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ON THE STABILITY OF A COLUMN SUBJECTED TO A TIME-DEPENDENT AXIAL LOAD*

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1. Introduction

The dynamic stability of a linear elastic column subjected to a time-varying axial load is investigated. The form of stability considered is almost sure asymptotic stability in the case that the load is assumed to be stochastic; if the load is assumed to be deterministic the stability is asymptotic stability in the sense of Liapunov.

Studies similar to the one presented here have been previously published by Caughey and Gray[1], Ariaratnam[2], and Lepore and Shah[3]. In these works stability conditions were obtained by the modal approach; Caughey and Gray obtained conditions for almost sure asymptotic stability, Lepore and Shah for asymptotic stability in the mean.

In this note we give conditions for the almost sure asymptotic stability of the column, since it seems that this is the more natural mathematical concept for the physical problem under consideration. Simultaneously with conditions for asymptotic stability we obtain measures of exponential decay of the solutions which are of interest in themselves. We first obtain these results by the modal approach, but then show that this procedure, which is somewhat questionable mathematically, is unnecessary and that the same estimates and conditions for almost sure asymptotic stability can be obtained in a simple and direct manner from the original partial differential equation. The results obtained are then compared to previously published results.

The approach of this paper represents an application and generalizations of the results of Infante[4] on linear differential equations.

2. Basic Equations

Consider the displacement $w(x,t)$ of a linear pinned column subjected to a time-varying axial load $p(t) = p_T + f(t)$ whose "average" over the time interval $[0,T]$ is p_T. Let β be the damping coefficient. In nondimensional form the equation for this displacement over the time interval of interest is given by

$$\frac{\partial^4 w}{\partial x^4} + [p_T + f(t)]\frac{\partial^2 w}{\partial x^2} + 2\beta\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial t^2} = 0, \quad 0 < x < 1,\ 0 < t \leq T \tag{1}$$

and by the boundary conditions

$$w(0,t) = \frac{\partial^2 w(0,t)}{\partial x^2} = w(1,t) = \frac{\partial^2 w(1,t)}{\partial x^2} = 0, \quad t \geq 0. \tag{2}$$

The nondimensional time-varying load $p(t)$ is assumed to be strictly stationary and ergodic in the case that it is assumed to be a stochastic process. The average of the load $p(t)$ over the desired time interval $[0,T]$ is defined in such a manner that

$$e_T\{f(t)\} = \frac{1}{T}\int_0^T f(t)dt = 0 \tag{3}$$

and it is assumed that $e_T\{f^2(t)\}$ exists and is positive. Defining

$E\{f(t)\} = \lim_{T\to\infty} e_T\{f(t)\}$, $E\{f^2(t)\} = \lim_{T\to\infty} e_T\{f^2(t)\}$ and $p = \lim_{T\to\infty} p_T$, we obtain the standard expectations over the infinite time interval. Our choice of measure for the function $f(t)$ is obviously motivated by energy considerations.

For the modal analysis, let

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin n\pi x \tag{4}$$

which, when substituted into (1), yields

$$\ddot{a}_n + 2\beta\dot{a}_n + \left[n^4\pi^4\left(1 - \frac{p_T}{n^2\pi^2}\right) - n^2\pi^2 f(t)\right]a_n = 0, \quad n = 1,2,\ldots,\ 0 < t \leq T \tag{5}$$

where $\dot{} = \frac{d}{dt}$. It is immediately noted that $p = \pi^2$ represents the Euler buckling load for this column.

To determine stability conditions for (1) or, in the modal approach, for (5), we follow the modification of the Liapunov approach described by Infante[4]. Hence, we seek a positive definite functional V for (1) (function for (5)) such as to obtain a differential inequality of the type $\dot{V}/V \leq \lambda(t)$. Such an inequality immediately allows the determination of stability conditions and of exponential bounds.

3. Exponential Bounds and Stability Conditions for the Modal Approach

Let us first consider equations (5) and determine exponential bounds for their behavior. For this purpose, let these equations be rewritten in canonical form as

$$\dot{x}_1^n = x_2^n$$
$$\dot{x}_2^n = -\left[n^4\pi^4\left(1 - \frac{p_T}{n^2\pi^2}\right) - n^2\pi^2 f(t)\right]x_1^n - 2\beta x_2^n, \; n = 1,2,\ldots, \; 0 < t \leq T \tag{6}$$

where obviously $x_1^n = a_n$, $x_2^n = \dot{a}_n$. Consider the quadratic, positive definite function defined on each of these modes by

$$V_T^n(x_1^n, x_2^n) = \left[\beta x_1^n + x_2^n\right]^2 + \sqrt{n^4\pi^4 e_T\{f^2(t)\} + \left[n^4\pi^4\left(1 - \frac{p_T}{n^2\pi^2}\right) - \beta^2\right]^2} \; x_1^{n^2} \tag{7}$$

and denote by V^n this expression with $E\{f^2(t)\}$ substituted for $e_T\{f^2(t)\}$. Consider now the total time derivative of this function along the solutions of (6):

$$\dot{V}_T^n(x_1^n, x_2^n, t) = -2\beta V_T^n(x_1^n, x_2^n) + [\beta^2 - \omega_T^n + \alpha_T^n + n^2\pi^2 f(t)](2\beta x_1^{n^2} + 2x_1^n x_2^n) \tag{8}$$

where

$$\omega_T^n = n^4\pi^4 - n^2\pi^2 p_T, \quad \alpha_T^n = [n^4\pi^4 e_T\{f^2(t)\} + (n^4\pi^4 - n^2\pi^2 p_T - \beta^2)^2]^{1/2}.$$

Estimating the quotient $(2\beta x_1^{n^2} + 2x_1^n x_2^n)/V_T^n(x_1^n, x_2^n)$ by a procedure such as the use of Lagrange multipliers, we obtain

$$|(2\beta x_1^{n^2} + 2x_1^n x_2^n)/V_T^n(x_1^n, x_2^n)| \leq (\alpha_T^n)^{-1/2} \tag{9}$$

from which (8) can be written in the form

$$\frac{\dot{V}_T^n(t)}{V_T^n(t)} \leq -2\beta + [(\beta^2-\omega_T^n+\alpha_T^n)^2 + 2(\beta^2-\omega_T^n+\alpha_T^n)n^2\pi^2 f(t) + n^4\pi^4 f^2(t)]^{1/2}(\alpha_T^n)^{-1/2} \quad (10)$$

Integrating this expression between 0 and T and using the Schwartz inequality yields

$$V_T^n(t) \leq V_T^n(0)e^{T\{-2\beta + [2(\beta^2-\omega_T^n+\alpha_T^n)]^{1/2}\}} . \quad (11)$$

This expression is a measure of the exponential decay of the oscillation of the modes; however, note that for every different time T the expression V_T^n is a different one.

A slightly different expression can be obtained by letting $T \to \infty$ in (10) and integrating between 0 and t. Using again the Schwarz inequality yields

$$V^n(t) \leq V^n(0) \exp t\left\{-2\beta+[2(\beta^2-\omega^n+\alpha^n)(1+n^2\pi^2 e_t\{f(t)\}/\alpha^n) + n^4\pi^4(e_t\{f^2(t)\} - E\{f^2(t)\})/\alpha^n]^{1/2}\right\} \quad (12)$$

where ω^n and α^n are the same as ω_T^n and α_T^n with p_T replaced by p and e_T by E. Upon noting that $\lim_{t\to\infty} e_t\{f(t)\} = 0$ and $\lim_{t\to\infty} e_t\{f^2(t)\} = E\{f^2(t)\}$ we obtain for sufficiently large t

$$V^n(t) \leq V^n(0)e^{t\left\{-2\beta+\sqrt{2\left[\beta^2-n^4\pi^4\left(1-\frac{p}{n^2\pi^2}\right)\right]+\sqrt{n^4\pi^4E\{f^2(t)\}+\left[n^4\pi^4\left(1-\frac{p}{n^2\pi^2}\right)-\beta^2\right]}}+\epsilon\right\}} \tag{13}$$

where ϵ is any positive number, as small as desired. Now, for asymptotic stability it is sufficient that the term multiplying t in the exponential be negative. A simple computation on the quantity in the brackets yields that this condition is

$$E\{f^2(t)\} \leq 4\beta^2\left(1 - \frac{p}{n^2\pi^2}\right) - \epsilon \tag{14}$$

where ϵ is any positive number. The most restrictive condition is the one for $n = 1$. Hence, for the asymptotic stability of all the modes we obtain as a sufficient condition that

$$E\{f^2(t)\} \leq 4\beta^2\left(1 - \frac{p}{\pi^2}\right) - \epsilon \tag{15}$$

which also shows that we must have $p < \pi^2$. It is of interest to note that the exponential estimate (12) is a rather sharp one, since if $f(t) \equiv 0$ (the system is autonomous) then we obtain the estimates

$$V^n(t) \leq V^n(0)e^{t\left\{-2\beta+\sqrt{2\left[\beta^2-n^4\pi^4\left(1-\frac{p}{n^2\pi^2}\right)+\left|n^4\pi^4\left(1-\frac{p}{n^2\pi^2}\right)-\beta^2\right|\right]}\right\}} \tag{16}$$

which are the exact exponential bounds for the autonomous system.

If we consider $f(t)$ as a random, strictly stationary ergodic process then the estimate (15) gives us a condition for almost sure asymptotic stability.

4. Direct Approach

In this section we derive the stability conditions and exponential bounds directly from the original equation of motion (1). Consider first the functional

$$V_T(w,v) = \int_0^1 \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - p_T\left(\frac{\partial w}{\partial x}\right)^2 + v^2 + 2\beta vw + 2\beta^2 w^2\right]dx \tag{17}$$

where $v = \partial w/\partial t$. For functions w satisfying the boundary conditions (2), one can use the calculus of variations to show that

$$\int_0^1 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \geq \pi^2 \int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx \tag{18}$$

so that

$$V_T(w,v) \geq \int_0^1 \left[\left(\pi^2 - p_T\right)\left(\frac{\partial w}{\partial x}\right)^2 + (v+\beta w)^2 + \beta^2 w^2\right]dx. \tag{19}$$

Assume that

$$p_T < \pi^2. \tag{20}$$

Then V_T is positive definite.

The total time derivative of V_T along the solutions of (1) is easily computed to be

$$\dot{V}_T(w,v,t) = -2\beta V_T(w,v) + 2\int_0^1 \left[2\beta^2 vw + 2\beta^3 w^2 + \beta f(t)\left(\frac{\partial w}{\partial x}\right)^2 - f(t)v\frac{\partial^2 w}{\partial x^2}\right]dx \quad (21)$$

Using the method of Lagrange multipliers, we can obtain the inequality

$$\left|\int_0^1 \left[2\beta^2 vw + 2\beta^3 w^2 + \beta f(t)\left(\frac{\partial w}{\partial x}\right)^2 - f(t)v\frac{\partial^2 w}{\partial x^2}\right]dx\right| \leq \gamma V_T(t), \quad (22)$$

where

$$\gamma = \max_{n=1,2,\ldots} \{[2\beta^2 - n^2\pi^2 f(t)]^2/(n^4\pi^4 - n^2\pi^2 p_T + \beta^2)\}^{1/2}. \quad (23)$$

Therefore

$$\frac{\dot{V}_T(t)}{V_T(t)} \leq -2\beta + \gamma. \quad (24)$$

Integrating this expression between 0 and T, and using (3) and the Schwartz inequality, we find that

$$V_T(t) \leq V_T(0)e^{T\{-2\beta + [(4\beta^4 + \pi^4 e_T\{f^2(t)\})/(\pi^4 - \pi^2 p_T + \beta^2)]^{1/2}\}}. \quad (25)$$

This expression gives an exponential bound for the functional V_T.

If we let $T \to \infty$ and then integrate (24) between 0 and t, we obtain for sufficiently large t

$$V(t) \leq V(0)e^{t\{-2\beta + [(4\beta^4 + \pi^4 E\{f^2(t)\})/(\pi^4 - \pi^2 p + \beta^2)]^{1/2} + \epsilon\}} \quad (26)$$

where V denotes the functional V_T with p_T replaced by p. The term multiplying t in the exponential will be negative if

$$E\{f^2(t)\} \leq 4\beta^2\left(1 - \frac{p}{\pi^2}\right) - \epsilon \tag{27}$$

where ϵ is any positive number. It follows that (27) is a sufficient condition for almost sure asymptotic stability of the column for the case of a strictly stationary and ergodic load $f(t)$, and for asymptotic stability in the case of a deterministic load. This condition is the same as that obtained by a modal analysis in the previous section.

We can obtain stronger exponential bounds than (25) and (26) if we consider the functional

$$W_T(w,v) = \int_0^1 \left\{(v+\beta w)^2 + 2w \sum_{n=1}^{\infty} K_n\left[\int_0^1 w(\bar{x},t)\sin n\pi\bar{x}\, d\bar{x}\right]\sin n\pi x\right\}dx \tag{28}$$

where

$$K_n = [(n^4\pi^4 - n^2\pi^2 p_T - \beta^2)^2 + n^4\pi^4 e_T\{f^2(t)\}]^{1/2}. \tag{29}$$

Following the same procedure as for V_T yields equations (11) and (13) with $n = 1$ and V_T^1 replaced by W_T.

5. Discussion of Results

We have derived exponential bounds and stability conditions for a pinned column subjected to a time-dependent axial load. In particular, if the load is given by $p + f(t)$ with $E\{f(t)\} = 0$, then equation (13) gives an exponential bound for each mode, with $n = 1$ providing a bound for any motion $w(x,t)$ of the column, and equations (15) and (27) give a sufficient condition for stability. This stability condition is shown in Fig. 1. For comparison, we have also depicted conditions on $E\{f^2(t)\}$ which can be obtained using the methods of references 1 and 3.

As can be seen from Fig. 1, the stability condition derived here is a significant improvement on the previously published results of other authors. If $p < \pi^2$, it is possible to let $E\{f^2(t)\} \to \infty$ as the damping increases, whereas previous results have indicated a limit to the value of $E\{f^2(t)\}$ no matter what the magnitude of damping.

We note again the direct approach of Section 4, which eliminates the mathematical uncertainty involved when judging the stability of a column by the stability of each mode separately. This approach is also of particular advantage when dealing with boundary conditions for which the modes are not as simple as $\sin n\pi x$.

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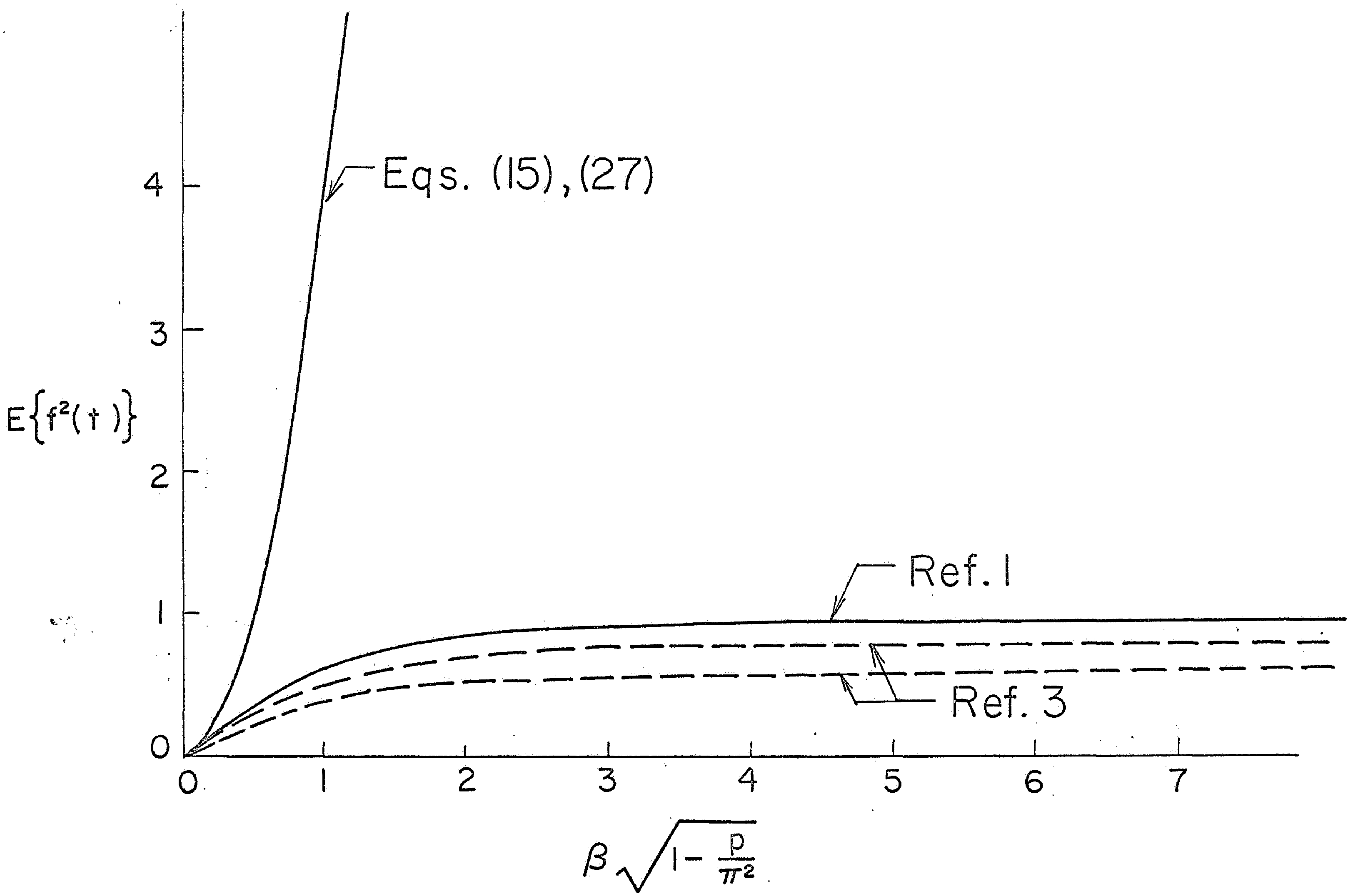
Eqs. (15), (27)
Ref. 1
Ref. 3
$E\{f^2(t)\}$
$\beta\sqrt{1-\frac{p}{\pi^2}}$
0
1
2
3
4
5
6
7

Caption for Illustration

Figure 1. Stability Conditions