FINAL REPORT ON NASA GRANT NGR-44-005-037

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BASIC RESEARCH IN TOPOLOGY

The research accomplished was with two purposes. One facet of the investigation was that of examining proximity spaces and topological spaces (not as general as Hausdorff spaces) with intent of establishing a theory of integration which does not require local compactness.

In the case of topological spaces, it was established that, in a regular, developable topological space it is possible to establish the usual theory of integration provided the space is a hedgehog in which closed sets are G_δ. This is a highly restricted space in which each point is either an interior point of an arc or an end point common to uncountably many arcs. While the theory remains consistent, there is serious question that application of value will be forthcoming since the space itself must be so highly restricted.

Similar results were found concerning proximity spaces. Each proximity space gives rise to a topological space such that the topology agrees with the proximity space. For a proximity space to give rise to a reasonable theory of integration, the topological space identified with the proximity space must be described as above. Thus, the proximity space is seen to be one which is so highly restricted that anticipated value of relaxation of the condition of local compactness did not find fruition.

Another facet of the investigation was that of investigating covering properties with particular attention given to the property of countable paracompactness. Earlier preprints submitted describe in more detail the results obtained.

Considerable attention was given to the question as to whether each normal, regular, developable topological space is a subspace of a complete, regular, developable normal space. Should this result be valid, it would establish that each normal Moore space is a subspace of a normal Moore space which contains a dense metric subspace. This would give additional insight into the problem of determining whether each normal Moore space is metrizable.

However, in the construction of completeness used, regularity was sacrificed to acquire completeness. The primary result then, was that each semi-metric space is a dense subspace of a strongly complete semi-metric space. Should the completion space have been regular, then it would have been established that each regular semi-metric space is a dense subspace of a complete Moore space and thus would be a dense subspace of a regular developable space which contains a dense metric subspace.

In addition, two NASA Trainees, working under the direction of the principal investigator earned their doctorate, partially satisfying the requirements for that degree by taking as dissertation topics theorems which deal with the problems discussed herein. Since the research effort was close between principal investigator and student, at least partial support must be credited to this grant. A list of the results obtained follows:

Suppose that X is a countably paracompact topological space and f is a proper mapping taking the topological space Y onto X. Then Y is countably paracompact.

The topological product of $X \times Y$ of a countably paracompact space X and a bicompact space Y is countably paracompact.

The topological product $X \times Y$ of a countably paracompact space X and a locally bicompact Lindelof space Y is countably paracompact.

The topological product $X \times Y$ of a locally bicompact separable metric space X and a paracompact normal topological space Y is normal.

Suppose that $\{X_i, \pi_j^i\}$ is an inverse system such that

(a) for some i_o, X_{i_o} is countably paracompact

(b) for each i, π_i^{i+1} is proper and onto. Then inv. lim. $\{X_i, \pi_j^i\}$ is countably paracompact.

The topological space X is normal if it is true that if H and K are mutually exclusive closed sets in X, then there is a sequence of domains $\{D_1, D_2, \ldots\}$ such that $H \subset \bigcap_{i \varepsilon J} D_i$ and $K \cap [\bigcup_{i \varepsilon J} cl(D_i)] = \emptyset$.

The topological space X is perfectly normal if and only if it is true that if H is a closed subset of X, then there is a sequence $\{D_1, D_2, \ldots\}$ such that $\bigcap_{i \varepsilon J} cl(D_i) = \bigcap_{i \varepsilon J} (D_i) = H$.

The following conditions are equivalent:

(a) If X is a countably paracompact T_2-space such that every closed subset of X is a G_δ set, then X is normal.

(b) If $\{X_i, \pi_j^i\}$ is an inverse system with bonding maps onto and, for each i, X_i is a countably paracompact T_2-space such that every closed subset of X_i is a G_δ set, then inv. lim. $\{X_i, \pi_j^i\}$ is hereditarily countably paracompact.

(c) If X is a countably paracompact T_2-space such that every closed subset of X is a G_δ set, then X is hereditarily countably paracompact.

Every hereditarily countably paracompact T_2-space S such that each closed subset of S is a G_δ set is normal if and only if it is true that if $\{X_i, \pi_j^i\}$ is an inverse system, such that for each i, X_i is hereditarily countably paracompact T_2-space such that each closed subset of X_i is a G_δ set with bonding maps onto, then inv. lim. $\{X_i, \pi_j^i\}$ is hereditarily countably paracompact.

If $2^{\aleph_0} < 2^{\aleph_1}$, then every separable countably paracompact Moore space is normal if and only if every separable hereditarily countably paracompact Moore space is normal.

The following conditions are equivalent:

(a) If X is a cb-space such that every closed subset of X is a G_δ set, then X is normal.

(b) If X is a cb-space such that every closed subset of X is a G_δ set, then X is hereditarily countably paracompact.

(c) If X is a cb-space such that every closed subset of X is a G_δ set, then every subspace of X is a cb-space.

(d) If $\{X_i, \pi_j^i\}$ is an inverse system such that each X_i is a cb-space such that each closed subset of X_i is a G_δ set, inv. lim. $\{X_i, \pi_j^i\}$ is hereditarily countably paracompact.

(e) If $\{X_i, \pi_j^i\}$ is an inverse system such that each X_i is a cb-space such that each closed subset of X_i is a G_δ set, with bonding maps onto, then every subspace of inv. lim. $\{X_i, \pi_j^i\}$ is a cb-space.

Every paracompact topological space is stongly paranormal.

Every paranormal space is countably paracompact.

Every paranormal T_2-space is a T_3-space.

Every compact paranormal space is bicompact.

Every paranormal, separable Moore space is metrizable.

Suppose that S is a locally separable, paranormal, and developable T_2-space and suppose that f is a closed mapping taking S onto a metric space X such that if $P \varepsilon X$ and if K $f^{-1}(P)$ such that $|K| > \aleph_1$, then K has a limit point. Then S is screenable.

Suppose that S is a locally separable, paranormal, and developable T_2-space and suppose that G is a basis for the topology on S. Then if $|G| \geqq \aleph_1$, S is screenable.

Every locally separable pre-Lindelof Moore space is metrizable.

Suppose that S is a connected, locally peripherally connected (locally connected), strongly paranormal, and developable T_2-space and that $\{G_a | a \varepsilon A\}$ is a monotone collection of domains covering S such that, for each $a \varepsilon A$, $\beta(G_a)$ is separable. Then some countable subcollection of $\{G_a | a \varepsilon A\}$ covers the space.

Suppose that S is a locally peripherally separable, locally peripherally connected (locally connected), connected, strongly paranormal, and developable T_2-space with a basis G such that $|G| \leqq \aleph_1$. Then S is a separable metric space.

Suppose that X is a (strongly) paranormal topological space and f is a proper mapping taking the topological space Y onto X. Then Y is (strongly) paranormal.

The topological product $X \times Y$ of a (strongly) paranormal space X and a bi-compact space Y is (strongly) paranormal.

The topological product $X \times Y$ of a (strongly) paranormal space X and a locally bi-compact and Lindelof space Y is (strongly) paranormal.

The following conditions are equivalent:

(a) If X is a paranormal T_2-space such that every closed subset of X is a G_δ set, then X is normal.

(b) If $\{X_i, \pi_j^i\}$ is an inverse system such that for each $j \varepsilon J$:

(i) X_j is a paranormal T_2-space.

(ii) Every closed subset of X_j is a G_δ set

and (iii) π_j^{j+1} is onto,

then inv. lim. $\{X_i, \pi_j^i\}$ is hereditarily countably paracompact.

(c) If X is a paranormal T_2-space such that every closed subset of X is a G_δ set, then X is hereditarily countably paracompact.

Suppose that H is a discrete collection of compact sets in the normal Moore space S such that S is not collectionwise normal with respect to H. Then there is a normal Moore space S' that is not collectionwise normal with respect to a discrete point set.

Suppose that H is a discrete collection of bicompact subsets of the topological space S. Then S is countably paracompact if and only if S/H is countably paracompact.

Suppose that $H = \{ h_a | a \in A\}$ is a discrete collection of bicompact sets in the T_3-space S and suppose that S/H is normal and collectionwise normal with respect to discrete subsets of S/H. Then S is normal.

Suppose that H is a discrete collection of closed and compact sets in the Moore space S. If S is not normal but S/H is normal, then S/H is a normal Moore space which is not collectionwise normal and S is a countably paracompact space which is not normal.

If S is a topological space, if G is an open cover of S, and H is a Property Q refinement of G, then some subcollection of H is a point-finite cover of S.

If a topological space has Property Q, it is pointwise paracompact, but not conversely.

If S is first countable, if G is an open cover of S, and if H is a Property Q refinement of G, then some subcollection of H is a locally finite cover of S.

A first countable topological space has Property Q if and only if it is paracompact.

Let S denote a first countable T_1 space and let G denote an open cover of S. If H is a strong cover compact refinement of G, H is a weak cover compact refinement of G.

Let S denote a first countable, F_σ-screenable T_1 space and let G denote an open cover of S. If H is a weak cover compact refinement of G, then some subcollection of H is a σ-closure preserving open cover of S.

In a semi-metric T_3 space S, the following statements are equivalent:

(1) S is paracompact.

(2) S is strongly screenable.

(3) S is fully normal.

(4) S is collectionwise normal.

(5) S has Property Q.

(6) S is strong cover compact.

(7) S is weak cover compact.

In a developable T_3 space, the above properties are equivalent to metrizability.

There exists a collectionwise normal, first countable T_3 space which is compact (hence weak and strong cover compact) but does not have Property Q.

There exists a collectionwise normal, first countable T_3 space which is neither weak nor strong cover compact.

In a first countable T_3 space S, the following statements are equivalent:

(1) S is paracompact.

(2) S is strongly screenable.

(3) S is fully normal.

(4) S has Property Q.

(5) S is collectionwise normal and pointwise paracompact.

(6) S is strong cover compact and pointwise paracompact.

(7) S is weak cover compact and pointwise paracompact.

If a compact topological space has Property Q, it is bicompact.

There exists a compact (hence weak and strong cover compact) T_3 space which is not normal.

If a locally compact T_3 space is strong cover compact, it is weak cover compact.

There exists a locally compact T_3 space which is weak cover compact but not strong cover compact.

In a locally compact T_3 space S, the following statements are equivalent and each implies that S has Property Q:

(1) S is paracompact.

(2) S is strongly screenable.

(3) S is fully normal.

(4) S is collectionwise normal and pointwise paracompact.

(5) S is strong cover compact and pointwise paracompact.

(6) S is weak cover compact and pointwise paracompact.

There exists a T_3 space which is pointwise paracompact, strong cover compact, and has Property Q, but which is neither normal nor weak cover compact.

If a first countable (or locally compact) T_4 space is weak cover compact, it is collectionwise normal.

If a first countable (or locally compact) T_4 space is strong cover compact, it is collectionwise normal.

There exists a T_4 space which is weak cover compact, is strong cover compact, and has Property Q, but which is not collectionwise normal.

There exists a T_4 space which is collectionwise normal and strong cover compact, but not weak cover compact.

RESEARCH IN ANALYSIS

1. Suppose BV is the linear family of left-continuous complex functions which are of bounded variation on the real line R and have limit 0 at minus infinity. If f is in BV the norm of f is the total variation (assumed finite) of f on R. Investigation was begun this summer on the following two questions:

a. Identify a class of bounded linear transformations T from BV into BV which have the following integral representation for some "kernel" K: For f in BV and s in R,

$$Tf(s) = \int K(s,x)\, d\ f(x)$$

where the integral is a Riemann-Stieltjes integral over R.

b. Identify a subclass of such linear transformations T for which it is true that if f and g are cumulative probability distribution functions then the sequences (Tf, TTf, TTTf, ...) and (Tg, TTg, TTTg, ...) converge in BV to the same cumulative distribution function.

2. A paper, "On the existence of Stieltjes integrals", was submitted for publication August 6, and is now being refereed. We mention it here because final preparation for publication took place this summer, and because it contains the result of interest for problem (a) above:

"Variants of the following theorem for the Riemann-Stieltjes integral are true for the interior, mean and Cauchy left Stieltjes integrals:

If f is bounded on [a,b] and g is of bounded variation on [a,b], then the Riemann-Stieltjes integral $\int_a^b f\, dg$ exists if and only if for each $\omega > 0$ and $\sigma > 0$ there exists a partition $D = \{[x_{i-1}, x_i]: i = 1, 2, \ldots, n\}$ of $[a,b]$ such that if $C \subset D$ and each interval of C contains numbers x and y with $|f(y) - f(x)| \geq \omega$ then $\sum_C [V(q) - V(p)] < \sigma$, where V is the total variation function for g on $[a,b]$ and the sum is over the intervals $[p,q]$ of C.

Theorems for the mean and interior integrals result on replacing 'contains' by 'has in its interior'. A theorem for the Cauchy left integral results on making the same replacement and also replacing '$V(p)$' by '$V(p+)$'."