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ABSTRACT

AN INVARIANCE PRINCIPLE FOR DYNAMICAL SYSTEMS IN HILBERT SPACES

by

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In recent years the extension of the second method of Liapunov to distributed parameter systems has gained much attention. Movchan [1] and Zubov [2] have given results paralleling Liapunov's classical theorems. In applications, however, these theorems are now always useful because of their rather stringent hypotheses. The approach taken to weaken the hypotheses of Liapunov's theorems for autonomous systems in finite dimensions has been to state an invariance principle [3]. Analogously, it has become necessary to investigate invariance principles for infinite dimensional problems i.e. a study of abstract dynamical systems [4,5,6,7]. In certain mixed initial boundary value problems, difficulties in applying recent results have made new outlooks seem valuable.

One such outlook is the approach taken in this paper. Specifically the new concept of a weak dynamical system has been added to the concept of a dynamical system.

Let R^+ denote the interval $[0,\infty)$ and $\mathscr{B}$ a Banach space with $\|\phi\|_{\mathscr{B}}$ denoting the norm of an element $\phi \in \mathscr{B}$.

<u>Definition 1</u>. We say u is a <u>dynamical system</u> on a Banach space $\mathscr{B}$ if i) u is a mapping of $R^+ \times \mathscr{B}$ into $\mathscr{B}$, ii) $u(t,\phi)$ is continuous in (t,ϕ), iii) $u(0,\phi) = \phi$, and iv) $u(t+\tau,\phi) = u(t,u(\tau,\phi))$ for $t,\tau \geqq 0$, $\phi \in \mathscr{B}$. The <u>positive orbit</u> $O^+(\phi)$ through $\phi \in \mathscr{B}$ is defined as $O^+(\phi) = \bigcup_{t\geq 0} u(t,\phi)$.

Definition 2. We say u is a weak dynamical system in a Banach space $\mathscr{B}$ if i) u is a mapping of $R^+ \times \mathscr{B}$ into $\mathscr{B}$, ii) $u(t,\phi)$ is weakly continuous, i.e. if $|t-t_n| \to 0$ and $\phi_n \xrightarrow{w} \phi$ then $u(t_n,\phi_n) \xrightarrow{w} u(t,\phi)$, iii) $u(0,\phi) = \phi$, and iv) $u(t+\tau,\phi) = u(t,u(\tau,\phi))$ for $t,\tau \geq 0$, $\phi \in \mathscr{B}$. The positive orbit is as in Definition 1.

Definition 3. A set M in $\mathscr{B}$ is an invariant set of the weak dynamical system u if for each ϕ in M, there exists a function $U(t,\phi)$, $U(0,\phi) = \phi$, defined and in M for $t \in (-\infty,\infty)$, such that for any $\sigma \in (-\infty,\infty)$, $u(t,U(\sigma,\phi)) = U(t+\sigma,\phi)$ for $t \in R^+$. (In particular, for $t \in R^+$, $u(t,\phi) = U(t,\phi)$, $\phi \in M$).

With this structure an invariance principle may be stated.

Let $\mathscr{B}$ and $\mathscr{L}$ be separable Hilbert spaces and assume $I: \mathscr{B} \to \mathscr{L}$ is completely continuous. Let $u: R^+ \times \mathscr{B} \to \mathscr{B}$ be a weak dynamical system on $\mathscr{B}$. Let V be a continuous scalar functional in the larger Hilbert space $\mathscr{L}$ and define

$$\mathring{V}(\varphi) = \overline{\lim_{t\to 0^+}} \frac{1}{t}[V(u(t,\varphi)) - V(\varphi)].$$

Definition 4. We say a scalar functional V is a Liapunov functional on a set G of $\mathscr{B}$ if i) V is a continuous functional (in the $\mathscr{L}$ topology), ii) V is bounded from below on G and, iii) $\mathring{V}(\varphi) \leq 0$ for all φ in G.

Definition 5. Let $R = \{\varphi$ in $\mathscr{B}$: $\mathring{V}(\varphi) = 0\}$ and denote by M the largest set in R invariant with respect to the weak dynamical system.

Then:

Theorem 1 (Invariance Principle): If i) V is a Liapunov functional in $G \subset \mathscr{B} \subset \mathscr{L}$, ii) an orbit $0^{+}(\varphi)$ belongs to G, G is a closed and bounded set in $\mathscr{B}$, then weak $\text{dist}_{\mathscr{B}}(u(t,\varphi),M) \to 0$ as $t \to \infty$. Here weak $\text{dist}_{\mathscr{B}}(u(t,\varphi),M) \triangleq \inf_{m \in M} \rho(u(t,\varphi),m)$, where ρ represents the metrized weak topology of closed, bounded sets in $\mathscr{B}$.

In applications it is always assumed that the system under consideration is both a dynamical system and a weak dynamical system on $\mathscr{B}$. As motivation for this assumption the following theorem is proven.

Theorem 2: The wave equation describes a weak dynamical system on $W_2^{(1)}(\Omega) \times L_2(\Omega)$.

The two major applications given in the paper are

1. Tunnel diode system [4], described by the system of equations

$$\begin{aligned} Li_t &= -v_x - Ri, \\ -Cv_t &= i_x + Gv, \end{aligned} \qquad 0 \leq x \leq 1, \quad t > 0$$

and boundary conditions

$$0 = E - v(0,t) - R_o i(0,t),$$
$$-C_1 v(1,t)_t = -i(1,t) + f(v(1,t));$$

where $f(v)$ has the graph of the form shown in Figure 1.

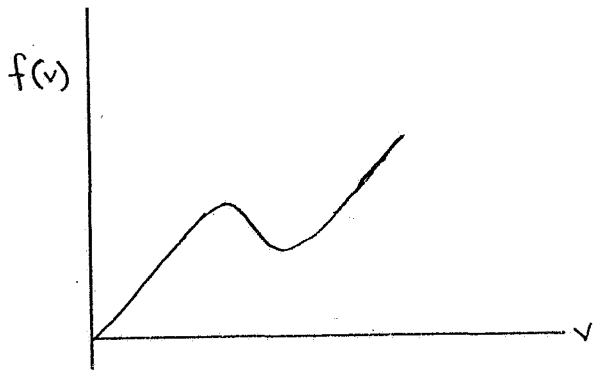

Figure 1.

In this example Theorem 1 is used to give sufficient conditions for complete stability.

2. Asymptotic behavior of the equations of linear thermoelasticity [8], in this case considered in the form

$$\rho \ddot{u}_i = (C_{ijk\ell} u_{k,\ell})_{,j} - (m_{ij} T)_{,j}$$
$$\rho c_D \mathring{T} + m_{ij} \theta_o \dot{u}_{i,j} = (K_{ij} T_{,j})_{,i} \qquad t > 0$$
$$T = 0 \text{ on } \partial\Omega, \quad u_i = 0 \text{ on } \partial\Omega.$$
$$\Omega \text{ finite n-dimensional domain}$$

In this example Theorem 1 gives sufficient conditions for the system to approach the equilibrium state

$$((u_i, \dot{u}_i, T) \in W_2^{(2)} \times W_2^{(1)} \times W_2^{(2)} \Big| T = 0,\ \int_\Omega m_{ij} u_{i,j}(x,t)\,dx = 0)$$

$$\text{as} \quad t \to \infty.$$

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