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NON-LINEAR ACOUSTICS

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ABSTRACT

Non-linear acoustics problems are first defined as those problems which require non-linear equations for their descriptions. Mass, momentum, and energy conservation equations are then derived with particular attention paid to the introduction of first, second and bulk viscosity coefficients. The equation of state is briefly introduced and the assumptions made on the nature of acoustical processes (adiabatic, isentropic, etc.) are analyzed and classified. The assumptions made to derive Burgers' equation for plane non-linear waves are then listed. Some solutions of Burgers' equations are reported and their range of practical validity pointed out. An equation for cylindrical, spherical and exponential horn waves, equivalent to Burgers' equation for plane waves and including it as a special case, is then derived. The solution of this equation for the case of exponential horn is given in proper terms of the solution of Burgers' equation. It is also suggested that the numerical solution of this equation for spherical and cylindrical waves would help interpretation of the nature of shocklets observed close to jet exhausts. Finally the important diffusion coefficient is studied again with the object of explaining the way it should be determined.

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FIGURE 1. ENERGY ATTENUATION PER WAVELENGTH, $\alpha\lambda$, FOR SOUND WAVES IN CO_2 AT 23° C., PLOTTED AGAINST FREQUENCY-PRESSURE RATIO IN MC./ATM. 21

LIST OF SYMBOLS

a = speed of sound

A = cross sectional area

C_p = specific heat at constant pressure; C_v = specific heat at constant volume

e = specific internal energy

f = logarithmic derivative of the cross sectional area function

F = auxiliary function

$\underset{\sim}{i}$ = unit vector in the x-direction

$\underset{\sim}{j}$ = unit vector in the y-direction

$\underset{\sim}{k}$ = unit vector in the z-direction; k = coefficient of heat conduction

K = heat transfer coefficient

$\underset{\sim}{n}$ = unit vector

p = pressure

r = auxiliary variable

R = control volume; R = perfect gas constant

s = specific entropy; auxiliary variable

S = control surface

t = time

T = temperature

u = particle velocity in the x-direction

v = particle velocity in the y-direction; excess wavelet velocity

∇ = particle velocity vector

w = particle velocity in the z-direction

x = rectangular coordinate

X = auxiliary variable

y = rectangular coordinate

z = rectangular coordinate

LIST OF SYMBOLS (Continued)

Greek Alphabet

γ = ratio of the specific heats

δ = diffusion coefficient

θ = auxiliary function

λ = second viscosity coefficient; wave length

μ = first viscosity coefficient; μ_v = bulk viscosity coefficient

ρ = density

Φ = auxiliary function

ω = sound angular frequency

1.0 INTRODUCTION

This report represents an initial effort to organize the field of non-linear acoustics of fluids starting from basic physical principles. Such organization helps clarify the usefulness and limitations of some existing solutions to some non-linear problems while suggesting rational patterns for further studies.

1.1 Definition of Non-linear Acoustics

Let us first define non-linear acoustics. The propagation of perturbances in uniform gases, in layered media, in humid atmosphere, and in hot jet gases are some of the problems of the acoustics of fluids. These physical realities are first reduced to physical models. These models are translated into mathematical equations and the equations are then solved or studied. A physical problem is defined non-linear when its equations are non-linear. Thus, non-linear acoustics cover those acoustic problems whose equations are non-linear. A differential equation or a system of differential equations is non-linear when any of its terms is non-linear in any of its dependent variables, or its derivatives, or when any of its terms contain products of dependent variables or of their derivatives. In the following instance, terms in parentheses are non-linear:

$$\frac{\partial \rho}{\partial t} + \left(v \frac{\partial \rho}{\partial x}\right) = - \left(\rho \frac{\partial v}{\partial x}\right)$$

$$\frac{\partial v}{\partial t} + \left(v \frac{\partial v}{\partial x}\right) = - \left(\frac{1}{\rho} \frac{\partial p}{\partial x}\right) + \mu \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial e}{\partial t} + \left(v \frac{\partial e}{\partial x}\right) = - \left(\frac{p}{\rho} \frac{\partial v}{\partial x}\right) + \left(\frac{\mu}{\rho}\left(\frac{\partial v}{\partial x}\right)^2\right) + \left(\frac{k}{\rho}\frac{\partial^2 T}{\partial x^2}\right)$$

The main feature of non-linear equations is that their solutions cannot be superimposed. Thus, if the conditions of one phenomenon can be seen as the sum of the conditions of two separate phenomena, the solution for the first phenomenon is not necessarily equal to the sum of the solutions for the two separate phenomena. In this report the term quasi-linear will not be used. A diffential equation or a system of differential equations is called quasi-linear if it is linear in its highest order derivatives (say the nth) with coefficients depending on the independent variables or on the dependent ones or on their derivatives up to order $n - 1$. According to this definition, the above three equations are quasi-linear. The second and third equations, for example, are quasi-linear since their highest order derivatives (2nd) are multiplied by either a constant (μ) or a dependent variable ($1/\rho$) but not by any second order derivative.

1.2 Kinetic Versus Continuum Theory

Classically, acoustics has long been considered as that branch of science dealing with small perturbations in relatively standard media, and the continuum theory has therefore been considered more than adequate. However, phenomena like relaxation, shock formation and sound propagation through high temperature gases, which are now considered part of non-linear acoustics are either in the field of kinetic theory (relaxation) or at the boundary between kinetic and continuum theories. In spite of this, the continuum theory is still the only one used in acoustics, with at most the introduction of some correcting constants, such as the "Bulk Viscosity Coefficients", of Lighthill (Reference 1), to take into account, to a first approximation, the kinetic phenomenon of molecular relaxation. The use of continuum theory, in a situation where its validity is questionable, suggests that the results obtained will have limited accuracy. This will allow less rigorous definition of the constants and the method of solution. Specifically, for example, we will attempt to show that the viscosity coefficient and diffusion coefficient should be allowed to vary more widely, in studying the propagation of sound in atmosphere or the formation of shock waves from high intensity sound, and should essentially be determined a posteriori by experimental results. It is a waste of time to try to assign them exact values in the continuum theory when their principal components are explained only on molecular considerations.

It is proper therefore, to discuss the subject of kinetic versus continuum theory, even if this entire report is based on continuum theory. The actual physical process which occurs when a fluid is perturbed is the transfer of both energy and momentum. The transfer is accomplished through molecular collisions which are considered by the kinetic theory through the Boltzmann equation. This transfer can be assumed to occur in a continuum medium through the action of stresses and heat flux dependent respectively upon the spatial derivatives of the velocities and of temperature, in which case the dependency must be determined empirically or from the use of the kinetic theory. The main reason for using the continuum theory is that the kinetic theory is too complex when applied to most of the practical problems and would require assumptions as questionable as that of using the continuum theory. Moreover, according to W.D. Hayes (Reference 2) "The Navier-Stokes relations (here taken to include the Fourier heat conduction relation) appear naturally as the lowest dissipative approximation with either approach". It is important to notice that the Navier-Stokes relations (continuum theory) consider at most the lowest dissipative effects. Since the dissipative effects in the continuum theory are increasing functions of the spatial derivatives of the velocities and of the temperature, the Navier-Stokes equations, in their usual form, can be expected to yield decreasing accuracy as the cited derivatives increase, as in shock formation or high frequency sound propagation. It is now evident why attention has been concentrated on this dependency of the dissipative effects on the spatial derivatives of the velocities and the temperatures in the efforts to use the Navier-Stokes equations for phenomena just out of their acceptable range of application. Thus, for instance, classically it was assumed that the dissipative term of the momentum equation was linearly dependent on the spatial

derivative of the velocity, but when the structure of moderately strong shocks was to be studied, the need was felt for considering also a term proportional to the square of the velocity gradient (Reference 3) or, according to others, to the second derivatives of the velocity (Reference 4). Others (Reference 1) still modified the standard coefficients of the linear viscosity to consider at first approximation the rotational relaxation phenomenon. Considering this, when using the continuum equations to study non-linear phenomena, one should keep an open mind in selecting the form of the dissipative terms and be prepared to change it if experience suggests so.

2.0 THE BASIC EQUATIONS IN CONTINUUM THEORY

The dependent variables of a three-dimensional flow are: u, v, w, p, ρ, T, and e. The independent variables are x, y, z, and t. Therefore, to describe complet a three-dimensional flow, seven equations plus boundary and initial conditions are necessary. The mass conservation principle provides one equation. Three equations are derived from the conservation of momentum principle applied in the three directions. A fifth equation is provided by the conservation of energy principle. The equation of state of the matter which flows is the sixth equation.

It can be seen that for all physical problems one equation is still missing. The engineer must furnish the last equation and it will always be an assumption that he makes on the manner in which the particular process under consideration occurs or a measured property of the medium in which the process occurs. When an assumption is used it is called a process assumption. Common instances of process assumptions are that of adiabatic and isentropic processes.

In the following pages the seven equations will be studied with the purpose of pointing out the significance of their terms which in turn should help to simplify the equations themselves. The simplification will be necessary if the system of equations must be solvable in closed form.

2.1 Mass Conservation Equation

Consider a control volume R, enclosed by a control surface S, fixed in the moving medium. If ds is an infinitesimal area of the control surface, the mass flowing through it in the unit of time is $\rho \underset{\sim}{V} \cdot \underset{\sim}{n}\, ds$, where $\underset{\sim}{V}$ is the local particle velocity vector and $\underset{\sim}{n}$ is the unit vector normal to ds. Thus $\underset{\sim}{V} \cdot \underset{\sim}{n}$ is the particle velocity normal to ds and is assumed to be positive when directed outward. Thus, in the unit of time the mass flowing out of the control volume is:

$$\iint_S \rho \underset{\sim}{V} \cdot \underset{\sim}{n}\, ds \tag{1}$$

Since the total mass must be constant, the mass which flowed out of the control volume in the unit time must be equal to the decrease of mass within the volume in the same unit time:

$$\iint_S \rho \underset{\sim}{V} \cdot \underset{\sim}{n}\, ds = - \frac{\partial}{\partial t} \iiint_R \rho\, dr \tag{2}$$

Applying the divergence theorem to the left-hand side of the above equation, the surface integral of $\rho \underset{\sim}{V} \cdot \underset{\sim}{n}$ is replaced by the volume integral of $\nabla \cdot (\rho \cdot \underset{\sim}{V})$:

$$\iiint_R \nabla \cdot (\rho \underset{\sim}{V})\, dr = - \frac{\partial}{\partial t} \iiint_R \rho\, dr \tag{3}$$

Since the limits of the space integral are not dependent on time, the time derivative can be taken into the volume integral thus obtaining:

$$\iiint_R \left[\nabla . (\rho \underset{\sim}{V}) + \frac{\partial \rho}{\partial t} \right] = 0 \tag{4}$$

The above equation must hold for any volume no matter how small and, therefore, the integrand must be identically zero:

$$\rho_t + \nabla . (\rho \underset{\sim}{V}) = 0 \tag{5}$$

Alternative forms of the mass conservation equation are given below:

$$\rho_t + \rho \nabla . \underset{\sim}{V} + \underset{\sim}{V} . \nabla \rho = 0 \tag{6}$$

$$D\rho/Dt + \rho \nabla . \underset{\sim}{V} = 0 \tag{7}$$

$$\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0 \tag{8}$$

A useful expression can be derived for the total derivative of any scalar property (f) of a flow; like entropy, enthalpy, energy, velocity component etc., from the mass conservation principle. Thus, using the definition of total derivative and adding a term equivalent to zero because of the mass conservation principle (equation 5), we have:

$$\rho \frac{Df}{Dt} = \rho \left(\frac{\partial f}{\partial t} + \underset{\sim}{V} . \nabla f\right) + f \left(\frac{\partial \rho}{\partial t} + \nabla . (\rho \underset{\sim}{V})\right) =$$

$$= \rho \frac{\partial f}{\partial t} + f \frac{\partial \rho}{\partial t} + \rho \underset{\sim}{V} . \nabla f + f \nabla . (\rho \underset{\sim}{V}) =$$

$$= \frac{\partial}{\partial t} (\rho f) + \nabla . (\rho f \underset{\sim}{V}) \tag{8a}$$

Equation 8a can be rearranged as follows and compared with equation 5.

$$(f\rho)_t + \nabla . (f \rho \underset{\sim}{V}) = \rho \frac{Df}{Dt}$$

The above equation tells us the physical meaning of the total derivative of a scalar property. The value of the total derivative is the rate at which the scalar property (energy, for example) is produced by a source or absorbed by a sink within the control volume. The first term on the left-hand side of the above equation gives the rate at which the scalar property changes within the control volume, the second term gives the flux of the scalar property through the control colume. If their sum is different from zero and equal to $\rho Df/Dt$ then the scalar property was generated or absorbed

within the control volume at the rate of $\rho Df/Dt$. In the case in which $\rho Df/Dt = 0$ we admit no source or sink for the scalar property. This is the case of equation 5 where no mass is generated or absorbed. However, the condition $\rho Df/Dt = 0$ does not imply that the scalar property f is everywhere constant within the control volume. Thus, for instance $ds = 0$ implies constant entropy everywhere (isentropic process) while $\rho Ds/Dt = 0$ implies that there is no sink or source of entropy within the control volume (the flow is adiabatic and reversible) but the entropy can have different values at different points and can vary with time at any specific point of the control volume.

2.2 The Momentum Conservation Equations

The momentum conservation equation is derived by requiring that the time rate of increase of the momentum of the mass within a fixed volme is equal to the rate at which momentum flows into the volume plus the forces acting on it. The first step is, therefore, to define the forces acting on this volume. Besides mass forces, proportional to the mass on which they act like gravitational or centrifugal forces, there are surface forces, proportional to the area on which they act like pressure and the viscous forces, which have molecular origin but are accounted for in the continuum theory by specific terms. The terms are not of evident meaning and therefore they will be considered in some detail.

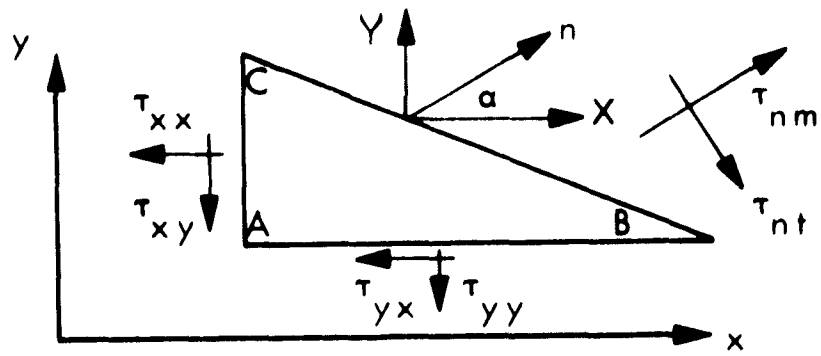
It was stated in the introduction that at first approximation dissipative terms are assumed to be proportional to the first space derivatives of the velocity and, when these space derivatives become large, to the second space derivatives or according to others, to the square of the first space derivatives. The linear dependency is introduced in the momentum equation through the following steps. First, it is assumed that when these friction forces are referred to principal axes, they are proportional to both the divergency of the velocity and the space derivatives of the velocities in the direction of the principal axes. Then new axes are chosen and the new stresses and space derivatives of the velocities are derived from the principal stresses and the derivatives of the principal velocities using the direction cosines of the new axes with respect to the principal ones.

Finally the direction cosines are eliminated and the stresses are expressed only as functions of the space derivatives of the velocities in the arbitrary reference system. This process is repeated, with more detail, in the next section.

2.2.1 The Principal Stresses

Consider a small volume in equilibrium within a certain medium. Assume that the mass outside this volume can exert normal and tangential forces on the mass inside it through its surface. For any set of such forces there will be three orthogonal axes with respect to which the normal and tangential forces reduce down to normal forces only.

For simplicity this property will be derived for the two dimensional case only. The following sketch defines the symbols:



A = Area of CB

In the above sketch X and Y are the stresses on the face BC referred to the x and y axes respectively while τ_{nm} and τ_{nt} are the stresses on the face BC, but referred to the axis n. There are two values for x which satisfy the conditions: $\tau_{nt} = 0$. These two values give a set of perpendicular axes called principal axes. For the equilibrium of the forces in the vertical and horizontal directions it is necessary that:

$$AX = \tau_{xx} A \cos a + \tau_{yx} A \sin a$$

$$AY = \tau_{yy} A \sin a + \tau_{xy} A \cos a$$

Thus, the stresses on the BC face can be expressed as follows:

$$\tau_{nm} = X \cos a + Y \sin a = \tau_{xx} \cos^2 a + \tau_{yy} \sin^2 a + 2\tau_{xy} \sin a \cos a$$

$$\tau_{nt} = -X \cos a + Y \cos a = \tau_{xy} (\cos^2 a - \sin^2 a) + (\tau_{yy} - \tau_{xx}) \sin a \cos a$$

From the second equation it is found that $\tau_{nt} = 0$ when a satisfies the following relationship:

$$\frac{\tau_{xy}}{\tau_{xx} - \tau_{yy}} = \frac{\sin a \cos a}{\cos^2 a - \sin^2 a} = \frac{1}{2} \tan 2 a$$

Thus proving the existence of the principal axes with respect to which τ_{nm} is sufficient to describe the equilibrium of the element. Similarly for the three-dimensional case, through any point in a flow in equilibrium, three principal planes can be found over which the shearing stresses vanish leaving only three principal normal stresses:

$$\tau_1, \tau_2, \tau_3.$$

From the equilibrium to the rotation around the z-axis, and neglecting infinitesimals

of higher order, it could also be proved that:

$$\tau_{xy} = \tau_{yx}$$

When the above reasoning is extended to the three-dimensional case the following six stresses are found necessary and sufficient to describe the equilibrium of an infinitesimal volume:

$$\tau_{xx};\quad \tau_{yy};\quad \tau_{zz};\quad \tau_{zx} = \tau_{xz};\quad \tau_{xy} = \tau_{xy};\quad \tau_{yz} = \tau_{zy} \tag{9}$$

So far the reasoning and the conclusions are exactly the same as those of the physics of solids.

It is striking that the equilibrium properties: $\tau_{yz} = \tau_{zy}$; $\tau_{zx} = \tau_{xz}$; $\tau_{xy} = \tau_{yx}$; valid for static or uniform motion situations, are later assumed to hold for dynamic situations. It seems as if the above equilibrium properties are more rightfully used in the physics of solids where semi-static loads are commonly found, than in the physics of fluids where dynamic motions are of prime interest.

2.2.2 The Viscosity Coefficients

In the previous section we have seen how at any point of a medium at rest or in uniform motion, three directions can be determined with respect to which the shearing stresses vanish and the surface forces reduce down to three normal stresses τ_1, τ_2, τ_3. In the static case these normal stresses are the familiar static pressure: $-p$. It is at this point that the main assumption on the nature of the stresses in a medium with arbitrary motion is introduced. It is assumed that each of the principal stresses in a medium in arbitrary motion (Is it proper to associate principal stresses in a medium in arbitrary motion?) differ from the static pressure by two factors: One proportional to the divergency of the principal velocity and the second one to a specific space derivative of the principal velocity components:

$$\left.\begin{aligned}
\tau_1 &= -p + \lambda\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'}\right) + 2\mu\frac{\partial u'}{\partial x'}\\
\tau_2 &= -p + \lambda\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'}\right) + 2\mu\frac{\partial v'}{\partial y'}\\
\tau_3 &= -p + \lambda\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'}\right) + 2\mu\frac{\partial w'}{\partial z'}
\end{aligned}\right\} \tag{10}$$

The above equations are assumptions confirmed experimentally or better deduced from experimental finding on fluid in slowly changing non-uniform motions. As previously stated, the above equations are a semiempirical way to account, in

continuum theory, for molecular transfer phenomena which belong to kinetic theory. The above assumptions can easily be defined as the most important point to understand, criticize and improve in the continuum theory equations for viscous flows. Some comments are quoted on the above assumption. H. Lamb (Reference 5 page 574)".... p_1, p_2, p_3 (the principal stresses) differ from - p (the static pressure) by quantities depending on the motion of distortion, which must therefore be functions of a', b', c', (the space derivati ves of the velocities only). The simplest hypothesis we can frame on this point is that these functions are linear .... (equations 10 follow) ... where λ, μ are constants depending on the nature of the fluid and on its physical state, this being the most general assumption consistent with the above suppositions and with symmetry". F.V. Hunt (Reference 6 page 3-31): "The residual stress tensor (whose components are the terms proportional to the space derivatives of the velocities in equations 10) is given, at first approximation, by the linear terms of an expansion in powers of the viscosity coefficients".
M.J. Lighthill (Reference 1 page 259): " if the fluid velocity, pressure and density vary only slightly over a distance of one mean free path, then the departure from equilibrium is small and the viscous stress may be supposed to depend linearly on velocity gradient".

After having chosen for an arbitrary motion the dependency of the principal stresses on the space derivatives of the principal velocities, the normal and tangential stresses are now expressed as functions of the space derivatives for any arbitrary reference system. Let the new coordinate system be determined by the following direction cosines with respect to the principal axis:

$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & \ell_1 & m_1 & n_1 \\ y' & \ell_2 & m_2 & n_2 \\ z' & \ell_3 & m_3 & n_3 \end{array} \qquad (11)$$

It could be shown (Reference 7) only on mathematical grounds with no further assumptions that the new stresses and velocities are related to the principal stresses and principal velocities by the above direction cosines according to the following equations:

$$
\left.
\begin{aligned}
\tau_{xx} &= \tau_1 \ell_1^2 + \tau_2 \ell_2^2 + \tau_3 \ell_3^2 \\
\tau_{yy} &= \tau_1 m_1^2 + \tau_2 m_2^2 + \tau_3 m_3^2 \\
\tau_{zz} &= \tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2 \\
\tau_{yz} = \tau_{zy} &= \tau_1 m_1 n_1 + \tau_2 m_2 n_2 + \tau_3 m_3 n_3 \\
\tau_{zx} = \tau_{xz} &= \tau_1 n_1 \ell_1 + \tau_2 n_2 \ell_2 + \tau_3 n_3 \ell_3 \\
\tau_{xy} = \tau_{yx} &= \tau_1 \ell_1 m_1 + \tau_2 \ell_2 m_2 + \tau_3 \ell_3 m_3
\end{aligned}
\right\} \tag{12}
$$

$$
\left.
\begin{aligned}
\frac{\partial u}{\partial x} &= \ell_1^2 \frac{\partial u'}{\partial x'} + \ell_2^2 \frac{\partial v'}{\partial y'} + \ell_3^2 \frac{\partial w'}{\partial z'} \\
\frac{\partial v}{\partial y} &= m_1^2 \frac{\partial u'}{\partial x'} + m_2^2 \frac{\partial v'}{\partial y'} + m_3^2 \frac{\partial w'}{\partial z'} \\
\frac{\partial w}{\partial z} &= n_1^2 \frac{\partial u'}{\partial x'} + n_2^2 \frac{\partial v'}{\partial y'} + n_3^2 \frac{\partial w'}{\partial z'}
\end{aligned}
\right\} \tag{13}
$$

The direction cosines can be eliminated in equations 12 and 13 using the following relationships:

$$
\left.
\begin{aligned}
\ell_i^2 + m_i^2 + n_i^2 &= 1 \\
\ell_1^2 + \ell_2^2 + \ell_3^2 &= 1 \\
m_1^2 + m_2^2 + m_3^2 &= 1 \\
n_1^2 + n_2^2 + n_3^2 &= 1
\end{aligned}
\right\} \tag{14}
$$

After the direction cosines have been eliminated (Reference 7) the following relationships between normal and tangential stresses and space derivatives of the velocities are found:

$$
\begin{aligned}
\tau_{xx} &= -p + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu \frac{\partial u}{\partial x} \\
\tau_{yy} &= -p + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu \frac{\partial v}{\partial y} \\
\tau_{zz} &= -p + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu \frac{\partial w}{\partial z} \\
\tau_{yz} &= \tau_{zy} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\
\tau_{zx} &= \tau_{xz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\
\tau_{xy} &= \tau_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)
\end{aligned}
\tag{15}
$$

Thus, within the validity of the assumption made so far the stresses acting on three elementary surfaces parallel to the reference axes, at any point of a fluid in arbitrary motion, are given by equations 15. Therefore the force acting on any other elementary surface of area A passing by the same point but randomly oriented with respect to the reference axis is given by:

$$
\begin{aligned}
\underset{\sim}{f} = &\left[\tau_{xx} (\underset{\sim}{n}\cdot\underset{\sim}{i}) A + \tau_{yx} (\underset{\sim}{n}\cdot\underset{\sim}{j}) A + \tau_{zx} (\underset{\sim}{n}\cdot\underset{\sim}{k}) A\right] i + \\
&\left[\tau_{xy} (\underset{\sim}{n}\cdot\underset{\sim}{i}) A + \tau_{yy} (\underset{\sim}{n}\cdot\underset{\sim}{j}) A + \tau_{zy} (\underset{\sim}{n}\cdot\underset{\sim}{k}) A\right] j + \\
&\left[\tau_{xz} (\underset{\sim}{n}\cdot\underset{\sim}{i}) A + \tau_{yz} (\underset{\sim}{n}\cdot\underset{\sim}{j}) A + \tau_{zz} (\underset{\sim}{n}\cdot\underset{\sim}{k}) A\right] k
\end{aligned}
\tag{16}
$$

In the above equation $\underset{\sim}{n}$ is the unit vector normal to A so that $[(\underset{\sim}{n}\cdot\underset{\sim}{y}) A]$, for example, is the projection of the area A on the x = 0 plane and the product $[\tau_{xx} (\underset{\sim}{n}\cdot\underset{\sim}{i}) A]$ is the force in the x - direction due to stress τ_{xx}.

The components of the stress vector ($\underset{\sim}{\tau} = \underset{\sim}{f}/A$) on the area A are then given by:

$$
\left.
\begin{aligned}
\underset{\sim}{\tau}\cdot\underset{\sim}{i} &= \underset{\sim}{f}\cdot\underset{\sim}{i}/A = (\tau_{xx}\underset{\sim}{i} + \tau_{yx}\underset{\sim}{j} + \tau_{zx}\underset{\sim}{k})\cdot\underset{\sim}{n} \\
\underset{\sim}{\tau}\cdot\underset{\sim}{j} &= \underset{\sim}{f}\cdot\underset{\sim}{j}/A = (\tau_{xy}\underset{\sim}{i} + \tau_{yy}\underset{\sim}{j} + \tau_{zy}\underset{\sim}{k})\cdot\underset{\sim}{n} \\
\underset{\sim}{\tau}\cdot\underset{\sim}{k} &= \underset{\sim}{f}\cdot\underset{\sim}{k}/A = (\tau_{xz}\underset{\sim}{i} + \tau_{yz}\underset{\sim}{j} + \tau_{zz}\underset{\sim}{k})\cdot\underset{\sim}{n}
\end{aligned}
\right\}
\tag{17}
$$

Before deriving the familiar form of the momentum conservation equation by direct use of equations 17, we want to discuss further the viscosity terms of equation 15. We first notice that we could have assumed equations 15 directly thus avoiding the

use of the principal axis concept. The terms of equations 15 are indeed the ones measured. In a classical experiment, a fluid is confined between two parallel plates of area A. One of the plates is at rest and the other is moved at constant speed V. The velocity is kept small so that the flow between the plates is laminar. The fluid is at rest on the steady plate and moves with the velocity V on the moving one. Under equilibrium conditions a force F is necessary and sufficient to keep one plate moving at velocity V. This force is found to be proportional to the area of the plates A, to the velocity V and inversely proportional to the distance between the two plates Y:

$$F = \mu AV/Y$$

and in differential form:

$$\tau_{yx} = -\mu \frac{\partial v_x}{\partial y} \tag{18}$$

The above equation is part of the last of equations 15. This equation is known as Newton's equation of viscosity, and fluids which obey this equation are called Newtonian fluids.

Some considerations on the coefficients λ and μ of equations 15 follow. The first source of confusion is their name. Table 1 clarifies this point.

TABLE 1

VISCOSITY COEFFICIENTS

Symbols	Name	References
μ	Coefficient of Viscosity	Stokes (Reference 5); Rayleigh (Reference 5); H. Lamb (Reference 5)
λ	No name : $\lambda = -\frac{2}{3}\mu$	Kuethe, A.M. and Schetzer, J.D. (Reference 7); Rohsenow, W.H. and Choi, H. (Reference 8); Burnett, W.A. and Ackerman, E. (Reference 9)
η	μ First Viscosity Coefficient or Conventional Shear Viscosity Coefficient	F.V. Hunt (Reference 6)
η'	λ Second Viscosity Coefficient or Dilatational Viscosity Coefficient	
η_B	$= \lambda + \frac{2}{3}\mu = \mu' = \mu_v$ Bulk Viscosity	
ν	$= \frac{4}{3} + \frac{\eta_B}{\eta}$ Viscosity Number	
μ	Coefficient of Viscosity	
μ_v	$= \mu' = \eta_B$ Bulk Viscosity	
λ	$= \mu_v - \frac{2}{3}\mu$ Second Viscosity Coefficient (Not recommended)	M.J. Lighthill (Reference 1)
μ	Conventional Shear Viscosity Coefficient	
μ'	$= \eta_B = \mu_v = \lambda + 2\mu/3$ Bulk Viscosity	W.D. Hayes (Reference 2)
μ''	$= \frac{4}{3}\mu + \mu' = \left(\frac{4}{3} + \frac{\mu'}{\mu}\right)\mu = \nu\mu =$ $= 2\mu + \lambda$ Longitudinal Coefficient of Viscosity	
μ	$= \eta$ First Viscosity Coefficient	
λ	$= \eta' = \mu_v - \frac{2}{3}\mu$ Second Viscosity Coefficient	...ions used in this report
μ_v	$= \mu' = \eta_B = \lambda + \frac{2}{3}\mu$ Bulk Viscosity Coefficient	

Essentially the differences and the confusion are about the bulk viscosity coefficient. First the simple "viscosity coefficient" μ was introduced (see equation 18). Then a second viscosity coefficient (λ) was introduced to include all the linear combinations of space derivatives of the velocities (see equation 15) in analogy with the physics of solids. However, this second coefficient (λ) was expressed immediately in terms of the first one ($\lambda = -2\mu/3$) since it was assumed that the pressure is a function only of the density and not of the rate of change of the density. Indeed summing the first three of equations 15 it is found

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = -3p + 3\lambda\, \nabla . \underset{\sim}{V} + 2\mu\, \underset{\sim}{\nabla} . V$$

and using the continuity equation 7

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = -3p - (3\lambda + 2\mu)\frac{1}{\rho}\frac{D\rho}{Dt}$$

This equation states that the average of the normal siresses differs from the static pressure by a factor proportional to the derivatives of the density. Therefore for the pressure to be a function of the density only, the following condition must be satisfied

$$3\lambda + 2\mu = 0 \qquad \text{Hence} \quad \lambda = -2\mu/3 \tag{19}$$

The above can be called the "old approach".
It will presently be shown that following the old approach, the viscosity coefficient to be used in the momentum equation for the 1 - dimensional case is $4\mu/3$ (equation 27). The old approach was satisfactorily as long as semistationary flows were considered, but was found insufficient to describe fast changing flows (shocks, high frequency sound). W.D. Hayes writes about it (Reference 2): "The Stokes relation (equation 19) has not been shown to be correct for all monatomic gases and is completely false for most other gases and liquids (Truesdell Reference 10). Moreover, for the purpose of the theory of shock structure it is unnecessary and unhelpful".

Then the condition 19 was dropped and a new viscosity coefficient was introduced: the bulk viscosity (new approach):

$$\mu_v = \lambda + \frac{2}{3}\mu \tag{20}$$

Evidently whenever the bulk viscosity is set equal to zero we go back to the old approach. After having introduced the bulk viscosity coefficient, an explanation for it was felt necessary, and Lighthill (Reference 1) suggested that the additional viscosity term takes into account at first approximation the effects of rotational relaxation otherwise completely ignored in the continuum theory. Using previous findings of the kinetic theory, Lighthill further shows that for nitrogen $\mu_v \simeq 0.8\mu$ but for relatively dry air and for frequencies in the higher part of the audible range, the bulk viscosity can be of the order of ten to a hundred times μ (Reference 1). We can easily conclude that by the introduction of the bulk viscosity coefficient modern

acoustics has freed itself from any traditional value of the viscosity coefficient, practically claiming that the viscosity should be accounted for by that factor which gives the best results. It will presently be shown that following the new approach, the viscosity coefficient to be used in the momentum equation for the 1 - dimensional case is $4\mu/3 + \mu_v$ or $\mu\,(4/3 + \mu_v/\mu)$ or $\nu\mu$, where ν is called the "viscosity number" and it is a positive multiplier of the old viscosity coefficient (equation 28).

Table 1 further shows how old researchers and authors of modern college textbooks present the simple old approach, while the new researchers all agree on the use of the bulk viscosity coefficient, even if they disagree on its symbol and on its meaning. According to the author, the conclusion about the viscosity term (or more generally diffusion terms) to be used in the continuum theory equations (Navier-Stokes) to study flows in which velocity, pressure and density vary significantly over distances of the order of several free path lengths, is rather simple. For the above conditions the continuum theory equation does not hold accurately, therefore any correction factor which makes the results in better agreement with the experimental ones is acceptable. More specifically, it is suggested to consider the diffusion terms by any or a combination of the following terms:

(a) (constant) . (First space derivative of the velocity component)

(b) (constant) . (Second space derivative of the velocity component)

(c) (constant) . (Square of the first space derivative of the velocity component)

where the constants are to be determined experimentally. The reasons for the above choices follow. A combination of (a) and (b) would mean considering the first two terms of a Taylor expansion for the unknown diffusion function. A combination of (a) and (c) would mean considering the first two terms of a power expansion for the unknown diffusion function. Moreover, the analogy with the physics of solids should be remembered. Equations 15 are exactly equal to the stress-strain relationships of the physics of solids (Hook's Law) if the strains are substituted for the space derivatives of the velocities (instead of strains we have ratio of strains). This analogy has the physical meaning that solids resist change of shape while fluids resist only rate of change of shape. In the physics of solids when the plastic range is considered, the stresses are no longer assumed to be functions of a linear combination of the strains but of either a Taylor or a power series of the strains. Finally, as mentioned in the introduction, the need of a term proportional to the square of the velocity gradient (Reference 3) or to the second space derivatives of the velocity (Reference 4) has already been felt by other investigators (Reference 2).

2.2.3 The Momentum Conservation Equations

In classical acoustics the momentum equations were written in fully expanded forms which allowed the evaluation of each term (Reference 6). In more recent works the momentum equation is written in a more concise form, in which both static pressure

and viscosity terms are incorporated in one single term called the stress tensor (Reference 11). The old approach has the advantage of leading to closed form solutions, after a series of carefully evaluated approximations. The more recent method yields more general information through the use of dimensional analysis, but it doesn't lead to a solution. The more recent approach is more consistent with respect to the viscosity terms: we don't really know their exact form, and in this approach it is not necessary.

The classical derivation proceeds similarly to the derivation of the continuity equation (Reference 7). The momentum conservation principle states that the rate of increase of momentum in a specific direction (x direction, for example) of a mass within a control volume R, enclosed by a control surface S, fixed in a moving medium, is equal to the rate at which momentum is flowing into R (given by: $\rho \underset{\sim}{V} \cdot \underset{\sim}{n} u$), plus the forces acting on the fluid within R both through its mass (mass forces: $\rho \underset{\sim}{g} \cdot \underset{\sim}{i}$) and its surface S (pressure and viscosity forces: $\underset{\sim}{\tau} \cdot \underset{\sim}{i}$). Thus, for the conservation of momentum in the x direction the above concept is expressed by the following equation:

$$\frac{\partial}{\partial t} \iiint_R \rho u \, dR = -\iint_S (\rho \underset{\sim}{V} \cdot \underset{\sim}{n}) u \, ds + \iiint_R \rho \underset{\sim}{g} \cdot \underset{\sim}{i} \, dR + \iint_S \underset{\sim}{\tau} \cdot \underset{\sim}{i} \, ds \tag{21}$$

In the above equation $\underset{\sim}{n}$ is the unit vector normal to S, and $\underset{\sim}{\tau} \cdot \underset{\sim}{i}$ is the component of the stress vector in the x - direction as defined by the first of equations 17. Substituting $\underset{\sim}{\tau} \cdot \underset{\sim}{i}$ by the first of equations 17, applying the divergence theorem to the surface integrals on the right-hand side of equation 21 and recalling that the limits of the space integral are not dependent on time, equation 21 gives:

$$\iiint_R \left[\frac{\partial}{\partial t} \rho u + \nabla \cdot (\rho u \underset{\sim}{V}) - \rho \underset{\sim}{g} \cdot \underset{\sim}{i} - \nabla \cdot (\tau_{xx} i + \tau_{yx} j + \tau_{zx} k) \right] dR = 0$$

The above equation must hold no matter how small is the volume of integration and therefore the integrand must vanish:

$$\frac{\partial}{\partial t} \rho u + \nabla \cdot (\rho u V) - \rho X - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \tau_{zx}}{\partial z} = 0$$

In the above equation, $X = \underset{\sim}{g} \cdot \underset{\sim}{i}$ is the component of the mass force in the x direction. Using equation 8a with $f = u$, the above equation gives:

$$\rho \frac{Du}{Dt} = \rho X + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{22}$$

Substituting for the stresses the expressions given by equations 15 it is found:

$$\rho \frac{Du}{Dt} = \rho X + \left[\lambda (u_x + v_y + w_z)\right]_x + 2\left[\mu u_x\right]_x + \left[\mu (v_x + u_y)\right]_y + \left[\mu (u_z + w_x)\right]_z \quad (23)$$

The above equation is written below in vector notations together with the other two momentum conservation equations in the y -, and z - direction respectively:

$$\left.\begin{aligned} \rho \frac{Du}{Dt} &= \rho X - p_x + (\lambda \nabla . \underset{\sim}{V})_x + \nabla . (\mu \underset{\sim}{V}_x) + \nabla . (\mu \nabla u) \\ \rho \frac{Dv}{Dt} &= \rho Y - p_y + (\lambda \nabla . \underset{\sim}{V})_y + \nabla . (\mu \underset{\sim}{V}_y) + \nabla . (\mu \nabla v) \\ \rho \frac{Dw}{Dt} &= \rho Z - p_z + (\lambda \nabla . \underset{\sim}{V})_z + \nabla . (\mu \underset{\sim}{V}_z) + \nabla . (\mu \nabla w) \end{aligned}\right\} \quad (24)$$

For the one-dimensional case, for which v = w = 0, equations 24 give:

$$\rho \frac{Du}{Dt} = \rho X - p_x + (\lambda u_x)_x + 2(\mu u_x)_x \quad (25)$$

If the viscosity coefficients are assumed to be constant we have:

$$\rho \frac{Du}{Dt} = \rho X - p_x + \lambda u_{xx} + 2\mu u_{xx} \quad (26)$$

If we set $\lambda = -2\mu/3$ according to the "old approach" we have:

$$\rho \frac{Du}{Dt} = \rho X - p_x + \frac{4}{3} \mu u_{xx} \quad (27)$$

If we set $\lambda = \mu_v - 2\mu/3$ according to the "new approach" we have:

$$\rho \frac{Du}{Dt} = \rho X - p_x + \left(\frac{4}{3} + \frac{\mu_v}{\mu}\right) \mu u_{xx} \quad (28)$$

In the more recent approach all the stresses are included in one term: the stress tensor. The reference axes are first called x_1, x_2, x_3, and the velocity components v_1, v_2, v_3 instead of u, v, w. Equations 15 are then combined into the following equation:

$$\tau_{ij} = -p\,\delta_{ij} + \lambda (e_{kk}\,\delta_{ij}) + 2\mu\, e_{ij} \quad (29)$$

where:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \text{rate of strain} \tag{30}$$

$$\epsilon_{kk} = \Delta = \nabla . \underset{\sim}{V} = \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) = \text{dilatation} \tag{31}$$

$$\delta_{ij} = 1 \text{ if } i = j \text{ and } \delta_{ij} = 0 \text{ if } i \neq j = \text{Kronecker delta} \tag{32}$$

Then equation 22 and the other two equations in the y - and z - direction are expressed simply by the following equation:

$$\rho \frac{Du_i}{Dt} = \rho F_i + \frac{\partial \tau_{ij}}{\partial x_j} \tag{33}$$

Equation 33 is exactly equal to equation 22 and similar even if it might look quite different.

If in equation 25, λ is set equal to $\mu_v - 2\mu/3$ according to the "new approach", we get the stress tensor given by Lighthill [Reference 1, equation 11, if τ_{ij} is set equal to his $(-p_{ij})$] :

$$\tau_{ij} = -p\,\delta_{ij} + (\mu_v - \frac{2}{3}\mu)(e_{kk}\,\delta_{ij}) + 2\mu\, e_{ij}$$

Also

$$p_{ij} = -(p - \mu_v \Delta)\,\delta_{ij} - 2\mu(e_{ij} - \frac{1}{3}\Delta\,\delta_{ij}) \tag{34}$$

If the bulk viscosity coefficient (μ_v) is set equal to zero, we get the stress tensor according to the "old approach".

2.3 Energy Conservation Equation

The energy conservation equation can again be written in an explicit form where all the viscosity terms are listed, or in an implicit form where only one viscosity term appears. Consider a control volume R, enclosed by a control surface S, fixed in a moving medium. The rate of change of the energy within the volume must be equal to the energy flowing in, in the unit of time, plus the energy increase in the unit of time due to eventual temperature gradients, plus the work done by the viscous forces in the unit of time. This is expressed by the following equation:

$$\frac{\partial}{\partial t}\iiint_R \rho E\, dR = -\iint_S (\rho \underset{\sim}{V} . \underset{\sim}{n}) E\, ds + \iint_S k \underset{\sim}{n} . \nabla T\, ds + \iint_S \underset{\sim}{\tau} . \underset{\sim}{V}\, ds \tag{35}$$

The term $\underset{\sim}{\tau}.\underset{\sim}{V}$ is equal to the sum of the products of the components of $\underset{\sim}{\tau}$ in the three directions and the relative components of $\underset{\sim}{V}$ in those directions. The components of $\underset{\sim}{\tau}$ in the three directions are given by equations 17 so that the product $\underset{\sim}{\tau}.\underset{\sim}{V}$ can be written as follows:

$$u(\tau_{xx}\underset{\sim}{i} + \tau_{yx}\underset{\sim}{j} + \tau_{zx}\underset{\sim}{k}).\underset{\sim}{n} + v(\tau_{xy}\underset{\sim}{i} + \tau_{yy}\underset{\sim}{j} + \tau_{zy}\underset{\sim}{k}).\underset{\sim}{n} + w(\tau_{xz}\underset{\sim}{i} + \tau_{yz}\underset{\sim}{j} + \tau_{zz}\underset{\sim}{k}).\underset{\sim}{n} \tag{36}$$

and setting

$$\underset{\sim}{f}_1 = \tau_{xx}\underset{\sim}{i} + \tau_{yx}\underset{\sim}{j} + \tau_{zx}\underset{\sim}{k}$$

$$\underset{\sim}{f}_2 = \tau_{xy}\underset{\sim}{i} + \tau_{yy}\underset{\sim}{j} + \tau_{zy}\underset{\sim}{k}$$

$$\underset{\sim}{f}_3 = \tau_{xz}\underset{\sim}{i} + \tau_{yz}\underset{\sim}{j} + \tau_{zz}\underset{\sim}{k}$$

equation 36 gives:

$$u\underset{\sim}{f}_1 \cdot \underset{\sim}{n} + v\underset{\sim}{f}_2 \cdot \underset{\sim}{n} + w\underset{\sim}{f}_3 \cdot \underset{\sim}{n} = (u\underset{\sim}{f}_1 + \underset{\sim}{v} f_2 + w\underset{\sim}{f}_3) \cdot \underset{\sim}{n}$$

Similarly to the derivation of momentum and mass conservation, we can substitute equation 36 into equation 35, apply the divergence theorem to the surface integral, take the time derivation under the sign of integration and impose the condition that the integrand must vanish and we find:

$$\frac{\partial}{\partial t}\rho E + \nabla.(\rho E\underset{\sim}{V}) - \nabla.(k\nabla T) - \nabla.\left[u\underset{\sim}{f}_1 + v\underset{\sim}{f}_2 + w\underset{\sim}{f}_3\right] = 0 \tag{37}$$

The following relationships can be substituted into equation 37. The first is from the definition of energy, the second is from equation 8a with f = E, the third, fourth, and fifth are from the momentum equation 22 and similar, and the last one is from the definitions of velocity and displacement:

$$E = e + \frac{V^2}{2} - \underset{\sim}{g}.\underset{\sim}{s} \quad \text{where } \underset{\sim}{s} \text{ is the dispacement vector}$$

$$\frac{\partial}{\partial t}\rho E + \nabla.(\rho E\underset{\sim}{V}) = \rho\frac{DE}{Dt}$$

$$\left.\begin{aligned}
u\nabla.\underset{\sim}{f}_1 &= u\left(\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial x}\right) = \rho u\frac{Du}{Dt} - u\rho X \\
v\nabla.\underset{\sim}{f}_2 &= v\left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z}\right) = \rho v\frac{Dv}{Dt} - v\rho Y \\
w\nabla.\underset{\sim}{f}_3 &= w\left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z}\right) = \rho w\frac{Dw}{Dt} - w\rho Z
\end{aligned}\right\}\rho\frac{D}{Dt}\left(\frac{V^2}{2}\right)\rho\underset{\sim}{V}.\underset{\sim}{g}$$

$$\frac{D}{Dt}(\underset{\sim}{g}\cdot\underset{\sim}{s}) = \underset{\sim}{g}\cdot\frac{D}{Dt}\underset{\sim}{s} = \underset{\sim}{V}\cdot\underset{\sim}{g}$$

making the substitution we find:

$$\rho\frac{D}{Dt}\left[e + \frac{\underset{\sim}{V}^2}{2} - \underset{\sim}{g}\cdot\underset{\sim}{s}\right] - \nabla\cdot(k\nabla T) - \left\{\rho\frac{D}{Dt}\left(\frac{\underset{\sim}{V}^2}{2}\right) - \rho\underset{\sim}{V}\cdot\underset{\sim}{g} + (\underset{\sim}{f}_1\cdot\nabla u + \underset{\sim}{f}_2\cdot\nabla v + \underset{\sim}{f}_3\cdot\nabla w)\right\} = 0$$

and simplifying we find:

$$\rho\frac{De}{Dt} - \nabla\cdot(k\nabla T) - \left[\underset{\sim}{f}_1\cdot\nabla u + \underset{\sim}{f}_2\cdot\nabla v + \underset{\sim}{f}_3\cdot\nabla w\right] = 0 \qquad (38)$$

Substituting $\underset{\sim}{f}_1$, $\underset{\sim}{f}_2$, and $\underset{\sim}{f}_3$ with their expressions in terms of stresses and substituting the stresses with their expressions in terms of static pressure, velocity derivatives and viscosity coefficients (equations 15), equation 38 gives:

$$\rho\frac{De}{Dt} = \nabla\cdot(k\nabla T) - p\nabla\cdot\underset{\sim}{V} + \lambda(\nabla\cdot\underset{\sim}{V})^2 + 2\mu\left[u_x^{\,2} + v_y^{\,2} + w_z^{\,2}\right] + \mu\left[(v_x + u_y)^2 + (u_z + w_x)^2 + (w_y + v_z)^2\right] \qquad (39)$$

again λ is equal to $\mu_v - 2\mu/3$ or to $-2\mu/3$ according to whether the bulk viscosity coefficient is used or not. No further general simplification can be introduced in equation 39. Some of the velocity derivatives in the viscous terms can be dropped when specific problems are considered. Generally it can be noticed that the viscosity terms in the energy conservation equation are roughly proportional to the square of the velocity components. In most problems it will therefore be sufficient to retain a few terms directly proportional to the square of those space derivatives of the velocity components which, on physical grounds, are estimated to be the largest ones. For the particular case of one dimensional problems, equation 39 reduces to the following one:

$$\rho(e_t + u e_x) = (k T_x)_x - p u_x + (\lambda + 2\mu) u_x^{\,2} \qquad (40)$$

2.4 State Equation

Only the equation of state of gases will be briefly considered. It is concluded that in acoustics the perfect gas equation of state is satisfactory for most gases (air) but might be insufficient to describe some of them (CO_2).

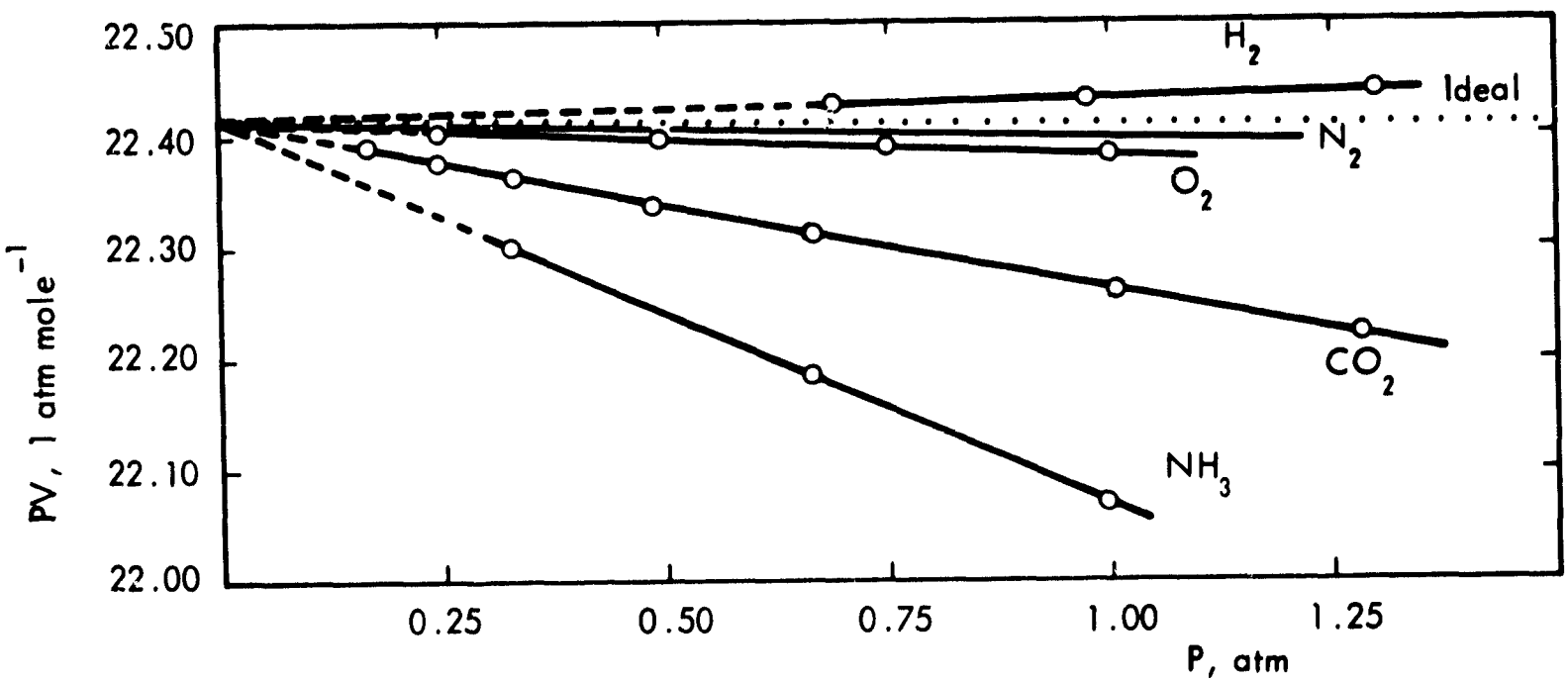
Boyle (1662) first observed for air that pressure and volume are approximately in inverse proportion at constant temperature and mass:

PV = constant (T, M constants)

Regnault (1850) noticed that the product PV is not exactly constant with T but increases linearly as the pressure increases:

$PV = A + BP$ (T constant and referred to one mole of gas) (41)

In equation 41, A and B are empirical functions of the temperature only and they are characteristic of each gas. For some gases the deviation from Boyle's law is appreciable also at low pressure as the following figure shows (Reference 12)



One of these gases is CO_2 for which the following equation holds (Reference 12):

$PV = 22.4133 - 0.4163\,P \quad (0 < P < 4 \text{ At.})$ (ℓ At. mole)

Air on the contrary, follows Boyle's law rather accurately at low pressures. For higher pressures the linear equation suggested by Ragnault is not sufficient and a power

series for PV in either P or $1/V$ with empirically determined constants (the virial equation of state) is often used:

$$PV = A + BP + CP^2 + DP^4 \qquad \text{(T constant and referred to one mole of gas)} \qquad (42)$$

or

$$PV = A' + B'/V + C'/V^2 + D'/V^4 \qquad \text{(T constant and referred to one mole of gas} \qquad (43)$$

It turns out that the above equations fit the data better if the odd powers of P or $1/V$ after the first are omitted (Reference 12).

The linear dependency of PV on the temperature was proved by Gay-Lussac (1800). This constant of proportionality was found to be equal for all gases if mole-weights of each gas are considered:

$$pv = R'T \qquad \text{(Referred to one mole of gas)} \qquad (44)$$

Comparing equation 44 with equation 43 (or 42) the following equation of state, accurate for most practical purposes for all gases, is found:

$$PV = R'T + B'(T)/V + C'(T)/V^2 + D'(T)/V^4 \qquad \text{(Referred to one mole of gas)} \qquad (45)$$

In acoustics only the first term on the right-hand side of equation 45 is actually retained, even if for some gases (CO_2) and some problems (noise from jet exhaust) it is not so evident that the approximation is satisfactory. If a unit weight instead of a mole of a specific gas is considered the universal constant R' is divided by the molecular weight of that gas to give a new constant R characteristic of that gas. In which case the equation of state as used in acoustics is as follows:

$$PV = RT \qquad \text{(Referred to the unit weight of a specific gas)} \qquad (46)$$

2.5 Fluid Properties and Process Assumptions

As explained at the beginning of Section 2.0 it is always necessary to use some experimental property of the matter under consideration, or to make some assumption about the physical nature of the process, to produce one more equation, since the number of equations available from the application of the basic principals would not otherwise be sufficient to determine a gas dynamic problem.

2.5.1 Polytropic Substances

An experimental property of the gases is that their internal energy is a function of their temperature only, and for low temperatures it is simply proportional to it ($e = C_v T$). A substance whose internal energy is proportional to the temperature is called polytropic. This property furnishes the missing relationship to solve any dynamic

problem involving gases. When the polytropic relationship is used together with the perfect gas equation of state, the following well known relationship is found:

$$e = C_v T \quad \frac{C_v p v}{R}$$

But $\gamma = C_p/C_v$ and $R = C_p - C_v$ so that $R/C_v = \gamma - 1$ and substituting:

$$e = \frac{pv}{\gamma - 1} \tag{47}$$

It is very important to notice that equation 47 comes only from an experimental property of the gases and does not imply any assumption about the process undergone by them. Equation 47 however, does not simplify the solution of the gas dynamic equations. For this reason simplifying assumptions are often made. Acoustic processes are often assumed to be adiabatic and reversible, or isentropic. The meaning and the range of validity of these process assumptions will shortly be discussed. But first, some considerations on which aspects of a physical problem are presently under consideration.

It is important to specify whether the microscopic aspects of a process will be considered or only the macroscopic ones. The process assumptions will be determined by this choice. For the example of a standing shock in a pipe, the interest may lie in the trajectory of the particles and in the structure of the shock, or alternatively in the conditions of the flow far downstream of the shock in terms of the far upstream conditions.

In the first case we must consider the generation and exchange of heat between particles (diabatic process) since they are essential elements to the formation of a shock. In the second case we can make the usual assumption that there is no heat exchange through the wall of the pipe and consider the shock thermally insulated (adiabatic process). Thus, in general the relationship between an elementary particle of the flow and its neighboring particles is not necessarily equal to the relationship of the entire flow to its surrounding. Hence, generally, different process assumptions hold for the microscopic and for the macroscopic aspects of the same phenomenon.

We have been using differential equations which apply to the motion of each elementary particle and therefore our process assumptions must be consistent with the microscopic aspects of any phenomenon which will be considered.

Acoustic processes are considered to be either diabatic, or diabatic and reversible, or isentropic. Some confusion exists about the specific implications of those terms.

2.5.2 Diabatic Process

We are considering an elementary mass in the flow which undergoes certain transformations. Heat can be given to this elementary mass through its surface by heat transfer

processes or can be generated around it and added to it by the viscous forces acting again on its surface. Heat can also be generated from the inside of this elementary mass by chemical reactions. Thus in all cases in which heat transfer, viscous forces and chemical reactions are considered, the heat of the particle varies and the macroscopic process is diabatic. No simple equation comes from the knowledge that a process is diabatic. Consider the following thermodynamic definition of entropy applied to an elementary mass within a polytropic perfect gas which receives the heat dq:

$$ds = \frac{dq}{T} = \frac{de}{T} + \frac{p}{T}\,dv = C_v \frac{dT}{T} + R\,\frac{dv}{v} \tag{48}$$

and integrating

$$s - s_o = C_v \ln \frac{T}{T_o} + R \ln \frac{v}{v_o} \rightarrow e^{s-s_o} = \left(\frac{T}{T_o}\right)^{C_v} \left(\frac{v}{v_o}\right)^{R} = \left(\frac{T}{T_o}\right)^{C_v} \left(\frac{\rho_o}{\rho}\right)^{R}$$

and from the last of the above equations:

$$e^{(s-s_o)/C_v} = \frac{T}{T_o}\left(\frac{\rho_o}{\rho}\right)^{R/C_v} = \left(\frac{p\,\rho_o}{p_o\,\rho}\right)\left(\frac{\rho_o}{\rho}\right)^{\gamma-1} = \frac{p}{p_o}\left(\frac{\rho_o}{\rho}\right)^{\gamma} \tag{49}$$

The diabatic process equation 49 introduces a new variable (s = entropy) which is, however, defined in terms of the other variables so far used by equation 48. In the case of diabatic processes, nevertheless, the process equation 49 is seldom used, instead the equation derived directly from the polytropic property of gases ($e = pv/(\gamma - 1)$) is preferred.

2.5.3 Isentropic Process

The assumption that a process is isentropic leads to a very commonly used equation ($p/p_o = (\rho/\rho_o)^{\gamma}$) when applied to a perfect gas, but it is also very restrictive and applies only to a well defined class of acoustic phenomena. Isentropic means that the entropy has a constant value (s_o) for all elementary masses, everywhere in the flow and at all times. This can occur only if there is no heat exchange whatsoever ($dq = 0$ in equation 48). Therefore no chemical reactions or heat transfer of viscous effects are assumed to exist to any significant extent. In this case the left-hand side of equation 49 is identically equal to one and the right-hand side gives:

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} \tag{50}$$

Equation 50 applies to all linear acoustics problems such as the propagation of weak sound in non-reactive media. It also applies to some non-linear acoustics problems when the non-linearity is due only to transport phenomena such as the propagation of weak sound in non-uniform non-reactive media. It does not apply to any high intensity sound problems where viscous and thermal processes play determining roles.

2.5.4 Diabatic and Reversible Process

The diabatic and reversible process is between the diabatic and the isentropic one. Consider again an elementary mass within a flow. Suppose that this mass exchanges heat with the neighboring masses through heat transfer, or chemical reaction phenomena but that these exchanges are so small and slow that they can be considered reversible. Moreover, suppose that the flow as a whole does not exchange heat with the surroundings. In this case the entropy of the flow as a whole remains constant but the entropy of each elementary mass within the flow may change as the mass moves from one place to another and as the time changes. As explained in Section 2.1, the total derivative of the scalar property s (entropy) will be equal to zero:

$$\frac{Ds}{Dt} = 0 \tag{51}$$

Equation 51 is sufficient to complete the set of thermodynamic equations since the entropy can again be expressed in terms of the other flow variables using equation 48. Equation 51 is widely used (Reference 13 and 14) to study flows without viscous forces but involving chemical reactions and internal heat transfer phenomena. Viscous forces generate basically unreversible phenomena while chemical reactions and heat transfer can, in principle, be reversible. Equation 51 is often called "the adiabatic process equation" (Reference 13). Adiabatic in this case refers to the flow as a whole only. For each particle the flow is not adiabatic; if it were, the entropy would be everywhere constant ($dq = 0$ in equation 48) and equation 51 would be identically satisfied ($s(x, y, z, t) = s_o$).

In Table 2, the considerations of this Section are summarized and the conclusion is easily reached that for the great majority of acoustic problems in any gas in near-to-ordinary conditions, equation 47 $\left[e = pv/(\gamma - 1)\right]$ can be used. On the other hand, equation 50 $\left[p = p_o/(\rho/\rho_o)^{\gamma}\right]$ should not be used in all cases in which heat transfer chemical reactions and viscous effects are taken into consideration. However, when only approximate equations are of interest, the first approximation is always that of using $p = p_o/(\rho/\rho_o)^{\gamma}$ even when heat transfer, chemical reactions and viscous effects are known to be present (see Section 3.1). This use is justified by the fact that the error introduced by the use of equation 50, instead of equation 47, is of higher order with respect to other assumptions which must still be made to reduce the difficulty of the equations (see Section 3).

TABLE 2

STATE EQUATIONS FOR THE PROPAGATION OF SOUND IN PERFECT, POLYTROPIC GASES

(Most Gases Under Near-To-Ordinary Conditions)

Restrictions and Example of Practical Acoustic Problems	Equations	Process Undergone by the Elementary Mass	Process Undergone by the Flow as a Whole
Small amplitude sound in non-reactive, thermally non-conductive, non-viscous, uniform or non-uniform gases (weak sound in non-uniform atmosphere).	$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma}$	Isentropic $\left[dq = 0\right]$	Isentropic $\left[\iiint_V dq = 0\right]$
Sound in reactive, thermally conductive, non-viscous, reversible, uniform or non-uniform gases (weak sound in reactive medic)	$\frac{Ds}{Dt} = 0$ $\left[\text{or } e = \frac{pv}{\gamma-1}\right]$	Diabatic and Reversible $\left[dq \neq 0 \text{ and Reversible}\right]$	Adiabatic $\left[\iiint_V dq = 0\right]$
Sound in reactive, thermally conductive, viscous, irreversible, uniform or non-uniform gases (high intensity sound).	$\frac{p}{p_o} = e^{\frac{S-S_o}{C_v}} \frac{\rho^{\gamma}}{\rho_o}$ $\left[\text{or } e = \frac{pv}{\gamma-1}\right]$	Diabatic $\left[dq \neq 0 \text{ and Irreversible}\right]$	$\left[\iiint_V dq \neq 0\right]$
Sound in any perfect polytropic gas	$e = \frac{pv}{\gamma-1}$	Independent of the nature of the process	Independent of the nature of the process

3.0 APPROXIMATE NON-LINEAR ACOUSTICS EQUATIONS AND THEIR SOLUTIONS

In Section 2.0, the five necessary equations (mass, momentum, energy conservation equations, state equation, and fluid property equation or process assumption equation) were derived and discussed. The primary goals of the exercise were: to point out the intrinsic limitations of those equations; to clarify the origin and the meaning of key terms (like the viscosity coefficients); to give sufficient background for the consistent choice of equations for specific problems. The final results were expressed by equation 8 (mass conservation), 24 (momentum conservation), 39 (energy conservation), 46 (state equation), 47, 50, or 51 (fluid property equation or process assumptions). The first obvious and well known conclusion is that the above equation cannot be solved in their original form. The second not-so-obvious and well known conclusion is that such a solution would be an inconsistency since the viscosity terms are much more complicated than the simple assumptions, on which they were derived, justifies. In this section therefore, the above equations will be simplified, solved and the practical limits of application of the solutions indicated. First we will see the most common simplification for constant cross sectional area propagations, i.e., Burgers' equation solved by Hopf (Reference 15), Cole (Reference 16) and more recently by Rodin (Reference 17). Then we will derive similar equations for the case in which the cross sectional area varies with the distance (spherical, cylindrical, exponential horn waves) and we will study the solution of one particular case. Finally, we will summarize our considerations on the diffusion coefficient (δ) which is the most important parameter in non-linear acoustics.

3.1 Burgers' Equation For Constant Cross Sectional Area

A consistent set of equations for plane wave (constant cross sectional area) of strong perturbances in gases is given by:

$$\rho_t + u\rho_x = -\rho u_x \tag{8}$$

$$u_t + uu_x = -p_x/\rho + \left(\frac{4}{3}\mu + \mu_v\right) u_{xx}/\rho \tag{28}$$

$$e_t + ue_x = -pu_x/\rho + \left(\frac{4}{3}\mu + \mu_v\right) u_x^2/\rho + (kT_x)_x/\rho \tag{40}$$

$$p = RT/\rho \tag{46}$$

$$e = C_vT = p/\rho(\gamma-1) \tag{47}$$

The only limitations of the above equations are: the uncertainty about the values of μ_v for some gas mixtures and/or frequency ranges due primarily to relaxation phenomena (air-water-vapor mixture, for example); the validity of equation of state (46) for some peculiar gases under relatively high pressures (for CO_2 at 10 At. the ratio p/ρ is off ~ 20% from that calculated from RT. This could be significant for the study of sound

oscillations in the combustion chamber of a rocket engine); the cross sectional area must be constant (this limits their practical usefulness considerably); the distances under consideration should be much longer than the mean free path (thus these equations do not describe the shock front profile).

In order to derive Burger's equations the following limitations are further introduced. The first limitation is introduced by setting

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} = \left(\frac{a}{a_o}\right)^{\frac{2\gamma}{\gamma-1}} \tag{50}$$

With the above simplification one ceases to be consistent unless we also set $\mu = \mu_r = k = 0$. For practical purposes however, equation 50 gives valid results for shocks with Mach numbers up to 1.5 to 2.0 and therefore can be accepted also in acoustics.

The second limitation is introduced by neglecting to consider explicitly the effect of heat transfer. The temperature and internal energy variables are not considered and equations 40 and 47 will not be used. Some account of the heat transfer will be taken by adding one more term to the viscous terms. This term is generally $k\,(\gamma-1)/C_p$ (Reference 19), thus the following equations are now left:

$$\rho_t + u\rho_x + \rho u_x = 0 \tag{52}$$

$$u_t + uu_x + p_x/\rho = \delta\, u_{xx} \tag{53}$$

$$\rho = \rho_o\,(a/a_o)^{\frac{2}{\gamma-1}}; \qquad p = p_o\,(a/a_o)^{\frac{2\gamma}{\gamma-1}} \tag{54}$$

where

$$\delta = \frac{1}{\rho_o}\left[\frac{4}{3}\mu + \mu_v + (\gamma-1)\frac{k}{C_p}\right] \tag{55}$$

Equation 55 defines the diffusion coefficient (δ) and one more approximation was made since ρ_o was used to define δ instead of ρ. The diffusivity coefficient will be considered again in Section 3.3, for the present time, δ is assumed to be constant.

Equations 54 establish a direct correspondence between ρ, p and the local speed of sound a. This allows the following substitutions:

$$\left.\begin{aligned}
\rho_t &= \rho_a a_t = \frac{2}{\gamma-1}\,\frac{\rho_o}{a_o}\left(\frac{a}{a_o}\right)^{\frac{2}{\gamma-1}-1} a_t \\
\rho_x &= \rho_a a_x = \frac{2}{\gamma-1}\,\frac{\rho_o}{a_o}\left(\frac{a}{a_o}\right)^{\frac{2}{\gamma-1}-1} a_x \\
p_x &= p_a a_x = \frac{2\gamma}{\gamma-1}\,\frac{p_o}{a_o}\left(\frac{a}{a_o}\right)^{\frac{2\gamma}{\gamma-1}-1} a_x
\end{aligned}\right\} \tag{56}$$

When equations 56 and 54 are substituted into equations 52 and 53, it is found:

$$a_t + u a_x + \frac{\gamma-1}{2} a u_x = 0 \tag{57}$$

$$u_t + u u_x + \frac{2}{\gamma-1} a a_x = \delta u_{xx} \tag{58}$$

Burnett and Ackerman (Reference 18) linearized equations 57 and 58 and found the first and second order linear solutions. Their work, however, will be considered in Section 3.3. With change of variables and one more approximation, equations 57 and 58 are now transformed into Burgers' equation. Set:

$$r = \frac{a}{\gamma-1} + \frac{u}{2} \qquad s = \frac{a}{\gamma-1} - \frac{u}{2} \tag{59}$$

from which:

$$a = \frac{\gamma-1}{2}(r+s)\,; \qquad u = r - s\,; \qquad a + u = \frac{\gamma+1}{2} r + \frac{\gamma-3}{2} s \tag{60}$$

substituting equations 60 into equations 57 and 58 it is found:

$$\begin{aligned}
r_t + (a+u)\, r_x &= \frac{1}{2}\delta\,(r_{xx} - s_{xx}) \\
s_t - (a-u)\, s_x &= \frac{1}{2}\delta\,(s_{xx} - r_{xx})
\end{aligned} \tag{61}$$

The first of equations 61 represent a wave which moves in the positive x direction while the second represent a curve moving in the opposite direction and if $\delta = 0$ these curves will grow steeper and steeper due to the convective terms (non-linear terms in the left hand side of equations 61). If $\delta \neq 0$, the two waves are coupled and the steeping process checked by the diffusion terms in the right hand side of equations 61. Lighthill (Reference 1) suggested uncoupling the two equations by setting $s = s_o$ and

studying only the wave moving in the positive direction:

$$r_t + (a + u)\, r_x = \frac{1}{2}\, \delta\, r_{xx} \tag{62}$$

changing again the variables by setting:

$$v = a + u - a_o \simeq \frac{\gamma+1}{2} r + \frac{\gamma-3}{2} s_o - a_o \rightarrow r = \frac{2v}{\gamma+1} - \frac{\gamma-3}{\gamma+1} s_o + \frac{2a_o}{\gamma+1} \tag{63}$$

$$X = x - a_o t \tag{64}$$

it is found:

$$v_t + v v_X = \frac{\delta}{2}\, v_{XX} \tag{65}$$

Equation 65 is Burgers' equation. Equation 63 indicates that v is the excess wavelet velocity and equation 64 means that the X coordinate is measured in a reference system moving at the undisturbed speed of sound, a_o, in the same direction of the perturbation.

To solve equation 45, set:

$$v = \phi_X \tag{66}$$

where ϕ is a function of θ and $\theta(X; t)$ is a solution of the heat equation:

$$\phi = F(\theta) \tag{67}$$

$$\theta_t = \frac{\delta}{2}\, \theta_{XX} \tag{68}$$

The function $F(\theta)$ must now be determined. Substitute equation 67 into equation 65 and integrate once with respect to X:

$$\Phi_t + \frac{1}{2} (\Phi_X)^2 = \frac{\delta}{2}\, \Phi_{XX} \tag{69}$$

Substituting equation 67 into equation 69:

$$F_\theta \theta_t + \frac{1}{2} (F_\theta\, \theta_X)^2 = \frac{\delta}{2} (F_{\theta\theta}\, \theta_X^2 + F_\theta\, \theta_{XX}) \tag{70}$$

Recall condition 68 and divide through by θ_X^2:

$$F_\theta^2 = \delta \, F_{\theta\theta} \tag{71}$$

The solution of 71 is:

$$F = -\delta \ln\left|\theta + c_1\right| + c_2 \tag{72}$$

Using equation 66, 67 and 72 the excess wavelet velocity is then given by:

$$v(X, t) = \Phi_X - F_\theta \, \theta_X = -\delta \frac{\theta_X}{\theta} \tag{73}$$

In equation 73, θ is a solution of equation 68 selected with the use of the initial and boundary conditions on $v(X, t)$. For the case in which $v(X, t)$ is given at $t = 0$ and it is desired at any later time, Lighthill (Reference 1) suggests La Place solution of equation 68:

$$\theta(X, t) = \frac{1}{(2\pi\delta t)^{1/2}} \int_{-\infty}^{+\infty} \theta(\xi, 0) \exp -\left[\frac{(X-\xi)^2}{2\delta t}\right] d\xi \tag{74}$$

which gives for $v(X, t)$ using equation 73:

$$v(X, t) = \frac{\displaystyle\int_{-\infty}^{+\infty} \frac{X-\xi}{t} \exp \frac{1}{\delta}\left[\int_{\xi}^{+\infty} v(\xi, 0)\, d\xi - \frac{(X-\xi)^2}{2t}\right] d\xi}{\displaystyle\int_{-\infty}^{+\infty} \exp \frac{1}{\delta}\left[\int_{\xi}^{+\infty} v(\xi, 0)\, d\xi - \frac{(X-\xi)^2}{2t}\right] d\xi} \tag{75}$$

Similarly, for the perturbance generated by a piston moving at velocity v_1 in a constant cross section tube and suddenly started, the initial conditions on $v(X, t)$ are (Reference 1):

$$\begin{aligned} v(X, 0) &= \frac{1}{2}(\gamma + 1) v_1 = v_o \quad \text{for } X < 0 \\ v(X, 0) &= 0.0 \qquad\qquad \text{for } X > 0 \end{aligned} \tag{76}$$

and v at any later time is given by (Reference 1):

$$v(X,t) = \frac{\frac{1}{2}(\gamma+1)v_o}{+\exp\left[\frac{v_o\left(X-\frac{v_o t}{2}\right)}{\delta}\right]\frac{\int_{-X}^{\infty}\exp\left[-\xi^2/2\delta t\right]d\xi}{\int_{X-v_o t}^{\infty}\exp\left[-\xi^2/2\delta t\right]d\xi}} \tag{77}$$

As far as the validity of Burgers' equation, according to Lighthill (Reference 1) all the approximations which have been made to derive Burgers' equation are consistent and equivalent to neglecting terms of the order of $(\nu\omega/a^2)(U/a)$ with respect to terms of the order of $(\nu\omega/a^2)$ or (U/a), ν being μ/ρ, and U being the velocity amplitude. In turn these last two terms are calculated to be of comparable order for sound frequencies and intensities in the audible range. Lighthill (Reference 1) gives the following Table:

Frequency Hz.	$\frac{\nu\omega}{a^2}$	Threshold of Hearing U/a	Threshold of Pain U/a
30	2.10^{-8}	4.10^{-7}	3.10^{-4}
100	8.10^{-8}	10^{-8}	3.10^{-4}
300	2.10^{-7}	10^{-9}	3.10^{-4}
1000	8.10^{-7}	2.10^{-10}	3.10^{-4}
3000	2.10^{-6}	10^{-10}	2.10^{-4}
10000	8.10^{-6}	5.10^{-10}	3.10^{-4}

As far as the validity of the above solutions of Burgers' equation, Rodin (Reference 17) has recently objected that the initial and boundary value problems have not been treated with mathematical rigor. Apparently (Reference 16) for the solution of equation 75 only the initial value function, $v(X;0)$, and two initial conditions, $v(X_1,t)=0$, $v(X_2,t)=0$ are sufficient. The solution of equation 68, however, generally requires two initial value functions, $v(X;0)$ and $v_X(X,0)$ for example, and two initial conditions. According to Rodin, the necessary and sufficient initial value functions to solve Burgers' equation are $v(0,t)$ and $v_X(0,t)$. It is likely that Rodin's initial values are sufficient, but it is unlikely that they are necessary. In this case both the solutions presented in this report and that of Rodin would be valid for different physical problems. For the propagation of plane waves generated by a piston moving in the x-direction according to $x = g(t)$, Rodin suggests as initial value functions:

$$v(0,t) = 0.0$$

$$v_X(0,t) = \dot{g}(t)/g(t)$$

For the propagation of plane waves generated by a piston oscillating about $x = 0$ the initial value functions for Rodin's approach might possibly be:

$$v(0,t) = V \sin(\omega t)$$
$$v_X(0,t) = 0.0 \tag{78}$$

Equations 78 are equivalent to substituting for the oscillating piston an ideal forcing function which keeps the particle velocity in oscillating motion at $X = 0$ and to assuming that the wave length of the oscillation is long enough so that the particle velocity does not change appreciably within a small distance about the $X = 0$.

In conclusion the solution of Burgers' equation for the deformation of a wave of initially known form is given by equation 75, for the propagation of a wave preceding a moving piston is given by equation 77 and for wave generated by an oscillating infinitesimal piston might be achieved through the approach indicated by Rodin (Reference 17) as previously explained. All these solutions are subject to the following limitations:

1. exclusively plane waves
2. limited frequency band so that δ can be assumed to be constant with respect to the frequencies within the band.
3. the value of δ can be estimated to different accuracies for different media but its final value needs experimental verification for each medium and frequency range by measurements of the absorption coefficient (see Section 3.3) or by comparison of calculated and measured distorted waves. This approach to the choice of δ is believed to take care of the following assumptions made in deriving Burgers' equation for which corrections would otherwise be very difficult to estimate:

   a) independency of δ from the frequency

   b) perfect, isentropic gas

   c) approximate account for the heat transfer

   d) assumption that $s = s_o$

Of the above limitations the first one is by far the most restrictive. Solutions for plane waves cannot be extrapolated to non-plane waves due to the non-linearity of the process which establishes coupling between the effects of the individual variables.

The second and third limitations will be further substantiated by the considerations of Section 3.3 where estimated values for δ will also be given.

3.2 Equation For Varying Cross Sectional Area

Burgers' equation (65) is valid only for plane waves. In this section an equation for a wave traveling in a pipe whose cross sectional area (A) is a function only of its distance from the origin (A (x)) will be derived. Plane, cylindrical, spherical and exponential horn waves are particular cases of the equation. Some considerations on the solution of this equation will also be presented.

For the flow of a perfect gas in a pipe of area A (x) the following equations can be written, instead of equations 52, 53 and 54:

$$A\rho_t + (A\rho u)_x = 0$$

$$A\rho(u_t + uu_x) = -Ap_x + \delta(Au_x)_x \rho \tag{79}$$

$$\rho = \rho_o (a/a_o)^{\frac{2}{\gamma-1}}; \quad p = p_o (a/a_o)^{\frac{2\gamma}{\gamma-1}}$$

$$\text{where} \quad \delta = \frac{1}{\rho_o}\left[\frac{4}{3}\mu + \mu_v + (\gamma-1)\frac{k}{C_p}\right]$$

the above equations can be rewritten as follows:

$$\rho_t + u\rho_x + \rho u_x = -\rho u A_x/A \tag{80}$$

$$u_t + uu_x + p_x/\rho = \delta u_{xx} + \delta u_x A_x/A \tag{81}$$

Equations 80 and 81 are similar to equations 57 and 58 so that with the use of equations 56 they become:

$$a_t + ua_x + \frac{\gamma-1}{2} au_x = -\frac{A_x}{A}\left(\frac{\gamma-1}{2}\right) au \tag{82}$$

$$u_t + uu_x + \frac{2}{\gamma-1} aa_x = \delta u_{xx} + \delta \frac{A_x}{A} u_x \tag{83}$$

defining $f = A_x/A$ the following particular cases are noticed:

$$\begin{aligned} f &= 0 && \text{for plane waves} \\ f &= 1/x && \text{for cylindrical waves} \\ f &= 2/x && \text{for spherical waves} \\ f &= \beta && \text{for exponential horn waves } A = A_o \exp\left[\beta x\right] \end{aligned} \tag{84}$$

In equations 82 and 83, changing the variables from a, u to r, s using equations 59 and then taking s_t from equation 83 and substituting it into equation 82 it is found:

$$r_t + (a + u)\, r_x = \frac{\delta}{2}\left[r_{xx} - s_{xx} + f\,(r_x - s_x)\right] - \frac{f}{2}\left(\frac{\gamma-1}{2}\right)(r^2 - s^2) \tag{85}$$

Again, making the assumption that $s \simeq s_o$ if terms of the order of $(\nu\omega/a^2)(U/a)$ are neglected with respect to either $(\nu\omega/a^2)$ or (U/a), r and s (from equations 59) are then given by:

$$\begin{aligned} s \simeq s_o &= \frac{a_o}{\gamma-1} \longrightarrow s_x = s_{xx} = 0.0 \\ r &= \frac{a}{\gamma-1} + \frac{u}{2} \end{aligned} \tag{86}$$

Equation 85 now becomes:

$$r_t + (a + u)\, r_x = \frac{\delta}{2}\left[r_{xx} + f r_x\right] - \frac{f}{2}\left(\frac{\gamma-1}{2}\right)(r^2 - s_o^{\,2}) \tag{87}$$

Introducing again the excess wavelet velocity defined as in equation 63 it is found:

$$v = a + u - a_o \simeq \frac{\gamma+1}{2}\, r + \frac{\gamma-3}{2}\, s_o - a_o = \frac{\gamma+1}{2}\, r - \frac{\gamma+1}{\gamma-1}\,\frac{a_o}{2} \tag{88}$$

From equation 88, r is defined as follows:

$$r = \frac{2\,v}{\gamma+1} + \frac{a_o}{\gamma-1} \tag{89}$$

Using equation 89, equation 87 becomes:

$$v_t + (a_o + v)\, v_x = \frac{\delta}{2}\left[v_{xx} + f v_x\right] - \frac{f}{2}\left[\frac{\gamma-1}{\gamma+1}\, v^2 + a_o v\right] \tag{90}$$

Before rearranging the terms of equation 90, it is useful to examine in detail their origin and their importance. First we recall that v is the excess particle velocity $(v = u + a - a_o)$ equal to the particle velocity (u) plus the local variation of the speed of sound $(a - a_o)$ originally due to the non-linear terms $u\,\rho_x$, $\rho\, u_x$, $u\, u_x$ and p_x/ρ of equations 80 and 81. Thus v is at most of the order of 2u, i.e. much smaller than a_o. Hence, v^2 will be neglected with respect to $a_o v$ and this is proper and consistent with the other approximations made so far. It is also evident that the terms multiplied by f are due to the change in cross sectional area; f = 0 reduces equation 90 to

equation 62 for the constant cross sectional area if r is introduced using equation 63.

The effect of a changing area comes both through the mass conservation equation 82 and the momentum conservation equation 83 . In equation 90 the term multiplied by δf comes from the momentum while those multiplied by f only come from the mass conservation equation. If we think of v as an approximate sine wave we know that $v_{xx} = -(2\pi)^2 v/\lambda^2$ where λ is the wave length. Thus the effect of an area increase, as introduced through the mass conservation $(-fa_o v/2)$, has the same effect as an increase of diffusivity in a constant area propagation $(\delta v_{xx}/2)$ but while the effect of diffusivity decreases as the wave length (λ) increases, the effect of an area change, as introduced through the mass conservation, depends only on the area change and not on the frequency of the sound $(-fa_o v/2)$. On the other hand the effect of the area change as introduced through the momentum conservation $(\delta f v_x/2)$ is out of phase with respect to the undisturbed wave and contributes directly to its distortion. However, this effect also decays with increasing wave lengths. Thus, if we conclude that:

$$\begin{aligned} \frac{\delta}{2} v_{xx} &= 0\left[\frac{\delta}{2}\left(\frac{2\pi}{\lambda}\right)^2 v\right] \\ \frac{\delta}{2} f v_x &= 0\left[\frac{\delta}{2} f \left(\frac{2\pi}{\lambda}\right) v\right] \\ \frac{f}{2} a_o v &= 0\left[\frac{f}{2} a_o v\right] \end{aligned} \tag{91}$$

then, defining n, the sound frequency in cycles per second, in equation 90 we could neglect $\delta v_{xx}/2$ if:

$$\frac{2\pi}{\lambda} \ll f \longrightarrow n \ll \frac{f a_o}{2\pi} \tag{92}$$

we could neglect $\delta v_{xx}/2$ and $\delta f v_x/2$ if:

$$\frac{\delta 2\pi}{\lambda} \ll a_o \longrightarrow n \ll \frac{a_o^2}{2\pi\delta} \tag{93}$$

while we could neglect $fa_o v/2$ if:

$$\frac{\delta 2\pi}{\lambda} \gg a_o \longrightarrow n \gg \frac{a_o^2}{2\pi\delta}$$

or

$$\delta\left(\frac{2\pi}{\lambda}\right)^2 \gg f a_o \longrightarrow n \gg \frac{a_o^{3/2}}{2\pi}\left(\frac{f}{\delta}\right)^{1/2} \tag{94}$$

Concluding the equation for the propagation of high intensity sound in a pipe of

varying cross sectional area (A (x)) corresponding to Burgers' equation for a constant cross sectional area (A_x 0) is:

$$v_t + (a_o + v - \frac{\delta A_x}{2A}) v_x \quad \frac{\delta}{2} v_{xx} - \frac{A_x}{2A} a_o v \tag{95}$$

where A_x/A for plane, cylindrical, spherical and exponential horn waves are evidently given by equations 84. Equation 95 will be solved in closed form for some particular cases of exponential horn waves. But first, some general remarks are presented.

The ratio A_x/A introduces explicitly the independent variable x in equation 95 in all cases except for the plane and exponential horn. This further complicates the search for closed form solutions of equation 95. This equation, with one single dependent variable, should be integrated very easily on a computer, particularly since its solution for the plane wave case is known and could be used as the first guess for an iterative solution. One possible application of equation 95 is as follows:
In the near field of a stationary rocket exhaust, conical shocklets have been observed. The axes of the cones coincide with the axes of the exhaust gases and the cones expand out while moving downstream. The conical shocklets could be considered as the envelopes of strong spherical waves emitted by a spherical source which moves downstream with the exhaust gases, at supersonic speed with respect to the surrounding air. The strong spherical wave should be calculated exactly by equation 95 and a means to include the effects of motion devised. A better approach perhaps, would be to use the analogy between the hypersonic flow around blunt bodies and cylindrical blast waves. Consider the intersection of the conical shock generated by a blunt body in a uniform hypersonic motion with a stationary plane normal to the flight path. This intersection is a circle which expands out, similar to the cross sectional area of a cylindrical blast. The energy in this case is that dissipated through the drag at the nose of the blunt body (Reference 20). In our case, the conical shocklet could be considered as being generated by a supersonic noise source, and its properties at its intersection with a normal stationary plane calculated by solving equation 95 for a strong cylindrical wave of energy equal to the acoustic energy of the moving change.

Now we want to study the closed form solution of equation 95 for two particular cases of waves propagating in an exponential horn. For this case $f = A_x/A = \beta$ where β is a constant. If β is very small the area of the horn increases little with the distance and at the limit of $\beta = 0$ we have again a constant section and a plane wave.

Consider the case in which the sound frequency,

$$n \gg \frac{a_o^{3/2}}{2\pi} \left(\frac{\beta}{\delta}\right)^{1/2} \tag{94}$$

This case can be realized in practice but we must use high frequency sound in a very slowly diverging exponential horn ($\beta \ll 1$) possibly in a high viscous medium. As previously explained, in this case the effect of the slowly increasing area is described by the term $\delta f v_x/2$ of equation 90 or the equivalent expression $\delta v_x A_x/2A$ of equation 95 becoming $\delta\beta v_x/2$. Thus equation 95 reduces to:

$$v_t + (a_o + v - \frac{\delta\beta}{2}) v_x = \frac{\delta}{2} v_{xx} \tag{96}$$

since β is a constant, set:

$$X = x - (a_o - \frac{\delta\beta}{2}) t \tag{97}$$

and substituting in 96 we get again

$$v_t + v v_x = \frac{\delta}{2} v_{XX} \tag{98}$$

Equation 98 is Burgers' equation again for which the solutions have been discussed in previous sections. However, notice the difference between equations 97 and 64. As previously anticipated, the slowly increasing area over a distance of a wave length introduces a distortion in the wave form. Equation 97 tells us that this distortion is equivalent to that observed from a reference system which is moving with the wave but slowly lagging behind it [we have $(a_o - \delta\beta/2)\, t$ instead of $a_o t$]. Equation 97 together with solutions for Burgers' equation gives a closed form solution for this particular case and could be verified by experiments.

The second case we want to consider is that in which β is arbitrary as is the frequency, but the intensity is not so high that the shape of the distorted wave is close to an N-wave. In this case the previous considerations on the properties of a sine wave lead us to believe that we can set

$$-\frac{\beta a_o}{2} v \simeq \frac{\beta a_o}{2} \left(\frac{\lambda}{2\pi}\right)^2 v_{xx} \tag{99}$$

Then using equation 99, equation 95 gives:

$$v_t + (a_o + v - \frac{\delta\beta}{2}) v_x = \left[\frac{\delta}{2} + \frac{\beta a_o}{2} \left(\frac{\lambda}{2\pi}\right)^2\right] v_{xx} \tag{100}$$

In summary, to describe the propagation and distortion on a sine wave of arbitrary frequency and of intensity up to the threshold of pain in an arbitrary exponential horn, the following equations are believed to be sufficient:

$$
\left.
\begin{aligned}
& v_t + v v_X = \frac{\delta^*}{2} v_{XX} \\
& X = x - \left(a_o - \frac{\delta\beta}{2}\right) t \\
& v = a + u - a_o \\
& \delta^* = \left[\delta + \beta a_o \left(\frac{\lambda}{2\pi}\right)^2\right] \\
& \delta = \frac{1}{\rho_o}\left[\frac{4}{3}\mu + \mu_v + (\gamma - 1)\frac{k}{C_p}\right]
\end{aligned}
\right\} \qquad (101)
$$

In the above equations the distorting effect of a varying area is taken into account exactly by the definition of X while the smoothing effect is taken into approximate account by an increased diffusivity coefficient: δ^* . By exact, we still mean neglecting terms of the order of $(\nu\omega/a^2)$ or (U/a) with respect to terms of the order of $(\nu\omega/a^2)$ or (U/a). The uncertainty about the value of δ, which is further explained in the next section, is believed to justify amply all the assumptions made in this and in the preceding sections.

3.3 Final Thoughts on the Diffusion Coefficient

In this section we will consider the diffusion coefficient as defined by equation 55. But we will acknowledge that the diffusion coefficient is not always assumed to have the form given by equation 55. Burnett and Ackerman (Reference 18) used:

$$
\delta = \frac{1}{\rho_o}\left[\mu + (\gamma - 1)\nu \frac{\rho_o^{\gamma-2}}{C_o^2}\right]
$$

For our purpose the above equation shows emphasis on the heat conduction term of the diffusivity, while the bulk viscosity is left out. The validity of this equation versus equation 55 is difficult to judge due to the apparent continuous inconsistency of the units used by Burnett and Ackerman. The coefficient ν, for example, is defined as a "normalized" coefficient of heat conduction but no units are given for it, and when these units are inferred from different equations, they are found to be different. (From equation 9 of Reference 18, $\nu \equiv k$ but from equation 10 $\nu \equiv k/C_p$ for example). The work of Burnett and Ackerman (Reference 18) is interesting because it is complete, in that they give a final graph of sound pressure levels versus frequency where areas of validity of linear, non-linear, and shock theories are singled out. However, their solution is a linearized one. Non-linear equations, practically the same as equations 52, 53, 54, were linearized and solved up to the second order terms. This approach allowed them to break sounds into their harmonics, to study the propagation of the

individual harmonics and to rebuild the original sound afterwards. We prefer the non-linear solution obtainable from Burgers' equation even if this equation is only an approximate representation of equations 52, 53 and 54. The two solutions however, should not differ considerably if the diffusion coefficients were the same.

Going back to the diffusion coefficient defined by equation 55, defining α (absorption coefficient) as the ratio of the decay of energy flux per unit distance to the energy flux, for a plane sinusoidal wave of frequency ω, it is found:

$$\alpha = \frac{\delta\omega^2}{a_o^3} = \frac{\omega^2}{\rho_o a_o^3}\left[\frac{4}{3}\mu + \mu_v + (\gamma - 1)\frac{k}{C_p}\right] \tag{102}$$

It was explained how the coefficient μ is based on equilibrium conditions on similarities with the physics of solids, and its values on laminar flow experiments (Section 2.2.1 and 2.2.2). Thus, it is no surprise that its values become less and less accurate as the rate of change of the flow properties increases. Thus, for high frequency sound, alterations of the standard values of μ are justified. Some standard values of μ are given in Table 3, taken partially from Reference 7. As far as μ_v is concerned it was proved that it should be equal to zero whenever the static pressure depends on the density but not on the time rate of change of the density (equation 19). This can be expected to occur only for monatomic gases. Good agreement is found between the measured values of α and the values calculated by equation 102 for monatomic gases, if μ_r is set equal to zero. Equation 102 with $\mu_r = 0$ has been known for a long time and is called the Stokes- Kirchhoff formula. Taking liberties, we might say that the accuracy of the Stokes-Kirchhoff formula for monatomic gases justifies the validity of the term $(\gamma - 1)\, k/C_p$ by which the effects of heat transfer are reduced to viscosity effects. Going back to polyatomic gases, we do not expect the static pressure to be independent of the time rate of change of the density. The molecule of a monatomic gas is similar to a very small sphere which can store energy only through translational motion. A diatomic molecule can be considered as two small spheres connected by a spring. This system can store energy not only by translational motion but also through rotation and vibration. Thus, the total internal energy will be the sum of the translational, rotational and vibrational components. Now consider a gas in equilibrium in a cylinder between a piston and a closed end. When the piston is moved, the translational position of a molecule will be changed by its first impact on the piston or by another molecule which has already hit the piston. It is intuitive that the rotational and vibrational properties of that molecule will adjust to the new situation only after more collisions. During this time, called the relaxation time, the effective internal energy would be smaller because only its translational component has been activated. Hence, the pressure change will lag with respect to the density change ($p = (\gamma - 1)\, \rho e$), during this period. If we keep moving the piston, thus changing the density, at a rate greater than the relaxation time, the pressure will continue to lag. Hence we have established a relationship between the static pressure and the rate of change of density and the coefficient μ_v must be different from zero.

TABLE 3. SOME PROPERTIES OF GASES AT ATMOSPHERIC PRESSURE

Gases	T (°F)	ρ (lb_m/ft^3)	C_p (Btu/lb_mF)	k (Btu/hr ft F)	μ (lb_m/hr ft)	ν (ft^2/hr)	α (ft^2/hr)	Pr	γ	μ_v/μ	δ (ft^2/hr)
Dry Air	0	0.0863	0.239	0.013	0.040	0.457	0.633	0.722	1.4	0.8	1.23
	60	0.0763	0.240	0.015	0.043	0.56	0.799	0.712			1.51
	100	0.0709	0.241	0.016	0.046	0.648	0.919	0.706			1.75
	200	0.0601	0.242	0.018	0.052	0.864	1.25	0.694			2.34
	500	0.0413	0.248	0.025	0.068	1.63	2.40	0.680			4.43
	800	0.0315	0.257	0.030	0.081	2.56	3.74	0.685			6.95
	1500	0.0202	0.277	0.041	0.106	5.22	7.37	0.709			14.1
Carbon Dioxide	0	0.13	0.184	0.0076	0.0316	0.241	0.313	0.77	1.28	-	0.410
	100	0.11	0.203	0.010	0.0378	0.352	0.455	0.77			0.597
	500	0.063	0.247	0.012	0.060	0.957	1.27	0.75			1.63
	2000	0.025	0.309	0.050	0.12	4.81	6.55	0.74			8.23
Helium	-200	0.021	1.24	0.054	0.030	1.44	2.05	0.70	1.4	0	2.74
	0	0.012	1.24	0.078	0.044	3.68	5.29	0.70			7.02
	200	0.0083	1.24	0.098	0.056	6.71	9.49	0.71			12.7
	1000	0.0038	1.24	0.16	0.093	24.4	34.0	0.72			46.2
Hydrogen	0	0.0060	3.39	0.094	0.0194	3.20	4.62	0.70	1.4	0	6.11
	200	0.0042	3.44	0.122	0.0249	5.94	8.45	0.69			11.3
	2000	0.0011	3.76	0.307	0.059	51.8	74.2	0.72			98.6
Steam	212	0.0372	0.45	0.0145	0.0314	0.842	0.870	0.96	1.3	-	1.38
	500	0.0258	0.47	0.0228	0.0455	1.76	1.88	0.94			2.91
	1000	0.0169	0.51	0.0388	0.0691	4.07	4.50	0.91			6.77
	2000	0.0100	0.60	0.076	0.109	10.9	12.7	0.86			18.2

Notice $\nu = \mu/\rho$; $\alpha = k\rho/C_p$; $P_2 = \nu/\alpha$; $\delta = \nu\left(\frac{4}{3} + \frac{\mu_v}{\mu}\right) + (\gamma - 1)\alpha$

Its value however will depend on the relationship between the frequency at which the density is changed and the relaxation time. However, the relaxation time will be a characteristic of the molecule, and for complex molecules, a discrete band of relaxation times will be available, since many modes of vibration are possible. Thus μ_v will be a function of the frequency of the sound and of the nature of the medium. Therefore a will not be simply proportional to the square of the frequency, as equation 102, with μ_v equal to zero or to a constant, would suggest. It is clear that in Figure 1 (from Reference 1) for CO_2, $a\lambda = \delta 2\pi\omega/a_o^2$ is not proportional to ω. Table 4 (from Reference 19) shows the same variety of a/ω^2 for humid air, proving that δ is anything but a constant in this case. Theoretically δ is further complicated by classical diffusion and radiation effect (Reference 9).

While the above discussion shows that a simple definition of δ is not possible, it also shows that, for relatively narrow frequency ranges, a value of μ_r, for a given medium, can be measured so that equation 102 becomes meaningful and relatively accurate.

Thus, we conclude this report by suggesting that in applying the solutions of Burgers' equation for plane waves, or the given solutions for exponential horn waves, or in the numerical integration of equation 95 for cylindrical or spherical waves, an open mind must be kept in the choice of the values for the diffusion coefficient δ. Indicative values can be found in Table 3 or estimated by equation 55 but accurate values can only be measured for each frequency range and for each medium. For the experimental determination of δ, equation 102 can be used but must be applied to plane waves.

TABLE 4. ABSORPTION COEFFICIENT OF AIR FOR SOME FREQUENCIES AND HUMIDITIES

Absorption (dB/mile)

Frequency Hz / Humidity %	1	2	5	10	20	50	100	200	500	1000
100	6.7×10^{-5}	1.4×10^{-4}	3.5×10^{-4}	7.2×10^{-4}	1.5×10^{-3}	4.6×10^{-3}	1.2×10^{-2}	3.3×10^{-2}	1.6×10^{-1}	5.6×10^{-1}
70	6.8×10^{-5}	1.4×10^{-4}	3.5×10^{-4}	7.4×10^{-4}	1.6×10^{-3}	5.2×10^{-3}	1.4×10^{-2}	4.2×10^{-2}	2.2×10^{-1}	7.9×10^{-1}
50	6.8×10^{-5}	1.4×10^{-4}	3.6×10^{-4}	7.9×10^{-4}	1.8×10^{-3}	6.3×10^{-3}	1.8×10^{-2}	6.0×10^{-2}	3.2×10^{-1}	1.2×10^{0}
30	7.0×10^{-5}	1.4×10^{-4}	4.0×10^{-4}	9.4×10^{-4}	2.4×10^{-3}	1.0×10^{-2}	3.4×10^{-2}	1.2×10^{-1}	7.1×10^{-1}	2.8×10^{0}
20	7.2×10^{-5}	1.6×10^{-4}	4.7×10^{-4}	1.2×10^{-3}	3.5×10^{-3}	1.7×10^{-2}	6.2×10^{-2}	2.3×10^{-1}	1.4×10^{0}	5.6×10^{0}
10	8.6×10^{-5}	2.1×10^{-4}	8.1×10^{-4}	2.6×10^{-3}	8.9×10^{-3}	5.1×10^{-2}	2.0×10^{-1}	7.7×10^{-1}	4.7×10^{0}	18.3×10^{0}
7	1.0×10^{-4}	2.7×10^{-4}	1.2×10^{-3}	4.2×10^{-3}	1.5×10^{-2}	9.1×10^{-2}	3.6×10^{-1}	1.4×10^{0}	8.5×10^{0}	30.5×10^{0}
5	1.3×10^{-4}	3.8×10^{-4}	1.9×10^{-3}	6.8×10^{-3}	2.6×10^{-2}	1.6×10^{-1}	6.1×10^{-1}	2.4×10^{0}	13.7×10^{0}	41.7×10^{0}

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