ON THE IDENTIFICATION OF OBSERVABLE ORBIT PARAMETERS, WITH APPLICATION TO LUNAR ORBITER TRACKING[†]

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Abstract

A technique is developed for identifying those linear combinations of orbit parameters which can be estimated from a given span of tracking data. It is shown that there are class I observable parameters, which affect the data strongly and must be included in the orbit determination solution, and class II observable parameters, which affect the data weakly and can be ignored. The method is numerically applied to two selected lunar orbiter tracking situations. A theoretical explanation of the results is developed to show that one would expect 2n + 4 combinations of potential terms to be class I observable if an n^{th} order potential model is postulated. A Fourier analysis technique for efficient orbit determination is suggested.

1. INTRODUCTION

The parameters which determine a satellite orbit are usually obtained from a maximum likelihood (weighted least squares) fit to the tracking data. The theoretical formulation is well known, but practical difficulties arise when a very large number of parameters can affect the orbit. For example, it is obviously impossible to estimate all of the coefficients of the spherical harmonics describing the gravitational potential of the central body, yet any or all of them might have a significant effect upon the solution of the orbit determination problem. In practice one usually defines the parameters to be solved for from physical considerations and intuitive judgement. Theoretically, deletion of parameters from the solution can only be justified by an examination of the observability and apriori uncertainty of the system.

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That is, a parameter or linear combination of parameters may be deleted from consideration if it is unobservable, which means it does not affect the data (the partial derivative of data with respect to parameter is zero), or if its apriori variance is zero, which means that it is perfectly known. It has been suggested that the problem might be the attacked by computing the eigenvalues and eigenvectors of the normal matrix (the inverse covariance matrix obtained from the data). The linear combinations of parameters formed as the inner products of the eigenvectors with the orbit parameters then become an uncoupled set of new parameters, and those linear combinations which correspond to negligibly small eigenvalues are supposed to be deleted. This approach cannot be strictly correct, however, for an arbitrary scaling of the original parameters will yield an arbitrary set of eigenvalues and eigenvectors.

It is the purpose of this paper to describe a theoretically justifiable method for identifying the linear combinations of parameters to be solved for and deleted. The analysis will be applied to a tracking study of two selected lunar orbits (reference 1), in order to show which terms are significant if a 4^{th} order potential model is postulated. A modified form of the potential function will be developed to explain the numerical results and to show that, in general, one would expect at most (2n + 4) combinations of potential terms to be significant for an n^{th} order potential model. A technique for efficient data reduction by a Fourier analysis will also be suggested.

2. CLASSES OF OBSERVABLE PARAMETERS

In this section we will establish the theoretical justification for considering only certain linear combinations of parameters in orbit determination analysis. The approach is based upon the following classification of orbit parameters:

Unobservable parameters: those linear combinations of orbit parameters which have no effect upon the data, and which can be deleted from consideration in the orbit determination solution.

Class I observable parameters: those linear combinations of orbit parameters which affect the data strongly, and which must be included in the orbit determination solution.

Class II observable parameters: those linear combinations of orbit parameters which affect the data weakly, and which can be ignored in the orbit determination solution. These parameters can be determined from the data residuals resulting from the Class I parameter solution, however.

The analysis proceeds as follows: Suppose there is a data vector (single data type) which, after linearization about a nominal trajectory, takes the form

$$\delta\bar{z} = A\delta\bar{x} + \bar{n} \tag{1}$$

where $\delta\bar{x}$ is a m-dimensional vector composed of variations from nominal of unknown orbit parameters, A is the partial derivative matrix $[\partial\bar{z}/\partial\bar{x}]$, and $\bar{n}$ is data noise. Let the apriori estimates of $\delta\bar{x}$ and $\bar{n}$ be zero, and the apriori variances be

$$E[\delta\bar{x}\delta\bar{x}^T] = \Lambda \tag{2}$$

$$E[\bar{n}\bar{n}^T] = \sigma^2 I \tag{3}$$

where I is the identity matrix, and $E[\cdots]$ indicates the statistical expectation. Define

$$\delta\bar{y} = L^T\Lambda^{-\frac{1}{2}}\delta\bar{x} \tag{4}$$

where L is an orthogonal transformation such that

$$L^T\Lambda^{-\frac{1}{2}}A^TA\Lambda^{-\frac{1}{2}}L = D = \begin{bmatrix} d_1^2 & & 0 \\ & \ddots & \\ 0 & & d_m^2 \end{bmatrix} \tag{5}$$

Thus the δy_i are linear combinations of the δx_j, formed by scaling the δx_j by the inverse of their apriori standard deviations and rotating the axis of the δx_j coordinate system. The δy_i have unit apriori variances, for we have

$$E[\delta\bar{y}\delta\bar{y}^T] = L^T\Lambda^{-\frac{1}{2}}\Lambda\Lambda^{-\frac{1}{2}}L = I \tag{6}$$

Equation (1) becomes

$$\delta\bar{z} = B\delta\bar{y} + \bar{n} \tag{7}$$

where

$$B = (A\Lambda^{\frac{1}{2}}L) \tag{8}$$

Applying the well-known formula, the maximum likelihood (minimum variance) estimate of $\delta\bar{y}$ given $\delta\bar{z}$ is

$$\delta\bar{y}^* = \frac{1}{\sigma^2}[I + \frac{1}{\sigma^2}B^TB]^{-1}B^T\delta\bar{z}$$

$$= \frac{1}{\sigma^2}[I + \frac{1}{\sigma^2}D]^{-1}B^T\delta\bar{z} \tag{9}$$

since, from (5) and (8), $B^TB = D$. Thus the $i^{\underline{th}}$ component of $\delta\bar{y}^*$ is

$$\delta y_i^* = \frac{1}{\sigma^2}\left(1 + \frac{d_i^2}{\sigma^2}\right)^{-1}\bar{b}_i^T\,\delta\bar{z} \tag{10}$$

where $\bar{b}_i$ is the $i^{\underline{th}}$ column of the B matrix. Since $|\bar{b}_i| = d_i$, we make the following definition:

<u>Definition</u>: A parameter δy_i is said to be unobservable if $d_i^2 = 0$.

A parameter δy_i is said to be Class I observable if $\left(\frac{d_i}{\sigma}\right)^2 \geq \varepsilon$, where ε is a number which determines the estimation error one is willing to accept (say $\varepsilon = 0.01$); a parameter δy_i is said to be Class II observable if $\left(\frac{d_i}{\sigma}\right)^2 < \varepsilon$.

This definition is motivated by the fact that deleting the Class II and unobservable parameters from consideration will cause a data residual of order ϵ, which is supposed to be negligible. The $\left(\frac{d_i}{\sigma}\right)$ can be thought of as "signal-to-noise ratios" describing the information content of the data relative to the (unit) a priori information. Thus we seek to identify the Class I observable parameters for the orbit determination problem.

3. NUMERICAL RESULTS FOR TRACKING OF LUNAR ORBITERS

Orbit determination results from tracking of lunar orbiters (reference 1) have indicated that poor solutions are obtained when a relatively small number of parameters are solved for, in the sense that large systematic data residuals and poor orbit predictions are obtained. The effect of neglected gravitational potential terms is the most probable cause of the difficulty. The results might be improved by developing larger computer programs capable of solving for more parameters, but this could be a formidable task. In this Section the theory developed above will be applied to identify those linear combinations of potential terms which are significant for short tracking intervals.

It is clear from the previous discussion (and intuitively obvious) that the significant combinations of parameters to be identified and solved for depend upon the assumed values of the elements of the apriori covariance matrix,* that is, an apriori judgement is required. Two ways of defining the potential term variances were developed for the purpose of the study described here. In both cases it was assumed that the potential terms are uncorrelated. The "gross variances" were some rather large numbers chosen by increasing some published results (reference 1) by approximately an order of magnitude. The resulting standard deviations (variance$^{\frac{1}{2}}$) are shown in Table I. The "theory variances" were smaller numbers obtained from a formula developed by Kaula (reference 2), who extrapolated geodesy data by arguing that the moon might be 35 times as rough as the earth. Accounting for appropriate normalizing factors (reference 3, pp. 7) the formula for the standard deviations are

$$\sigma(C_{nm} S_{nm}) = \left[\frac{(2 - \delta_{om})(n - m)!(2n + 1)}{(n + m)!}\right]^{\frac{1}{2}} \left(\frac{10^{-9}}{n^4}\right)^{\frac{1}{2}} \tag{11}$$

where C_{nm}, S_{nm} are the coefficients of the spherical harmonics for the moon, and δ_{om} is the Kronecker delta (equal to 1 if m = 0, equal to zero otherwise). The results of this calculation for $n \leq 8$ are shown in Table 2.

It was assumed that the complete solution vector was composed of the potential terms, the initial position and velocity of the spacecraft, the station location errors of a single doppler tracking station (Woomera, Australia), and a doppler bias. The standard deviations for the non-potential parameter were arbitrarily chosen to be

Gross Standard Deviations

σ (initial position, each component)	=	10,500 ft.
σ (initial velocity, each component)	=	10.5 ft/sec
σ (doppler bias)	=	.001 ft/sec
σ (station latitude error)	=	210 ft.
σ (station longitude error)	=	21 ft.
σ (station altitude error)	=	105 ft.

Theoretical Standard Deviations

σ (initial position, each component)	=	1,050 ft.
σ (initial velocity, each component)	=	1.05 ft/sec
σ (doppler bias)	=	.001 ft/sec
σ (station latitude error)	=	210 ft.
σ (station longitude error)	=	21 ft.
σ (station altitude error)	=	105 ft.

The data noise standard deviation was $\sigma = 0.213$ ft/sec = 0.065 m/sec in both cases, which corresponds to a 1 cycle/sec error for a one minute count of counted doppler.

Table 1: Gross Apriori Standard Deviations

n	m	From reference 1		Assumed gross value × 10^4
		$\sigma_{Cnm} \times 10^4$	$\sigma_{Snm} \times 10^4$	
1	0	-	-	1.0
	1	-	-	0.5
2	0	.141	-	1.0
	1	.051	.039	0.5
	2	.029	.025	0.3
3	0	.180	-	1.0
	1	.048	.053	0.5
	2	.028	.033	0.3
	3	.015	.018	0.1
4	0	.128	-	1.0
	1	.036	.028	0.3
	2	.010	.013	0.1
	3	.008	.006	0.1
	4	.003	.003	0.03

Table 2: Theoretical Apriori Standard Deviations

n	m	$\sigma \times 10^5$
2	0	2.0
	1	1.0
	2	0.5
3	0	0.9
	1	0.4
	2	0.1
	3	0.05
4	0	0.6
	1	0.2
	2	.04
	3	.02
	4	.006
5	0	0.4
	1	0.1
	2	0.02
	3	.004
	4	.001
	5	$3(10^{-4})$

n	m	$\sigma \times 10^5$
6	0	0.3
	1	0.07
	2	0.01
	3	.002
	4	$3(10^{-4})$
	5	$7(10^{-5})$
	6	$2(10^{-5})$
7	0	0.3
	1	0.05
	2	0.006
	3	0.001
	4	$1(10^{-4})$
	5	$2(10^{-5})$
	6	$4(10^{-6})$
	7	$1(10^{-6})$

n	m	$\sigma \times 10^5$
8	0	0.2
	1	0.03
	2	0.004
	3	$5(10^{-4})$
	4	$6(10^{-5})$
	5	$9(10^{-6})$
	6	$1(10^{-6})$
	7	$3(10^{-7})$
	8	$6(10^{-8})$

The Class I observable parameters were determined for eight lunar tracking situations: a high altitude and low altitude nominal trajectory was chosen; each of these trajectories and the associated partial derivatives was computed with a nominal lunar potential, consisting of reasonable estimates of lunar potential terms, and a spherical lunar potential, with only the central body term present, and, for each case, both the gross and theoretical variances were applied. The characteristics of the high and low orbit tracking situations were:

Low orbit: semi-major axis = 1968 Km, eccentricity = 0.04, inclination = 20.9°, single station (Woomera) tracking with range rate data at rate of one point per minute for 75 minutes

High orbit: semi-major axis = 2722 Km, eccentricity = 0.31, inclination = 20.9°, single station (Woomera) tracking with range rate data, at rate of one point per minute for 120 minutes

The calculations described in Section 2 were carried out for a parameter vector consisting of 34 terms: 3 initial position components, 3 initial velocity components, the 24 gravitational potential terms through 4th order (the n = 1 terms are to be interpreted as lunar ephemeris error), 3 station location errors, and a range rate bias. The eigenvalues d_i^2 were determined, and the Class I observable parameters were defined to be those δy_i corresponding to $\left(\frac{d_i}{\sigma}\right)^2 > 0.01$, where $\sigma = 0.213$ ft/sec. It was found that in none of the eight cases studied were there more than 10 Class I observable parameters. Tables 3 - 10 list the values of $\left(\frac{d_i}{\sigma}\right)^2$ associated with the major eigenvectors, and coefficients c_{ij} which are the major components of the eigenvectors. Thus theoretically one should estimate the parameter combinations

$$\delta y_i = \sum_{j=1}^{34} c_{ij} \left(\frac{\delta x_j}{\sigma_j}\right) \qquad i = 1, \cdots N \leq 10 \tag{12}$$

but, as a simplifying approximation, it might suffice to include only those x_j for which $|c_{ij}| \geq$ some small number $\hat{\varepsilon}$. Rounding off $|c_{ij}|$ to two significant figures, and setting $\hat{\varepsilon} = 0.20$, the potential terms which then remain as components of any of the y_i are listed in Table 11. Since Tables 3-10 indicate that all initial position and velocity components are significant, a reasonable fit to the tracking data might be obtained by solving for these parameters plus the potential terms listed in Table 11. This approach would yield approximately the same data residuals obtained by solving for the parameter combinations $\{y_i\}$. Note that the sectorial potential terms (C_{nn}, S_{nn}) seem to be the most important, and that the results for the nominal and spherical potential cases are almost identical.

Table 3: Numerical Results for Low Orbit, Gross Variances, Nominal Potential

$\left(\frac{d_i}{\sigma}\right)^2$	Major components and associated coefficients of major eigenvectors															
.453	x	$\dot{z}$	$\dot{y}$	$\dot{x}$	z											
E7	.580	.525	.496	.342	.120											
.126	$\dot{x}$	$\dot{y}$	$\dot{z}$	x	z	y										
E6	-.670	.454	.388	-.279	-.266	-.177										
.115	y	z	$\dot{z}$	x	$\dot{x}$	C_{43}	C_{32}									
E5	.816	.371	-.285	-.212	-.144	-.102	-.101									
.549	$\dot{y}$	x	$\dot{x}$	$\dot{z}$	z											
E4	-.645	.509	-.389	.323	-.163											
.835	C_{44}	C_{43}	C_{33}	C_{32}	x	$\dot{x}$										
E3	.703	-.539	.345	-.175	-.123	.103										
.798	S_{44}	S_{43}	S_{33}	S_{32}	C_{43}											
E3	.744	-.543	.316	-.154	.109											
.397	S_{44}	S_{43}	S_{33}	C_{43}	S_{22}	$\dot{z}$	x	$\dot{x}$	S_{32}	S_{42}	C_{32}	y				
E3	.496	.452	-.370	.356	-.286	.225	-.185	.170	.142	.121	-.120	-.112				
.174	C_{44}	C_{33}	$\dot{z}$	$\dot{x}$	$\dot{y}$	C_{43}	x	S_{33}	y	S_{32}	C_{32}	C_{22}	z	S_{22}	S_{43}	
E3	.559	-.379	-.320	-.285	.249	.220	.196	-.194	.185	.183	.147	-.144	.127	-.120	-.107	
.698	S_{33}	S_{43}	C_{32}	C_{44}	S_{32}	C_{33}	x	S_{22}	C_{43}	$\dot{z}$	S_{42}	S_{44}				
E1	.676	.350	.308	.275	-.228	-.184	-.170	-.165	.163	.134	.107	-.106				
.219	C_{33}	$\dot{z}$	x	$\dot{x}$	S_{43}	S_{32}	S_{33}	C_{32}	$\dot{y}$	C_{43}	S_{22}	y				
E1	.615	-.371	.312	-.285	.278	-.214	.187	-.186	.178	.174	-.126	.108				

Table 4: Numerical Results for Low Orbit, Theory Variances, Nominal Potential

$\left(\frac{d_i}{\sigma}\right)^2$	Major components and associated coefficients of major eigenvectors														
.464	x	$\dot{z}$	$\dot{y}$	$\dot{x}$	S_{11}	z									
E5	.574	.519	.490	.339	.157	.119									
.127	$\dot{x}$	$\dot{y}$	$\dot{z}$	x	z	y									
E4	-.665	.459	.393	-.272	-.265	-.178									
.119	y	z	$\dot{z}$	C_{11}	x	$\dot{x}$									
E3	.799	.360	.286	.246	-.219	-.155									
.564	$\dot{y}$	x	$\dot{x}$	$\dot{z}$	z	J_2	C_{22}	S_{11}							
E2	-.650	.484	-.397	.311	-.158	.143	-.133	-.123							
.231	S_{22}	$\dot{z}$	$\dot{x}$	x	C_{33}	S_{33}	$\dot{y}$	y	C_{21}	C_{22}	z	C_{44}	C_{32}	S_{43}	S_{11}
E1	.488	-.446	-.339	.331	-.248	.223	.221	.171	.170	-.164	.137	-.123	.120	-.119	-.104
.133	S_{22}	S_{33}	C_{22}	C_{33}	$\dot{z}$	S_{43}	S_{21}	C_{43}	S_{32}	$\dot{y}$	$\dot{x}$	x	S_{31}		
E1	-.613	-.367	-.324	-.320	-.195	.192	.192	.189	.163	.160	-.157	.120	-.101		
.403	S_{44}	S_{33}	C_{43}	C_{44}	S_{43}	C_{33}	S_{22}	C_{22}	C_{32}	$\dot{z}$	x				
E0	.634	.384	.379	-.277	-.235	-.215	-.182	-.146	.140	.110	-.106				
.276	C_{44}	S_{44}	S_{43}	C_{33}	S_{22}	C_{43}	$\dot{z}$	$\dot{x}$	$\dot{y}$	x	S_{42}				
E0	.655	.356	-.307	.269	.257	-.218	-.209	-.176	.127	.153	.102				

Table 5: Numerical Results for Low Orbit, Gross Variances, Spherical Potential

$\left(\frac{d_i}{\sigma}\right)^2$	Major components and associated coefficients of major eigenvectors														
.454	x	$\dot{z}$	$\dot{y}$	$\dot{x}$	z										
E7	.580	.525	.496	.342	.120										
.126	$\dot{x}$	$\dot{y}$	$\dot{z}$	x	z	y									
E6	-.670	.454	.388	-.279	-.266	-.176									
.115	y	z	$\dot{z}$	x	$\dot{x}$	C_{33}	C_{32}								
E5	.816	.371	.286	-.212	-.144	-.102	-.101								
.550	$\dot{y}$	x	$\dot{x}$	$\dot{z}$	z										
E4	-.645	.509	-.389	.322	-.164										
.834	C_{44}	C_{43}	C_{33}	C_{32}	x	$\dot{x}$									
E3	.703	-.540	.345	-.176	-.123	.103									
.797	S_{44}	S_{43}	S_{33}	S_{32}	C_{43}										
E3	.743	-.545	.316	-.154	.107										
.396	S_{44}	S_{43}	S_{33}	C_{43}	S_{22}	$\dot{z}$	x	$\dot{x}$	S_{32}	S_{42}	C_{32}	y			
E3	.498	.450	-.370	.356	-.287	.224	-.184	.169	.142	.121	-.119	-.111			
.173	C_{44}	C_{33}	$\dot{z}$	$\dot{x}$	$\dot{y}$	C_{43}	x	S_{33}	y	S_{32}	C_{32}	C_{22}	z	S_{22}	S_{43}
E3	.559	-.380	-.321	-.286	.249	.218	.196	-.192	.185	.183	.148	-.144	.127	-.119	-.108
.692	S_{33}	S_{43}	C_{32}	C_{44}	S_{32}	C_{33}	x	S_{22}	C_{43}	$\dot{z}$	S_{42}	S_{44}			
E1	.676	.351	.307	.274	-.229	-.181	-.170	-.166	.164	.134	.107	-.105			
.217	C_{33}	$\dot{z}$	x	$\dot{x}$	S_{43}	S_{32}	C_{32}	S_{33}	$\dot{y}$	C_{43}	S_{22}	y			
E1	.616	-.371	.313	-.286	.276	-.212	-.187	.184	.178	.174	-.126	.107			

Table 6: Numerical Results for Low Orbit, Theory Variances, Spherical Potential

$\left(\frac{d_i}{\sigma}\right)^2$	Major components and associated coefficients of major eigenvectors														
.464 E5	x .574	$\dot{z}$ .519	$\dot{y}$ .490	$\dot{x}$ .339	S_{11} −.157	z .119									
.126 E4	$\dot{x}$ .665	$\dot{y}$ .459	$\dot{z}$ .393	x −.272	z −.265	y −.178									
.119 E3	y .799	z .360	$\dot{z}$ .287	C_{11} .246	x −.219	$\dot{x}$ −.155									
.564 E2	$\dot{y}$ −.649	x .484	$\dot{x}$ −.397	$\dot{z}$ .310	z −.159	J_2 .143	C_{22} −.133	S_{11} −.123							
.230 E1	S_{22} .489	$\dot{z}$ −.445	$\dot{x}$ −.338	x .330	C_{33} −.248	S_{33} .225	$\dot{y}$ .221	y .171	C_{21} .170	C_{22} −.164	z .136	C_{44} −.124	C_{32} .121	S_{43} −.120	S_{11} −.104
.133 E1	S_{22} −.612	S_{33} −.366	C_{22} −.324	C_{33} −.321	$\dot{z}$ −.196	S_{43} .192	S_{21} .191	C_{43} .189	S_{32} .163	$\dot{y}$ .160	$\dot{x}$ −.157	x .121	S_{31} −.101		
.402	S_{44} .636	S_{33} .384	C_{43} .378	C_{44} −.274	S_{43} −.236	C_{33} −.213	S_{22} −.184	C_{22} −.145	C_{32} .139	$\dot{z}$ .110	x −.106				
.276	C_{44} .657	S_{44} .353	S_{43} −.306	C_{33} .269	S_{22} −.256	C_{43} −.220	$\dot{z}$ −.209	$\dot{x}$ −.177	x .154	$\dot{y}$ .128	S_{42} .102				

Table 7: Numerical Results for High Orbit, Gross Variances, Nominal Potential

$\left(\frac{d_i}{\sigma}\right)^2$	Major components and associated coefficients of major eigenvectors											
.872	$\dot{x}$	$\dot{z}$	x	$\dot{y}$								
E7	.770	.432	.333	.319								
.298	$\dot{y}$	$\dot{z}$	$\dot{x}$	x								
E6	.629	.562	-.477	-.236								
.128	$\dot{z}$	$\dot{y}$	y	x	$\dot{x}$							
E5	.671	-.662	.189	.189	-.167							
.742	y	z	$\dot{y}$									
E4	.848	.483	.143									
.419	C_{44}	S_{44}	C_{43}	C_{33}	x	C_{32}	$\dot{x}$	S_{22}				
E3	.567	.426	-.422	.379	-.281	-.161	.114	-.111				
.370	S_{44}	S_{43}	C_{44}	S_{33}	S_{32}	S_{22}						
E3	.618	-.518	-.383	.370	-.184	.115						
.147	x	C_{44}	S_{43}	C_{43}	$\dot{x}$	S_{22}	S_{33}	S_{44}	$\dot{y}$	$\dot{z}$		
E3	-.570	-.451	.415	.308	.231	-.181	-.171	.160	-.132	.123		
.107	S_{44}	C_{43}	x	S_{22}	C_{44}	S_{33}	C_{22}	C_{32}	$\dot{x}$	C_{33}	$\dot{y}$	
E3	.518	.518	.340	-.301	.249	-.219	-.194	.159	-.139	-.115	.106	
.129	S_{43}	C_{33}	x	C_{44}	S_{33}	$\dot{x}$	S_{32}	C_{43}	S_{22}	C_{22}		
E2	.500	.486	.460	-.280	.256	-.206	-.130	-.128	-.119	.105		
.236	S_{33}	C_{32}	S_{32}	C_{44}	C_{33}	S_{43}	S_{22}	x	C_{22}	S_{42}		
E1	.614	.418	-.338	.255	-.250	.231	-.214	-.199	-.137	.102		

Table 8: Numerical Results for High Orbit, Theory Variances, Nominal Potential

$\left(\frac{d_i}{\sigma}\right)^2$	Major components and associated coefficients of major eigenvectors												
.880	$\dot{x}$	$\dot{z}$	x	$\dot{y}$									
E5	.766	.430	.332	.318									
.299	$\dot{y}$	$\dot{z}$	$\dot{x}$	x									
E4	.629	.563	-.472	-.234									
.129	$\dot{z}$	$\dot{y}$	y	x	$\dot{x}$								
E3	.667	-.656	.200	.185	-.171								
.787	y	z	C_{11}	$\dot{y}$									
E2	.818	.469	.254	.158									
.175	x	$\dot{x}$	C_{22}	S_{22}	C_{33}	S_{11}	$\dot{y}$	$\dot{z}$					
E1	-.710	.318	.312	-.281	.245	.196	-.180	.159					
.877	S_{22}	S_{33}	C_{22}	S_{43}	C_{44}	S_{21}	x	S_{32}					
	.777	.366	.340	-.179	-.155	-.152	-.135	-.109					
.229	C_{33}	S_{44}	x	C_{44}	C_{43}	C_{22}	S_{33}	S_{22}	S_{43}	$\dot{x}$	C_{32}		
	.501	.412	.354	.341	-.246	.244	.200	-.191	-.175	-.160	-.133		
.149	S_{44}	C_{44}	S_{33}	C_{43}	S_{22}	S_{43}	C_{22}	C_{33}	C_{32}	S_{32}			
	.497	-.442	.409	.346	-.310	-.213	-.167	-.164	.147	-.145			

Table 9: Numerical Results for High Orbit, Gross Variances, Spherical Potential

$\left(\frac{d_i}{\sigma}\right)^2$	Major components and associated coefficients of major eigenvectors											
.872	$\dot{x}$	$\dot{z}$	x	$\dot{y}$								
E7	.770	.432	.333	.319								
.298	$\dot{y}$	$\dot{z}$	$\dot{x}$	x								
E6	.629	.562	-.477	-.236								
.128	$\dot{z}$	$\dot{y}$	y	x	$\dot{x}$							
E5	.670	.662	.189	.189	-.167							
.741	y	z	$\dot{y}$									
E4	.848	.483	.144									
.417	C_{44}	S_{44}	C_{43}	C_{33}	x	C_{32}	$\dot{x}$	S_{22}				
E3	.568	.425	-.422	.379	-.281	-.161	.114	-.111				
.147	x	C_{44}	S_{43}	C_{43}	$\dot{x}$	S_{22}	S_{33}	S_{44}	$\dot{y}$	$\dot{z}$		
E3	-.571	-.451	.414	.307	.232	-.180	-.171	.160	-.133	.123		
.107	S_{44}	C_{43}	x	S_{22}	C_{44}	S_{33}	C_{22}	C_{32}	$\dot{x}$	C_{33}	y	
E3	.518	.518	.339	-.302	.248	-.219	-.194	.159	-.138	-.115	.106	
.129	S_{43}	S_{44}	x	C_{44}	S_{33}	$\dot{x}$	S_{32}	C_{43}	S_{22}	C_{22}	C_{32}	
E2	.500	.486	.460	-.281	.255	-.206	-.130	-.128	-.119	.105	-.100	
.235	S_{33}	C_{32}	S_{32}	C_{44}	C_{33}	S_{43}	S_{22}	x	C_{22}	S_{42}		
E1	.615	.417	-.339	.255	-.247	.231	-.214	-.199	-.137	.102		

Table 10: Numerical Results for High Orbit, Theory Variances, Spherical Potential

$\left(\frac{d_i}{\sigma}\right)^2$	Major components and associated coefficients of major eigenvectors												
.880	$\dot{x}$	$\dot{z}$	x	$\dot{y}$									
E5	.766	.430	.331	.318									
.299	$\dot{y}$	$\dot{z}$	$\dot{x}$	x									
E4	.629	.563	-.472	-.234									
.129	$\dot{z}$	$\dot{y}$	y	x	$\dot{x}$								
E9	.667	-.656	.201	.185	-.171								
.787	y	z	C_{11}	$\dot{y}$									
E2	.818	.469	.254	.159									
.174	x	$\dot{x}$	C_{22}	S_{22}	C_{33}	S_{11}	$\dot{y}$	$\dot{z}$					
E1	-.710	.318	.313	-.280	.245	.196	-.180	.159					
.877	S_{22}	S_{33}	C_{22}	C_{44}	S_{21}	x	S_{32}						
	.777	.366	.340	-.155	-.152	-.134	-.109						
.229	C_{33}	S_{44}	x	C_{44}	C_{43}	C_{22}	S_{33}	S_{22}	S_{43}	$\dot{x}$	C_{32}		
	.502	.412	.354	.340	-.246	.244	.200	-.191	-.175	-.160	-.133		
.149	S_{44}	C_{44}	S_{33}	C_{43}	S_{22}	S_{43}	C_{22}	C_{33}	C_{32}	S_{32}			
	.497	-.442	.409	.346	-.310	-.213	-.167	-.163	.146	-.145			

TABLE 11: SIGNIFICANT* POTENTIAL TERMS

			J_1	J_2	J_3	J_4	C_{11}	S_{11}	C_{21}	S_{21}	C_{31}	S_{31}	C_{41}	S_{41}	C_{22}	S_{22}	C_{32}	S_{32}	C_{42}	S_{42}	C_{33}	S_{33}	C_{43}	S_{43}	C_{44}	S_{44}
LOW ORBIT	NOMINAL POTENTIAL	GROSS														X	X	X			X	X	X	X	X	X
		THEORY					X								X	X					X	X	X	X	X	X
	SPHERICAL POTENTIAL	GROSS														X	X	X			X	X	X	X	X	X
		THEORY					X								X	X					X	X	X	X	X	X
HIGH ORBIT	NOMINAL POTENTIAL	GROSS														X	X	X			X	X	X	X	X	X
		THEORY					X	X							X	X					X	X	X	X	X	X
	SPHERICAL POTENTIAL	GROSS														X	X	X			X	X	X	X	X	X
		THEORY					X	X							X	X					X	X	X	X	X	X

* Coefficient of major eigenvector less than 0.20

4. A THEORETICAL EXPLANATION OF THE NUMERICAL RESULTS

The numerical results indicate that at most 10 linear combinations of orbit parameters are significant for all cases studied, and that the combinations are similar. In this section we will seek a theoretical explanation of this phenomenon.

Let the nominal orbit be a Keplerian ellipse,* and apply Lagrange's equations of motion (reference 3, page 29) to obtain the variation of the orbital elements in the form

$$\delta a(t) = \delta a(o) + \frac{2}{Na} \int_o^t \left(\frac{\partial R_n}{\partial M}\right) ds \tag{13}$$

$$\delta e(t) = \delta e(o) + \left[\frac{1 - e^2}{Na^2 e}\right] \int_o^t \left(\frac{\partial R_n}{\partial M}\right) ds - \left[\frac{\left(1 - e^2\right)^{\frac{1}{2}}}{Na^2 e}\right] \int_o^t \left(\frac{\partial R_n}{\partial \omega}\right) ds \tag{14}$$

$$\delta\omega(t) = \delta\omega(o) - \left(\frac{\cot i}{Na^2(1-e^2)^{\frac{1}{2}}}\right) \int_o^t \left(\frac{\partial R_n}{\partial i}\right) ds + \frac{(1-e)^{\frac{1}{2}}}{Na^2 e} \int_o^t \left(\frac{\partial R_n}{\partial e}\right) ds \tag{15}$$

$$\delta i(t) = \delta i(o) + \left(\frac{\cot i}{Na^2(1-e^2)^{\frac{1}{2}}}\right) \int_o^t \left(\frac{\partial R_n}{\partial \omega}\right) ds - \left[\frac{1}{Na^2(1-e^2)^{\frac{1}{2}} \sin i}\right] \int_o^t \left(\frac{\partial R_n}{\partial \Omega}\right) ds \tag{16}$$

$$\delta\Omega(t) = \delta\Omega(o) + \left[\frac{1}{Na^2(1-e)^{\frac{1}{2}} \sin i}\right] \int_o^t \left(\frac{\partial R_n}{\partial i}\right) ds \tag{17}$$

$$\begin{aligned} \delta M(t) = {} & \delta M(o) - \frac{3N}{2a}\left[\delta a(o)t + \frac{2}{Na} \int_o^t (t - s)\left(\frac{\partial R_n}{\partial M}\right) ds\right] \\ & - \left[\frac{1-e^2}{Na^2 e}\right] \int_o^t \left(\frac{\partial R_n}{\partial e}\right) ds - \frac{2}{Na} \int_o^t \left(\frac{\partial R_n}{\partial a}\right) ds \end{aligned} \tag{18}$$

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* That is, to first order we assume that the orbital elements (except M) are constants equal to their initial values.

where $(a, e, \omega, i, \Omega, M)$ are the Keplerian elements, N is the mean angular motion, and R_n is the gravitational disturbing function for an $n^{\underline{th}}$ order potential model, which, for small eccentricity, is approximately given by (reference 3)

$$R_n = \sum_{\ell=1}^{n} \sum_{m=0}^{\ell} \left(\frac{\mu a_M^{\ell}}{a^{\ell+1}}\right) \sum_{p=0}^{\ell} F_{\ell mp}(i) \sum_{q=-1}^{+1} G_{\ell pq}(e)\, S_{\ell mpq} \tag{19}$$

The μ is the gravitational constant of the moon, a_M is the mean radius of the moon, $F_{\ell mp}(i)$ and $G_{\ell pq}(e)$ are, respectively, the inclination and eccentricity functions defined in reference 3, and

$$S_{\ell mpq}(\omega,M,\Omega,\theta) = \begin{bmatrix} C_{\ell m} \\ \\ -S_{\ell m} \end{bmatrix}_{(\ell-m)\text{odd}}^{(\ell-m)\text{even}} \cos\left[(\ell - 2p)\,\omega + (\ell - 2p + q)\,M + m(\Omega - \theta)\right]$$

$$+ \begin{bmatrix} S_{\ell m} \\ \\ C_{\ell m} \end{bmatrix}_{(\ell-m)\text{odd}}^{(\ell-m)\text{even}} \sin\left[(\ell - 2p)\,\omega + (\ell - 2p + q)\,M + m(\Omega - \theta)\right] \tag{20}$$

where $\left\{C_{\ell m}, S_{\ell m}\right\}$ are the coefficients of the spherical harmonics, and θ is moon's rotation rate multiplied by tracking time. The assumption of small eccentricity is used in the evaluation of $G_{\ell mp}(e)$, for it can be shown that the index q ranges from -1 to +1 rather than from $-\infty$ to $+\infty$ if the e^2 terms are negligible. Suppose we consider tracking arcs sufficiently short to cause θ to be negligible ($\theta \simeq \frac{8\pi}{3{,}000} \simeq .01$ for a tracking time of one hour), and rewrite R_n in the form

$$R_n = \sum_{k=0}^{n+1} a_k \cos k\,M + b_k \sin k\,M \tag{21}$$

where the a_k and the b_k are combinations of the $\left\{C_{nm}, S_{nm}\right\}$ and the elements a, e, ω, i, Ω, and the mean anomaly is $M = N(t-t_o)$ with t_o = time of periapsis passage. Such a representation is valid if θ and e^2 are negligible, for it can be seen from (19) and (20) that R_n is then a combination of sines and cosines of kM, where k ranges from 0 to n+1. Denoting the orbital elements by q_i, where $q_1 = a$, $q_2 = e$, $q_3 = \omega$, $q_4 = i$, $q_5 = \Omega$, $q_6 = M$, we use equation (21) in equations (13) - (18) to obtain an expression for deviations from the nominal orbit in the form

$$\delta q_i(t) = \delta q_i(o) + \sum_{k=0}^{n+1} \alpha_{ik} [\sin kM(t) - \sin kM(o)] + \beta_{ik}[\cos kM(t) - \cos kM(o)] \qquad i = 1,\cdots 5 \qquad (22)$$

$$\delta q_6(t) = \delta q_6(o) - \frac{3N}{2a}\int_o^t \delta q_1(s)\, ds + \sum_{k=0}^{n+1} \alpha_{6k}[\sin kM(t) - \sin kM(o)] + \sum_{k=0}^{n+1} \beta_{6k}[\cos kM(t) - \cos kM(o)] \qquad (23)$$

where the $\{\alpha_{ik}, \beta_{ik}\}$ are linear combinations of the $\{a_k, b_k\}$ with coefficients which depend upon the $\{q_i(o)\}$.

Suppose there is a continuous data type z(t), such as doppler, which is of the form

$$\delta z(t) = H(t)\, \delta\bar{q}(t) + n(t) = A(t)\, \delta\bar{x} + n(t) \qquad (24)$$

where

$$\delta\bar{x}^T = [\delta q_1(o) \cdots \delta q_6(o), \alpha_{11}, \beta_{11}, \cdots \alpha_{ij}, \beta_{ij} \cdots]$$

and H(t) and A(t) are row matrices given by $\left[\frac{\partial z(t)}{\partial \bar{q}(t)}\right]$ and $\left[\frac{\partial z(t)}{\partial \bar{x}}\right] = \left[\frac{\partial z(t)}{\partial \bar{q}(t)}\right]\left[\frac{\partial \bar{q}(t)}{\partial \bar{x}}\right]$, respectively. If H(t) is approximately constant over the (short) tracking interval, equations (22) and (23) show that the $\{q_i\}$ and the $\{C_{nm}, S_{nm}\}$ terms of an n^{th} order potential model generate elements of the A(t) matrix composed of at most 2n + 4 linearly independent functions: n + 1 cosine functions, n + 1 sine functions, a constant, and a (constant) x (time). Since only N linear combinations of parameters can be observed (determined) from data composed of N linearly independent functions*, we conclude that at most $2n + 4$ linear combinations of the potential coefficients and initial conditions can be determined from a short arc of tracking data. This conclusion indicates that a significant reduction in the number of solution parameters can be achieved, for with an n^{th} order potential model there are $n(n + 2)$ potential terms to be considered plus the six state variable components. The theory is in agreement with the numerical results obtained with a fourth order potential model ($n = 4$), for at most 10 parameter combinations were found to be significant and theoretically we have $2n + 4 = 12$.

It should be noted that the number of observable parameters increases if $H(t)$ is not almost constant over the tracking interval. For example, if any of the $H_i(t)$ were of the form

$$H_i(t) = h_{i0} + h_{i1} \cos M + h_{12} \sin M + h_{i3} t$$

then the product $A(t)\delta\bar{q}(t)$ would generate sin kM and cos kM terms up to $k = n + 2$ as well as terms of the type (t cos kM) and (t sin kM) up to $k = n + 1$ and a (constant) $x(t^2)$ term. This would indicate as many as $(4n + 9)$ observable parameters. The numerical results seem to indicate that the non-constant portion of $H(t)$ is not significant, however.

5. FOURIER ANALYSIS OF DATA RESIDUALS

The analytical treatment developed in Section 4 could be applied to determine the potential terms from a Fourier analysis of data residuals. Suppose some small number of orbit parameters have been estimated (such as initial conditions only), resulting in data residuals of the form

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* This statement of the relationship between linear dependence and observability is easily verified, and will not be demonstrated here.

$$\rho(t) = [z(t) - H(t)\bar{q}^*(t)] = H(t)\,\delta\bar{q}(t) + n(t) \quad (25)$$

where $\delta\bar{q}(t) = [\bar{q}(t) - \bar{q}^*(t)]$. Suppose we assume that the previously obtained solution for initial conditions defines a set of mean orbital elements* which are to be held fixed, so that $\rho(t)$ can be put into the form

$$\rho(t) = \left\{\sum_{k=0}^{n+1} \alpha_k \cos kM + \beta_k \sin kM\right\} + n(t) \quad (26)$$

where $\left\{\alpha_k, \beta_k\right\}$ are combinations of the $\left\{\alpha_{ij}, \beta_{ij}\right\}$ and the constant components of H(t), and we have considered only the constant portion of the H(t) matrix to sum to n + 1 (see Part 4). The $\rho(t)$ can therefore be thought of as data containing information about the 2n + 3 significant combinations of potential terms.

The functions $\left\{\cos kM, \sin kM\right\}$ are not orthogonal over the tracking interval T because T does not, in general, correspond to an arc length of 2π radius. An alternative representation of $\rho(t)$ can be obtained as

$$\sum_{k=0}^{n+1} \alpha_k \cos kM + \beta_k \sin kM = \sum_{m=0}^{\infty} \hat{\alpha}_m \cos mu + \hat{\beta}_m \sin mu \quad (27)$$

where

$$u(t) = \left[\frac{2\pi(M + N\,t_o)}{NT}\right] = \left|\frac{2\pi t}{T}\right| \quad (28)$$

The right hand series is composed of functions orthogonal over the interval T, and

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* That is, we suppose that mean elements rather than osculating elements at t = 0 are estimated in the orbit determination procedure. The secular terms are then eliminated and only the periodic terms remain.

$$\hat{\alpha}_m = \sum_{k=0}^{n+1} \frac{1}{\pi} \int_0^{2\pi} [\cos mu][\alpha_k \cos kM + \beta_k \sin kM]\, du \tag{29}$$

$$\hat{\beta}_m = \sum_{k=0}^{n+1} \frac{1}{\pi} \int_0^{2\pi} [\sin mu][\alpha_k \cos kM + \beta_k \sin kM]\, du \tag{30}$$

Thus the $\left\{\hat{\alpha}_m, \hat{\beta}_m\right\}$ are linear combinations of the $\left\{\alpha_k, \beta_k\right\}$, and can be experimentally determined from the residual data by

$$\hat{\alpha}_m^* = \frac{2}{T} \int_0^T [\cos mu(s)][\rho(s)]\, ds \tag{31}$$

$$\hat{\beta}_m^* = \frac{2}{T} \int_0^T [\sin mu(s)][\rho(s)]\, ds \tag{32}$$

It can be shown that the $\left\{\hat{\alpha}_m^*, \hat{\beta}_m^*\right\}$ of equations (31) and (32) are the minimum variance estimates, and, assuming no apriori information (infinite apriori variance) the estimation error has variance

$$E[\hat{\alpha}_m^* - \alpha_m]^2 = E[\hat{\beta}_m^* - \beta_m]^2 = E[n]^2 = \sigma^2 \tag{33}$$

Only (2n + 3) values of $\left\{\hat{\alpha}_m^*, \hat{\beta}_m^*\right\}$ are sufficient to recover the (2n + 3) numbers $\left\{\alpha_k^*, \beta_k^*\right\}$ if the corresponding (2n + 3) x (2n + 3) dimensional matrix $\left[\dfrac{\partial(\hat{\alpha}_m, \hat{\beta}_m)}{\partial(\alpha_k, \beta_k)}\right]$ has full rank. Denoting this subset of the $\left\{\hat{\alpha}_m^*, \hat{\beta}_m^*\right\}$ as $(\hat{\bar{\alpha}}^*, \hat{\bar{\beta}}^*)$, the minimum variance estimates of the potential parameters $\left\{C_{nm}, S_{nm}\right\}$ can also be recovered. Construct the rectangular matrix

$$K = \left[\frac{\partial(\hat{\bar{\alpha}}, \hat{\bar{\beta}})}{\partial(\bar{\alpha}, \bar{\beta})}\right]\left[\frac{\partial(\bar{\alpha}, \bar{\beta})}{\partial(\bar{C}, \bar{S})}\right] = \left[\frac{\partial(\hat{\bar{\alpha}}, \hat{\bar{\beta}})}{\partial(\bar{C}, \bar{S})}\right] \tag{34}$$

so that

$$\begin{bmatrix} \hat{\bar{\alpha}} \\ \hat{\bar{\beta}} \end{bmatrix} = K \begin{bmatrix} \bar{S}_{nm} \\ \bar{C}_{nm} \end{bmatrix} = \begin{bmatrix} \hat{\bar{\alpha}}^* \\ \hat{\bar{\beta}}^* \end{bmatrix} - \bar{\varepsilon} \tag{35}$$

where $\bar{\varepsilon}$ is the estimation error which, according to (33), has variance σ^2 x (the identity). If P is the apriori variance of the $\left\{C_{nm}, S_{nm}\right\}$, the minimum variance estimates become

$$\begin{bmatrix} \bar{S}^*_{nm} \\ \bar{C}^*_{nm} \end{bmatrix} = \frac{1}{\sigma^2}\left[P^{-1} + \frac{1}{\sigma^2} K^T K\right]^{-1} K^T \begin{bmatrix} \hat{\bar{\alpha}}^* \\ \hat{\bar{\beta}}^* \end{bmatrix} \tag{36}$$

This is the desired result.

Note that some simplification of the estimation equations results if the residual data can be interpreted as acceleration, so that we have

$$[\text{residual acceleration}] = C(t)[\dot{\bar{q}}(t) - \dot{\bar{q}}^*(t)] + \text{noise}$$

where C(t) is a matrix resolving the acceleration vector into a line-of-sight direction. Such a data type could be constructed by differentiating doppler residuals or by solving for an unknown acceleration vector from the original data. The estimation procedure is essentially the same as described above.

6. CONCLUSION

In this paper we have shown that a relatively small number of combinations of orbit parameters are observable for a short arc of lunar tracking data, and suggested several ways of exploiting this fact for orbit determination purposes:

(1) Identify, either numerically or analytically, the observable parameter combinations and solve for these quantities. With this approach the solution parameters are linear combinations of all the parameters introduced in the initial problem formulation.

(2) Identify, either numerically or analytically, the major components of the observable parameter combinations and solve for these quantities. With this approach one solves for a certain "best" subset of the solution parameters introduced in the initial problem formulation.

(3) Analytically represent the observable parameter combinations as coefficients of a Fourier series, and numerically determine these coefficients from a spectral analysis of data residuals.

With any of these methods the estimates of any or all of the parameters introduced in the initial problem formulation (e.g., the C_{nm}, S_{nm} potential coefficients) can be recovered by the method described in Section 5.

REFERENCES

(1) Robert H. Tolson and John P. Gapcynski, "An Analysis of the Lunar Lunar Gravitational Field as Obtained from Lunar Orbiter Tracking Data", paper presented at IQSY/COSPAR Assemblies, London, England, July, 1967 (also NASA Langley Research Center, Hampton, Va.)

(2) William M. Kaula, "Plans for Analysis of Lunar Satellite Orbits", Publication No. 430, Institute of Geophysics and Planetary Physics, University of California, Los Angeles.

(3) William M. Kaula, Theory of Satellite Geodesy (book), Blaisdell Publishing Company, Waltham, Massachusetts, 1966.