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PROPAGATION OF ONE AND TWO DIMENSIONAL WAVES OF FINITE AMPLITUDE

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FOREWORD

The present report can be regarded as an extension of the previous one by the author (see Bibliography, (1)), which dealt with the propagation of waves of finite amplitude in thermo-viscous media, but only in one space dimension. The natural extension of those results, which contained many fundamentally new facts and relations, is to a physically more realistic, but mathematically much more difficult, higher dimension. We thus give an outline below, section by section, underlining those results of ours which seem to be as significant as the ones obtained in the one-dimensional case.

1.0 This introductory section gives a new derivation of Burgers' equation for a one dimensional framework of reference; treating, however, the case where fluid density is not constant. Thus, a stream function or potential is introduced, for the purpose of motivation in the higher dimensional case, and a measure of the deviation from the exact results is obtained.

2.0 Section two contains a concise collection of those known analytical results which are used in the sequel.

3.0 As a by-product of the present investigation, it was found possible to obtain an exact analytical characterization of finite amplitude wave propagation in a periodically driven tube. The result is reduced to a form which shows its relation to and connection with the well known results for propagation in a lossless medium.

4.0 As a first logical step in extending one dimensional results to higher dimensions, we are considering propagation in pipes of variable cross sections. Thus, here we derive the necessary extensions of Burgers' equation to the use of propagation in a pipe the cross-section of which is location-dependent. These equations are found to be generally "Burgers'-type"; that is, second order, parabolic and nonlinear, but with non-constant coefficients and also an additional term. These seemingly slight differences, however, change the nature of the equations to such an extent, that no analytical treatment seems to be presently possible. This difficulty is discussed in detail in Appendix II.

5.0 In an effort to find an analytical method for handling variable cross sections, we are discussing here the time dependent case; where the cross sectional area changes with time. An explicit solution is computed here, in terms of a rather complicated infinite series. The limiting cases are discussed in particular; it is shown that the behavior of the model solution is very realistic.

6.0 This section is devoted to showing why, despite a statement by one of the original solvers of Burgers' equation to the contrary, it is unrealistic to treat Burgers' equation as a model in the three (or even two) dimensional cases.

7.0 Here we are deriving, somewhat analogously to Section 1, the governing Navier-Stokes type equations for propagation in two dimensions.

8.0 Section 8 is devoted to a completely new and physically inspired generalization of Burgers' equation. Solutions are obtained and discussed; and it is shown that in this case elliptic functions make their appearance.

9.0 Another entirely new, and this time three-dimensional (although axially symmetric) pair of equations is given here as a quasi-linear approximation. The method and the solutions, which are given explicitly, are couched in terms of a potential function, similar to the one obtained in Section 1. The main purpose of this section is to give a somewhat simple illustration of the development in Section 10.

10.0 As a culmination of our effort, we are giving here a new way of generalizing Burgers' equation, for frameworks of reference similar to those in Section 9. As one of the most important features of this model, it is shown that a stochastic interpretation is possible; and that one can prescribe the initial and/or boundary conditions for the acoustical field in terms of random functions. As it happens so often in the case of a physically correct model, a rather simple, and quite new method of solution is also derived.

SUMMARY

The present report, an extension of "Propagation of Waves of Finite Amplitude in Thermo-viscous Media," (NASA CR-643, November 1966) by the author, consists of the analytical description of three subjects: a) the propagation of piston driven periodic acoustic waves of finite amplitude in a cylindrical tube, b) the same waves in a pipe of variable cross section (where the cross section is both time and location dependent) and c) an extension of the analytical methods employed for the one dimensional case to two spatial dimensions. Several explicit solutions are given and discussed.

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1.0 INTRODUCTION

In our previous report (Reference 1), we have given rational arguments to show why Burger's equation,

$$v_t + v v_x = \delta v_{xx} \qquad (1)$$

is an appropriate analytical vehicle for the description of the propagation of waves of finite amplitude in thermo-viscous media. One of the most important reasons was that Equation (1) is one of the very - very few nonlinear partial differential equations for which an exact and complete analysis can be given (References 2 and 3).

The essential purpose of this Introduction, at the head of a report which still relies heavily on Burgers' equation, is to give yet another analysis of how one can estimate the usefulness of this equation and the magnitude of the errors introduced by using it, in preference to a more complete equation or set of equations.

Let us start from the pair of equations

$$v_t + v v_x + \frac{2}{\gamma - 1} a a_x = \epsilon v_{xx} \qquad (2)$$

$$a_t + a_x v + \frac{\gamma - 1}{2} a v_x = 0 \qquad (3)$$

This pair was used by Riemann (Reference 4) and by Earnshaw (Reference 5), in the case $\epsilon = 0$, to obtain the description of nonlinear one dimensional propagation in a lossless medium; and it was arrived at by Lighthill (Reference 6) in his now classical paper, as the next-to-the-last step in his sequence of approximations to obtain an appropriate analytical vehicle for the examination of the structure of weak shocks in thermo-viscous media. Lighthill, in his analysis, reduced the pair (2)-(3) to Equation (1). This is exactly what we shall do also; however, in contrast to his physical order of magnitude arguments, we shall give a more mathematical analysis.

Let us begin with our definitions; for the details, see our previous report (Reference (1). The meaning of the various symbols in Equations (1)-(3) is the following:

v	particle velocity
a	local sound speed
x	displacement
t	time

γ ratio of specific heats

ϵ, ξ :iffusion (thermo-viscous) constants

The subscripts, as usual, denote partial differentiations with respect to the independent variable involved.

The substitution

$$a = a_o \left(\frac{\rho}{\rho_o}\right)^{\frac{\gamma - 1}{2}}$$

in Equations (2)-(3), replacing the local sound speed functions by a density function ρ and in terms of the ambient values a_o and ρ_o, reduces (2)-(3) to the system

$$v_t + v v_x + k^2 \rho^{\gamma - 2} \rho_x = \epsilon v_{xx} \tag{4}$$

$$\rho_t + (\rho v)_x = 0 \tag{5}$$

where the non-negative quantity k^2 is given by

$$k^2 = a_o^2 \rho_o^{1-\gamma} .$$

Equation (5) gives the impetus to introduce a stream function ϕ, defined essentially by a desire to satisfy (5):

$$\phi: \begin{cases} \rho = \phi_x \\ v = - \dfrac{\phi_t}{\phi_x} \end{cases} \tag{6}$$

Then, of course, Equation (5) is completely satisfied; we now have to satisfy (4).

With the foreknowledge available in connection with the solution of Equation (1), we now assume that our stream function ϕ satisfies the linear diffusion equation

$$\phi_t = c_1 \phi_{xx} , \tag{7}$$

for some diffusion constant c_1 .

Because of the first part of our definition (6), however, the density function ρ also satisfies this diffusion equation

$$\phi_{xt} = c_1 \phi_{xxx} ,$$

i.e.,

$$\rho_t = c_1 \rho_{xx}$$

Furthermore, for the particle velocity function v we now find from this last relation and from (6) and (7)

$$v = - \frac{\phi_t}{\phi_x} = - c_1 \frac{\phi_{xx}}{\phi_x} = - c_1 \frac{\rho_x}{\rho} \tag{8}$$

This is a most reasonable and encouraging result. For if $k^2 = 0$ in Equation (4) - which is the one that we are seeking to satisfy - then the value of $c_1 = 2\epsilon$ in (8) satisfies (4) exactly. This we know from knowing the exact solution of Burger's equation. On the other hand, if $k^2 \neq 0$, it is reasonable to assume that the combination of ρ_x and ρ as given in (8), will give an acceptable approximation for the last term on the right hand side of (4), for some value of c_1; particularly since the most interesting values of γ lie between 1 and 2 ($\gamma = 1.41$ for air).

If we use v, as given by (8), in Equation (4), than the following arrangement is possible:

$$- c_1 \frac{\rho\rho_{xt} - \rho_x \rho_t}{\rho^2} + c_1^2 \frac{\rho_x}{} \frac{\rho\rho_{xx} - \rho_x^2}{\rho^2} + k^2 \rho^{\gamma-2} \rho_x = - \epsilon c_1 \frac{\rho^2 \rho_{xxx} - 3\rho\rho_x\rho_{xx} + 2\rho_x^3}{\rho^3}$$

$$\rho^2 \rho_{xt} - \rho\rho_x \rho_t - c_1 \left(\rho\rho_x \rho_{xx} - \rho_x^3\right) - \frac{k^2}{c_1} \rho^{\gamma+1} \rho_x = \epsilon \left(\rho^2 \rho_{xxx} - 3\rho\rho_x \rho_{xx} + 2\rho_x^3\right)$$

$$\rho^2 \left(\rho_{xt} - \epsilon \rho_{xxx}\right) - \rho\rho_x \left(\rho_t + (c_1 - 3\epsilon)\rho_{xx}\right) + (c_1 - 2\epsilon) \rho_x^3 - \frac{k^2}{c_1} \rho^{\gamma+1} \rho_x = 0$$

$$(c_1 - \epsilon) \rho^2 \rho_{xxx} - (2c_1 - 3\epsilon) \rho\rho_x \rho_{xx} + (c_1 - 2\epsilon) \rho_x^3 = \frac{k^2}{c_1} \rho^{\gamma+1} \rho_x \tag{9}$$

The values

$$c_1 = \frac{n}{2}\epsilon \quad , \quad n = 0, 1, 2, 3, 4$$

lead to special cases, with correspondingly special physical assumptions. Note, however, that we used assumption (7) only in passing from the next to the last line in (9) to the last one. It is possible to proceed otherwise; for instance, to rewrite the third line of (9) in the form

$$\left(\frac{\rho_t - \epsilon\,\rho_{xx}}{\rho}\right)_x - (c_1 - 2\epsilon)\left[\rho_x\,\rho_{xx} - \frac{\rho_x^3}{\rho}\right] = \frac{k^3}{c_1}\;\rho^\gamma\;\rho_x$$

If we take $c_1 = 2\epsilon$ here, then we see that instead of (7), ρ now has to satisfy a nonlinear parabolic equation

$$\rho_t - \epsilon\,\rho_{xx} = \frac{k^2}{2\,c_1\,(\gamma+1)}\;\rho^{\gamma+2} \quad .$$

While this would yield an exact solution, the analysis of the last equation is quite difficult and therefore we shall abandon this avenue of investigation.

By assumption (7), we have that ρ satisfies

$$\rho_{xt} = c_1\,\rho_{xxx} \quad , \quad c_1 > 0$$

Let us recall that, for increasing values of c_1, the solutions of the heat equation approach bounded values in a manner which is usually of the exponentially damped type. Furthermore, derivatives ρ_x of the solution approach 0 for increasing c_1. With this in mind, let us write (9) in the form

$$\rho^2\rho_{xxx} - \frac{2\,c_1 - 3\epsilon}{c_1 - \epsilon}\;\rho\rho_x\,\rho_{xx} + \frac{c_1 - 2\epsilon}{c_1 - \epsilon}\;\rho_x^3 = \frac{k^2}{(c_1-\epsilon)\,c_1}\;\rho^{\gamma+1}\,\rho_x \quad , \tag{10}$$

and consider the effect of letting c_1 increase. Clearly, since by the preceding argument the numerator of the right hand side of (10) is bounded, it becomes very small for c_1 large; in fact, it is of the order $o\,(c_1^\alpha)$, where $\alpha < 2$. Therefore, choosing

$$c_1 = \delta > \left[1;\ \epsilon;\ k^2\ M^{\gamma+1}\ M_x\right] \quad , \tag{11}$$

where

$$M = \max\ \rho \text{ in some region } R$$

$$M_x = \max\ \rho_x \text{ in the same region}$$

we reduce (10) to

$$\rho^2 \rho_{xxx} - 2\rho\rho_x \rho_{xx} + \rho_x^3 = 0 \tag{12}$$

This expression can be integrated; using the letter p for the upcoming functions (of t) of integration, we obtain

$$\rho\rho_{xx} - \rho_x^2 = \rho\, p_1(t) \ ,$$

or

$$\rho \left(\frac{\rho_x}{\rho}\right)_x = p_1(t) \ . \tag{13}$$

Letting now

$$\rho = e^u$$

(13) becomes

$$u_x u_{xx} = p_1(t)\, e^{-u}\, u_x$$

which can again be integrated:

$$\frac{1}{2} u_x^2 = -p_1(t)\, e^{-u} + p_2(t)$$

and thus

$$\frac{u_x}{\sqrt{p_2(t) - p_1(t)\, e^{-u}}} = \sqrt{2} \tag{14}$$

One more integration, and a transformation from u back to ρ gives us

$$\rho = \frac{p_1(t)}{p_2(t)} \sin^2 \left[\frac{\sqrt{-p_2(t)}}{2} \left(\sqrt{2}\, x + p_3(t) \right) \right] \tag{15}$$

where we must have $p_1(t) < 0$ and $p_2(t) < 0$.

This function then is the justification for the assumptions made on ρ a priori, for an appropriate choice of the arbitrary functions $p_i(t)$. In particular, let us note that ρ, as given by (15), is a non-negative function; a fact which further corroborates the appropriateness of our choice of ρ as a solution of the diffusion equation; for as we mentioned in our previous report (Reference 1), the value of v in terms of the logarithmic derivative of ρ (formula (8)) pre-supposes a non-negative ρ.

Our conclusion, therefore, is the following: if ρ satisfies

$$\rho_{xt} = \delta\, \rho_{xxx}$$

with δ given by (11), and

$$v = -\delta\, \frac{\rho_x}{\rho} ,$$

then Equation (5) is satisfied identically. Furthermore v then satisfies the Burgers' equation

$$v_t + 2\, v\, v_x = \delta\, v_{xx} ,$$

while the magnitude of error in using an approximation like this for Equation (4) is

$$E = \left| \frac{v^2}{2} + \frac{k^2}{\gamma - 1} \exp\left[- \frac{\gamma - 1}{\delta} \int v\, dx \right] + (\delta - \epsilon)\, v_x + G(t) \right| ,$$

where G may be freely chosen, (i.e., if the error is small at any given instant, it will remain small - for the particular x under consideration - for all time). We have also shown here that E is negligible for a δ chosen large enough, in some region of interest R.

With this, we have now the above additional argument for the use of Burgers' equation, as the analytical vehicle for the description of propagation of waves of finite amplitude in thermo-viscous media.

2.0 PREVIOUS RESULTS

It will be convenient to reassemble here, in a rather compact form, the results of our previous study (Reference 1). We shall do this with a minimum of elaboration, mentioning only those aspects of that study which will be used in the sequel.

Let us consider a pipe of arbitrarily large radius, one end of which contains a large piston. The pipe itself extends to infinity in both directions. If one prescribes the motion of the piston by

$$x = g(t) \tag{16}$$

and assumes that it moves with at most sonic velocity, then, if the fluid velocity function is $v = v(x, t)$, we have a boundary condition of the type

$$v(g(t), t) = g'(t) \tag{17}$$

The expansion of v in (17) about $x = 0$ in a Taylor series yields

$$v(g(t), t) = v(0, t) + g(t)\, v_x(0, t) + \sum_{n=2}^{\infty} \left\{ \frac{\partial^n}{\partial_x^n} v(0,t) \frac{g^n(t)}{n!} \right\} = g'(t) \tag{18}$$

In [1], we estimated the error incurred by neglecting the summation in the center of equality (18). In this manner we are led to a boundary condition of the form

$$v(0, t) + g(t)\, v_x(0,t) = g'(t) \tag{19}$$

for our Equation (1).

This development naturally led us to inquire about solutions of (1) satisfying

$$\begin{aligned} & v_t + v v_x = \delta v_{xx} \\ & v(0,t) = a(t), \quad v_x(0,t) = b(t) \end{aligned} \tag{20}$$

The most general solutions then turned out to be

$$v(x,t) = \frac{-2\delta \sum_{n=1}^{\infty} \frac{K^{(n)}(t)}{\delta^n} \frac{x^{2n-1}}{(2n-1)!} + \sum_{n=0}^{\infty} \frac{[a(t)\,K(t)]^{(n)}}{\delta^n} \frac{x^{2n}}{(2n)!}}{\sum_{n=0}^{\infty} \frac{K^{(n)}(t)}{\delta^n} \frac{x^{2n}}{(2n)!} - \frac{1}{2\delta} \sum_{n=0}^{\infty} \frac{[a(t)\,K(t)]^{(n)}}{\delta^n} \frac{x^{2n+1}}{(2n+1)!}} \tag{21}$$

where

$$G^{(n)}(t) \equiv \frac{d^n G}{d t^n} ,$$

$$K(t) = \exp\left[\frac{1}{2\delta} \int^t D(\tau)\, d\tau\right] , \tag{22}$$

with

$$D(t;\delta) = D(t) = \det \begin{vmatrix} a(t) & 2\delta \\ b(t) & a(t) \end{vmatrix} . \tag{23}$$

Now the formula (21) was shown to be applicable, in particular if the condition (19) was simplified by taking $v(0,t) = 0$, to a large variety of situations. For example, if one assumed piston motion described by

$$G(t) = \left[1 + 2 \sum_{k=1}^{\infty} e^{-k^2 t}\right]^{-1} , \tag{24}$$

it was possible to derive the classical solution of Fay from our results, as a very special case. The formula that we used - as a reduction of (21) - was

$$v(x,t) = -2\delta \frac{\partial}{\partial x} \left\{ \ln\left[\sum_{n=0}^{\infty} [(G(t))^{-1/2}]^{(n)} \frac{x^{2n}}{\delta^n (2n)!} \right] \right\} , \tag{21'}$$

in which G represents the piston displacement.

3.0 PROPAGATION OF PERIODIC PISTON MOTION

The results of Fay which we mentioned previously were quite limited in the following sense: he sought only the most stable periodic wave forms in the propagation. No wonder, therefore, that such waveforms arise from as "gentle" a piston motion as that described by (24). However, it is natural to inquire what the situation is when the piston motion is periodic; a task to which we are devoting our present section. Since we are not interested in shocks, we shall take

$$G(t) = \exp\left[-\alpha \cos \omega t\right] \quad . \tag{24'}$$

Then, aiming at obtaining the form of (21'), we have

$$\sum_{n=0}^{\infty} \frac{\left[G^{-\frac{1}{2}}(t)\right]^{(n)}}{\delta^{n}} \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{\left(\exp\left[\frac{\alpha}{2} \cos \omega t\right]\right)^{(n)}}{\delta^{n}} \frac{x^{2n}}{(2n)!} \tag{25}$$

In order to perform the differentiations in (25), we write $\exp\left[\frac{\alpha}{2} \cos \omega t\right]$ as a series of modified Bessel functions, utilizing the formula

$$\exp\left[z \cos \omega t\right] = I_0(z) + 2\sum_{k=1}^{\infty} I_k(z) \cos(k\omega t) \; . \tag{26}$$

We shall also need the integral of this expression with respect to t :

$$\int_0^t \exp\left[z \cos \omega \tau\right] d\tau = t\, I_0(z) + 2\sum_{k=1}^{\infty} \frac{I_k(z)}{k\,\omega} \sin(k\omega t), \tag{26'}$$

which is a well defined expression even for $\omega = 0$. We rewrite now (25) in terms of (26), to obtain

$$\sum_{n=0}^{\infty} \left\{\left[I_0\left(\frac{\alpha}{2}\right) + 2\sum_{k=1}^{\infty} I_k\left(\frac{\alpha}{2}\right) \cos k\omega t\right]^{(n)} \frac{x^{2n}}{\delta^{n}(2n)!}\right\} =$$

$$= \exp\left(\frac{\alpha}{2}\cos\omega t\right) + 2\sum_{n=1}^{\infty}\left\{\sum_{k=1}^{\infty}(-1)^n I_k\left(\frac{\alpha}{2}\right)\left[(k\omega)^{2n-1}\sin k\omega t + \right.\right.$$

$$\left.\left. + (k\omega)^{2n}\cos k\omega t\right]\frac{x^{2n}}{\delta^n (2n)!}\right\}$$

Because of the uniform convergence, we may interchange the order of the summations, and rearrange the series:

$$= \exp\left(\frac{\alpha}{2}\cos\omega t\right) + 2\sum_{k=1}^{\infty}\frac{I_k\left(\frac{\alpha}{2}\right)\sin(k\omega t)}{k\omega}\left\{\sum_{n=1}^{\infty}(-1)^n\frac{(k\omega x)^{2n}}{\delta^n (2n)!}\right\} +$$

$$+ 2\sum_{k=1}^{\infty} I_k\left(\frac{\alpha}{2}\right)\cos(k\omega t)\left\{\sum_{n=1}^{\infty}(-1)^n\frac{(k\omega x)^{2n}}{\delta^n (2n)!}\right\} .$$

Both inner series yield the same cosine function:

$$= \exp\left(\frac{\alpha}{2}\cos\omega t\right) + 2\sum_{k=1}^{\infty}\left\{I_k\left(\frac{\alpha}{2}\right)[\cos(k\omega\Delta x) - 1]\left[\frac{\sin(k\omega t)}{k\omega} + \cos(k\omega t)\right]\right\} \tag{27}$$

(We are using here the notation: $\delta^{-\frac{1}{2}} \equiv \Delta$). Multiplication of the two bracketed terms in this series allows us to write it as the sum of four series; in particular, the two arising from the products of (- 1) with $[\sin(k\omega t)]/k\omega$ and with $\cos(k\omega t)$, yields from (26) and (26'),

$$t\left(I_0\left(\frac{\alpha}{2}\right) + 1\right) - \exp\left[\frac{\alpha}{2}\cos\omega t\right] - \int_0^t \exp\left[\frac{\alpha}{2}\cos\omega\tau\right]d\tau \tag{28}$$

To reduce the trigonometric products, we use the identities:

$$\cos(k\omega\Delta x)\ \sin(k\omega t) = \frac{1}{2}\Big[\sin k\omega\,(t+\Delta x) + \sin k\omega\,(t-\Delta x)\Big]$$

$$\cos(k\omega\Delta x)\ \cos(k\omega t) = \frac{1}{2}\Big[\cos k\omega\,(t+\Delta x) + \cos k\omega\,(t-\Delta x)\Big]\ .$$

Thus, we obtain for the sum of the other two series the expression

$$-I_0\left(\frac{a}{2}\right)[t+1] + \frac{1}{2}\left\{\exp\left[\frac{a}{2}\cos\omega\,(t+\Delta x)\right] + \exp\left[\frac{a}{2}\cos\omega\,(t-\Delta x)\right]\right\} +$$

$$+\int_0^t \left[\exp\left[\frac{a}{2}\cos\omega\,(\tau+\Delta x)\right] + \exp\left[\frac{a}{2}\cos\omega\,(\tau-\Delta x)\right]\right] d\tau\ . \tag{29}$$

Therefore, (27) can be written as the sum of the following: the first term in (27), together with (28) and (29). Some further simplifications yield in this way an expression which can be written in the following symmetric form:

$$t - I_0\left(\frac{a}{2}\right) + \frac{1}{2}\Big[\Big(F(t+\Delta x) - F(t)\Big) + \Big(F(t-\Delta x) - F(t)\Big)\Big]$$

$$+\frac{1}{2}\int_0^t \Big[\Big(F(\tau+\Delta x) - F(\tau)\Big) + \Big(F(\tau-\Delta x) - F(\tau)\Big)\Big]\, d\tau\ , \tag{30}$$

where

$$F(s) = \exp\left[\frac{a}{2}\cos\omega s\right]\ . \tag{31}$$

To recapitulate: expression (30) is the transformed form of (25), with $G(t)$ given by (24). Thus, the solution of (1), with the piston condition (24'), is given, according to (16), by the product of (-2δ) with the logarithmic x-derivative of (30). This takes the form

$$v(x,t) = \frac{\frac{2}{\Delta}\Big[\Big[F'(t-\Delta x) - F'(t+\Delta x)\Big] + \Big[F(t-\Delta x) - F(t+\Delta x)\Big]\Big]}{M + \Big[\Big(F(t+\Delta x) - F(t)\Big) + \Big(F(t-\Delta x) - F(t)\Big)\Big] + \int_0^t \Big[\Big(F(\tau+\Delta x) - F(\tau)\Big) + \Big(F(\tau-\Delta x) - F(\tau)\Big)\Big] d\tau} \tag{32}$$

where

$$M = 2\left[t - I\left(\frac{a}{2}\right)\right]$$

The suggestiveness of (32) is obvious: the two types of waves, so well known from linear theory, are both present. However, because of the viscous mechanism, they contain a "damping" - in fact, it is clear here how increased viscosity ($\delta \to \infty \Rightarrow \Delta \to 0$) reduces the propagation to zero. Moreover, the nonlinearity of the mechanism and/or of the medium are represented in a conceptually rather simple form. Note that shocks depend on the values of t and a in particular.

4.0 GENERALIZED ONE DIMENSIONAL EQUATIONS

As a first step in generalizing results obtained for the one dimension, we would like to discuss the question of propagation in a pipe with variable cross section. We derive the necessary equations here.

Burgers' equation is valid for non-linear waves in constant cross-section pipes:

$$v_t + v v_X = \frac{\delta}{2} v_{XX} \tag{33}$$

where

$v = a + u - a_o$ — excess wavelet velocity

$X = x - a_o t$ — moving reference system

$\delta = \frac{1}{\rho_o}\left[\frac{4}{3}\mu + \mu_v + (\gamma - 1)\frac{k}{c_p}\right]$ — diffusivity

We shall now derive an equation for waves traveling in a pipe whose cross-sectional area (A) is a function only of its distance from the origin (A (x)). Plane, cylindrical, spherical, and exponential horn waves, are particular cases of this equation.

For the flow of a perfect gas in a pipe of area A (x) the following equations can be written:

$$A \rho_t + (A \rho u)_x = 0 \tag{34}$$

$$A \rho (u_t + u u_x) = - A p_x + \rho \delta (A u_x)_x \tag{35}$$

$$\rho = \rho_o (a/a_o)^{\frac{2}{\gamma-1}} \; ; \quad p = p_o (a/a_o)^{\frac{2\gamma}{\gamma-1}} \tag{36}$$

The above equations can be rewritten as follows:

$$\rho_t + u \rho_x + \rho u_x = - \rho u A_x / A \tag{37}$$

$$u_t + u u_x + p_x/\rho = \delta u_{xx} + \delta u_x A_x / A \tag{38}$$

Using Equations (36), Equations (37) and (38) become:

$$a_t + u a_x + \frac{\gamma - 1}{2} a u_x = - \frac{A_x}{A} \left(\frac{\gamma - 1}{2} \right) a u \tag{39}$$

$$u_t + u u_x + \frac{2}{\gamma - 1} a a_x = \delta \left(u_{xx} + \frac{A_x}{A} u_x \right) \tag{40}$$

Defining $f = A_x / A = (\ln A)_x$, the following particular cases are noticed:

$$\frac{A_x}{A} = f = 0 \qquad \text{Plane waves} \tag{41}$$

$$\frac{A_x}{A} = f = 1/x \qquad \text{Cylindrical waves} \tag{42}$$

$$\frac{A_x}{A} = f = 2/x \qquad \text{Spherical waves} \tag{43}$$

$$\frac{A_x}{A} = f = \beta \qquad \beta = \text{constant Exponential horn waves:} \quad A = A_o \, \text{EXP} [\beta x] \tag{44}$$

Setting:

$$r = \frac{a}{\gamma - 1} + \frac{u}{2} \; ; \qquad s = \frac{a}{\gamma - 1} - \frac{u}{2}$$

Substituting a , u with r , s in Equations (39) and (40) and then taking s_t from Equation (40) and substituting it into Equation (39) it is found:

$$r_t + (a + u) r_x = \frac{\delta}{2} \left[r_{xx} - s_{xx} + f (r_x - s_x) \right] - \frac{f}{2} \left(\frac{\gamma - 1}{2} \right) (r^2 - s^2) \tag{45}$$

Making the assumption that $s \simeq s_o$ if terms of order $(\nu \omega / a^2)(V/a)$ are neglected with respect to the terms of order $(\nu \omega / a^2)$ or (U/a) as Lighthill suggested, r and s reduce to:

$$r = \frac{a}{\gamma - 1} + \frac{u}{2} \tag{46}$$

$$s \simeq s_o \simeq \frac{a_o}{\gamma - 1} \quad \text{Hence,} \; s_x = s_{xx} = 0 \tag{47}$$

Introducing the excess wavelet velocity (v):

$$v = a + u - a_o \simeq \frac{\gamma+1}{2} r + \frac{\gamma-3}{2} s_o - a_o = \frac{\gamma+1}{2} r - \frac{\gamma+1}{\gamma-1} \frac{a_o}{2} \tag{48}$$

So that:

$$r = \frac{2v}{\gamma+1} + \frac{a_o}{\gamma-1} \tag{49}$$

and finally, substituting Equations (57) and (55) in Equation (53), it is found:

$$v_t + (a_o + v)\, v_x = \frac{\delta}{2}\left[v_{xx} + f v_x\right] - \frac{f}{2}\left[\frac{\gamma-1}{\gamma+1} v^2 + a_o v\right] \tag{50}$$

Before rearranging the terms of Equation (50) it is useful to make some considerations on their origin and their importance. First, we recall that v is the excess particle velocity ($v = u + a - a_o$) equal to the particle velocity (u) plus the local variation of the speed of sound ($a - a_o$) originally due to the non-linear terms $u\,\rho_x$, $\rho\, u_x$, $u\, u_x$ and p_x/ρ of Equations (37) and (38). Thus v is at most of the order of $2u$, i.e., much smaller than a_o hence, v^2 will be neglected with respect to $a_o v$. This is proper and consistent with the other approximations made so far. It is also evident that the terms multiplied by f are due to the change in cross-sectional area; $f = 0$ reduces Equation (50) to Equation (33) for the constant cross-sectional area if $X = x - a_o t$ is introduced in Equation (50). The effect of a changing area comes both through the mass conservation equation (37) and the momentum conservation equation (38). In Equation (50) the term multiplied by δf comes from the momentum while those multiplied by f only come from the mass conservation equation. If we think of v as an approximate sine wave we know that $v_{xx} = -(2\pi)^2\, v/\lambda^2$ where λ is the wave length. Thus, the effect of an area increase as introduced through the mass conservation ($-f a_o v/2$) has the same effect as an increase of diffusivity in a constant area propagation ($\delta v_{xx}/2$) but while the effect of diffusivity decreases as the wave length (λ) increases, the effect of an area change, as introduced through the mass conservation, depends only on the area change and not on the frequency of sound ($-f a_o v/2$). On the other hand the effect of the area change as introduced through the momentum conservation equation ($\delta f\; v_x/2$) is out of phase with respect to the undistorted wave and contributes directly to its distortion. However, also this effect decays with increasing wave lengths. Thus, if we conclude that:

$$\begin{cases} \frac{\delta}{2} \; v_{xx} = O\left[\frac{\delta}{2}\left(\frac{2\pi}{\lambda}\right)^2 \; v\right] \\ \frac{\delta}{2} \; f v_x = O\left[\frac{\delta}{2} \; f\left(\frac{2\pi}{\lambda}\right) v\right] \\ \frac{f}{2} \; a_o v = O\left[\frac{f}{2} \; a_o v\right] , \end{cases} \tag{51}$$

then , defining (c.p.s.) the sound frequency in cycles per second, we could neglect $\delta v_{xx}/2$ if:

$$\frac{2\pi}{\lambda} << f \implies (c.p.s.) << \frac{f a_o}{2\pi}$$

We could neglect $\delta v_{xx}/2$ and $\delta f v_x/2$ if:

$$\frac{\delta 2\pi}{\lambda} << a_o \implies (c.p.s.) << \frac{a_o^2}{2\pi\delta}$$

While we could neglect $f a_o v/2$ if:

$$\frac{\delta 2\pi}{\lambda} >> a_o \implies (c.p.s.) >> \frac{a_o^2}{2\pi\delta}$$

or

$$\delta\left(\frac{2\pi}{\lambda}\right)^2 >> f a_o \implies (c.p.s.) >> \frac{a_o^{3/2}}{2\pi}\left(\frac{f}{\delta}\right)^{1/2}$$

Concluding, the equation for the propagation of high intensity sound in a pipe of varying cross-sectional area (A (x)) is:

$$v_t + \left(a_o + v - \frac{\delta A_x}{2A}\right) v_x = \frac{\delta}{2} \; v_{xx} - \frac{A_x}{2A} \; a_o v \tag{52}$$

For plane waves, using Equation (41)

$$v_t + (a_o + v) \; v_x = \frac{\delta}{2} \; v_{xx} \tag{53}$$

For cylindrical waves, using Equation (42)

$$v_t + \left(a_o + v - \frac{\delta}{2x}\right) v_x = \frac{\delta}{2} v_{xx} - \frac{a_o}{2x} v \qquad (54)$$

For spherical waves, using Equation (43)

$$v_t + \left(a_o + v - \frac{\delta}{x}\right) v_x = \frac{\delta}{2} v_{xx} - \frac{a_o}{x} v \qquad (55)$$

For exponential horn waves, using Equation (44)

$$v_t + \left(a_o + v - \frac{\beta\delta}{2}\right) v_x = \frac{\delta}{2} v_{xx} - \frac{\beta a_o}{2} v \qquad (56)$$

The substitution $X = x - a_o t$ transforms Equation (53) to Burgers' equation and it can approximately be used in the solution of Equation (56), but it is not useful in Equations (54) and (55) because there the independent variable x appears explicitly. In any case, however, it is clear that the solutions of Equations (54)-(56) are involved in almost insurmountable difficulties; and such is the case, a fortiori, for Equation (40). Thus, in the next section we shall discuss an approach based on a different generalization; one which shows more promise of being tractable. However, in Appendix II, we are giving a brief discussion of why even an apparently very simple case such as (56) involves great difficulties.

5.0 THE CASE OF VARIABLE GEOMETRY

In Section 3.0 we have obtained an explicit expression for propagation induced by a periodic piston. However, all of the preceding solutions were obtained under the assumption that we have a simple geometry: that is, flow in a one dimensional lossless pipe. The question arises, however, as to what the situation is when the pipe has a variable, or indeed a time-dependent geometry. Note that Section 4.0 treated the location dependent situation and, in general, that is simpler than the time-dependent one, discussed here. For example, suppose that we have a solid fuel rocket, in which the waves propagate down the "pipe" which is the central core; however, in the course of this propagation the walls of this "pipe" get used up, so that the geometry changes continuously. It is because of such considerations that the present section is included here. The main connection between what we have here and the other parts of our work is that here too we are using Burgers' equation; furthermore, in deriving it, we shall follow once again the guidelines given by Lighthill.

Let us consider one dimensional flow in a vessel of time-varying cross-sectional area $A(t)$. As our flow quantity, let us choose the flow velocity u, which we shall assume to depend on the independent variables $X = (x - ct)$ and t, so that we shall have a moving frame of reference. Thus, at time $t = t_0$ and at the point $x = x_0$, the velocity will be given by

$$u_0 \quad = \quad u(x_0 - ct_0 , t_0)$$

and at that (moving) point $X_0 = (x_0 - ct_0)$ the cross sectional area will be $A_0 = A(t_0)$. In order to make our theory agree with the customary linear theory, we shall take

$$c \quad = \quad \text{local soundspeed.}$$

We are interested in waves of finite amplitude, and thus have to take convection into account. This means that

$$u \quad = \quad \frac{dX}{dt}$$

will not vanish along individual wavelets; rather, it will be proportional to the fluid velocity itself. Now the energy density in the system is

$$v^2 = A u^2 ,$$

so that, following Lighthill, we obtain the appropriate relation

$$v_T + v v_X \approx 0 , \tag{57}$$

where $\quad v = A u \quad$ (57')

and $\quad T = \int \frac{dt}{\sqrt{A(t)}} . \quad$ (58)

If we consider dissipation also, the right hand side of (57) has to be replaced by a quantity proportional to it, i.e.,

$$v_T + v v_X = \delta(T) v_{XX} \tag{59}$$

We followed here the derivation given by Lighthill, so that we shall be able to trace back the significance of the various quantities.

As we have seen, the solution of (59), for $\delta(T) = \delta_0 =$ constant, is given by

$$v = -2\delta_0 \frac{\theta_X}{\theta} \tag{60}$$

where θ is a positive solution of

$$\theta_T = \delta_0 \theta_{XX} \tag{61}$$

Without any regard to possible boundary and/or initial conditions, let us take for a solution of (61) the function

$$\theta(X,T) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2[\alpha + \delta T]} \cos nX , \tag{62}$$

which is Jacobi's 4th Theta function, with α an arbitrary (non negative) constant. The advantage in taking this solution is that there is a rather simple expression for the logarithmic derivative of it; i.e., using (62), (60) becomes

$$v(X,T) = -2\delta \sum_{n=1}^{\infty} \left\{ \cosh \left[n(\alpha + \delta T) \right] \sin nX \right\} , \tag{63}$$

with α still arbitrary. (In any specific situation it could be used to satisfy an initial condition, for instance.)

In order to analyze expression (63), we must relate it to an experimentally observable quantity, such as the pressure P. Let us assume therefore an adiabatic type compression, so that

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0} \right)^{\gamma} ,$$

where, as before, γ is the adiabatic constant and ρ is density. Let us also write

$$P = p + P_0 ,$$

where p is excess, while P_0 ambient pressure. Furthermore, if in this one dimensional system we denote particle displacement by y, then

$$y_T = u . \tag{64}$$

Also, from the principle of conservation of mass we have that

$$\frac{\rho}{\rho_0} = \frac{1}{y_X}$$

Connecting all these relationships, we arrive at

$$p = P_0 \left[(y_X)^{-\gamma} - 1 \right] \tag{65}$$

Using now (57'), (64) and (65), we note that

$$p = P_0 \left\{ \left[\frac{\partial}{\partial X} \int^t \frac{v(X,T)}{A(t)} \, dT \right]^{-\gamma} - 1 \right\} \tag{66}$$

We can perform the integration indicated in (66), for an average value A of A(t), from (63), and then differentiate the result, to obtain

$$P = P_0 \left\{ \left[f(X) + \sum_{n=1}^{\infty} n \, \ell n \left[\tanh \frac{n}{2}(\alpha + \delta T) \right] \cos n X \right]^{-\gamma} - 1 \right\} \tag{67}$$

with f an arbitrary function of location. Note that the condition of zero excess pressure can be satisfied by this function; for we can set it equal to the negative of the summation + 1 at time $t = 0$. Furthermore, the arbitrary constant α can be used to satisfy yet another condition. However, to see the situation somewhat more clearly, let us introduce a normalizing f and take $\alpha = 0$. We can then rewrite (67) as an infinite product:

$$P = P_0 \left\{ \left[\ell n \prod_{n=1}^{\infty} \left[\frac{\tanh \, n\delta T/2}{\tanh \, n\delta/2} \right]^{n \cos n X} \right]^{-\gamma} - 1 \right\} \tag{67'}$$

Let us observe the following: As

$$A(t) \longrightarrow 0$$

$$T = \int \frac{dt}{\sqrt{A(t)}} \longrightarrow \infty$$

$$\text{Tanh } T \longrightarrow 1$$

$$\ell n \; [\tanh \; T] \longrightarrow 0$$

also. We see then, from (67), that because $\gamma > 0$, P increases without bound; so that

Cross Sectional area $\longrightarrow 0$

implies

Excess pressure $\longrightarrow \infty$

which is certainly a reasonable conclusion. On the other hand, as

$$A(t) \longrightarrow \infty$$

$$T = \int \frac{dt}{\sqrt{A(t)}} \longrightarrow 0$$

$$\tanh T \longrightarrow 0^{+}$$

$$\ell n \; (\tanh T) \longrightarrow -\infty$$

This, in turn, implies that as

Cross sectional area $\longrightarrow \infty$

Excess pressure $\longrightarrow$ Absolute minimum.

Of course, there are several other conclusions that could be drawn from this model - however, more research seems to be indicated. In particular, the complicated nature of the nonlinear mechanism. as exhibited in (67) and in (67') would indicate the necessity of a parametric study by computer. A first step was indeed taken in this direction; some results are given in Appendix I .

6.0 THE THREE-DIMENSIONAL CASE: BURGERS' EQUATION

The two basic papers in connection with Burgers' equations are those of E. Hopf and J. D. Cole, mentioned in the Bibliography. In the latter, the following statement is made: Since

$$v = -2\delta \frac{\theta_x}{\theta} ,$$

where

$$\theta_t = \theta_{xx} ,$$

is a solution of Burgers' equation

$$v_t + v v_x = \delta v_{xx} , \quad v = v(x,t) ,$$

we can extend this to the three-dimensional realm quite simply; however under certain restrictions. Namely,

$$u = -2\delta \frac{\Delta\phi}{\phi} , \quad \Delta\phi \equiv \phi_x + \phi_y + \phi_z ,$$

where

$$\phi_t = \delta \Delta^2\phi , \quad \Delta^2 \phi \equiv \phi_{xx} + \phi_{yy} + \phi_{zz}$$

is a solution of the three-dimensional Burgers' equation

$$u_t + u\Delta u = \delta \Delta^2 u, \quad u = u(x, y, z, t)$$

Unfortunately, it turns out that the restrictions mentioned by Cole are quite severe - indeed so much so, that the three dimensional solution given above has no practical value. Thus, this section is devoted to illustrate and to prove that point; and, in doing so, would point out a need for the introduction of a new approach. To approach this question formally, let us state and prove the following

Theorem:

If $\phi = \phi(x, y, z, t)$ is a function such that

$$\phi_t = \phi_{xx} + \phi_{yy} + \phi_{zz}$$

and $u = (x, y, z, t)$ is a function such that

$$u_t = u_{xx} + u_{yy} + u_{zz} - u u_x - u u_y - u u_z ,$$

then these two partial differential equations are in general <u>not</u> connected by

$$u = -2 \; \frac{\phi_x + \phi_y + \phi_z}{\phi}$$

<u>Proof:</u>

We shall use a concise vector notation. Thus, let

$$\vec{e} = (1, 1, 1)$$

$$\vec{\Delta}_u = (u_x, u_y, u_z)$$

$$\Delta^2_u = u_{xx} + u_{yy} + u_{zz}$$

then we have

$$\phi_t = \Delta^2 \phi$$

$$u_t = \Delta^2 u - u (\vec{e} \cdot \vec{\Delta} u)$$

and

$$u = - \frac{2}{\phi} (\vec{e} \, \vec{\Delta} \phi)$$

Now

$$u_t = - \frac{2}{\phi} (\vec{e} \cdot \vec{\Delta} \phi_t) + \frac{2}{\phi^2} (\vec{e} \cdot \vec{\Delta} \phi) \phi_t$$

$$= - \frac{2}{\phi} (\vec{e} \cdot \vec{\Delta} (\Delta^2 \phi)) + \frac{2}{\phi^2} (\vec{e} \cdot \vec{\Delta}_\phi) \Delta^2 \phi$$

$$= - \frac{2}{\phi} \Delta^2 (\vec{e} \cdot \vec{\Delta} \phi) + \frac{2}{\phi^2} \Delta^2 \phi \, (\vec{e} \cdot \vec{\Delta} \phi)$$

and

$$\vec{\Delta}\, u = -2\,\vec{\Delta}\left(\vec{e}\cdot\frac{\vec{\Delta}\phi}{\phi}\right)$$

$$= +\frac{2}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\phi)\,\vec{\Delta}\phi - \frac{2}{\phi}\,\vec{\Delta}\,(\vec{e}\cdot\vec{\Delta}\phi)$$

by direct expansion and recombination.

$$\vec{e}\cdot\vec{\Delta}\,u = \frac{2}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\phi)^2 - \frac{2}{\phi}\,\vec{e}\cdot\vec{\Delta}\,(\vec{e}\cdot\vec{\Delta}\phi)$$

$$-u\,(\vec{e}\cdot\vec{\Delta}\,u) = +\frac{4}{\phi^3}\,(\vec{e}\cdot\vec{\Delta}\phi)^3 - \frac{4}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\phi)\,(\vec{e}\cdot\vec{\Delta}\,(\vec{e}\cdot\vec{\Delta}\phi))$$

$$\Delta^2 u = -2\,\vec{\Delta}\cdot\vec{\Delta}\left(\vec{e}\cdot\frac{\vec{\Delta}\phi}{\phi}\right)$$

$$= \vec{\Delta}\cdot\left(\frac{2}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\phi)\,\vec{\Delta}\phi\right) - \vec{\Delta}\cdot\left(\frac{2}{\phi}\,\vec{\Delta}\,(\vec{e}\cdot\vec{\Delta}\phi)\right)$$

$$= \frac{2}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\phi)\,\Delta^2\phi + \vec{\Delta}\phi\cdot\vec{\Delta}\left(\frac{2}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\phi)\right)$$

$$- \frac{2}{\phi}\,\Delta^2\,(\vec{e}\cdot\vec{\Delta}\phi) - \vec{\Delta}\,(\vec{e}\cdot\vec{\Delta}\phi)\cdot\vec{\Delta}\left(\frac{2}{\phi}\right)$$

$$= \frac{2}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\phi)\,\Delta^2\phi - \frac{4}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\phi)\,\vec{\Delta}\phi\cdot\vec{\Delta}\phi$$

$$+ \frac{2}{\phi^2}\,\vec{\Delta}\phi\cdot\vec{\Delta}\,(\vec{e}\cdot\vec{\Delta}\phi) - \frac{2}{\phi}\,\Delta^2\,(\vec{e}\cdot\vec{\Delta}\phi) +$$

$$+ \frac{2}{\phi^2}\,\vec{\Delta}\phi\cdot\vec{\Delta}\,(\vec{e}\cdot\vec{\Delta}\phi)$$

$$= -\frac{2}{\phi}\,\Delta^2\,(\vec{e}\cdot\vec{\Delta}\,\phi) + \frac{2}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\,\phi)\,\Delta^2\phi \; +$$

$$+\frac{4}{\phi^2}\,\vec{\Delta}\phi\cdot\vec{\Delta}(\vec{e}\cdot\vec{\Delta}\,\phi) - \frac{4}{\phi^3}\,(\vec{e}\cdot\vec{\Delta}\phi)\,\vec{\Delta}\phi\cdot\vec{\Delta}\phi\,.$$

Summarizing then, (if $\phi_t = \Delta^2\phi$)

$$u_t = -\frac{2}{\phi}\,\Delta^2\,(\vec{e}\cdot\vec{\Delta}\,\phi) + \frac{2}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\,\phi)\,\Delta^2\phi$$

$$-u\,(\vec{e}\cdot\vec{\Delta}\,u) = +\frac{4}{\phi^3}\,(\vec{e}\cdot\vec{\Delta}\phi)^3 - \frac{4}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\phi)\,(\vec{e}\cdot\vec{\Delta}\,(\vec{e}\cdot\vec{\Delta}\,\phi))$$

$$\Delta^2 u = -\frac{2}{\phi}\,\Delta^2(\vec{e}\cdot\vec{\Delta}\,\phi) + \frac{2}{\phi^2}\,(\vec{e}\cdot\vec{\Delta}\,\phi)\,\Delta^2\phi$$

$$-\frac{4}{\phi^3}\,(\vec{e}\cdot\vec{\Delta}\phi)\,\vec{\Delta}\phi\cdot\vec{\Delta}\phi + \frac{4}{\phi^2}\,\vec{\Delta}\phi\cdot\vec{\Delta}(\vec{e}\cdot\vec{\Delta}\phi)$$

Before analyzing these equations we note that if we are dealing with <u>one</u> space dimension; i.e., E_1,

$$u_t = -\frac{2}{\phi}\,\phi_{xxx} + \frac{2}{\phi^2}\,\phi_x\,\phi_{xx}$$

$$-u\,(\vec{e}\cdot\vec{\Delta}\,u) = -u\,u_x = +\frac{4}{\phi^3}\,(\phi_x)^3 - \frac{4}{\phi^2}\,\phi_x\,\phi_{xx}$$

$$\Delta^2 u = u_{xx} = -\frac{2}{\phi}\,\phi_{xxx} + \frac{2}{\phi^2}\,\phi_x\,\phi_{xx}$$

$$-\frac{4}{\phi^3}\,(\phi_x)^3 + \frac{4}{\phi^2}\,\phi_x\,\phi_{xx}$$

which, as Hopf has shown, are perfectly consistent. However, for more than one space variable, consider those terms which have ϕ^3 in their denominators. We must have,

$$- \frac{4}{\phi^3} (\vec{e} \cdot \vec{\Delta} \phi)^2 = \frac{4}{\phi^3} (\vec{e} \cdot \vec{\Delta} \phi) \vec{\Delta} \cdot \vec{\Delta} \phi \quad ; \quad \phi \neq 0$$

$$\Rightarrow \qquad (\vec{e} \cdot \vec{\Delta} \phi)^2 = \vec{\Delta} \phi \cdot \vec{\Delta} \phi = || \vec{\Delta} \phi ||^2$$

But, we know that

$$(\vec{e} \cdot \vec{\Delta} \phi)^2 = (\phi_x + \phi_y + \phi_z)^2$$

$$\neq \phi_x^2 + \phi_y^2 + \phi_z^2 = || \vec{\Delta} \phi ||^2$$

Thus, Hopf's substitution may not be directly extended into two and three space variables. In fact, this can be done only if the flow field is assumed to be completely irrotational.

We thus conclude that the extensions of the solutions of Burgers' equation to a three dimensional domain is not feasible. Alternatives are discussed in the next two sections.

7.0 TWO-DIMENSIONAL NON-LINEAR WAVE EQUATIONS

We shall start our discussion of problems in a dimension higher than the first by an appropriate derivation of the equations employed.

Having a flow which, for physical reasons, admits an axis of symmetry, let x be the distance along this axis measured from a reference point and y the distance from this axis. By definition, there will be no flow normal to both x and y and the mass and momentum conservation equations reduce to the following ones:

$$\rho_t + u\,\rho_x + v\,\rho_y + \rho\,v_x + \rho\,v_y = 0 \tag{68}$$

$$u_t + u\,u_x + v\,u_y + p_x/\rho = \delta\,(u_{xx} + u_{yy}) \tag{69}$$

$$v_t + u\,u_x + v\,v_y + p_y/\rho = \delta\,(v_{xx} + v_{yy}) \tag{70}$$

$$\delta = \frac{1}{\rho_o}\left[\frac{4}{3}\,\mu + \mu_r + (\gamma - 1)\,\frac{k}{c_p}\right] \tag{71}$$

The above equations are exact except for the definition of the diffusion coefficient (δ). By introducing δ, the energy conservation equation is not considered explicitly. The diffusive effect of heat transfer is however taken under consideration by the factor: $(\gamma - 1)\,k/c_p\,\rho_o$. Lighthill has shown that the above approximation is equivalent to neglecting terms of order $(\nu\,\omega/a^2)\,(V/a)$ with respect to terms of order $(\nu\,\omega/a^2)$ or (V/a), where $\nu = \mu/\rho$, ω is the sound frequency, a is the local speed of sound, and V the sound velocity amplitude. All the following developments of Equations (68), (69), and (70) will imply this approximation.

The diffusion coefficient (δ) was amply discussed from basic principles in our previous report. The conclusions are that δ can be estimated accurately enough for engineering purposes, but the accuracy of the estimate is different for different media. Its final value, however, needs experimental verification for each medium and frequency range which can be achieved by measuring the absorption coefficient or by comparing calculated and distorted plane waves.

The next approximation is to assume isentropic process in a perfect gas:

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} = \left(\frac{a}{a_o}\right)^{\frac{2\gamma}{\gamma - 1}} \tag{72}$$

The above assumption is in contradiction with a $\delta \neq 0$. However it is a well established fact that entropy generation is a second or third order phenomenon from which it follows that Equation (73) holds even for shocks with Mach numbers up to 1.5 to 2.0. Equation (72) establishes a direct correspondence between ρ, p, and the local speed of sound a. This leads to the following relationships:

$$\rho_t = \rho_a a_t = \frac{2}{\gamma - 1} \frac{\rho_o}{a_o} \left(\frac{a}{a_o}\right)^{\frac{2}{\gamma - 1} - 1} a_t$$

$$\rho_x = \rho_a a_x = \frac{2}{\gamma - 1} \frac{\rho_o}{a_o} \left(\frac{a}{a_o}\right)^{\frac{2}{\gamma - 1} - 1} a_x$$

$$p_x = p_a a_x = \frac{2\gamma}{\gamma - 1} \frac{p_o}{a_o} \left(\frac{a}{a_o}\right)^{\frac{2\gamma}{\gamma - 1} - 1} a_x \qquad (73)$$

Similar relationships can also be written for ρ_y and p_y. Substituting Equation (73) and similar into Equations (68), (69), and (70), it is found:

$$a_t + u a_x + v a_y + \frac{\gamma - 1}{2} a \left[u_x + v_y\right] = 0 \qquad (74)$$

$$u_t + u u_x + v u_y + \frac{2}{\gamma - 1} a a_x = \delta \left[u_{xx} + u_{yy}\right] \qquad (75)$$

$$v_t + u v_x + v v_y + \frac{2}{\gamma - 1} a a_y = \delta \left[v_{xx} + v_{yy}\right] \qquad (76)$$

Evidently the above equations constitute a system of three nonlinear partial differential equations in the variables $a = a(x; y; t)$; $u = u(x; y; t)$; $v = v(x; y; t)$, where a is the local speed of sound and u and v are the components of the local particle velocity.

The above system can be reduced to a system of two nonlinear partial differential equations in two unknowns (the excess wavelet velocity in the x and y directions) which, for the one dimensional case reduce to Burgers' equation. Again, the error introduced can be shown to be of order $(\nu\, \omega/a^2)$ (V/a) with respect to orders $(\nu\, \omega/a^2)$ or (V/a); we demonstrated this in a previous report.

The method is similar to that used to derive the equation for spherical and cylindrical waves (Section 4.0) but somewhat more complicated. The result is:

$$V_t + (U + a_o)\, U_x - \frac{\delta}{2}\, U_{xx} + \left\{ \left(\frac{V}{2} + \frac{a_o}{2} \right) V_y + \frac{V U_y}{\gamma + 1} - \frac{\varepsilon}{2}\, U_{yy} \right\} = 0 \tag{77}$$

$$V_t + (V + a_o)\, V_y - \frac{\delta}{2}\, V_{yy} + \left\{ \left(\frac{U}{2} + \frac{a_o}{2} \right) U_x + \frac{U V_x}{\gamma + 1} - \frac{\delta}{2}\, V_{xx} \right\} = 0 \tag{78}$$

where $U = a + u - a_o$ excess wavelet velocity in the x direction

$V = a + v - a_o$ excess wavelet velocity in the y direction

Changing the fixed reference system $(x\,;y)$ to a system moving with the wave $(X = x - a_o t;\ Y = y - a_o t)$ it is found:

$$U_t + U U_X - \frac{\delta}{2}\, U_{XX} + \left\{ \left(\frac{V}{2} + \frac{a_o}{2} \right) V_y + \left(\frac{V}{\gamma + 1} - a_o \right) U_y - \frac{\delta}{2}\, U_{YY} \right\} = 0 \tag{79}$$

$$V_t + V V_Y - \frac{\delta}{2}\, V_{YY} + \left\{ \left(\frac{U}{2} + \frac{a_o}{2} \right) U_X + \left(\frac{U}{\gamma + 1} - a_o \right) V_X - \frac{\delta}{2}\, V_{XX} \right\} = 0 \tag{80}$$

It can be noticed that Equations (79) and (80) reduce to Burgers' equation for the one dimensional case. It should also be noticed that, the excess wavelet velocities cannot be neglected in comparison with the speed of sound in nonlinear problems in general.

8.0 A DIFFERENT GENERALIZATION

As a method of approach to any of the pairs of equations (75-76), (77-78) or (79-80), one might try the following.

We have seen in Section 6.0 that no physically meaningful solutions can be constructed for the equation

$$v_t + v v_x + v v_y = \delta v_{xx} + \delta v_{yy}$$

On the other hand, the above mentioned pairs of equations suggest that one try to solve, as an interim step, the equation

$$v_t + v v_x = \delta v_{xx} + \delta v_{yy} \tag{81}$$

We would like to discuss this problem briefly here.

If the second term on the right hand side of (81) is missing, then we have a Burgers' equation which we know how to solve. Furthermore, it is precisely this last term which shows the dependence of v on a second spatial variable y. This suggests that we try to find solutions for (81) in terms of solutions of Burgers' equation. In particular, let us assume that

$$v = k \frac{\theta_x}{\theta} \tag{82}$$

where k is constant, while θ is same as yet unspecified positive function. Then, after many transformations, (81) becomes

$$\left(\frac{\theta_t - \delta \theta_{xx}}{\theta}\right)_x + (k + 2\delta) \left(\frac{\theta_x}{\theta}\right)^3 = \delta \left(\frac{\theta_x}{\theta}\right)_{yy} \tag{83}$$

It is clear from (84) why, in the absence of its right hand side, the assumptions

$$\theta_t = \delta \theta_{xx} \tag{84}$$

$$k = -2\delta \tag{85}$$

in (82) solve Burgers' equation.

Suppose, however, that in (83) we keep assumptions (84), but dispense with (85) and leave the constant k undetermined for the moment; then denoting

$$k_1 = \frac{k + 2\delta}{\delta} , \tag{86}$$

Equation (83) becomes

$$w'' = k_1 w^3 , \tag{87}$$

where

$$w = \frac{\theta_x}{\theta} \tag{88}$$

and where the primes denote differentiations with respect to y. Without going into details of the solution, let us observe that (87) can be solved in the following way:

$$w(y) = w_o \, C_n\left(\frac{1}{\sqrt{2}}, \sqrt{k_1} \; w_o y\right) \tag{89}$$

Here w_o is the value of w at $y = 0$, while C_n is the elliptic cosine function. From (89), thus, w_o is an arbitrary function of X and of t, so that - with 88 - we have

$$w(y) = \frac{\theta_x(x, y; t)}{\theta(x, y; t)} = w_o(x,t)\, C_n\left[\frac{1}{\sqrt{2}}, \sqrt{k_1} \; w_o(x,t)\, y\right] \tag{90}$$

From (90),

$$\theta = F(y,t) \exp\left[\int^x w_o(\zeta, t)\, C_n\left[\frac{1}{\sqrt{2}}, k_1 w_o(\zeta, t)\, y\right] d\zeta\right] , \tag{91}$$

where F is an arbitrary function. In order for this analysis to be complete, we must choose F and w_o so that (91) will satisfy its assumed Equation (84).

Note, however, that we need only take $k = 1$ in (82) in order for $w = v$. Thus, (90) says that if on the line $y = 0$ there exists a periodic boundary excitation, then it will remain periodic. Furthermore, if it is not periodic on $y = 0$, the elliptic cosine functions will make it nearly so at large distances from the plane of $y = 0$. Finally, this periodicity, as well as an exponential type - although quite complicated - damping exists also; the first with respect to x, the second with respect to t. (This can be seen by noting that θ is a solution of the heat Equation (84); and by noting that the logarithmic derivatives of such solutions behave very much like the Jacobian Zeta functions, which are periodic in x and damped for $t \to \infty$).

What this qualitative analysis can show, then, is that an investigation of Equation (81) from a quantitative point of view might be quite rewarding; and that, furthermore, Equation (81) is an acceptable analytical vehicle on a posteriori grounds.

9.0 FIRST PROPAGATION MODEL EQUATION

In the past several years a great deal of effort, including that presented here, has been expended on Burgers' equation. It is well, we think, to recall the circumstances why this happened.

Burgers' introduced his model in 1942. As a matter of fact, he gave as a model a pair of equations, involving partial and ordinary derivatives. He then obtained some approximate results from those, after reducing them to the simple equation that is named for him today. These results were well received, but not too much was done with them, because the exact solution of that equation was unknown. It was only after 1956 and 1957, when Cole and Hopf, respectively, published their solutions, that so much work began to be based on this equation. There exists work in a great many fields, which has as its aim the reduction of problems to Burgers' equation.

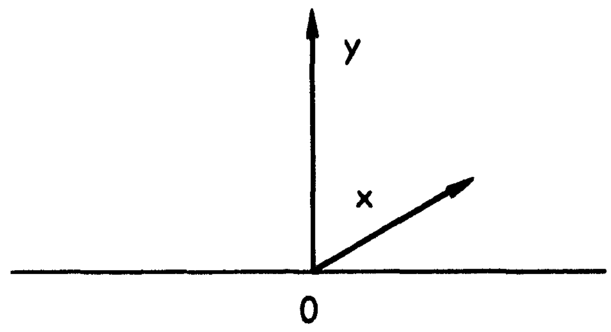
This is of course not surprising. After all, the number of physically significant nonlinear partial differential equations, for which an exact solution is known, is extremely small - probably less than a dozen.

The success of Burgers' was due to two factors. First, the equations that he postulated turned out to be excellent analytical vehicles. Secondly, the approximate solutions he had originally obtained were also very good.

Let us put a stress on the point that he postulated his equations, instead of deriving them from, say, the Navier-Stokes equations. In the meanwhile, of course, the justification of a derivation has been completed also. However, since no equations are exact, it seems to be a very reasonable approach to postulate equations, note what the consequent solutions are and then on these a posteriori grounds, to lay a proper formulation under the analytical vehicle used.

It is exactly this which we shall address ourselves to now: namely, the postulations of equations and a brief examination of their solutions. Thus, in this section, we will give the brief description of a particular set of equations, intended to serve as a model for other, more appropriate, sets. It was chosen for illustration because of its particular simplicity; for more details, see the paper by Kampé de Feriet.

Let us consider the physical model of sound radiation from a uniformly distributed source on the ground, with say, radial coordinate x, vertically upwards, in the direction of increasing y. All motion will be assumed to take place in a plane parallel to the plane of x y.



In a nondimensional system of units, under the assumptions of incompressibility and non-viscousness, an irrotational motion will define a velocity potential ϕ (see formula (6), for example, which is a quasi-potential) such that the two velocity components u, v are related to it by

$$u(x, y, t) = -\phi_x(x, y, t) \tag{92}$$

$$v(x, y, t) = -\phi_y(x, y, t), \tag{93}$$

while ϕ itself satisfies

$$\phi_{xx} + \phi_{yy} = 0, \tag{94}$$

because of the incompressibility condition.

Furthermore, since we expect the vertical velocity component to be of periodic type in time, while undergoing exponential damping with increasing altitude, we also postulate for ϕ the equation

$$\phi_y = \phi_{tt}, \tag{95}$$

which, incidentally, also agrees with the linear approximation.

Let us observe now that the function ϕ, which depends on the three variables x, y, t, $\phi = \phi(x, y, t)$, is considered as a function of two variables (x and y) in (94), and of two <u>other</u> variables (y and t) in (95). Thus, as a <u>consequence</u>, ϕ also satisfies

$$\phi_{tttt} + \phi_{yy} = 0; \tag{96}$$

an equation known in connection with the transverse vibrations of a bar.

It is rather simple to obtain solutions for the pair (94-95). One can start from the four simplest ones, i.e.,

$$\phi = \begin{cases} e^{-\lambda y} \cos \lambda x \cos \sqrt{\lambda} t \\ e^{-\lambda y} \cos \lambda x \sin \sqrt{\lambda} t \\ e^{-\lambda y} \sin \lambda x \sin \sqrt{\lambda} t \\ e^{-\lambda y} \sin \lambda x \cos \sqrt{\lambda} t \end{cases} \tag{97}$$

Linear combinations of these give two superposed waves, well known from linear acoustical theory, with one going towards $x = \infty$, and the other in the opposite direction. That is, if

$$\phi = e^{-\lambda y} \left\{ \cos \sqrt{\lambda} t \left[A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x \right] + \right.$$

$$\left. + \sin \sqrt{\lambda} t \left[C(\lambda) \cos \lambda x + D(\lambda) \sin \lambda x \right] \right\} , \tag{98}$$

then

$$\phi = \phi_1 + \phi_2 , \tag{99}$$

where

$$\phi_1 = \frac{A(\lambda) + D(\lambda)}{2} \cos (\lambda x - \sqrt{\lambda} t) + \frac{B(\lambda) - C(\lambda)}{2} \sin (\lambda x - \sqrt{\lambda} t)$$

$$\phi_2 = \frac{A(\lambda) - D(\lambda)}{2} \cos (\lambda x + \sqrt{\lambda} t) + \frac{B(\lambda) + C(\lambda)}{2} \sin (\lambda x - \sqrt{\lambda} t) \tag{100}$$

(Compare these, for instance, to the results we obtained for the velocity functions, with particles excited by a periodic-type piston motion; formula (32)).

Thus, the general solution of the pair of equations (94-95) can be written as

$$\phi_1 (x, y, t) = \int_0^{\infty} \left[e^{-\lambda y} \cos \lambda x \cos \sqrt{\lambda t} \right] f(\lambda) \, d\lambda \ , \tag{101}$$

where the integrand contains in brackets the first function of (97); and also a ϕ_2, ϕ_3, ϕ_4, which are identical to (101) except that in the integrand the other functions of (97) are used. In fact, it is possible to combine all these, so that, using complex notations, we have

$$\phi (x, y, t) = \phi_1 + \phi_2 + \phi_3 + \phi_4 =$$

$$= \int_{-\infty}^{\infty} \exp \left[i y \lambda - t \left| \lambda \right| - x \lambda^2 \right] f(\lambda) \, d\lambda \ . \tag{102}$$

This is a rather good representation, for it can be shown that every non-negative solution of (94-95) which is Lebesgue integrable on

$$\begin{aligned} -\infty < X < \infty \\ y > 0 \\ t > 0 \end{aligned} \tag{103}$$

can be expressed in the form (102).

According to this, then, we have from (92), (93), and (102) that the velocity components will be given by integrals of the form

$$u (x, y, t) = \int_{-\infty}^{\infty} \exp \left[i y \lambda - t \left| \lambda \right| - x \lambda^2 \right] \lambda^2 f_1(\lambda) \, d\lambda \tag{104}$$

$$v (x, y, t) = \int_{-\infty}^{\infty} i \exp \left[i y \lambda - t \left| \lambda \right| - x \lambda^2 \right] \lambda f_2(\lambda) \, d\lambda \tag{105}$$

Note that, in general, the real component of (104) will be the same as the imaginary component of (105) and vice versa. This means, of course, that while one component contains in it motion which is essentially periodic propagation, the other will contain damping.

With this, we are now ready to pass on to our next model.

10.0 SECOND PROPAGATION MODEL EQUATION

We will now consider an axisymmetric flow, dependent on radial and longitudinal locations as well as on time. We will use for our velocity components u and v:

$$\begin{aligned} u &= u(r, y, t) = \text{longitudinal component} \\ v &= v(r, y, t) = \text{radial component} \end{aligned} \tag{106}$$

We will further assume that v is completely determined by u but not vice versa. In particular, on the basis of previous developments, we shall postulate the relationship connecting u and v:

$$v(r, y, t) = -2\delta \frac{u_r(r, y, t)}{u(r, y, t)} , \tag{107}$$

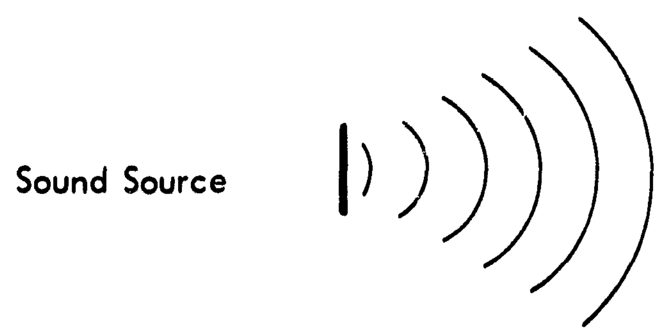
which of course is in keeping with our previous results. As a consequence, we also have, for δ = constant,

$$u(r, y, t) = K(y, t) \exp\left[-\frac{1}{2\delta} \int^r v(R, y, t)\, dR \right] , \tag{108}$$

where

$$K(y, t) = \text{arbitrary}, \tag{109}$$

so that the longitudinal component does not completely depend on the radial one. The physical meaning of this can be clarified by means of the following diagram:



In the neighborhood of the sound source there is a negligibly small vertical velocity component, and themotion is essentially one dimensional. However, as we proceed forward from the source, three dimensional dissipation, with its attendant vertical velocity component, takes place. The position where this begins to become significant is essentially dependent on what the horizontal component was initially; and therefore, the vertical one ought to be completely determined by it. On the other hand, the values of the horizontal component ought not depend on the nature or values of the particular three dimensional dissipation that is present, except in so far as the conservation of the total energy is concerned. These considerations are well reflected in formulas (107)-(108). Note that when the horizontal component is very large, $u = \infty$, then almost no radial motion exists (formula (107)), unless the radial component is such that

$$u_r \longrightarrow \infty \qquad \text{as} \qquad u \longrightarrow \infty$$

On the other hand, from formula (108) we have that, essentially the larger the total loss of energy from the horizontal component to the radial one is, the smaller the horizontal component (and, of course, its energy content) becomes. From the same formula we also see that if the medium is such that no radial component can be sustained, then the total energy remains and resides in the horizontal one.

We will now further assume, on the basis of our previous work, that the vertical component satisfies the equation

$$v_t + v v_r = \delta v_{rr} \tag{110}$$

for the same δ as in (107) and (108), and that the horizontal one is propagated according to the linear approximation,

$$u_{tt} = c^2 u_{rr} , \tag{111}$$

which is the linear wave equation.

We observe that once again, as in the previous model, both u and v are functions of the three variables r, y, and t. However, in their defining relationships (110) and (111) this is not apparent; (110) contains differentiations in time and radially, while (111) uses time and the longitudinal directions

Let us differentiate now (107), once with respect to t and twice with respect to r. We arrive at

$$u_t = \left[K_t(y,t) - \frac{1}{2\delta} K(y,t) \int^r v_t(R,y,t)\, dR\right] \exp\left[\int^r v(R,y,t)\, dR\right]$$

$$u_{rr} = \left[\frac{1}{4\delta^2} v^2(r,y,t) - \frac{1}{2\delta} v_r(r,y,t)\right] R(y,t) \exp\left[\int^r v(R,y,t)\, dR\right] ,$$

so that

$$u_t(r,y,t) - \delta\, u_{rr}(r,y,t) =$$

$$= -2\delta\left[\frac{K_t(y,t)}{K(y,t)} + \int^r v_t(R,y,t) + \frac{1}{2}v^2(r,y,t) - \delta\, v_r(r,y,t)\right]\left(\frac{K(y,t)}{-2\delta}\right) \times$$

$$\times \quad \exp\left[\int^r v(R,y,t)\, dR\right] \tag{112}$$

If we observe that interpation of (110) will yield

$$\int^r v_t(R,y,t) + \frac{1}{2}v^2(r,y,t) = \delta\, v_r(r,y,t) + A(y,t) \quad , \tag{110'}$$

with $A(y,t)$ an arbitrary function, and if we identify the first term in the first bracket of (112) as $A(y,t)$, then we see that the right hand side of (112) has to vanish. This implies then that

$$u_t = u_{rr} \; ; \tag{113}$$

so that the longitudinal velocity component u, in addition to satisfying (111), also satisfies (113). This part therefore points the way immediately for the computation of u and of v.

Let us collect these results and consider them in terms of a potential, in a way similar to the previous section: We say that there exists a potential $\phi = \phi(r,y,t)$, such that

$$\begin{cases} u(r,y,t) = \phi(r,y,t) \\ v(r,y,t) = -2\delta\,\dfrac{\phi_r(r,y,t)}{\phi(r,y,t)} \end{cases} \tag{114}$$

where ϕ satisfies <u>both</u>

$$\begin{cases} \phi_{tt} = c^2 \phi_{yy} \\ \phi_t = \delta \, \phi_{rr} \end{cases} , \tag{115}$$

for some viscosity number δ and wave number c.

It is rather interesting to compare (115) to (6), where we were considering one dimensional propagation with velocity component v and with a density function ρ. There we had a potential ϕ_x also, satisfying only the second equation of (115):

$$(\phi_x)_t = \delta \, (\phi_x)_{xx}$$

and with

$$\rho \, (x, t) = \phi_x \, (x, t)$$

$$v \, (x, t) = - \delta \, \frac{\phi_{xx} (x, t)}{\phi_x (x, t)}$$

which indeed is a striking similarity.

Let us see now how we could obtain solutions for (115). It seems that we can proceed as in the previous section, and write for our fundamental solutions

$$\phi \, (r, y, t) = e^{-kt} \, \begin{matrix} \cos \\ \sin \end{matrix} \left(\sqrt{\frac{k}{\delta}} \, r \right) \, \begin{matrix} \cosh \\ \sinh \end{matrix} \left(\frac{k}{c} \, y \right) , \tag{116}$$

where in (116) we mean any of the four variations possible. The general solution can then be written as

$$\delta \, (r, y, t) = \sum_{i=1}^{i=4} \int e^{-\zeta t} \, \begin{matrix} \cos \\ \sin \end{matrix} \left(\sqrt{\frac{\zeta}{\delta}} \, r \right) \, \begin{matrix} \cosh \\ \sinh \end{matrix} \left(\frac{\zeta}{c} \, y \right) \, P_i \, (\zeta) \, d\zeta \tag{117}$$

From (117) we obtain the formulas for the velocity components u and v:

$$u\,(r,y,t) = \sum_{i=1}^{i=4} \int e^{-\zeta t} \genfrac{}{}{0pt}{}{\cos}{\sin}\left(\sqrt{\frac{\zeta}{\delta}}\; r\right) \genfrac{}{}{0pt}{}{\cosh}{\sinh}\left(\frac{\zeta}{c}\, y\right) P_i\,(\zeta)\, d\zeta \tag{118}$$

and

$$v\,(r,y,t) = -\,2\sqrt{\delta}\; \frac{\displaystyle\sum_{i=1}^{i=4} \int \sqrt{\zeta}\; e^{-\zeta t} \genfrac{}{}{0pt}{}{(-\sin)}{(\ \cos)}\left(\sqrt{\frac{\zeta}{\delta}}\; r\right) \genfrac{}{}{0pt}{}{\cosh}{\sinh}\left(\frac{\zeta}{c}\, y\right) P_i\,(\zeta)\, d\zeta}{\displaystyle\sum_{i=1}^{i=4} \int e^{-\zeta t} \genfrac{}{}{0pt}{}{\cos}{\sin}\left(\sqrt{\frac{\zeta}{\delta}}\; r\right) \genfrac{}{}{0pt}{}{\cosh}{\sinh}\left(\frac{\zeta}{c}\, y\right) P_i\,(\zeta)\, d\zeta} \tag{119}$$

To illustrate one of the features of the solution, in the absence of any boundary conditions, let us return to the solutions which are the simplest ones conceptually, i.e., 116. If we use one term only - which is extremely crude, because it does not allow, among other things, any nonlinear interaction - then we have

$$\phi\,(r,\ y,\ t) = e^{-kt} \cos\left(\sqrt{\frac{k}{\delta}}\; r\right) \cosh\left(\frac{k}{c}\; y\right) \quad ,$$

with k some characteristic number. Then we have

$$\left\{\begin{aligned} u\,(r,\ y,\ t) &= e^{-kt} \cos\left(\sqrt{\frac{k}{\delta}}\; r\right) \cosh\left(\frac{k}{c}\; y\right) \\ v\,(r,\ y,\ t) &= 2\sqrt{k\,\delta}\; \tan\left(\sqrt{\frac{k}{\delta}}\; r\right) \end{aligned}\right. \tag{120}$$

The one important feature which is illustrated here is that along the central axis of the sound source, normal to the source, there is a zero radial component at all times. In the general solution this would of course become a funnel, perhaps of exponential type, such that at the throat it is just a point, but widens out as it proceeds. Throughout the entire length of this funnel, the propagation is essentially one dimensional.

To give a somewhat more complete analysis of the general solution (118)-(119) is difficult in the absence of boundary and/or initial restrictions, which are tasks for the future. However, let us point out the following generalizations which are possible within this rather simple mathematical framework of reference:

1) In the first equations of (115), we can take $c = c(r)$, because the differentiations there are not in the radial directions. This would mean that we are not assuming the uniform applicability of the wave equation throughout the entire width of that we are considering; rather, the type of wave motion would depend on our relative location in the cross-section.

2) Similarly, in the second equation of (115) we may take the diffusion constant δ as a function of y, for the same reason that we would allow c to depend on r. Physically, this would allow us to treat flows where dimensional diffusion takes place selectively; for instance, there may be very little scattering of the wave at the source, with significant increases at farther distances.

As a concluding point on possible generalizations let us remember that as early as 1872, Boussinesq - followed later by Reynolds - announced his view that most flows of the nonlinear nature ought to be handled by statistical methods. The latter even gave appropriate equations for such frameworks of reference.

Now there are many possibilities in obtaining appropriate probability distributions for so-called random solutions. Random solutions themselves can arise as consequences of either a probabilistic equation, or probabilistic boundary conditions, or probabilistic forcing functions; or, indeed, from any combinations of those.

It seems very clear, that while one has to depend to a large extent on deterministic descriptors, it is nevertheless necessary, in order to understand the nature of physical phenomena, that at least the boundary conditions for many of these things ought to be stochastically described. This is because one can, in the strict sense, correlate experiment and theory only by means of averages; for it is averages which are observable. It so happens that if the typical integral in (117),

$$\phi(r, y, t) = \int e^{-\zeta t} \cos\left(\sqrt{\frac{\zeta}{\delta}}\, r\right) \cosh\left(\frac{\zeta}{c}\, y\right) P(\zeta)\, d\zeta \qquad (121)$$

is replaced by

$$\overline{\phi}(r, y, t) = \int e^{-\zeta t} \cos\left(\sqrt{\frac{\zeta}{\delta}}\, r\right) \cosh\left(\frac{\zeta}{c}\, y\right) dP(\zeta), \qquad (122)$$

where (122) is a Stieltjes-Riemann Integral, then (122) still satisfies our equation. However, it does so in a greatly generalized sense; for on the one hand it admits quite discontinuous, indeed even discrete, boundary and/or initial values; and, if the function P is of bounded variation, then (122) can be interpreted as a stochastic generalized integral and the solutions obtained from it as probabilistic ones.

It is interest to point out here that besides the new physico-mathematical formulation given to our problem in this section, another novelty arises also: and this in the method of the general solutions. To show this, let us consider again the set (115):

$$\begin{cases} \text{a)} & \phi_{tt} = c^2 \phi_{yy} \\ \text{b)} & \phi_t = \delta \phi_{rr} \end{cases} \tag{115}$$

It is well known that all of the solutions of (115a) can be expressed in the form

$$\phi(y, t) = \overline{F}(y + ct) + \overline{G}(y - ct) \quad , \tag{123}$$

where $\overline{F}$ and $\overline{G}$ are arbitrary functions. This is true when ϕ depends on y and t alone. Here, however, while r does not appear in (115a) explicitly, it is nevertheless an independent variable; so that actually (123) should be

$$\phi(r; y, t) = F(r; y + ct) + G(r; y - ct) \tag{123'}$$

We separated r by a semicolon here, to show that in this context it appears as a parameter only.

Let us introduce the change of variables

$$\begin{aligned} y + ct &= z_1 \\ y - ct &= z_2 \quad , \end{aligned} \tag{124}$$

in order to rewrite (123') further as

$$\phi(r; z_1, z_2) = F(r; z_1) + G(r; z_2) \tag{125}$$

Now, however, the ϕ of (125) has to satisfy (115b) also; which implies that F and G have to satisfy it too. Thus, because of (124),

$$\phi_t = c\phi_{z_1} - c\phi_{z_2} \tag{126}$$

and thus (115b) becomes

$$c\left(\phi_{z_1} - \phi_{z_2}\right) = \delta \phi_{rr} \tag{127}$$

In terms of the constituent functions F and G of ϕ (127) can be written as

$$c\left(F_{z_1} - G_{z_2}\right) = \delta\left(F_{rr} + G_{rr}\right), \tag{128}$$

because F does not depend on z_2, nor does G depend on z_1. Let us make (128) more explicit, while rewriting it:

$$\left[F_{z_1}(r; z_1) - \frac{\delta}{c} F_{rr}(r; z_1)\right] - \left[G_{z_2}(r; z_2) + \frac{\delta}{c} G_{rr}(r; z_2)\right] = 0 \tag{129}$$

Evidently, in (129), the first bracket does not depend on z_2, nor does the second bracket depend on z_1. Then as far as these two variables are concerned, each bracket is constant; i.e., they are functions of r alone. In this way we obtain a separation of variables, which in explicit form says that F and G have to satisfy, respectively, the heat equation and its adjoint:

$$F_{z_1} - \frac{\delta}{c} F_{rr} = P_{(r)} \tag{130}$$

$$G_{z_2} + \frac{\delta}{c} G_{rr} = P_{(r)} \tag{131}$$

A solution of these equations, with appropriate boundary conditions imposed on them, is the future task we mentioned above.

11.0 DIRECTIONS FOR FUTURE RESEARCH

The intention of the present report was to extend the results of our previous one (see Reference 1) from the analytical treatment of the one dimensional nonlinear acoustical field to that of higher physical dimensions. In the course of this, as it often happens, several new by products of significance - connected with the previous research - arose, and they are included here. This is why we are proposing, as one possible extension of our one dimensional results,

1. The exact solution (now possible) of the unapproximated piston problem.

As a second avenue in one-dimensional investigation, we would take the results of Section 5, and analyze them more carefully; both by computers and analytically. However, to return to the higher dimensional case: the task of

2. The solution of Equation (81), which was shown to be possible (section 8.0) ought to be carried out.

This solu on would give a physically significant generalization of Burgers' equation, and because of the possibility of an analytical solution, it is expected that important insights into the effect of higher dimensions could be gained.

Finally, as the third large area of investigation, which seems to be of importance on account of the results of Section 10.0, is the investigation of the physical assumptions

3. Investigation of the Physical Assumptions leading to equations such as 107-110-111, and

4. A qualitative analysis of the solutions of the system of Equations (115), from the viewpoint of acoustical theory.

All four of the above endeavors seem to be important tasks. We would conclude by saying, however, that the most immediate - and most striking - results would probably emerge from carrying (2) of the program above.

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APPENDIX I

As a first attempt at evaluating the usefulness of formulas (67) for the excess pressure p in a pipe of variable cross section,

$$p = P_0 \left\{ \left[f(X) + \sum_{n=1}^{\infty} n \, \ell n \ \tanh \frac{n}{2}(\alpha + \delta T) \ \cos nX \right]^{-\gamma} - 1 \right\} , \quad (67)$$

a limited parametric study by computers was performed for the series in (67). The reason for this is that it is rather difficult to establish convergence properties of the series. It may converge, as it probably does, for certain ranges of T and X, depending on α and on δ. Clearly, the limit of the n^{th} term is zero, but this is only necessary for convergence and not sufficient; however, none of the well known tests seem to be applicable to this series. It also ought to be pointed out that divergent values of the series are acceptable because of the negative exponent; indeed, in equilibrium, the series must diverge.

We considered the series form (67), with $\alpha = 0$ (probably unrealistic) and $\delta = 10^{-2}$ (probably too big). The computer experiment was finite; Table 1 gives the flow chart. From the results it appears that $\alpha = 0$ corresponds to equilibrium conditions and that for this value the series tends to diverge. A few sample printout pages are reproduced, following the flow chart.

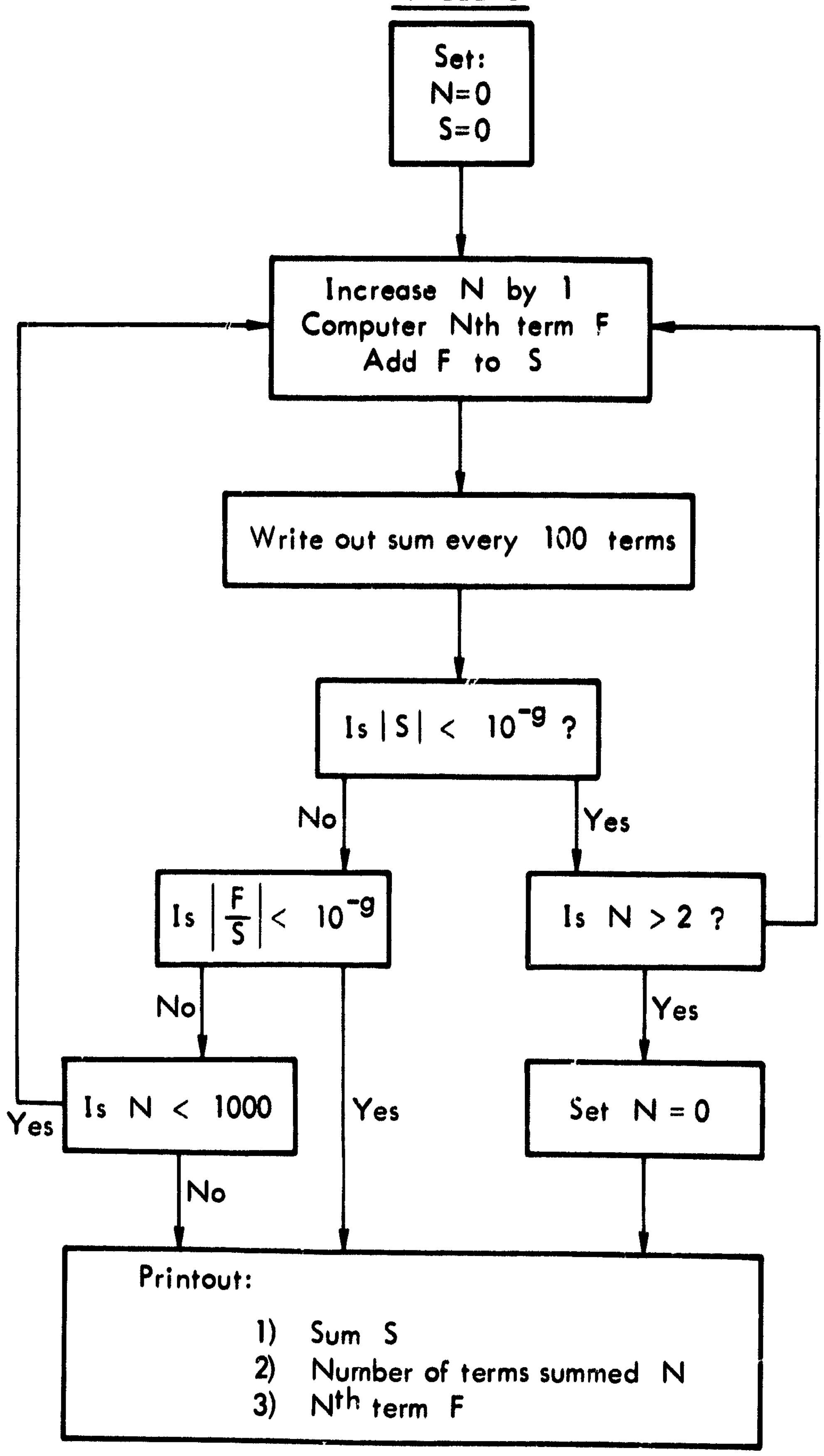
Thus, the series used was

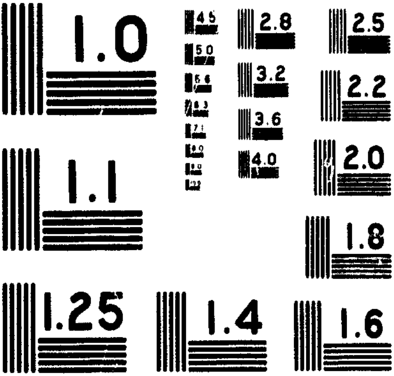
$$S = \sum_{n=1}^{\infty} n \, \ell n \left[\tanh \frac{n}{2} (.01T) \right] \cos nX$$

$$0 \leq X \leq 2\pi \qquad 0 < T < 1$$

The selected printouts show variation of T and of X on the range above.

TABLE I


Set:
N=0
S=0
Increase N by 1
Computer Nth term F
Add F to S
Write out sum every 100 terms
Is |S| < 10^{-g} ?
No
Yes
Is $\left|\frac{F}{S}\right| < 10^{-g}$
Is N > 2 ?
No
Yes
Yes
Is N < 1000
Yes
Set N = 0
Yes
No
Printout:
1) Sum S
2) Number of terms summed N
3) N^{th} term F



MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS—1963

```
CASE NUMBER     1  T=   0.99999999999999930-02   X=   0.0


SUM OF FIRST   0.99999999999999990D 02  TERMS IS  -0.29255891434269320D 05

SUM OF FIRST   0.20000000000000000D 03  TERMS IS  -0.10256355712713910D 06

SUM OF FIRST   0.29999999999999998D 03  TERMS IS  -0.21211755893056768D 06

SUM OF FIRST   0.39999999999999999D 03  TERMS IS  -0.35374885669115310D 06

SUM OF FIRST   0.50000000000000000D 03  TERMS IS  -0.52454445669457290D 06

SUM OF FIRST   0.59999999999999999D 03  TERMS IS  -0.72225859333163330D 06

SUM OF FIRST   0.70000000000000000D 03  TERMS IS  -0.94506247041590700D 06

SUM OF FIRST   0.79999999999999998D 03  TERMS IS  -0.11914125339712570D 07

SUM OF FIRST   0.89999999999999999D 03  TERMS IS  -0.14599742040021900D 07

SUM OF FIRST   0.99999999999999990D 03  TERMS IS  -0.17495718475023980D 07

S=  -0.17495718475023980 07  N=   0.99999999999999990D 03  F=  -0.29955512111766000 04

CASE NUMBER     2  T=   0.99999999999999930-02   X=   0.62831853071795700-01

S=  -0.10758890730447780 04  N=   0.25000000000000000 02  F=  -0.32058191664055590-12

CASE NUMBER     3  T=   0.99999999999999930-02   X=   0.12566370614359170 00


SUM OF FIRST   0.99999999999999990D 02  TERMS IS  -0.67066007377186950 02

SUM OF FIRST   0.20000000000000000 03  TERMS IS  -0.21901097483589660 03

SUM OF FIRST   0.29999999999999998D 03  TERMS IS  -0.36281032396105980 03

SUM OF FIRST   0.39999999999999999D 03  TERMS IS  -0.49706336684695990 03

SUM OF FIRST   0.50000000000000000D 03  TERMS IS  -0.62277794982229890 03

SUM OF FIRST   0.59999999999999999D 03  TERMS IS  -0.74102482387096910 03

SUM OF FIRST   0.70000000000000000D 03  TERMS IS  -0.85270185455321720 03

SUM OF FIRST   0.79999999999999998D 03  TERMS IS  -0.95853859348997300 03

SUM OF FIRST   0.89999999999999999D 03  TERMS IS  -0.10591300752662410 04

SUM OF FIRST   0.99999999999999990D 03  TERMS IS  -0.11549674608314000 04

S=  -0.11549674608314000 04  N=   0.99999999999999990D 03  F=  -0.29965651211176600 04
```

```
 CASE NUMBER    4  T=   0.9999999999999993D-02   X=   0.1884955592153874D 00

S=   0.1110391593929942D 04  N=   0.2500000000000000D 02   F=   0.3497257272442429D-12
 CASE NUMBER    5  T=   0.9999999999999993D-02   X=   0.2513274122871833D 00

SUM OF FIRST   0.9999999999999990D 02  TERMS IS  -0.2040762702743699D 03
SUM OF FIRST   0.2000000000000000D 03  TERMS IS  -0.3886685525148906D 03
SUM OF FIRST   0.2999999999999999D 03  TERMS IS  -0.5516677798745022D 03
SUM OF FIRST   0.3999999999999999D 03  TERMS IS  -0.6995575654426655D 03
SUM OF FIRST   0.5000000000000000D 03  TERMS IS  -0.8358510654969263D 03
SUM OF FIRST   0.5999999999999999D 03  TERMS IS  -0.9627394832719200D 03
SUM OF FIRST   0.7000000000000000D 03  TERMS IS  -0.1081719574640902D 04
SUM OF FIRST   0.7999999999999998D 03  TERMS IS  -0.1193878681385376D 04
SUM OF FIRST   0.8999999999999999D 03  TERMS IS  -0.1300042775156676D 04
SUM OF FIRST   0.9999999999999990D 03  TERMS IS  -0.1400860763153606D 04
S=  -0.1400860763153605D 04  N=   0.9999999999999990D 03  F=  -0.2996565121117660D 04
 CASE NUMBER    6  T=   0.9999999999999993D-02   X=   0.3141592653589791D 00

S=  -0.5037537742381037D 02  N=   0.4999999999999998D 01  F=  -0.1446642772566478D-13
```

```
CASE NUMBER  101  T=  0.19999999999999970-01  X=  0.0

SUM OF FIRST  0.99999999999999990D 02  TERMS IS  -0.25755519274395 70D 05
SUM OF FIRST  0.20000000000000000D 03  TERMS IS  -0.88632318743940910D 05
SUM OF FIRST  0.29999999999999998D 03  TERMS IS  -0.19082715911500280D 06
SUM OF FIRST  0.39999999999999999D 03  TERMS IS  -0.29917452780683680D 06
SUM OF FIRST  0.50000000000000000D 03  TERMS IS  -0.43776697213354130D 06
SUM OF FIRST  0.59999999999999999D 03  TERMS IS  -0.59736546996324020D 06
SUM OF FIRST  0.70000000000000000D 03  TERMS IS  -0.77514915778790130D 06
SUM OF FIRST  0.79999999999999998D 03  TERMS IS  -0.96958449839917700D 06
SUM OF FIRST  0.89999999999999999D 03  TERMS IS  -0.11793480070583510D 07
SUM OF FIRST  0.99999999999999990D 03  TERMS IS  -0.14032767180255500D 07
S=  -0.14032767180255500D 07  N=  0.99999999999999900D 03  F=  -0.23059106703521090D 04
CASE NUMBER  102  T=  0.19999999999999970-01  X=  0.62831853071795730D-01

S=  -0.97681938931828500D 03  N=  0.25000000000000000D 02  F=  -0.28733990611116410D-12
CASE NUMBER  103  T=  0.19999999999999970-01  X=  0.12566370614359170D 00

SUM OF FIRST  0.99999999999999990D 02  TERMS IS  -0.32414553784529540D 02
SUM OF FIRST  0.20000000000000000D 03  TERMS IS  -0.14972527472695860D 03
SUM OF FIRST  0.29999999999999998D 03  TERMS IS  -0.25891477062807080D 03
SUM OF FIRST  0.39999999999999999D 03  TERMS IS  -0.35858992552354980D 03
SUM OF FIRST  0.50000000000000000D 03  TERMS IS  -0.44976603906931520D 03
SUM OF FIRST  0.59999999999999999D 03  TERMS IS  -0.53352129264549420D 03
SUM OF FIRST  0.70000000000000000D 03  TERMS IS  -0.61076095498908020D 03
SUM OF FIRST  0.79999999999999998D 03  TERMS IS  -0.68222194942347490D 03
SUM OF FIRST  0.89999999999999999D 03  TERMS IS  -0.74850664654613350D 03
SUM OF FIRST  0.99999999999999990D 03  TERMS IS  -0.81011350345328980D 03
S=  -0.81011350345328980D 03  N=  0.99999999999999900D 03  F=  -0.23059106703521090D 04
```

```
CASE NUMBER  104  T=  0.1999999999999997D-01  X=  0.1884955592153974D 00

S=  0.9991662569575845D 03  N=  0.2500000000000000D 02  F=  0.3134617157576336D-12
CASE NUMBER  105  T=  0.1999999999999997D-01  X=  0.2513274122871833D 00

SUM OF FIRST  0.9999999999999999D 02  TERMS IS  -0.1694213547804029D 03
SUM OF FIRST  0.2000000000000000D 03  TERMS IS  -0.3193687068781727D 03
SUM OF FIRST  0.2999999999999998D 03  TERMS IS  -0.4477401818919124D 03
SUM OF FIRST  0.3999999999999999D 03  TERMS IS  -0.5610271752268907D 03
SUM OF FIRST  0.5000000000000000D 03  TERMS IS  -0.6627501909940483D 03
SUM OF FIRST  0.5999999999999999D 03  TERMS IS  -0.7551079616447847D 03
SUM OF FIRST  0.7000000000000000D 03  TERMS IS  -0.8396045483385277D 03
SUM OF FIRST  0.7999999999999998D 03  TERMS IS  -0.9173347719343063D 03
SUM OF FIRST  0.8999999999999999D 03  TERMS IS  -0.9891319506982725D 03
SUM OF FIRST  0.9999999999999990D 03  TERMS IS  -0.1055652322773748D 04
S=  -0.1055652322773749D 04  N=  0.9999999999999990D 03  F=  -0.2305910670352109D 04
CASE NUMBER  106  T=  0.1999999999999997D-01  X=  0.3141592653589791D 00

S=  -0.4651557758652311D 02  N=  0.4999999999999998D 01  F=  -0.1325547425736440D-13
```

```
 CASE NUMBER  201  T=   0.2999999999999998D-01   X=   0.0


SUM OF FIRST   0.9999999999999990D 02  TERMS IS  -0.2370806938648480D 05

SUM OF FIRST   0.2000000000000000D 03  TERMS IS  -0.8048415310392738D 05

SUM OF FIRST   0.2999999999999998D 03  TERMS IS  -0.1625288994519959D 06

SUM OF FIRST   0.3999999999999999D 03  TERMS IS  -0.2656830046795282D 06

SUM OF FIRST   0.5000000000000000D 03  TERMS IS  -0.3870477435194640D 06

SUM OF FIRST   0.5999999999999999D 03  TERMS IS  -0.5243953148954017D 06

SUM OF FIRST   0.7000000000000000D 03  TERMS IS  -0.6759184921941001D 06

SUM OF FIRST   0.7999999999999998D 03  TERMS IS  -0.8400998308469074D 06

SUM OF FIRST   0.8999999999999999D 03  TERMS IS  -0.1015634340279505D 07

SUM OF FIRST   0.9999999999999990D 03  TERMS IS  -0.1201379928796771D 07

S=  -0.1201379928796771D 07  N=   0.9999999999999990D 03  F=  -0.1904580857283373D 04

 CASE NUMBER  202  T=   0.2999999999999998D-01   X=   0.6283185307179570D-01


S=  -0.9182824549128681D 03  N=   0.2500000000000000D 02  F=  -0.2678946575595300D-12

 CASE NUMBER  203  T=   0.2999999999999998D-01   X=   0.1256637061435917D 00


SUM OF FIRST   0.9999999999999990D 02  TERMS IS  -0.1215140677118267D 02

SUM OF FIRST   0.2000000000000000D 03  TERMS IS  -0.1092271143800196D 03

SUM OF FIRST   0.2999999999999998D 03  TERMS IS  -0.1982224709396702D 03

SUM OF FIRST   0.3999999999999999D 03  TERMS IS  -0.2777566390906567D 03

SUM OF FIRST   0.5000000000000000D 03  TERMS IS  -0.3488572167985011D 03

SUM OF FIRST   0.5999999999999999D 03  TERMS IS  -0.4126145851109157D 03

SUM OF FIRST   0.7000000000000000D 03  TERMS IS  -0.4699460957388517D 03

SUM OF FIRST   0.7999999999999998D 03  TERMS IS  -0.5216006205821266D 03

SUM OF FIRST   0.8999999999999999D 03  TERMS IS  -0.5681923271113089D 03

SUM OF FIRST   0.9999999999999990D 03  TERMS IS  -0.6102313007340243D 03

S=  -0.6102313007340243D 03  N=   0.9999999999999990D 03  F=  -0.1904580857283373D 04
```

```
CASE NUMBER  204  T=   0.29999999999999980D-01   X=   0.18849555921538760D 00

S=   0.93410375577932880D 03   N=   0.25000000000000000D 02   F=   0.29224371733766910D-12

CASE NUMBER  205  T=   0.29999999999999980D-01   X=   0.25132741228718330D 00

SUM OF FIRST   0.99999999999999990D 02   TERMS IS   -0.14915217159271110D 03

SUM OF FIRST   0.20000000000000000D 03   TERMS IS   -0.27884691119362880D 03

SUM OF FIRST   0.29999999999999990D 03   TERMS IS   -0.38699451141593080D 03

SUM OF FIRST   0.39999999999999990D 03   TERMS IS   -0.48009979122283800D 03

SUM OF FIRST   0.50000000000000000D 03   TERMS IS   -0.56169341565410470D 03

SUM OF FIRST   0.59999999999999990D 03   TERMS IS   -0.63398847681512140D 03

SUM OF FIRST   0.70000000000000000D 03   TERMS IS   -0.69850057695532810D 03

SUM OF FIRST   0.79999999999999980D 03   TERMS IS   -0.75633653945958380D 03

SUM OF FIRST   0.89999999999999990D 03   TERMS IS   -0.80834163091479500D 03

SUM OF FIRST   0.99999999999999990D 03   TERMS IS   -0.85518386533099780D 03

S=   -0.85518386533099730D 03   N=   0.99999999999999900D 03   F=   -0.19045809572833730D 04

CASE NUMBER  206  T=   0.29999999999999980D-01   X=   0.31415926535897910D 00

S=   -0.44257739537894270D 02   N=   0.49999999999999980D 01   F=   -0.12548369950719130D-13
```

```
CASE NUMBER  301  T=  0.3999999999999999D-01  X=  0.0

SUM OF FIRST  0.9999999999999990D 02  TERMS IS  -0.2225542367058100D 05
SUM OF FIRST  0.2000000000000000D 03  TERMS IS  -0.7470409925194690D 05
SUM OF FIRST  0.2999999999999998D 03  TERMS IS  -0.1495519349320180D 06
SUM OF FIRST  0.3999999999999999D 03  TERMS IS  -0.2426483642867252D 06
SUM OF FIRST  0.5000000000000000D 03  TERMS IS  -0.3511068584244610D 06
SUM OF FIRST  0.5999999999999999D 03  TERMS IS  -0.4727152053675696D 06
SUM OF FIRST  0.7000000000000000D 03  TERMS IS  -0.6056851757639257D 06
SUM OF FIRST  0.7999999999999998D 03  TERMS IS  -0.7485215158236671D 06
SUM OF FIRST  0.8999999999999999D 03  TERMS IS  -0.8999450855435895D 06
SUM OF FIRST  0.9999999999999990D 03  TERMS IS  -0.1058842225037286D 07
S=  -0.1058842225037286D 07  N=  0.9999999999999990D 03  F=  -0.1622648183988879D 0
 CASE NUMBER  302  T=  0.3999999999999999D-01  X=  0.6283185307179570D-01

S=  -0.8767499220100252D 03  N=  0.2500000000000000D 02  F=  -0.2540981203837478D-12
 CASE NUMBER  303  T=  0.3999999999999999D-01  X=  0.1256637061435917D 00

SUM OF FIRST  0.9999999999999990D 02  TERMS IS   0.2218686502697720D 01
SUM OF FIRST  0.2000000000000000D 03  TERMS IS  -0.8052657095812872D 02
SUM OF FIRST  0.2999999999999998D 03  TERMS IS  -0.1552484599738546D 03
SUM OF FIRST  0.3999999999999999D 03  TERMS IS  -0.2205833171223039D 03
SUM OF FIRST  0.5000000000000000D 03  TERMS IS  -0.2775756995919554D 03
SUM OF FIRST  0.5999999999999999D 03  TERMS IS  -0.3273326840720095D 03
SUM OF FIRST  0.7000000000000000D 03  TERMS IS  -0.3707880123325456D 03
SUM OF FIRST  0.7999999999999998D 03  TERMS IS  -0.4087065369627548D 03
SUM OF FIRST  0.8999999999999999D 03  TERMS IS  -0.4417182129068642D 03
SUM OF FIRST  0.9999999999999990D 03  TERMS IS  -0.4703481746421447D 03
S=  -0.4703481746421447D 03  N=  0.9999999999999990D 03  F=  -0.1622648183988879D 04
```

CASE NUMBER 304 T= 0.3999999999999999D-01 X= 0.1884955592153874D 00

S= 0.8879413362184459D 03 N= 0.2500000000000000D 02 F= 0.2771979495095431D-12

CASE NUMBER 305 T= 0.3999999999999999D-01 X= 0.2513274122871833D 00

SUM OF FIRST 0.9999999999999990D 02 TERMS IS -0.1347737687680956D 03

SUM OF FIRST 0.2000000000000000D 03 TERMS IS -0.2501130534287335D 03

SUM OF FIRST 0.2999999999999998D 03 TERMS IS -0.3439458588657169D 03

SUM OF FIRST 0.3999999999999999D 03 TERMS IS -0.4227932983037579D 03

SUM OF FIRST 0.5000000000000000D 03 TERMS IS -0.4902053636310155D 03

SUM OF FIRST 0.5999999999999999D 03 TERMS IS -0.5484099573980920D 03

SUM OF FIRST 0.7000000000000000D 03 TERMS IS -0.5989400281484349D 03

SUM OF FIRST 0.7999999999999998D 03 TERMS IS -0.6429187352820744D 03

SUM OF FIRST 0.8999999999999999D 03 TERMS IS -0.6812073265987146D 03

SUM OF FIRST 0.9999999999999990D 03 TERMS IS -0.7144893995472874D 03

S= -0.7144893995472874D 03 N= 0.9999999999999990D 03 F= -0.1622648183988879D 04

CASE NUMBER 306 T= 0.3999999999999999D-01 X= 0.3141592653589791D 00

S= -0.4265577807226445D 02 N= 0.4999999999999998D 01 F= -0.1204667159606858D-13

CASE NUMBER 310 T= 0.3999999999999999D-01 X= 0.5654866776461627D 00

S= -0.2037633206456093D 03 N= 0.2500000000000000D 02 F= -0.1198791798038171D-11

CASE NUMBER 311 T= 0.39999999999999999D-01 X= 0.62831853071795860D 00

SUM OF FIRST 0.99999999999999990D 02 TERMS IS -0.18301525442654930D 03

SUM OF FIRST 0.20000000000000000D 03 TERMS IS -0.30753727077386770D 03

SUM OF FIRST 0.29999999999999990D 03 TERMS IS -0.40673152447404950D 03

SUM OF FIRST 0.39999999999999999D 03 TERMS IS -0.48936509934696550D 03

SUM OF FIRST 0.50000000000000000D 03 TERMS IS -0.55969718956489200D 03

SUM OF FIRST 0.59999999999999990D 03 TERMS IS -0.62026785859117890D 03

SUM OF FIRST 0.70000000000000000D 03 TERMS IS -0.67277925121678430D 03

SUM OF FIRST 0.79999999999999980D 03 TERMS IS -0.71845500979630050D 03

SUM OF FIRST 0.89999999999999990D 03 TERMS IS -0.75922126679694420D 03

SUM OF FIRST 0.99999999999999990D 03 TERMS IS -0.79280598906439610D 03

S= -0.79280598906439610D 03 N= 0.99999999999999990D 03 F= -0.15226481939889790D 04

CASE NUMBER 320 T= 0.39999999999999999D-01 X= 0.11938052083641160D 01

S= 0.10421290861304730D 03 N= 0.25000000000000000D 02 F= 0.62831535222163090D-11

CASE NUMBER 321 T= 0.39999999999999999D-01 X= 0.12566370614359120D 01

SUM OF FIRST 0.99999999999999990D 02 TERMS IS -0.19150547377504120D 03

SUM OF FIRST 0.20000000000000000D 03 TERMS IS -0.31733788522982180D 03

SUM OF FIRST 0.29999999999999980D 03 TERMS IS -0.41729649835879990D 03

SUM OF FIRST 0.39999999999999990D 03 TERMS IS -0.50047065290736370D 03

SUM OF FIRST 0.50000000000000000D 03 TERMS IS -0.57121768214704000D 03

SUM OF FIRST 0.59999999999999990D 03 TERMS IS -0.63212548177611730D 03

SUM OF FIRST 0.70000000000000000D 03 TERMS IS -0.68491917512795130D 03

SUM OF FIRST 0.79999999999999980D 03 TERMS IS -0.73083672691192340D 03

SUM OF FIRST 0.89999999999999990D 03 TERMS IS -0.77081351585242210D 03

SUM OF FIRST 0.99999999999999990D 03 TERMS IS -0.80558383472204590D 03

S= -0.80558383472204590D 03 N= 0.99999999999999990D 03 F= -0.16226481839888790D 04

CASE NUMBER 330 T= 0.39999999999999990-01 X= 0.18221237390920750 01

S= -0.47740345354758910 02 N= 0.25000000000000000 02 F= -0.13305501576459070-10

CASE NUMBER 331 T= 0.39999999999999990D-01 X= 0.18840555921528720 01

SUM OF FIRST 0.99999999999999900D 02 TERMS IS -0.19323239945810460 03

SUM OF FIRST 0.20000000000000000D 03 TERMS IS -0.31930112455598520 03

SUM OF FIRST 0.29999999999999990 03 TERMS IS -0.41939757475625790 03

SUM OF FIRST 0.39999999999999990 03 TERMS IS -0.50266905879069940 03

SUM OF FIRST 0.50000000000000000D 03 TERMS IS -0.57349110714621590 03

SUM OF FIRST 0.59999999999999990D 03 TERMS IS -0.63445969883263530 03

SUM OF FIRST 0.70000000000000000D 03 TERMS IS -0.68730430279883060 03

SUM OF FIRST 0.79999999999999980 03 TERMS IS -0.73126545993737020 03

SUM OF FIRST 0.89999999999999990 03 TERMS IS -0.77328021758861890 03

SUM OF FIRST 0.99999999999999900D 03 TERMS IS -0.80808400592709480 03

S= -0.80808400592709480 03 N= 0.99999999999999900D 03 F= -0.16226481839989790 04

CASE NUMBER 340 T= 0.39999999999999990-01 X= 0.24504422698000350 01

S= 0.26512181818037030 02 N= 0.25000000000000000 02 F= 0.12935904310445330-10

CASE NUMBER 341 T= 0.39999999999999990-01 X= 0.25132741228718290 01

SUM OF FIRST 0.99999999999999990D 02 TERMS IS -0.19379678939177980 03

SUM OF FIRST 0.20000000000000000D 03 TERMS IS -0.31994852151331350 03

SUM OF FIRST 0.29999999999999980 03 TERMS IS -0.42007756635443010 03

SUM OF FIRST 0.39999999999999990 03 TERMS IS -0.50337922708250060 03

SUM OF FIRST 0.50000000000000000D 03 TERMS IS -0.57422446856129550 03

SUM OF FIRST 0.59999999999999990D 03 TERMS IS -0.63521183520934300 03

SUM OF FIRST 0.70000000000000000D 03 TERMS IS -0.68807217149557150 03

SUM OF FIRST 0.79999999999999980 03 TERMS IS -0.73404680341291150 03

SUM OF FIRST 0.89999999999999990D 03 TERMS IS -0.77407329378523170 03

SUM OF FIRST 0.99999999999999900D 03 TERMS IS -0.80988742507892930 03

S= -0.80988742507892930 03 N= 0.99999999999999900D 03 F= -0.16226481839988790 04

```
CASE NUMBER  350  T=  0.39999999999999990-01  X=  0.30787608005179930 01

S=  0.31510270937625950 00  N=  0.25000000000000000 02  F=  -0.19958252364687090-10
CASE NUMBER  351  T=  0.39999999999999990-01  X=  0.31415926535897840 01


SUM OF FIRST  0.99999999999999900 02  TERMS IS  -0.19394140729975010 03
SUM OF FIRST  0.20000000000000000 03  TERMS IS  -0.32310150143588900 03
SUM OF FIRST  0.29999999999999980 03  TERMS IS  -0.42725119498762060 03
SUM OF FIRST  0.39999999999999990 03  TERMS IS  -0.50356027490700760 03
SUM OF FIRST  0.50000000000000000 03  TERMS IS  -0.57441129142510710 03
SUM OF FIRST  0.59999999999999990 03  TERMS IS  -0.63540337507542600 03
SUM OF FIRST  0.70000000000000000 03  TERMS IS  -0.68826764443680700 03
SUM OF FIRST  0.79999999999999980 03  TERMS IS  -0.73424564501845570 03
SUM OF FIRST  0.89999999999999990 03  TERMS IS  -0.77427506860524130 03
SUM OF FIRST  0.99999999999999900 03  TERMS IS  -0.80909178563882400 03
S=  -0.80909178563882400 03  N=  0.99999999999999900 03  F=  -0.16226481839988790 04
CASE NUMBER  401  T=  0.49999999999999980-01  X=  0.0


SUM OF FIRST  0.99999999999999900 02  TERMS IS  -0.21129739974399190 05
SUM OF FIRST  0.20000000000000000 03  TERMS IS  -0.70221942004663380 05
SUM OF FIRST  0.29999999999999980 03  TERMS IS  -0.13949227042901380 06

SUM OF FIRST  0.39999999999999990 03  TERMS IS  -0.22480036872274290 06
SUM OF FIRST  0.50000000000000000 03  TERMS IS  -0.32327531734197630 06
SUM OF FIRST  0.59999999999999990 03  TERMS IS  -0.43272484021162550 06
SUM OF FIRST  0.70000000000000000 03  TERMS IS  -0.55138497612162980 06
SUM OF FIRST  0.79999999999999980 03  TERMS IS  -0.67778855887123220 06
SUM OF FIRST  0.89999999999999990 03  TERMS IS  -0.81068872452942360 06
SUM OF FIRST  0.99999999999999900 03  TERMS IS  -0.94900812988121450 06
S=  -0.94900812988121450 06  N=  0.99999999999999900 03  F=  -0.14068291137472940 04
CASE NUMBER  402  T=  0.49999999999999980-01  X=  0.62831853071795700-01


S=  -0.84453487256580860 03  N=  0.25000000000000000 02  F=  -0.24339679394092590-12
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```
CASE NUMBER  403  T=  0.4999999999999998D-01  X=  0.1256637061435917D 00

SUM OF FIRST  0.9999999999999990D 02  TERMS IS   0.1335785316279536D 02
SUM OF FIRST  0.2000000000000000D 03  TERMS IS  -0.5829915621780455D 02
SUM OF FIRST  0.2999999999999999D 03  TERMS IS  -0.1220059077542412D 03
SUM OF FIRST  0.3999999999999999D 03  TERMS IS  -0.1764205226543260D 03
SUM OF FIRST  0.5000000000000000D 03  TERMS IS  -0.2226083340815558D 03
SUM OF FIRST  0.5999999999999999D 03  TERMS IS  -0.2616993397474663D 03
SUM OF FIRST  0.7000000000000000D 03  TERMS IS  -0.2946447606917117D 03
SUM OF FIRST  0.7999999999999998D 03  TERMS IS  -0.3222304593751313D 03
SUM OF FIRST  0.8999999999999999D 03  TERMS IS  -0.3451045919015623D 03
SUM OF FIRST  0.9999999999999990D 03  TERMS IS  -0.3638102162923040D 03
S=  -0.3638102162923040D 03  N=  0.9999999999999990D 03  F=  -0.1406829113747294D 04

CASE NUMBER  404  T=  0.4999999999999998D-01  X=  0.1884955592153874D 00

S=  0.8521351469496174D 03  N=  0.2500000000000000D 02  F=  0.2655237642991919D-12

CASE NUMBER  405  T=  0.4999999999999998D-01  X=  0.2513274122871833D 00

SUM OF FIRST  0.9999999999999990D 02  TERMS IS  -0.1236239197879500D 03
SUM OF FIRST  0.2000000000000000D 03  TERMS IS  -0.2278429704489764D 03
SUM OF FIRST  0.2999999999999999D 03  TERMS IS  -0.3106074715405083D 03
SUM OF FIRST  0.3999999999999999D 03  TERMS IS  -0.3784605037796920D 03
SUM OF FIRST  0.5000000000000000D 03  TERMS IS  -0.4349740757896197D 03
SUM OF FIRST  0.5999999999999999D 03  TERMS IS  -0.4823970557035262D 03
SUM OF FIRST  0.7000000000000000D 03  TERMS IS  -0.5222832102315686D 03
SUM OF FIRST  0.7999999999999998D 03  TERMS IS  -0.5557757780009404D 03
SUM OF FIRST  0.8999999999999999D 03  TERMS IS  -0.5837553613206208D 03
SUM OF FIRST  0.9999999999999990D 03  TERMS IS  -0.6069241083406431D 03
S=  -0.6069241083406431D 03  N=  0.9999999999999990D 03  F=  -0.1405829113747294D 04

CASE NUMBER  406  T=  0.4999999999999998D-01  X=  0.3141592653589791D 00

S=  -0.4141320035236458D 02  N=  0.4999999999999998D 01  F=  -0.1165752424147475D-13
```

APPENDIX II

In Section (4.0) we discussed the question of generalizing the one dimensional Burgers' equations, to include the possibility of propagation in a pipe of variable cross section. In this Appendix we point out the difficulties involved in attempting a solution for any but the simplest of even the regular geometries.

In that section we stated that if the area A is given as a function of location, $A = A(x)$, then Equation (52) replaces Burgers' equation:

$$V_t + \left(a_o + V - \frac{\delta}{2}\left(\frac{A_x}{A}\right)\right) V_x = \frac{\delta}{2} V_{xx} - \frac{\delta}{2} a_o \left(\frac{A_x}{A}\right) V \qquad \text{(A52)}$$

Also, we have four (4) particular cases of interest. They are:

Case 1	$\frac{A_x}{A} = \frac{2}{x}$		(Spherical)
Case 2	$\frac{A_x}{A} = \frac{1}{x}$		(Cylindrical)
Case 3	$\frac{A_x}{A} = c_o$		(Exponential)
Case 4	$\frac{A_x}{A} = 0$		(Planar)

We will treat cases 4 and 3. The former reduces to Burgers' equation, while the latter, simple as it is, remains intractable in our present framework of reference.

To simplify (A52), set $t \rightarrow Kt$

$V \rightarrow \rho V$;

$$V_t + \left(a_o K + K\rho V - \frac{K\delta}{2}\left(\frac{A_x}{A}\right)\right) V_x = \frac{K\delta}{2} V_{xx} - \frac{K\delta}{2} a_o \left(\frac{A_x}{A}\right) V$$

Now if $K = \frac{2}{\delta}$, then we obtain

$$V_t + \left(\frac{2\,a_o}{\delta} + \frac{2\,\rho}{\delta}\, V + \left(\frac{A_x}{A} \right) \right) V_x = V_{xx} - a_o \left(\frac{A_x}{A} \right) V$$

while if $\rho = \frac{\delta}{2}$, then

$$V_t + \left(\frac{2\,a_o}{\delta} + V + \left(\frac{A_x}{A} \right) \right) V_x = V_{xx} - a_o \left(\frac{A_x}{A} \right) V$$

or

$$V_t + V V_x = V_{xx} - \left(\frac{2\,a_o}{\delta} + \frac{A_x}{A} \right) V_x - a_o \left(\frac{A_x}{A} \right) V$$

which is a somewhat simpler form of (A52). We will use it as a working equation. In these terms then the 4 cases become the following:

Case 1: $$V_t + V V_x = V_{xx} - \left(\alpha + \frac{2}{x} \right) V_x - \frac{\beta}{x}\, V \qquad \text{(A53)}$$

Case 2: $$V_t + V V_x = V_{xx} - \left(\alpha + \frac{1}{x} \right) V_x - \frac{\beta}{x}\, V \qquad \text{(A54)}$$

Case 3: $$V_t + V V_x = V_{xx} - \alpha\, V_x - \beta\, V \qquad \text{(A55)}$$

Case 4: $$V_t + V V_x = V_{xx} - \alpha\, V_x \qquad \text{(A56)}$$

Let $X = \psi\,(x, t)$

$T = t$

Then the following relations hold:

$$dV = V_X\, dX + V_T\, dT$$

$$\begin{cases} V_t = V_X \psi_t + V_T \\ V_x = V_X \psi_x \\ V_{xx} = V_X \psi_{xx} + V_{XX} \psi_x \end{cases}$$

Let

$$X = \psi = x - \alpha t \quad ; \quad \psi_x = 1, \quad \psi_{xx} = 0$$

$$\psi_t = -\alpha$$

Thus,

$$V_T - \alpha V_X + V V_X = V_{XX} - \alpha V_X$$

$$V_T + V V_X = V_{XX}$$

and we succeeded in reducing this case to Burgers' equation.
Next consider

$$V_t + V V_x = V_{xx} - \alpha V_x - \beta V. \tag{A55}$$

To simplify, let us try

$$X = x - \alpha t$$

$$T = t \; ;$$

so that

$$\left. \begin{aligned} V_t &= V_T - \alpha V_X \\ V_x &= V_X \\ V_{xx} &= V_{XX} \end{aligned} \right\} \quad ;$$

and therefore

$$V_T + V V_X = V_{XX} - \beta V$$

Suppose we now let

$$V = e^{-\beta T} u ,$$

which implies

$$\left.\begin{aligned} V_T &= e^{-\beta T} u_T - \beta e^{-\beta T} u \\ V_X &= e^{-\beta T} u_X \\ V_{XX} &= e^{-\beta T} u_{XX} \end{aligned}\right\}$$

$$e^{-\beta T} u_T - \beta e^{-\beta T} u + e^{-2\beta T} u u_X = e^{-\beta T} u_{XX} - \beta e^{-\beta T} u$$

and finally

$$u_T + e^{-\beta T} u u_X = u_{XX}$$

a form which still has a variable coefficient.

If we rewrite our last equation as

$$u_T = \frac{\partial}{\partial X}\left(u_X - e^{-\beta T} \cdot \frac{u^2}{2}\right) \tag{A57}$$

then it is not hard to show that the trial transformations

$$u = \frac{\partial}{\partial X} F\left(\phi (X, T) g (T)\right) \tag{A57}$$

$$= \frac{\partial}{\partial X} F\left(\phi (X, T)\right) \cdot g' (T) \tag{A58}$$

are of no use in solving (A57); despite the fact that these would be the "natural" generalized Burgers' solutions. We note that

$$F_{\phi}\left(\phi_T - \phi_{XX} - \beta\phi\right) = \phi_x^2 \left\{ F_{\phi\phi} - F_{\phi}^2 \cdot \frac{e^{-\beta T}}{2} \right\} e^{-\beta T} + \mu(T)$$

$$g \quad e^{-\beta T} \,, \tag{A57}$$

while

$$F_{\phi}\left(\phi_T - \phi_{XX}\right) - F\beta = (\phi_X)^2 \left\{ F_{\eta\eta} - \frac{1}{\partial} F_{\eta}^2 \right\} + \mu(T) \tag{A58}$$

$$g \quad e^{-\beta T}$$

which implies that

$$F = 2 \ln \eta \,,$$

and this is of no use.

Other direct approaches lead to similar results. Thus, if we take

$$u = \frac{\partial}{\partial x} F\left(\phi\left(x, \rho(t)\right)\right) \; ; \quad u = \frac{\partial}{\partial x} F\left(\phi\left(x, \rho(t)\right) \cdot f(t)\right)$$

then a similar dead end is reached.

Let us try now a change of independent variables. Thus, let

$$X = \psi(x, t)$$
$$T = t$$

so that

$$\left.\begin{aligned} dV &= V_X\, dX + V_T\, dT \\ V_t &= V_X\, \psi_t + V_T \\ V_x &= V_X\, \psi_x \end{aligned}\right\}$$

Let

$$\psi(x, t) = x - a t = X,$$

which implies

$$\begin{cases} V_t = V_T - a V_X \\ V_x = V_X \\ V_{xx} = V_{XX} \end{cases}$$

This yields a somewhat simpler form of (A55); but still of no use; it contains a v-term:

$$V_T + \beta V = V_{XX} - V V_X$$

However, an obvious solution here is

$$V(X, T) = -2 \frac{e^{-\beta T}}{X} \tag{A59}$$

We would, of course, wish to consider something more general than Equation (A59), which was obtained by setting both the right and left hand sides of our equation equal to 0.

Hence, let

$$\begin{aligned} V &= F(f(X)\, g(T)) \\ &= F(\eta) \;;\quad \eta = f(X)\, g(T) \end{aligned}$$

Then we have the following development:

$$V_T + \beta V = F_\eta\, f(X)\, g'(T) + \beta F$$

$$V_X = F_\eta\, f'(X)\, g(T)$$

$$V_{XX} = F_\eta\, f''(X)\, g(T) + F_{\eta\eta}\, (f'(x))^2\, (g(T))^2$$

$$V V_X = F F_\eta\, f'(x)\, g(T)$$

$$V_T + \beta V = F_\eta \quad \eta \left(\frac{g'(t)}{g(t)} \right) + \beta F$$

$$V_X = F_\eta \quad \eta \left(\frac{f'(X)}{f(X)} \right)$$

$$V_{XX} = F_\eta \quad \eta \left(\frac{f''(X)}{f(X)} \right) + F_{\eta\eta} \quad \eta^2 \left(\frac{f'(x)}{f(x)} \right)^2$$

$$VV_X = FF_\eta \quad \eta \left(\frac{f'(x)}{f(x)} \right)$$

In view of the βF term, it seems unlikely that we may do more than let $F = \eta$. It seems to be clear from here, therefore, that even by use of this similarity transformation approach we cannot obtain anything better than setting $F = \eta$, which of course means a direct return to the original equation.

Several other approaches have been tried also. The one given here is perhaps typical of the kind of situations encountered. From this, the best that we can obtain is the following type of solution:

Taking

$$\frac{g'(t)}{g(t)} = c_1 \quad \text{and} \quad \frac{f'(x)}{f(x)} = c_2 \ ,$$

we also get (for c_1, c_2, constants)

$$\frac{f''(x)}{f(x)} = c_2^2$$

Thus, our equation becomes

$$\eta c_1 F' + \beta F + \eta c_2 F F' = \eta c_2^2 F' + \eta^2 \ c_2^2 \ F'' \quad ,$$

where

$$F' = \frac{dF}{d\eta}$$

and where

$$\eta = e^{c_1 \qquad c_2 x}$$

We can satisfy this ordinary nonlinear differential equation by letting F be some power of η; so that if

$$F = K \eta^k$$

where both K and k are constants, then we obtain the following relation (after concelling out common factors):

$$\beta + \left(c_1 - c_2^2\right) k + c_2 K k - c_2^2 k (k - 1) = 0$$

This yields the quadratic in k:

$$c_2^2 k^2 - (c_1 + K c_2) k - \beta = 0 ,$$

which has the solutions

$$k = \frac{c_1 + K c_2 \pm \sqrt{\left(c_1 + K c_2\right)^2 + 4 \beta c_2^2}}{2 c_2^2}$$

This will always yield two numbers; independently of K, provided only that $\beta > 0$ and $c_2 \neq 0$. Let the two numbers be denoted by k_1 and k_2; thus

$$k = k_1 (K) , \quad k = k_2 (K)$$

are both solutions of the quadratic. Thus, the solution is

$$F = K \left[\eta^{k_1(K)}\right]$$

or

$$F = K \left[\eta^{k_2(K)}\right]$$

Therefore, finally, the similarity solution here obtained is

$$v(x,t) = K\, e^{(c_1 t + c_2 x)\, k_1(K)}$$

or

$$v(x,t) = K\, e^{(c_1 t + c_2 x)\, k_2(K)}$$

These are solutions which, for appropriate choices of c_1 and c_2, can be periodic in either x or in t (or both), but they represent, at best, the stable oscillations and are thus merely limiting cases.