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EULER-LAGRANGE RELATIONSHIP FOR RANDOM DISPERSIVE WAVES

by

D. P. Hoult

April 1, 1968

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ABSTRACT

It is shown that the equations for the motion of a tagged fluid particle in a random wave field define a singular perturbation problem, characterized by a non-uniformity at large times. The uniformly valid asymptotic expansion to this problem, the Euler - Lagrange relationship for random dispersive waves, is obtained. As an application of these general results, an integral representation of the solution is worked out for the case of vertically propagating random acoustic waves in an isothermal atmosphere. It is shown that the non-uniformity of mediums leads to a wave generated diffusion process. The time and length scales over which the process is diffusive are determined, and a formula for the diffusion coefficient is presented.

CHAPTER I

INTRODUCTION

From the earliest studies of Taylor[1] and Richardson[2], it has been known that the statistics of the motion of tagged fluid particles in a turbulent velocity field are of central importance in theories of turbulent diffusion. Unless similarity considerations may be brought to bear (Batchelor[3]), or mixing length theory used (Taylor[4]), little can be said about the process of turbulent diffusion without understanding this problem. One way to phrase the problem is to ask for the relationship between the Eulerian statistics of the turbulent velocity field (which are measured at a fixed spatial point), and the Lagrangian statistics of the tagged particle (which is moving with the fluid). This question is known as the Euler - Lagrange problem. Recent contributions to the literature are due to Lumley[5], who gave a clear and general formulation to the problem, and Lumley and Corrsin[6], who studied a simple model of the problem as it applies to homogeneous isotropic turbulence.

This paper presents a general solution to the problem, when the velocity field may be considered composed of random dispersive waves. It might seem remarkable that such a general solution exists. However, there is a growing literature of very general results for flows which may be considered (to some approximation) as composed of random dispersive waves. Among these results are those of Benny and Saffmann[7], who showed that if the wave modes were weakly coupled to each other, there always exists a uniformly valid Gaussian closure scheme. This scheme, applied to water waves, leads to a Boltzmann-like equation when the waves are viewed as particles[8], Hoult[9] has given a simple rule for distinguishing random dispersive waves from strongly coupled turbulence (that is, a turbulence in which each eddy scatters on another before it travels one eddy length scale). Hasselmann[10] has shown that in dealing with the various weak nonlinear wave-wave interactions, one is led to a formalism very similar to that of quantum electrodynamics. That is not surprising, because, in quantum electrodynamics, the waves (electrons) are weakly coupled to the photons with which they interact.

It is typical that these results are all based on some form of perturbation theory, where the small parameter measures the departures from a linear theory. For water waves, the small parameter is the mean slope of the surface; for quantum electrodynamics, it is the fine structure constant. The present problem is also a problem in singular perturbation theory. However, the method used here differs significantly from past approaches. The present approach might be called the method of realizations. For each realization of a random wave field, the actual trajectory of the tagged particle is calculated. In one simple case, presented in section V, an integral representation of one such solution is derived. This representation relates the particle position to the statistics of the random sources of the wave field. Using the present method, one must derive, as an intermediate step, what amounts to a complete solution to the problem, before such physically useful results as diffusion coefficients can be obtained. We believe that it is the nonergodic property of random dispersive waves which leads to such an approach.

In the next section (See Sec. II) the problem is put in non-dimensional form, and the appropriate small parameter is defined. A naive attack on the problem fails because the results are not uniformly valid for large times. In Section III, the exact solution to a simple one-dimensional example is presented. This result, properly interpreted, shows that the difficulty is due to a streaming motion generated by the wave field. Section IV gives the general theory for arbitrary random wave fields. The general theory reduces to the simple example. Section V gives a one-dimensional example of the general theory. It is shown that random internal waves in an isothermal atmosphere are diffusive over certain length and time scales. A formula is obtained for the diffusion coefficient of this wave field.

CHAPTER II

DISCUSSION OF THE PROBLEM

Let times be measured by a characteristic period of the wave field, T, and lengths by a characteristic wave length λ. If the wave field has a characteristic velocity A, then

$$\varepsilon = \frac{AT}{\lambda}$$

is supposed to ba a small larameter. If this were not so, then each mode would be strongly scattered by other modes of the wave field, and it would not be possible to physically distinguish random dispersive waves. In short, the problem is physically consistent only if ε is small.[9]

Thus the problem to be solved is a nonlinear, stochastic set of total differential equations, which has the form

$$\frac{d\vec{x}}{dt} = \varepsilon \, \vec{u}(\vec{x}, t). \qquad (1)$$

Here $\vec{x}$ is the nondimensional position of the fluid particle, t is time, and $\vec{u}$ is the nondimensional velocity field, which has a zero mean value.

The simplest approach, which may be called linear theory, is to suppose the departures of x from an equilibrium position x(t=0) are small. The asymptotic expansion corresponding to this idea is

$$\vec{x}(t) \sim \vec{x}(0) + \varepsilon \, \vec{x}_1(t) + \varepsilon^2 \, \vec{x}_2(t) + \ldots \qquad (2)$$

Substitution into equation (1) yields

$$\frac{d\vec{x}_1}{dt} = \vec{u}(\vec{x}(0), t),$$

$$\frac{d\vec{x}_2}{dt} = \left.\left(\vec{x}_1 \cdot \frac{\partial}{\partial x} \vec{u}(x, t)\right)\right|_{\vec{x}=\vec{x}(0)}$$

The difficulty with this approach is that the time average of

$$\left.\left(\vec{x}_1 \cdot \frac{\partial}{\partial \vec{x}}\right) \vec{u}(\vec{x}, t)\right|_{\vec{x}=\vec{x}(0)} \qquad (3)$$

is non zero for anisotropic wave fields. (A simple example of this fact is discussed in the next section.) This implies that x_2 will oscillate about some mean value which grows with time, the rate of growth being proportional to the time average of (3).

Thus, as $t \rightarrow \infty$, x_2 is not bounded. The expansion (Eq. 2) is not valid for large times. It can be shown, by the methods which follow, that the method of linearizing the equation (1) of motion leads to correct results only when the wave field is isotropic. This is a serious disadvantage, because most physical systems which consist of weakly coupled random waves are anisotropic, and tend towards isotropy very slowly, if at all.

CHAPTER III

A SIMPLE EXAMPLE

The simplest example which exhibits the difficulties discussed in the previous section is that of the displacement due to a wave with a single frequency. The equation to be solved is

$$\frac{dx}{dt} = \varepsilon \cos(t - x), \quad x(t = 0) = 0 \tag{4}$$

This equation has an exact solution which results from the quadrature:

$$\int_0^{t-x} \frac{d\xi}{1 - \varepsilon \cos\xi} = t.$$

Evaluation of the integral gives

$$\frac{1}{\sqrt{1-\varepsilon^2}} \tan^{-1}\left[\frac{\sqrt{1-\varepsilon^2}\,\sin(t-x)}{\cos(t-x) - \varepsilon}\right] = t \tag{5}$$

To understand what this result implies, we expand the result in the limit $\varepsilon \to 0$, with t and x arbitrary. The result, after a rather long calculation, is

$$x \sim \frac{1}{2}\varepsilon^2 t + \varepsilon \sin(t - \frac{1}{2}\varepsilon^2 t) - \frac{\varepsilon^2}{4} \sin 2(t - \frac{1}{2}\varepsilon^2 t) + 0(\varepsilon^3) \tag{6}$$

The most remarkable feature* of this expansion is the steady drift, with nondimensional velocity, $1/2\ \varepsilon^2$, in the positive x direction. We

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* In the theory of water waves, this streaming motion is known as the mass transport effect. It was discovered by G.G. Stokes (1847) Trans.

Camb. Phil. Soc., 8, pp. 441 and has recently been studied by Longuet-Higgins, MS (1953) Phil. Trans. A., 245, pp. 535 and (1960) J. Fluid Mech., 8, pp. 293. These investigations are significantly different, in method, from present analysis. Stokes discovered the drift while analysing non-random finite amplitude waves, and Longuet-Higgins showed that near the free surface of the water, viscous forces ignored by Stokes, must be taken into account. None of the extensive literature on this effect is concerned explicitly with the effects of random waves, or the Euler-Lagrange problem.

may understand this drift in the following way. If x is increasing, the period of cos(t-x) (in Eq. 4) is slightly greater than 2π. Likewise, if x is decreasing, the period of cos(t-x) is slightly less than 2π. Hence the particle spends slightly more time moving in the direction of increasing x than in the direction of decreasing x. This results in a drift in the direction of increasing x.

If the wave had the form cos(t + x), the drift would be in the negative x direction. The drift is caused by small changes in phase due to changes in x; it is in the direction of the phase velocity of the wave, in this simple example.

Now consider ε to be a small random number, with zero mean. Then equation (6) represents the result of one realization of the random process. Considering the ensemble of all ε, the motion can be described as a drift with mean velocity $\frac{1}{2} < \varepsilon^2 >$ superimposed upon random oscillations with amplitude ε. Here, $< >$ denotes an ensembler average.

The non uniformity which invalidates the use of linear theory is simply due to the streaming motion generated by the wave field.

Perhaps a physical discussion of these results is warranted here. Suppose that the velocity field described by equation (4) is the velocity of a water wave at the surface of the water. Suppose that the wave is generated by the oscillatory motion of a wave maker at the end of a long wave tank. Then, assuming that the motion is exactly one dimensional, according to equation (6), the wave maker would slowly drift, with a velocity of magnitude $\frac{1}{2}$ ε times the wave maker velocity, in the positive x direction. In a real wave tank, the wave maker is fixed, but observations we have made in the M. I. T. random wave tank[11] show that away from the tank walls there is, in fact, a streaming motion of the correct magnitude and direction, as predicted in equation (6), on the free surface of the tank. There is a counterflow in the boundary layers on the walls of the tank. The real point of our discussion is not that a simple, one dimensional theory can accurately predict the drift velocity observed, but rather that such streaming motions are a general feature of random dispersive wave fields.

CHAPTER IV

THE GENERAL THEORY

The basic idea needed to solve equation (1) is a method of separating the drift motion from the oscillatory motion. To do this, we first notice that drift occurs on a time scale $\tau = \varepsilon^2 t$, whereas the oscillations occur on a time scale of t. This suggests that a two variable expansion is required[12]. Notice also that the oscillatory terms in equation (6) have a slow modulation, on a time scale τ. Finally, the solution, $x(t, \tau)$, viewed as a function of two variables, has the following properties: a) it is expandable in a power series in ε, b) $x(t, \tau)$ remains bounded in the limit $t \to \infty$, τ fixed.

These general features may be clarified by realizing that the position of a tagged particle in a velocity field composed of a sum of oscillatory modes will certainly oscillate with a time scale of a typical period of a typical mode. This oscillation will occur about some mean position, which, according to equation (6), may drift with time. However, each individual oscillation has an amplitude of order ε, according to equation (1). If drift is ignored (τ fixed), the oscillations (at a <u>fixed</u> mean position) are bounded in time. This is the same as statement b. The amplitude corrections to oscillations of amplitude ε are of order ε^2. This is the justification of statement a.

These remarks serve to specify the form of the asymptotic expansion:

$$\vec{x}(t) \sim \vec{x}_0(t, \tau) + \varepsilon\, \vec{x}_1(t, \tau) + \varepsilon^2\, \vec{x}_2(t, \tau) + \dots \qquad (7)$$

Let the velocity field be expanded in a Taylor series about $\vec{x}_0(t, \tau)$:

$$\vec{u}(\vec{x}, t) = \vec{u}(\vec{x}_0, t) + (\vec{x} - \vec{x}_0) \cdot \frac{\partial}{\partial \vec{x}}\, \vec{u}(\vec{x}, t)\Big|_{\vec{x}=\vec{x}_0} + \dots$$

Substituting these expansions into equation (1) and collecting like powers of ε produces a hierarchy of equations, the first three of which are given below:

$$\frac{\partial \vec{x}_0}{\partial t} = 0 \qquad , [0(1)]$$

$$\frac{\partial \vec{x}_1}{\partial t} = \vec{u}(\vec{x}_0, t) \qquad , [0(\varepsilon)]$$

$$\frac{\partial \vec{x}_0}{\partial \tau} + \frac{\partial \vec{x}_2}{\partial t} = \vec{x}_1 \cdot \frac{\partial}{\partial \vec{x}}\, \vec{u}(\vec{x}, t)\Big|_{\vec{x}=\vec{x}_0} \qquad , [0(\varepsilon^2)] .$$

The 0(1) and the 0(ε) equations yield

$$\vec{x}_0(t, \tau) = \vec{x}_0(\tau), \qquad (8)$$

$$\vec{x}_1 = \int^t \vec{u}(\vec{x}_0, t)\, dt + f(\tau). \qquad (9)$$

Hence the mean position of the particle, x_0, depends only on τ, the time scale for drift. $f(\tau)$ is a higher order correction to the mean position of the particle.

To determine x_0, the $0(\epsilon^2)$ equation must be solved. Let $\overline{g(\tau)}$ be the t average of $g(t, \tau)$:

$$\overline{g(\tau)} = \lim_{\substack{t \to \infty \\ \tau \text{ fixed}}} \frac{1}{t} \int_0^t g(t, \tau)\, dt \tag{10}$$

Now, using (8), (9), and (10), the equation for x_0 becomes

$$\frac{d\vec{x}_0}{d\tau} = \overline{\left(\int^t \vec{u}(\vec{x}_0, t)\, dt\right) \cdot \frac{\partial}{\partial \vec{x}_0} \vec{u}(\vec{x}_0, t)} \tag{11}$$

The remaining terms in equation (11), which are,

$$\overline{\frac{\partial x_2}{\partial t}}, \text{ and } \overline{f(\tau) \frac{\partial}{\partial \vec{x}_0} \vec{u}(\vec{x}_0, t)} ,$$

are zero by virtue of the boundness of x_2 in the $t \to \infty$, τ fixed, limit, and because $\vec{u}(\vec{x}_0, t)$ has zero mean value.

Equation (9) and the solution to equation (11) are in fact the general solution to the problem. An elementary calculation shows that if

$$u(x, t) = \cos(t - x),$$

then $$x_0 = \frac{1}{2}\tau, \text{ and } x_1 = \sin(t - \frac{1}{2}\tau)$$

in agreement with equation (6).

CHAPTER V

INTERNAL WAVES IN AN ISOTHERMAL ATMOSPHERE

Before applying these results (Eq. 9, 11) to a specific case, it is necessary to consider if any $0(\epsilon^2)$ effects due to the interaction between various modes of the velocity field are likely to change the general method. If such an interaction leads to a change in the power spectrum of the wave field (rather than coupled oscillations with a time scale T), then it would be necessary to include this slow variation in the velocity field. This effect may be taken account of, if, in the formulae of Section IV, the velocity field $u(x_0, t)$ is replaced by $u(x_0, t, \tau)$. Then the equations of Section IV would remain valid. However, it is known(13) that the interaction which results in a slow modification of the power spectrum of deep water waves with no surface tension is $0(\epsilon^3)$. We shall assume here that the same to be true for internal waves in an isothermal atmosphere.

For simplicity, we consider acoustic-gravity waves(14) propagating vertically upward in an isothermal atmosphere. Non-dimensionalize with the scale height of the atmosphere, and the speed of sound. Then the velocity field has the form(15) (z being positive upward)

$$\frac{dz}{dt} = \epsilon\, u(z, t) = \epsilon \int_{\omega_A}^{\infty} dA(\omega)\; e^{z/2}\; e^{i(kz - \omega t)}$$

$$k = \sqrt{\omega^2 - \omega_A^2} \quad . \qquad (12)$$

In equation (12), ω_A is the acoustic cutoff frequency, which has a value of 1/2 with the present nondimensionalization (times are measured by the time it takes a sound wave to propagate one scale height). The small parameter ε is the ratio of the fluid velocity in the wave to the speed of sound. For waves in the upper atmosphere ε is about 1/10. The solution for z_1 is, assuming $z(0) = 0$,

$$z_1(t, \tau) = -\int_{\omega_A}^{\infty} e^{z_0/2} \frac{dA(\omega)}{i\omega} e^{ikz_0} [e^{-i\omega t} - 1] + f(\tau) \qquad (13)$$

Letting $(\ )^*$ denote complex conjugate, equation (11) takes the following form:

$$\frac{dz_0}{d\tau} = \lim_{\substack{t \to \infty \\ \tau \text{ fixed}}} \left\{ \frac{1}{t} \int_0^t dt \left[\int_{\omega_A}^{\infty} \int_{\omega_A}^{\infty} e^{z_0 (\frac{1}{2} + ik)} e^{i(k-k')z_0} \frac{dA(\omega)dA^*(\omega')}{i\omega} \left\{ + e^{i(\omega' - \omega)t} - e^{-i\omega t} \right\} \right] \right\}$$

Now the time average of

$$+ e^{i(\omega' - \omega)t} - e^{-i\omega t}$$

has the following properties. It is zero if $\omega \neq \omega'$. If $\omega = \omega'$, the nonzero term arises from $+e^{i(\omega' - \omega)t}$, and has a value of +1. In this way we obtain, (since $dz_0/d\tau$ is real),

$$\frac{dz_0}{d\tau} = e^{z_0} \left| \int_{\omega_A}^{\infty} \sqrt{\frac{k}{\omega}} \, dA(\omega) \right|^2 \qquad (14)$$

The vertical bars, $| \; (\;) \; |$, denote the absolute value of ( ).
The solution to equation (14) is

$$z_0(\tau) = \log \left(\frac{1}{1 - \tau/\tau_\infty} \right) \qquad (15a)$$

where the time for a particle to drift from $z = 0$ to $z = \infty$ is

$$\tau_\infty = \frac{1}{\left| \int_{\omega_A}^{\infty} \sqrt{\frac{k}{\omega}} \, dA(\omega) \right|^2} \qquad (15b)$$

Now, from equation (13) and (15), the uniformly valid expansion for the motion of a single tagged particle, for one realization, is

$$z(t) \sim \log \left(\frac{1}{1 - \tau/\tau_\infty} \right) + \frac{\varepsilon}{\sqrt{1 - \tau/\tau_\infty}} \int_{\omega_A}^{\infty} \frac{dA(\omega)}{i\omega} \, e^{ikz_0} \left[e^{-i\omega t} - 1 \right]$$

$$+ \frac{\varepsilon f(\tau)}{(1 - \tau/\tau_\infty)^2} + 0(\varepsilon^2). \qquad (16a)$$

Clearly, times of physical interest are much shorter than τ_∞.

This equation becomes, for τ/τ_∞ small, and ignoring the second order effects in the drift (the term in $f(\tau)$),

$$z(t) \sim \tau/\tau_\infty + \epsilon(1 + \tau/2\tau_\infty) \int_{\omega_A}^{\infty} \frac{dA(\omega)}{i\omega} e^{ikz_0} [e^{-i\omega t} - 1] \qquad (16b)$$

$$+ 0(\tau/\tau_\infty)^2 \quad .$$

Physically, the particle drifts upward with a random drift velocity. As the particle drifts upward, the amplitude of the particle oscillations increase, due to the exponentially increasing amplitude of the acoustic wave. As time increases, the position of a particle becomes more uncertain, due to the random drift, and to the growing amplitude of the oscillatory motion. This means that the probability distribution of the particle becomes broader as time increases. Hence the process has a dispersive character.

If one models this dispersive process by a diffusion equation, the diffusion coefficient will in general vary with time. To calculate a diffusion coefficient, some assumptions about the statistics of random amplitudes, $A(\omega)$, are required. For the present, we suppose, consistent with a Gaussian approximation, that the ensemble average of triple products of $A(\omega)$ are identically zero. Then the mean square amplitude of z grows as

$$< z^2(t) > = < (\tau/\tau_\infty)^2 > + \qquad (16c)$$

$$\epsilon^2 < (1 + \tau/\tau_\infty) \left(\int_{\omega_A}^{\infty} \frac{4dA(\omega)\, dA^*(\omega)}{\omega^2} \sin^2 (\frac{\omega t}{2}) \right) >$$

Inspection of equation (16c) shows that $< z^2(t) >$ for short times increases as t^2, with oscillations on a time scale of t. It is physically plausible that the diffusive character of wave field should be the result of many such oscillations. On the other hand, in a physically meaningful process, the particle only drifts a finite distance. These remarks serve to define the diffusion limit of the wave field described by equation (12): $\tau/\tau_\infty \rightarrow 0$, $t \rightarrow \infty$. Diffusion occurs on a time scale such that $0(1) < t < 0(\frac{1}{\epsilon^2})$. On this time scale, $< z^2(t) >$ grows linearly with time,

$$< z^2(t) > \sim \epsilon^2 \int_{\omega_A}^{\infty} \frac{2dA(\omega)\, dA^*(\omega)}{\omega^2} + Dt + \ldots ,$$

with a diffusion coefficient D, which has a value

$$D = < \epsilon^2 \left[\epsilon^2 \int_{\omega_A}^{\infty} \int_{\omega_A}^{\infty} \frac{kdA(\omega)\, dA^*(\omega)}{\omega} \right] \int_{\omega_A}^{\infty} \int_{\omega_A}^{\infty} \frac{2dA(\omega)\, dA^*(\omega)}{\omega^2} >$$

D is simply the mean drift velocity (the term in square brackets) times the mean square amplitude of the oscillations at $\tau = 0$ divided by the scale height. The non-uniform medium causes the streaming motion to have a diffusive character over length scales of a scale height, and time scales of $0(1) < t < 0(1/\epsilon^2)$.

This example serves to show ho ffusion due to random dispersive waves may occur when the medium is non-uniform. If the medium were uniform, the exponential factor in equation (12) would be suppressed, there would be only a random drift with velocity equal to

$$\epsilon^2 \left| \int_{\omega_A}^{\infty} dA(\omega) \sqrt{\frac{k}{\omega}} \right|^2$$

The motion would still be dispersive, but no diffusion limit would exist in this case.

The expression obtained for D, when put into dimensional units, gives a simple estimate of the order of magnitude of wave diffusion:

$$D \sim \frac{2A^4T^3}{\lambda H}$$

Here H is the scale height. For the vertical diffusion coefficient in the atmosphere around 80 - 100 km., put

λ vertical $\approx$ 5 km, H = 8km.

A $\sim$ 20 cm./sec

T $\sim$ 10^4 sec

Then $D \sim 7 \times 10^5$ cm^2/sec. This calculation may give a simple explanation of the turbopause, for the ratio of D to the molecular diffusion coefficient is one at about 110 km., and is about 10 at 90 km. Thus, upon this mechanism, sodium vapor trails released from rockets would be diffused by waves below 110 km., and by molecular effects above 110 km. It should be carefully noted, however, that, as most of the random waves believed to exist in the 80 - 110 km region are internal gravity waves whose group velocity is upward, but whose phase velocity is downward, a full 3 dimensional calculation is required to find the diffusivity of the wave field. Nothing in our simple model would indicate that for all types of waves, D need always be positive, or proves that the order of magnitude estimate for D associated with acoustic waves is the same as for other random wave fields. The point is rather that one would in general expect random wave fields in a non-uniform media to have non zero diffusion coefficients, and that the present simple example indicates that such diffusion coefficients may be large enough to be geophysically important .

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