V = ship speed in kts

All the noise sources are assumed to have exponential correlation so that they can be modeled as

$$\dot{e} = -\beta e + w \tag{1}$$

where e = exponentially correlated noise source having correlation time of $1/\beta$ and mean square value of $q/2\beta$ ($=\sigma^2$)

$$E\{e(t)\, e(\tau)\} = \sigma^2 \exp(-\beta |t - \tau|)$$

q = power spectral density of white noise w.

$$E\{w(t)\, w(\tau)\} = q(t)\, \delta(t - \tau)$$

It is assumed that the noise sources are independent.

2.1 System Equations:

Let δV = velocity error in kts

δp = positon error in nm

g = acceleration of gravity = 6.857×10^4 nm/hr^2

e_a = accelerometer error in sec of arc

R = radius of the earth = 3436.4 nm

e_g = gyro drift rate error in sec of arc/hr

e_p = position fix error in nm

e_v = velocity measurement error in kts

$\delta\theta$ = platform tilt error in sec of arc

Using Fig. 1 system equations can be written as

$$\delta\dot{V} = -g(\delta\theta + e_a) \tag{2}$$

$$\delta\dot{p} = \delta v \tag{3}$$

$$\delta\dot{\theta} = \frac{\delta v}{R} + e_g \tag{4}$$

$$\dot{e}_a = -\beta_a e_a + w_a \tag{5}$$

$$\dot{e}_g = -\beta_g e_g + w_g \tag{6}$$

$$\dot{e}_p = -\beta_p e_p + w_p \tag{7}$$

$$\dot{e}_v = -\beta_v e_v + w_v \tag{8}$$

2.2 Measurements:

Position and velocity measurements are obtained on the system. These measurements are then used as input to an optimal smoother which comes out with best estimates of e_a, e_g, e_p and e_v. We shall consider 3 types of measurements.

(i) Velocity is measured continuously using EM log. Position fixes are made at discrete time intervals.

Let z_v = cont. velocity measurement (9)

z_p = discrete position measurement (10)

$$z_v = \delta V + e_v \qquad O \leq t \leq T \tag{11}$$

$$z_p = \delta p + e_p \;,\; t = t_i,\; i = 1, \ldots, T/t_i \tag{12}$$

Putting equations (2) to (12) in the state-vector form

$$\dot{x} = Fx + Gu \tag{13}$$

$$z_v = H_v x \tag{14}$$

$$z_p = H_p x \tag{15}$$

where x = n × 1 state vector of the system

u = r × 1 vector of random forcing functions

$$\underset{7 \times 1}{x} = \begin{bmatrix} \delta V \\ \delta p \\ \delta\theta \\ e_a \\ e_g \\ e_p \\ e_v \end{bmatrix} \qquad \underset{6 \times 1}{u} = \begin{bmatrix} w_a \\ w_g \\ w_p \\ w_v \end{bmatrix}$$

$$\underset{7 \times 7}{F} = \begin{bmatrix} 0, & 0, & -g, & -g, & \theta, & 0, & 0 \\ 1, & 0, & 0, & 0, & 0, & 0, & 0 \\ \frac{1}{R}, & 0, & 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & -\beta_a, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & -\beta_g, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & -\beta_p, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & -\beta_v \end{bmatrix}$$

$$\underset{7 \times 4}{G} = \begin{bmatrix} 0, & 0, & 0, & 0, \\ 0, & 0, & 0, & 0, \\ 0, & 0, & 0, & 0, \\ 1, & 0, & 0, & 0, \\ 0, & 1, & 0, & 0, \\ 0, & 0, & 1, & 0, \\ 0, & 0, & 0, & 1, \end{bmatrix}$$

$$\underset{1 \times 7}{H_v} = [1, \; 0, \; 0, \; 0, \; 0, \; 0, \; 1]$$

$$\underset{1 \times 7}{H_p} = [0, \; 1, \; 0, \; 0, \; 0, \; 1, \; 0]$$

(ii) Both velocity and position are measured continuously. This case is somewhat hypothetical, but the results obtained would provide lower bounds for the smoothing errors in case (i) as the frequency of the position fixes is allowed to increase.

Measurement equations are

$$z_v = \delta V + e_v \tag{16}$$

$$O \leq t \leq T$$

$$z_p = \delta p + e_p \tag{17}$$

Let

$$z = \begin{bmatrix} z_v \\ z_p \end{bmatrix} = Hx + v$$

where $\underset{2 \times 7}{H} = \begin{bmatrix} 1, & 0, & 0, & 0, & 0, & 0, & 1 \\ 0, & 1, & 0, & 0, & 0, & 1, & 0 \end{bmatrix}$

$$\underset{2 \times 1}{v} = \begin{bmatrix} e_v \\ e_p \end{bmatrix}$$

(iii) No position measurement is made. The equations are similar to case (i) and (ii) except that z_p and e_p equations are absent. The results for this case would provide upper bounds on results of case (i) as fix frequency is decreased.

3. Reducing Dimension of the State Vector

In section 2, correlated errors in the measurements were used to augment the state of the system. The resulting measurements for the augmented system are perfect and can be differentiated to obtain extra measurements. See Mehra and Bryson [10]. The perfect measurements are used to reduce the state of the system.

Differentiating (11) and substituting from (2) and (8)

$$\dot{z}_v = \delta\dot{V} + \dot{e}_v$$

$$\dot{z}_v = -g(\delta\theta + e_a) + (-\beta_v e_v + w_v) \qquad (18)$$

From Eq. (8)

$$e_v = z_v - \delta V$$

Substituting for e_v in (18)

$$\dot{z}_v = -g(\delta\theta + e_a) - \beta_v z_v + \beta_v \delta V + w_v$$

$$(\dot{z}_v + \beta_v z_v) = [\beta_v, 0, -g, -g, 0, 0, 0]\, x + w_v \qquad (19)$$

Eq. (19) is used as a new measurement equation with initial condition

$$z_v(0) = \delta V(0) + e_v(0)$$

Notice that Equations (2), (8) and (19) are linearly dependent. So Equation (8) can be dropped from the system equations.

Let x_1 be the reduced state vector

$$\underset{6\times 1}{x_1} = \begin{bmatrix} \delta V \\ \delta p \\ \delta_\theta \\ e_a \\ e_g \\ e_p \end{bmatrix}$$

New system equations are

$$\dot{x}_1 = F_1 x_1 + G_1 u_1$$

$$z_1 = H_1 x_1 + w_v$$

$$Z_p = [0, 1, 0, 0, 0, 1]\, x_1$$

where

$$\underset{6\times 6}{F_1} = \begin{bmatrix} 0, & 0, & -g, & -g, & 0, & 0 \\ 1, & 0, & 0, & 0, & 0, & 0 \\ \frac{1}{R}, & 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & -\beta_a, & 0, & 0 \\ 0, & 0, & 0, & 0, & -\beta_g, & 0 \\ 0, & 0, & 0, & 0, & 0, & -\beta_p \end{bmatrix}; \quad \underset{3\times 1}{u_1} = \begin{bmatrix} w_a \\ w_g \\ w_p \end{bmatrix}$$

$$\underset{6\times 3}{G_1} = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \\ 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{bmatrix}$$

$$\underset{1\times 6}{H_1} = [\beta_v,\ 0,\ -g,\ -g,\ 0,\ 0]$$

The power spectral density matrices of u_1 and w_v are as follows:

$$\underset{3\times 3}{Q_1} = \begin{bmatrix} 2\beta_a\sigma_a^2, & 0, & 0 \\ 0, & 2\beta_g\sigma_g^2, & 0 \\ 0, & 0, & 2\beta_p\sigma_p^2 \end{bmatrix}$$

$$\underset{1\times 1}{R_1} = 2\beta_v\sigma_v^2$$

Notice that

$$E\{u_1(t)u_1^T(\tau)\} = Q_1\,\delta(t-\tau)$$

Similarily in case (ii), we differentiate both (16) and (17) and then substitute from the system Equations (2) through (8)

$$\begin{bmatrix} \dot{z}_v + \beta_v z_v \\ \dot{z}_p + \beta_p z_p \end{bmatrix} = \begin{bmatrix} \beta_v, & 0, & -g, & -g, & 0 \\ 1, & \beta_p, & 0, & 0, & 0 \end{bmatrix} \begin{bmatrix} \delta V \\ \delta p \\ \delta\theta \\ e_a \\ e_g \end{bmatrix} + \begin{bmatrix} w_v \\ w_p \end{bmatrix}$$

Notice that the new state vector is 5 x 1

$$\underset{5\times 1}{x_2} = \begin{bmatrix} \delta V \\ \delta p \\ \delta\theta \\ e_a \\ e_g \end{bmatrix} \qquad \underset{2\times 1}{u_2} = \begin{bmatrix} w_a \\ w_g \end{bmatrix}$$

$$\underset{5\times 5}{F_2} = \begin{bmatrix} 0, & 0, & -g, & -g, & 0 \\ 1, & 0, & 0, & 0, & 0 \\ \frac{1}{R}, & 0, & 0, & 0, & 1 \\ 0, & 0, & 0, & -\beta_a, & 0 \\ 0, & 0, & 0, & 0, & -\beta_g \end{bmatrix}$$

$$\underset{2\times 5}{H_2} = \begin{bmatrix} \beta_v, & 0, & -g, & -g, & 0 \\ 1, & \beta_p, & 0, & 0, & 0 \end{bmatrix}$$

$$\underset{5\times 2}{G_2} = \begin{bmatrix} 0, & 0 \\ 0, & 0 \\ 0, & 0 \\ 1, & 0 \\ 0, & 1 \end{bmatrix}$$

Power spectral density matrix of white noise processes in the system equations is given as

$$\underset{2\times 2}{Q_2} = \begin{bmatrix} 2\beta_a\sigma_a^2, & 0 \\ 0 & 2\beta_g\sigma_g^2 \end{bmatrix}$$

Correlation matrix of white noise processes in measurements is given as

$$\underset{2\times 2}{R_2} = \begin{bmatrix} 2\beta_v\sigma_v^2, & 0 \\ 0 & 2\beta_p\sigma_p^2 \end{bmatrix}$$

4. Equations of an Optimal Smoother

The equations of an optimal smoother were first given in a recursive form by Bryson and Frazier [13]. Alternative forms of these equations were derived by Rauch [14], Mayne [11] and Fraser [5]. Mehra and Bryson [10] extended the above results to the case of colored noise in the measurements.

Denote by

$\hat{x}(t)$ = Filtering estimate of $x(t)$ based on measurements from t_o to t.

$\hat{x}(t/T)$ = Smoothing estimate of $x(t)$ based on measurements from t_o to T.

$P(t)$ = Covariance of error in the filtered estimate of $x(t)$

$$= E\{[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]^T\}$$

$P(t/T)$ = Covariance of error in the smoothed estimate of $x(t)$

$$= E\{[x(t) - \hat{x}(t/T)][x(t) - \hat{x}(t/T)]^T\}$$

Then the different filtering and smoothing equations are:

Kalman-Bucy Filter:

$$\dot{\hat{x}}(t) = F\hat{x} + K(z - H\hat{x}) \qquad (20)$$

$$K = PH^TR^{-1} \qquad (21)$$

$$\dot{P}(t) = FP + PF^T + GQG^T - PH^TR^{-1}HP \qquad (22)$$

$\hat{x}(t_o)$ and $P(t_o)$ are given.

Bryson-Frazier Smoother:

$$\hat{x}(t/T) = \hat{x}(t) - P(t)\lambda(t) \qquad (23)$$

$$\dot{\lambda}(t) = -(F - PH^TR^{-1}H)^T\lambda + H^TR^{-1}(z - H\hat{x}) \qquad (24)$$

$$\lambda(T) = 0 \qquad (25)$$

$$P(t/T) = P - P\Lambda P \qquad (26)$$

$$\dot{\Lambda} = -(F - PH^TR^{-1}H)^T\Lambda - \Lambda(F - PH^TR^{-1}H) - H^TR^{-1}H \qquad (27)$$

$$\Lambda(T) = 0 \qquad (28)$$

Rauch-Tung-Striebel Smoother:

$$\dot{\hat{x}}(t/T) = F\hat{x}(t/T) + N(t)(\hat{x}(t/T) - \hat{x}(t)) \qquad (29)$$

$$\hat{x}(T/T) = \hat{x}(T) \qquad (30)$$

$$N(t) = GQG^T P^{-1}(t) \qquad (31)$$

$$P(t/T) = (F + N)P(t/T) + \dot{P}(t/T)(F + N)^T - GQG^T \qquad (32)$$

$$P(T/T) = P(T) \qquad (33)$$

Two Kalman Filter Smoother (Mayne, Fraser):

$$\hat{x}(t/T) = P(t/T)[P^{-1}(t)\hat{x}(t) + P_b^{-1}(t)\hat{x}_b(t)] \qquad (34)$$

$$P^{-1}(t/T) = P^{-1}(t) + P_b^{-1}(t) \qquad (35)$$

$$P_b^{-1}(T) = 0 \qquad (36)$$

Subscript b stands for the backward Kalman filter.

The above form of equations hold for the case of white noise in the measurements. Mehra and Bryson [10] show that the same results can be used for the colored noise case if proper transformations are made. In the example considered here, the colored noise situation is handled by using in the above equations with matrices F_1, G_1, H_1, Q_1, R_1 and F_2, G_2, H_2, Q_2, R_2 in place of F, G, H, Q, R.

5. Numerical Solution of Smoothing Equations Using Automatic Synthesis Program

5.1 Choice of Smoothing Equations:

The error covariance equations for an optimal smoother are expressible in a number of different forms, as shown above. Fraser [5] discusses the relative merits of each from a computational point of view. However, if some of the measurements are discrete, only one form of the equations is suitable for numerical computations. This is the Rauch-Tung-Striebel form of equations. Eq. (29) - (33) show that in the case of discrete measurements, even though the filtering estimates have discontinuities, the smoothing estimates are continuous. If the Bryson-Frazier form of equations or the two Kalman filter approach is used, the numerical results obtained for the covariance of the smoothing errors are found to have discontinuities in them.* Therefore, it is necessary to use the Rauch-Tung-Striebel form of the equations, viz.,

$$\frac{d}{dt} P(t) = FP + PF^T + GQG^T - PH^T R^{-1} HP \qquad (22)$$

$P(t_o)$ given

$$\frac{d}{dt} P(t/T) = (F + GQG^T P^{-1}(t))\, P(t/T) + P(t/T)\,(F + GQG^T P^{-1}(t))^T - GQG^T$$

$$P(T/T) = P(T) \qquad (32)$$

---

* E.g. consider Equations (26) and (27). Suppose P(t) has finite discontinuities at finite number of points. From Equation (27), Λ is continuous. We also know that P(t/T) must be continuous. Therefore, the discontinuities in P and $P \Lambda P$ must cancel each other. This is numerically difficult to achieve.

For continuous measurements, any one of the smoothing equations can be used. All the three approaches were used on the problem and they were found to yield the same answers.

5.2 Automatic Synthesis Program (ASP) : [6]

Automatic Synthesis Program of Kalman and Englar is a multi-purpose matrix manipulation program. It has subroutines to add, subtract, multiply, invert and perform a number of other operations on matrices. The exponential routine provides the solution to a set of linear differential equations by evaluating e^{Ft}. A matrix riccati equation can be solved by converting it to an equivalent Hamiltonian system of linear equations. For example, suppose that we want to solve Equation (22) using ASP

$$\dot{P} = FP + PF^T + GQG^T - PH^T R^{-1} Hp$$

$P(t_o)$ given

Let $\quad \dot{x} = -F^T x + H^T R^{-1} HPx$

(This x is different from the state vector x).

and $\quad \lambda = Px$

Then $\quad \dot{\lambda} = \dot{P}x + P\dot{x}$

$$= FPx + PF^T x + GQG^T x - PH^T R^{-1} HPx - PF^T x + PH^T R^{-1} HPx$$

or $\quad \dot{\lambda} = F\lambda + GQG^T x$

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -F^T & H^T R^{-1} H \\ GQG^T & F \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \qquad (37)$$

Then P(t) is computed recursively using the following equation.

$$P(t+\tau) = [\phi_{\lambda x} + \phi_{\lambda\lambda} P(t)]\,[\phi_{xx} + \phi_{x\lambda} P(t)]^{-1}$$

where

$\begin{bmatrix} \phi_{xx} & \phi_{x\lambda} \\ \phi_{\lambda x} & \phi_{\lambda\lambda} \end{bmatrix}$ is the transition matrix of the x-λ system. (Equation (37)).

Equation (32) can be solved in a similar way.

Let $\dot{x} = -(F + GQG^T P^{-1}(t))^T x$

and $\lambda = Px$

Then
$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -(F + GQG^T P^{-1}(t))^T, & 0 \\ -GQG^T & , (F + GQG^T P^{-1}(t)) \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

As the boundry condition is specified at t = T, we let $\tau = (T - t)$ and solve the following equation:

$$\frac{d}{d\tau}\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} (F + GQG^T P^{-1}(t))^T, & \\ GQG^T & (F + GQG^T P^{-1}(t)) \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

starting from $\tau = t_o$.

5.3 Stationary Solution of Matrix Riccati Equation for a Constant Coefficient System:

If the system is observable [6], P(t) tends to a stationary limit which is the solution to the set of equations obtained by putting the right-hand-side of Equation (22) equal to zero. If this constant value of P(t) is used in Equation (32), the latter would also have a stationary solution.

It is useful to obtain these stationary values because they represent lower limits on the filtering and smoothing errors. They are valid for the case in which the time interval under consideration is large.

6. Results

6.1 Specifications:

A typical gyro used in cruise vehicles is chosen from Reference [7]. The a priori rms drift rate is .01 deg/hr with a correlation time of 5 hrs. Values of accelerometer errors and velocity errors due to ocean currents are taken from References [2] and [8]. Position fixes are made using LORAC C which has a position fix error of 0.1 nm RMS [9]. Table 1 gives all the numbers that are used to solve the smoothing equations numerically.

ship speed = 15 kts

Noise Source	RMS Error	Correlation Time
Accelerometer error, e_a	20 sec	30 hrs
Gyro drift rate error, e_g	.01 deg/hr	5 hrs
Velocity measurement error, e_v	0.6 kt	1 hr
Position fix error, e_p	0.1 nm	1/4 hr

Table 1 Showing Input Values

A number of cases were considered. The results obtained are as below.

6.2 Continuous Position and Velocity Measurements

Table 2 shows the stationary value of RMS errors in filtering and smoothing estimates of e_a, e_g, e_v and e_p. A comparison of Tables 1 and 2 shows that rms errors in e_g, e_v and e_p are significantly reduced. The accelerometer error e_a is almost unchanged.

Noise Source	Stationary Value of RMS Error in Filtered Estimate	Stationary Value of RMS Error in Smoothed Estimate
Accelerometer error, e_a	19.95 $\widehat{sec}$	19.9 $\widehat{sec}$
Gyro drift rate, e_g	0.0044 deg/hr	0.0024 deg/hr
Velocity measurement error (ocean currents), e_v	0.286 kt	0.1635 kt
Position measurement error, e_p	0.089 nm	0.067 nm

Table 2 Showing Stationary RMS Values of Smoothing and Filtering Errors for the Case of Continuous Position and Velocity Measurements

6.3 Continuous Velocity Measurement (No Position Fixes):

Position error is unbounded in this case. Table 3 shows the stationary RMS values of filtering and smoothing errors. It can be seen that they are much higher than the values in Table 2.

Noise Source	Stationary RMS Value of Filtering Error	Stationary RMS Value of Smoothing Error
Accelerometer error, e_a	20 $\widehat{sec}$	19.95 $\widehat{sec}$
Gyro drift rate, e_g	0.007 deg/hr	0.006 deg/hr
Velocity measurement error (ocean currents), e_v	0.45 kt	0.4 kt
Position measurement error, e_p	unbounded	unbounded

Table 3 Showing Stationary RMS Values of Smoothing and Filtering Errors for the Case of Continuous Velocity Measurements (No Position Fixes)

6.4 Continuous Velocity Measurements and Position Fixes Every One-Half Hour:

Filtering estimates have discontinuities, but the smoothing estimates are continuous. In Fig. 2, the propagation of errors in gyro drift rate are shown. The rms values are intermediate between the values in Tables 2 and 3. There is an initial discontinuity in the filered estimate of ocean currents due to the presence of perfect measurements. See Mehra and Bryson [10].

7. Methods to Reduce Smoothing and Filtering Errors

7.1 Ship Speed

Changing the speed of the ship changes the correlation time of ocean currents. It was found that reducing the ship speed reduced rms errors in ocean currents (e_v) and gyro drift rate (e_g). This is due to the fact that ocean currents look more and more like white noise if the ship speed is increased. Figure 3 shows rms smoothing errors (stationary values) of e_g vs. ship speed.

7.2 Correlation-Time of Accelerometer Errors

Changing the correlation time of accelerometer errors changes the smoothing and filtering errors in estimates of the noise sources. Figure 4 shows that smoothing error of e_g increases as the correlation time of e_a is reduced. The reason for this lies in the fact that the effect of e_a on system output is more pronounced when its time constant is comparable to the Schuler-loop frequency of 84 min. It is clear that for better recovery of gyro-drift rates, the spectrum of e_g should be different from that of e_a.

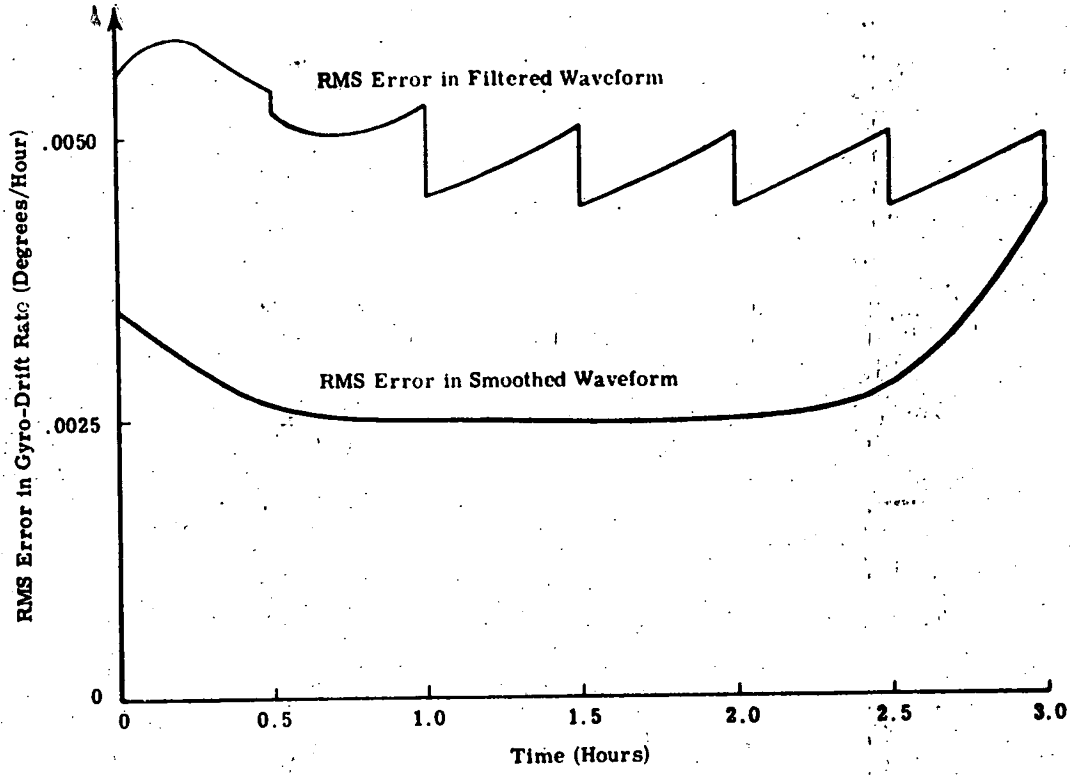

FIGURE 2 RMS ERRORS IN FILTERED AND SMOOTHED GYRO-DRIFT RATE WAVEFORMS--Continuous Velocity Measurement and Position Fixes Every One-Half Hour

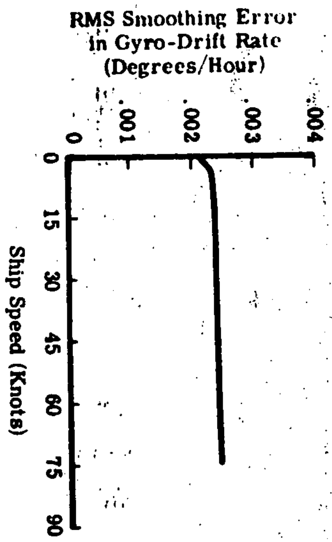

FIGURE 3 STATIONARY VALUE OF RMS SMOOTHING ERROR IN GYRO-DRIFT RATE WAVEFORM VERSUS SHIP SPEED

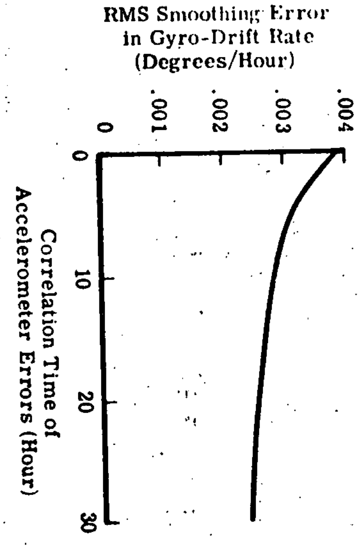

FIGURE 4 STATIONARY VALUE OF RMS SMOOTHING ERROR IN GYRO-DRIFT RATE VERSUS CORRELATION TIME OF ACCELEROMETER ERRORS

7.3 Including Ship's Dynamics with SINS:

Henrikson [2] has shown that by modeling the ship's motion and by adjoining it with the model for the SINS, the filtering error can be significantly reduced. The same holds true for the smoothing problem. The knowledge of position and velocity obtained from ship's dynamics acts as a measurement input to the SINS.

7.4 An Adaptive Scheme:

As the covariances of the noise sources are not known exactly, the smoothing technique being used is suboptimal. Furthermore, the calculated values of the covariances shown in Tables 1-3 and Figures 2-4 are not the actual values of covariances (Cf. Ref. [12]). But if the a priori values of noise covariances in Table 1 are upper limits on the actual values of noise covariances, then the calculated values given in Tables 2-3 and Figures 2-4 are upper limits on smoothing and filtering errors. See Mehra [12]. A comparison of suboptimal filters with the optimal filter shows that the errors in suboptimal filters increase as the difference between the assumed and actual values of the noise covariances increases. This suggests that we may profitably use an adaptive scheme to approach correct values of noise parameters and to recover the signals. We start with certain values for noise covariances which are different from the actual values. We update our estimate of these values by using smoothed waveforms as sample functions of the random processes. Then we use these new values of covariances and smooth the data again. This process is carried on till the changes in parameters become very small.

8. Summary and Conclusions

A Single-axis of a Ship's Inertial Navigation System (SINS) has been considered. The noise inputs into the system are the accelerometer errors, gyro drift rate errors, position fix errors and velocity measurement errors due to ocean currents. All the noise inputs are assumed to be exponentially correlated with unknown covariances. Continuous velocity and discrete position measurements are made on the system over a certain time interval. Assuming a priori values for noise covariances, it is shown that smoothing can significantly reduce the errors in the estimates. This is done by numerically solving the error covariance equations for the smoothed estimates. Since the covariance equations do not depend on the measurement vector or the state vector of the system, they can be solved before any data are collected. In fact, a parametric study can be made to examine the feasibility of satisfactory waveform recovery and to specify instrument accuracies for the test runs. There are a number of different forms of smoothing equations, but we show that if position fixes are discrete and the system is continuous, only the Rauch-Tung-Striebel form of equations is suitable for numerical computations. Since the measurements contain colored noise, we use results of Mehra and Bryson [10] to derive an equivalent reduced system with white noise in the measurements. Results of Ref. [12] are applied to evaluate the smoothing results in view of the fact that the smoother is no longer optimal if the noise covariances are not known exactly. A number of methods are suggested to improve the estimate of gyro drift rate errors. Finally, an adaptive scheme is suggested to obtain values of noise covariances.

ACKNOWLEDGEMENTS

The author is very grateful to Professor A. E. Bryson, Jr. of Harvard University and to Dr. A. Gelb and S. Levine of The Analytic Sciences Corporation, Winchester for their generous help and suggestions during the course of this research.

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Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Division of Engineering and Applied Physics Harvard University Cambridge, Massachusetts 02138	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP
3. REPORT TITLE APPLICATION OF STATISTICAL SMOOTHING TECHNIQUES TO INERTIAL NAVIGATION	
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Interim technical report	
5. AUTHOR(S) (First name, middle initial, last name) R. K. Mehra	
6. REPORT DATE April 1968	7a. TOTAL NO. OF PAGES 32 7b. NO. OF REFS 14
8a. CONTRACT OR GRANT NO. N00014-67-A-029800006 and b. PROJECT NO. NGR-22-007-068 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 560 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
10. DISTRIBUTION STATEMENT Reproduction in whole or in part is permitted for any purpose of the United States Government.	
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Office of Naval Research

13. ABSTRACT

All inertial navigation systems are excited by random error inputs such as gyro drift rates, accelerometer errors etc. These errors make the position and velocity errors of the inertial navigator grow unbounded with time if no external measurements are made. Statistical smoothing techniques of Bryson-Frazier [13] and Rauch-Tung and Striebel [14] provide methods of estimating these random inputs from test runs of the system. (Smoothing involves estimating the state of a system at time t using measurements made both before and after time t.) In the present report, a ship's inertial navigation system is considered. It is shown that by using smoothing techniques, uncertainties in the knowledge of random error sources, particularily, gyro drift rates, can be significantly reduced.

DD FORM 1 NOV 65 1473 (PAGE 1)
S/N 0101-807-6811

Unclassified
Security Classification

A-31408

Unclassified
Security Classification

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Smoothing						
Inertial navigation						

DD FORM 1 NOV 65 1473 (BACK)

Unclassified
Security Classification

A-31409