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ON THE GENERATION OF IDEAL COORDINATE SYSTEMS FOR THE ANALYSIS OF OPTIMAL TRAJECTORIES

by William F. Powers

Prepared by
THE UNIVERSITY OF MICHIGAN
Ann Arbor, Mich.

for

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ABSTRACT

A relatively well-known property of continuously thrusting optimal trajectories is that there exists a vector constant of the motion which is linear in the Lagrange multipliers. Since the relationship between the corresponding Lagrange multipliers for different sets of state variables is linear, the possibility exists that there is an ideal coordinate system such that three of the associated Lagrange multipliers are constants of the motion. It is shown that such a situation is impossible for more than one of the three constants of the motion. However, a method due to Whittaker is applicable to the problem of generating sets of state variables such that one of the corresponding Lagrange multipliers is a constant of the motion. It is shown that the system of variables generated by cylindrical coordinates possesses this property and, for a large class of problems, the remaining constants of the motion are used effectively to reduce from twelve to nine the number of differential equations which define three-dimensional, optimal trajectories

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I. INTRODUCTION

Two relatively well-known properties of continuously thrusting optimal trajectories are the following: (1) there exists a vector integral[1] similar to the angular momentum integral of the three-body problem; and (2) if the planar problem is formulated in polar coordinates, then one of the Lagrange multipliers is an integral. Actually (2) is a consequence of (1) in the plane, i.e., the vector integral reduces to a scalar integral in the plane and is equal to the Lagrange multiplier conjugate to the polar range angle.

In Reference 2, it is shown that the set of Lagrange multiplier transformations (associated with the set of nonsingular state transformations) forms a subgroup of the classical group of extended point-transformations[3]. Thus the relation between two sets of Lagrange multipliers for the same problem is linear. Since the vector integral, mentioned above, is linear with respect to the Lagrange multipliers, the following question is posed: "Does there exist a nonsingular state transformation such that three of the new Lagrange multipliers form the components of the vector integral?" This is a valid question since it is simply a generalization of the planar property mentioned above.

We shall show that the answer is no, but that there exist many state transformations which allow any one of the three integrals to be transformed into a new Lagrange multiplier. A classic result of Whittaker is shown to be directly applicable to this problem.

Finally, a basic reduction in the number of differential equations for the optimal trajectory problem is given. This sytem, which takes advantage of the vector integral, involves the integration of only nine differential equations as opposed to the usual twelve for three-dimensional, optimal trajectories.

II. BASIC THEORY

The basic theory and results necessary for the forthcoming analysis will be presented here for the sake of completeness.

A. Problem Formulation and the Vector Integral

Consider the problem of minimizing the flight time of a vehicle powered by a continuously thrusting engine. We shall assume that the thrust magnitude and the mass-flow rate are constant. In an inverse-square gravitational force field, the equations of motion in an inertial, cartesian coordinate system are (see Figure 1):

$$\begin{aligned} \ddot{x} &= -\frac{kx}{R^3} + \frac{T}{m} \cos\gamma \cos\alpha \\ \ddot{y} &= -\frac{ky}{R^3} + \frac{T}{m} \cos\gamma \sin\alpha \\ \ddot{z} &= -\frac{kz}{R^3} + \frac{T}{m} \sin\gamma \end{aligned} \tag{1}$$

where

$$m = m_0 + \dot{m}_0(t - t_0) \tag{2}$$

$$R \equiv \sqrt{x^2 + y^2 + z^2} \tag{3}$$

There exist numerous ways of formulating the necessary conditions for an optimal control program. Since we shall be making use of some aspects of Hamiltonian system theory, the Pontryagin maximum principle[4] is the most convenient for our purposes.

To apply the maximum principle, we must express Eqs (1) in a first-order form, i.e.

$$\dot{x}_i = x_{i+3} \qquad (i = 1, 2, 3)$$

$$\dot{x}_4 = -\frac{kx_1}{R^3} + \frac{T}{m} \cos\gamma \sin\alpha$$

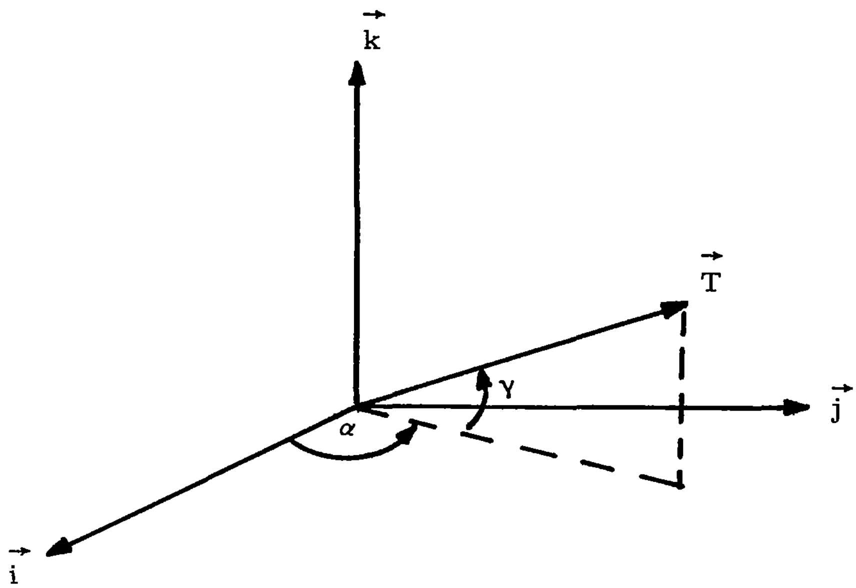

Figure 1. Control Angle Orientations

$$\dot{x}_5 = -\frac{kx_2}{R^3} + \frac{T}{m}\cos\gamma\sin\alpha \tag{4}$$

$$\dot{x}_6 = -\frac{kx_3}{R^3} + \frac{T}{m}\sin\gamma ,$$

where

$$x_1 \equiv x,\ x_2 \equiv y,\ x_3 \equiv z,\ x_4 \equiv \dot{x},\ x_5 \equiv \dot{y},\ x_6 \equiv \dot{z} .$$

Then, the maximum principle for a Mayer problem requires the introduction of a scalar function H*(the generalized Hamiltonian) and a six-vector of Lagrange multipliers:

$$H^* \equiv \sum_{i=1}^{6} \lambda_i f_i \left(x,\ \gamma,\ \alpha,\ \frac{T}{m}\right) , \tag{5}$$

where the f_i represent the right-hand sides of Eqs (4). Then, an optimal trajectory must satisfy the following conditions:

(C1.) Hamilton's equations: $\dot{x}_i = \dfrac{\partial H^*}{\partial \lambda_i}$, $\dot{\lambda}_i = -\dfrac{\partial H^*}{\partial x_i}$; (6)

(C2.) The Hamiltonian must be maximized with respect to the controls.

Assuming that the control region is an open set, (C2) can be expressed mathematically as follows:

$$
(C2)' \begin{cases} \dfrac{\partial H^*}{\partial \alpha} = \dfrac{\partial H^*}{\partial \gamma} = 0 , & (7) \\[2ex] \begin{bmatrix} \dfrac{\partial^2 H^*}{\partial \alpha^2} & \dfrac{\partial^2 H^*}{\partial \alpha \partial \gamma} \\[2ex] \dfrac{\partial^2 H^*}{\partial \gamma \partial \alpha} & \dfrac{\partial^2 H^*}{\partial \gamma^2} \end{bmatrix} \text{ is negative semi-definite.} & (8) \end{cases}
$$

Condition (C2)' implies the following relationships if the extremal is non-singular:

$$
\cos \alpha = +\lambda_4 (\lambda_4^2 + \lambda_5^2)^{-\frac{1}{2}} \quad , \quad \sin \alpha = +\lambda_5 (\lambda_4^2 + \lambda_5^2)^{-\frac{1}{2}} \tag{9}
$$

$$
\cos \gamma = +(\lambda_4^2 + \lambda_5^2)^{\frac{1}{2}} (\lambda_4^2 + \lambda_5^2 + \lambda_6^2)^{-\frac{1}{2}} \quad , \quad \sin \gamma = +\lambda_6 (\lambda_4^2 + \lambda_5^2 + \lambda_6^2)^{-\frac{1}{2}} . \tag{10}
$$

Consider the function

$$
H(x, \lambda, t) \equiv H^*[x, \lambda, \alpha(\lambda), \gamma(\lambda)]. \tag{11}
$$

Since $\dfrac{\partial H}{\partial \alpha} = \dfrac{\partial H}{\partial \gamma} = 0$, it follows that

$$
\frac{\partial H}{\partial x_i} = \frac{\partial H^*}{\partial x_i} = -\dot{\lambda}_i
$$

$$
\frac{\partial H}{\partial \lambda_i} = \frac{\partial H^*}{\partial \lambda_i} + \frac{\partial H^*}{\partial \alpha} \frac{\partial \alpha}{\partial \lambda_i} + \frac{\partial H^*}{\partial \gamma} \frac{\partial \gamma}{\partial \lambda_i} = \frac{\partial H^*}{\partial \lambda_i} = \dot{x}_i .
$$

Thus, $H(x, \lambda, t)$ is also a Hamiltonian for the problem. Since H does not depend on the controls, it is isomorphic to the Hamiltonian functions of classical mechanics. Therefore, the methods of canonical system theory can be applied to our problem.

Consider the following notation change

$$
\bar{q}_1 \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} , \quad \bar{q}_2 \equiv \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} , \quad \bar{p}_1 \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} , \quad \bar{p}_2 \equiv \begin{bmatrix} \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} \tag{12}
$$

Upon substitution of Eqs (9) and (10) into Eq. (5), we then have:

$$H = \overline{p}_1 \cdot \overline{q}_2 - \frac{k}{R^3} \overline{p}_2 \cdot \overline{q}_1 + \frac{T}{m} \sqrt{p_4^2 + p_5^2 + p_6^2} \ . \tag{13}$$

In vector form, Eqs (6) become

$$\dot{\overline{q}}_1 = \overline{q}_2 \quad , \quad \dot{\overline{q}}_2 = - \frac{k}{R^3} \overline{q}_1 + \frac{T}{m} (p_4^2 + p_5^2 + p_6^2)^{-\frac{1}{2}} \overline{p}_2 \tag{14}$$

$$\dot{\overline{p}}_1 = \frac{k}{R^3} \overline{p}_2 - \frac{3k}{R^5} (\overline{q}_1 \cdot \overline{p}_2) \overline{q}_1 \quad , \quad \dot{\overline{p}}_2 = -\overline{p}_1 , \tag{15}$$

or, in second-order form:

$$\ddot{\overline{q}}_1 = - \frac{k}{R^3} \overline{q}_1 + \frac{T}{m} (p_4^2 + p_5^2 + p_6^2)^{-\frac{1}{2}} \overline{p}_2 \tag{16}$$

$$\ddot{\overline{p}}_2 = - \frac{k}{R^3} \overline{p}_2 + \frac{3k}{R^5} (\overline{q}_1 \cdot \overline{p}_2) \overline{q}_1 \ . \tag{17}$$

Thus, operating with the vector product:

$$\ddot{\overline{q}}_1 \times \overline{p}_2 + \ddot{\overline{p}}_2 \times \overline{q}_1 = - \frac{k}{R^3} \overline{q}_1 \times \overline{p}_2 - \frac{k}{R^3} \overline{p}_2 \times \overline{q}_1 \equiv 0 . \tag{18}$$

But,

$$\frac{d}{dt} (\dot{\overline{q}}_1 \times \overline{p}_2 + \dot{\overline{p}}_2 \times \overline{q}_1) = \ddot{\overline{q}}_1 \times \overline{p}_2 + \ddot{\overline{p}}_2 \times \overline{q}_1 \equiv 0 .$$

Therefore,

$$\overline{A} \equiv \overline{q}_2 \times \overline{p}_2 - \overline{p}_1 \times \overline{q}_1 \tag{19}$$

is a vector integral for the optimal trajectory problem.

B. Extended Point-Transformations

The maximum principle, described by Eqs (5), (6), (7), and (8) in the last section, is valid for all coordinate systems. Thus, transformations between various formulations of the same problem preserve Hamiltonian form. Such transformations constitute the group of canonical transformations. In this section, certain basic definitions and properties of these transformations will be stated without proof. For a more thorough development of the subject, see References 2, 3, 5, or 6.

DEFINITION II.1: Let $\{X(x, \lambda, t), \Lambda(x, \lambda, t)\} \in C^2$ be a nonsingular transformation. If for "every" Hamiltonian $H(x, \lambda, t)$ there exists a Hamiltonian $K(X, \Lambda, t)$, then the transformation is said to be canonical.

Note that the word "every" is emphasized in the above definition. The definition does not say that each transformation which preserves Hamiltonian form is canonical, but only those which preserve Hamiltonian form and are independent of the Hamiltonian function. Also, Definition (II.1) is not a good "working" definition, i.e., one cannot check every Hamiltonian function. However, this definition leads to the following sufficient condition for a canonical transformation.

PROPERTY II.1: If the Lagrangians for two Hamiltonian systems differ, at most, by the total time derivative of an arbitrary scalar function, then the transformation between the two systems is canonical. That is,

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H(x, \lambda, t) = \sum_{i=1}^{n} \Lambda_i \dot{X}_i - K(X, \Lambda, t) + \frac{dS}{dt} \tag{20}$$

is a sufficient condition for the transformation $\{X(x, \lambda, t), \Lambda(x, \lambda, t)\}$ to be canonical.

With time as the independent variable, Eq. (20) can be expressed equivalently by the following two equations:

$$\delta S = \sum_{i=1}^{n} (\lambda_i \delta x_i - \Lambda_i \delta X_i) \tag{21}$$

$$K = \frac{\partial S}{\partial t} + H . \tag{22}$$

These equations are useful for defining the following class of canonical transformations.

DEFINITION II.2: A canonical transformation in which $\frac{\partial S}{\partial t} = 0$ and $\delta S = 0$ is called a homogeneous canonical transformation. Furthermore, a homogeneous canonical transformation in which n independent relations between $\{x_1, \ldots, x_n\}$ and $\{X_1, \ldots, X_n\}$ are specified is called an extended point-transformation.

The importance of extended point-transformations in the analysis of optimal control problems is demonstrated by the following property.

PROPERTY II.2: Let $x = \phi(X)$ be a nonsingular transformation between the coordinates of two Hamiltonian systems defined by $H = \sum_{i=1}^{n} \lambda_i f_i$, $K = \sum_{i=1}^{n} \Lambda_i F_i$, where $\dot{x} = f(x, \lambda, t)$ and $\dot{X} = F(X, \Lambda, t)$ are vector equations of motion. Then, the time independent Lagrange multiplier transformation between the two systems is defined by the n-equations

$$\Lambda_i = \sum_{j=1}^{n} \lambda_j \frac{\partial \phi_j}{\partial X_i} \quad . \quad (i = 1, \ldots, n) \tag{23}$$

<u>Proof</u>: With the assumed forms for H and K, Eq. (20) becomes

$$\sum_{i=1}^{n} \lambda_i (\dot{x}_i - f_i) = \sum_{i=1}^{n} \Lambda_i (\dot{X}_i - F_i) + \frac{dS}{dt}$$

or,

$$\frac{dS}{dt} \equiv 0 .$$

Since the transformation is time-independent, then $\delta S = 0$ and Eq. (21) gives

$$\sum_{i=1}^{n} (\lambda_i \delta x_i - \Lambda_i \delta X_i) = 0 . \tag{24}$$

But, $\delta x_i = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial X_j} \delta X_j$, so Eq. (24) reduces to the desired result

$$\Lambda_i = \sum_{j=1}^{n} \lambda_j \frac{\partial \phi_j}{\partial X_i} \quad , \ (i = 1, \ldots, n)$$

since the variations $\{\delta X_1, \ldots, \delta X_n\}$ are independent.

Property (II.2) has a number of important consequences. First of all, it tells us how to determine very simply the Lagrange multiplier transformation between any two coordinate formulations of the same optimal trajectory problem. Secondly, it tells us that these transformations are linear with respect to the Lagrange multipliers. This fact will be given more attention in Section III.

C. Integrals Linear in the Momenta

In analogy with classical mechanics, the Lagrange multipliers of the optimal trajectory problem posess the same properties as the generalized momenta of Hamiltonian system theory. Thus, a property of the momenta variables in classical mechanics implies a corresponding property of the Lagrange multipliers in trajectory analysis. In Reference 3 (Section 150), Whittaker presents a method for performing a canonical transformation which transforms a known integral, linear in the momenta, into a new momenta variable. Since Eq. (19) represents three integrals linear in the Lagrange multipliers, this method has an immediate application in trajectory analysis.

In Reference 3, the method is presented without motivation or proof. Thus, a more thorough treatment of the method will be given here.

Suppose that we have a Hamiltonian system

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \quad , \quad \dot{\lambda}_i = - \frac{\partial H}{\partial x_i} \quad (i = 1, \ldots, n)$$

which possesses an integral linear and homogeneous in the Lagrange multipliers, say

$$g_1(x)\lambda_1 + g_2(x)\lambda_2 + \cdots + g_n(x)\lambda_n \equiv \text{constant}. \tag{25}$$

We recognize that Eq. (25) is functionally similar to each of Eqs (23), i.e., the multiplier transformation defined by an extended point-transformation $x = \phi(X)$ such that one of the new multipliers is Eq. (25). We shall show that this is indeed the case.

Without loss of generality, assume that Eq. (25) is Λ_n in the new $\{X, \Lambda\}$-system, which is to be defined by an extended point-transformation, i.e.,

$$\Lambda_n \doteq g_1(x)\lambda_i + \cdots + g_n(x)\lambda_n \tag{26}$$

$$\Lambda_i = \sum_{j=1}^{n} \lambda_j \frac{\partial \phi_j}{\partial X_i} \quad . \quad (i = 1, \ldots, n) \tag{27}$$

In order that these equations hold, we must then have

$$g_j(x) = \frac{\partial \phi_j}{\partial X_n}, \qquad (j = 1, \ldots, n) \tag{28}$$

where $x = \phi(X)$ is the point-transformation which is to be determined.

Equations (28) represent a system of n partial differential equations which are to be solved for the n dependent functions $\phi_j(X_1, \ldots, X_n)$. The existence of these functions is guaranteed by first noting that

$$dx_i = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial X_j} dX_j \equiv \sum_{j=1}^{n} \phi_{ij} dX_j, \tag{29}$$

and then applying the following integrability theorem for a system of total differential equations (see Reference 7 for the proof).

PROPERTY II. 3: The necessary and sufficient condition for the system of total differential equations

$$dx_i = \sum_{j=1}^{n} \phi_{ij} dX_j \quad (i = 1, \ldots, n) \tag{30}$$

to be completely integrable (i.e., there exist functions $\phi_i(X_1, \ldots, X_n)$ for each i) is that

$$\frac{\partial \phi_{ij}}{\partial X_k} - \frac{\partial \phi_{ik}}{\partial X_j} + \sum_{m=1}^{n} \left[\frac{\partial \phi_{ij}}{\partial x_m} \phi_{mk} - \frac{\partial \phi_{ik}}{\partial x_m} \phi_{mj} \right] = 0, \tag{31}$$

$$(i, j, k = 1, \ldots, n)$$

where the ϕ_{ij} are assumed to be continuously differentiable.

Since Eqs (28) only depend upon $x_1, \ldots, x_n$, and since $\phi_{ij} = \phi_{ik} \equiv \phi_{in}$, then the integrability conditions are satisfied trivially. Thus, there exist solutions $x_1 = \phi_1(X), \ldots, x_n = \phi_n(X)$ to Eqs (28). To determine a set of solutions, we construct the method of Whittaker.

Note that for each $i = 1, \ldots, n$:

$$\frac{dx_i}{dX_n} = \frac{\partial \phi_i}{\partial X_n} \frac{dX_n}{dX_n}$$

or,

$$dx_i = \frac{\partial \phi_i}{\partial X_n} dX_n = g_i(x)\, dX_n \tag{32}$$

Thus,

$$\frac{dx_1}{g_1} = \frac{dx_2}{g_2} = \cdots = \frac{dx_n}{g_n} = dX_n \tag{33}$$

The only real restriction on the point-transformation $x = \phi(X)$ is that $\frac{\partial \phi_i}{\partial X_n} = g_i$ for each $i = 1, \ldots, n$. Because of this, there exist many point-transformations which satisfy this criterion. If any one of the g_i's can be expressed as a function $g_i(x_i)$, then X_n is defined by a quadrature, i.e.,

$$X_n = \int \frac{dx_i}{g_i(x_i)} \quad . \tag{34}$$

In general, though, each $g_i(x)$ will depend upon each element of $\{x_1, \ldots, x_n\}$. Thus, the following procedure can be used in this case:

(i) Find $n - 1$ integrals of the system (33), and denote these integrals as X_i's, $i = 1, \ldots, n - 1$. Thus,

$$\psi_i(x_1, \ldots, x_n) = \text{constant} \equiv X_i \quad (i = 1, \ldots, n - 1) \tag{35}$$

(ii) Use Eqs (35) to express $n - 1$ elements of the set $\{x_1, \ldots, x_n\}$ as functions of the X_i's and the remaining element of the set, say x_k. Then,

$$x_i = \phi_i(X_1, \ldots, X_{n-1}, x_k) \cdot \begin{cases} i = 1, \ldots, n \\ i \neq k \end{cases} \tag{36}$$

(iii) Use Eqs (36) to obtain $g_k(x_k; X_1, \ldots X_{n-1})$. Since the X_i $(i = 1, \ldots, n - 1)$ are constants for the system (33), then

$$X_n = \int \frac{dx_k}{g_k(x_k)} \equiv \psi_n[x_k; \psi_1(x), \ldots, \psi_{n-1}(x)] . \tag{37}$$

(iv) Then, Eqs (35) and (37) define the desired transformation. If Eq. (37) is inverted to form $x_k = \phi_k(X_1, \ldots, X_n)$, then this equation along with Eqs (36) define the new multipliers, i.e.,

$$\Lambda_i = \sum_{j=1}^{n} \lambda_j \frac{\partial \phi_j}{\partial X_i}, \quad (i = 1, \ldots, n)$$

where, of course, Λ_n is Eq. (26).

After this method has been applied, a new Hamiltonian system $\{X, \Lambda\}$, with Hamiltonian

$$K(X, \Lambda, t) \equiv H[x(X), \lambda(X, \Lambda), t],$$

is defined. In this system, $\frac{\partial K}{\partial X_n} = -\dot{\Lambda}_n = 0$, so X_n does not appear in $K(X, \Lambda, t)$. Thus, X_n will not appear in any of the Hamilton's equations, so one need not even integrate the $\dot{X}_n$-equation if the time-history of X_n is not a necessary part of the problem.

III. APPLICATIONS IN TRAJECTORY ANALYSIS

The developments of Section II now will be applied to the optimal trajectory problem. First, an important negative result will be presented.

THEOREM III. 1: Let $A_1 \bar{i} + A_2 \bar{j} + A_3 \bar{k}$ be the vector integral of Eq. (19). There does not exist an extended point-transformation such that two of the new Lagrange multipliers are independent linear combinations of the A_i's.

Proof: Assume the contrary, i.e., there exists a point-transformation $x = \phi(X)$ such that, without loss of generality,

$$\begin{aligned} \Lambda_1 &= a_{11} A_1 + a_{12} A_2 + a_{13} A_3 \\ \Lambda_2 &= a_{21} A_1 + a_{22} A_2 + a_{23} A_3, \end{aligned} \tag{38}$$

where the a_{ij} are real numbers, and the two expressions are linearly independent. Since the transformation is an extended point-transformation, we must satisfy

$$\Lambda_i = \sum_{j=1}^{6} \lambda_j \frac{\partial \phi_j}{\partial X_i} \qquad (i = 1, 2) \tag{39}$$

By Eq. (19), each of the A_i's is linear in the multipliers, so

$$A_i = \sum_{j=1}^{6} u_{ij}(x) \lambda_j , \quad (i = 1, 2, 3) \tag{40}$$

where each u_{ij} depends upon only one x_k. Thus, Eqs (38) become

$$\Lambda_i = \sum_{j=1}^{6} \sum_{\ell=1}^{3} a_{i\ell} u_{\ell j} \lambda_j . \quad (i = 1, 2) \tag{41}$$

A comparison of Eqs (39) and (41) shows that

$$\phi_{ji} \equiv \frac{\partial \phi_j}{\partial X_i} = \sum_{\ell=1}^{3} a_{i\ell} u_{\ell j}(x) . \quad \begin{cases} i = 1, 2 \\ j = 1, \ldots, 6 \end{cases} \tag{42}$$

Equations (42) represent a system of twelve partial differential equations, so they must satisfy the integrability conditions of Property (II.3). Since none of the X_i's appear in the right-hand sides of Eqs (42), the integrability conditions of Eqs (31) reduce to

$$\sum_{m=1}^{6} \left[\phi_{mk} \frac{\partial \phi_{ij}}{\partial x_m} - \phi_{mj} \frac{\partial \phi_{ik}}{\partial x_m} \right] = 0 , \tag{43}$$

for $i = 1, 2, \ldots, 6$ and $j, k = 1, 2$.

Equations (43) are satisfied trivially if $k = j$, so assume $k = 1, j = 2$, and substitute Eqs (42) into Eqs (43):

$$\sum_{m=1}^{6} \left[\left(\sum_{\ell=1}^{3} a_{1\ell} u_{\ell m} \right) \frac{\partial}{\partial x_m} \left(\sum_{\ell=1}^{3} a_{2\ell} u_{\ell i} \right) - \left(\sum_{\ell=1}^{3} a_{2\ell} u_{\ell m} \right) \frac{\partial}{\partial x_m} \left(\sum_{\ell=1}^{3} a_{1\ell} u_{\ell i} \right) \right] = 0 . \tag{44}$$

Let $i = 1$. Then, the following expression is obtained

$$(a_{11} a_{22} - a_{12} a_{21}) x_2 + (a_{11} a_{23} - a_{13} a_{21}) x_3 = 0. \tag{45}$$

But, x_2 and x_3 are independent variables, so in order to satisfy Eq. (45)

$$a_{11}a_{22} = a_{12}a_{21} \tag{46}$$

$$a_{11}a_{23} = a_{13}a_{21} \tag{47}$$

Let i = 2. Then, from Eq. (44)

$$(a_{12}a_{21} - a_{11}a_{22})x_1 + (a_{12}a_{23} - a_{13}a_{22})x_3 = 0 .$$

This equation implies the additional requirement that

$$a_{12}a_{23} = a_{13}a_{22} . \tag{48}$$

We shall now show that the conditions of Eqs (46), (47), and (48) imply that Eqs (38) are linearly dependent. This will give us the necessary contradiction.

First, assume $a_{21} = 0$. Then, $a_{11} = 0$ and/or $a_{22} = a_{23} = 0$. If $a_{11} \neq 0$, then $a_{22} = a_{23} = 0$. But this is not possible since it implies that $\Lambda_2 \equiv 0$. Therefore, assume $a_{11} = 0$. If any one of the a_{ij}'s in Eq. (48) is zero, then Eqs (38) are linearly dependent since such an assumption implies either $\Lambda_1 = a_{1k}A_k$ and $\Lambda_2 = a_{2k}A_k$ (no sum on k) or one of Eqs (38) is identically zero. Thus, each element of Eq. (48) must be nonzero and $a_{12} = (a_{13}/a_{23})a_{22}$. But, this implies $\Lambda_2 + (-a_{23}/a_{13})\Lambda_1 = 0$. Therefore, the assumption that $a_{21} = 0$ always leads to linearly dependent relationships between Λ_1 and Λ_2.

Finally, assume $a_{21} \neq 0$. Then, from Eqs (46) and (47), $a_{12} = (a_{11}/a_{21})a_{22}$ and $a_{13} = (a_{11}/a_{21})a_{23}$. Clearly $a_{11} \neq 0$, for otherwise $\Lambda_1 \equiv 0$. Thus, $\Lambda_2 + (-a_{21}/a_{11})\Lambda_1 = 0$, which again implies linear dependence. Since neither $a_{21} = 0$ nor $a_{21} \neq 0$ is possible, Eqs (38) cannot exist, and the theorem is proved.

An alternate proof of this theorem can be constructed by applying the theory of Poisson's brackets[3]. The Poisson brackets indicate that there does not exist a canonical transformation in which two of the new variables (either two new coordinates, two new multipliers, or one of each) are linear combinations of the known integrals. Thus, to obtain three new canonic variables which are also integrals, nonlinear combinations of the known integrals must be used. Theorem 1 indicates this result indirectly since one can use a simple canonical transformation to redefine either Λ_1 or Λ_2 as X_1 or X_2 (or both) without affecting the proof of the theorem.

Since no two linear combinations of the A_i's can be new canonic variables, then no three linear combinations of the A_i's can be new canonic variables either. But, as shown in Section II.C any one of the three A_i's can be a new Lagrange multiplier, and there exist many extended point-transformations which include a specified A_i as a new multiplier. The procedure of Section II.C now will be used to define a canonical transformation which includes

$$A_3 = -x_2\lambda_1 + x_1\lambda_2 - x_5\lambda_4 + x_4\lambda_5 \tag{49}$$

as a new multiplier.

To generate a point-transformation which causes Eq. (49) to be a new multiplier, five integrals of the system

$$\frac{dx_1}{-x_2} = \frac{dx_2}{x_1} = \frac{dx_3}{0} = \frac{dx_4}{-x_5} = \frac{dx_5}{x_4} = \frac{dx_6}{0} = d\tau\ , \tag{50}$$

where $\tau \in \{X_1, \ldots, X_6\}$, must be determined. Two immediate integrals of Eq. (50) are

$$X_3 = x_3\ , \qquad X_6 = x_6 \tag{51}$$

The three remaining integrals of Eq. (50) must satisfy the following system of differential equations

$$\begin{aligned} \frac{dx_1}{d\tau} &= -x_2 & \qquad \frac{dx_4}{d\tau} &= -x_5 \\ \frac{dx_2}{d\tau} &= x_1 & \qquad \frac{dx_5}{d\tau} &= x_4\ . \end{aligned} \tag{52}$$

Since

$$\frac{1}{2}\frac{d}{d\tau}(x_1^2 + x_2^2) = x_1\frac{dx_1}{d\tau} + x_2\frac{dx_2}{d\tau} \equiv 0\ , \tag{53}$$

then

$$X_1 \equiv \sqrt{x_1^2 + x_2^2} \tag{54}$$

is an integral of the system. Alternatively, we could have defined X_1 to be $x_1^2 + x_2^2$, $e^{(x_1^2 + x_2^2)}$, $\cos(x_1^2 + x_2^2)$, etc. Each of these choices implies a different set of new multipliers. However, in each set, one of the new multipliers must be defined by Eq. (49).

Another integral can be formed by manipulating the $\frac{dx_4}{d\tau}$ - and $\frac{dx_5}{d\tau}$ - equations in the same way that the $\frac{dx_1}{d\tau}$ - and $\frac{dx_2}{d\tau}$ - equations were manipulated in Eq. (53). The integral obtained in this manner is the magnitude of the velocity in the x_1x_2-plane. Also, note that since x_6 is a constant, another possibility is $(x_4^2 + x_5^2 + x_6^2)^{\frac{1}{2}}$, i.e., the magnitude of the velocity. However, with some foresight, we shall develop the second integral as follows:

$$x_1 \frac{dx_4}{d\tau} + x_5 \frac{dx_2}{d\tau} + x_4 \frac{dx_1}{d\tau} + x_2 \frac{dx_5}{d\tau} \equiv 0 ,$$

so

$$\frac{d}{d\tau}(x_1x_4 + x_2x_5) = 0 .$$

Thus, define: $\tilde{X}_4 \equiv x_1x_4 + x_2x_5$. Since X_1 is a constant and has the dimensions of length, $\tilde{X}_4$ can be modified to form a velocity variable, i.e.,

$$X_4 \equiv \tilde{X}_4/X_1 = (x_1x_4 + x_2x_5)/X_1 . \tag{55}$$

In a similar manner another integral can be formed by observing that

$$x_1 \frac{dx_5}{d\tau} + x_5 \frac{dx_1}{d\tau} - x_4 \frac{dx_2}{d\tau} - x_2 \frac{dx_4}{d\tau} \equiv 0 ,$$

so

$$\frac{d}{d\tau}(x_1x_5 - x_2x_4) = 0 .$$

Again we shall form a velocity variable by dividing by X_1, i.e.,

$$X_5 \equiv (x_1x_5 - x_2x_4)/X_1 . \tag{56}$$

The only remaining point-transformation variable to be defined is X_2, so let $\tau = X_2$. Then,

$$X_2 = \int \frac{dx_1}{-x_2} = \int \frac{dx_1}{\mp\sqrt{X_1^2 - x_1^2}} = \pm \tan^{-1}\left(\frac{\sqrt{X_1^2 - x_1^2}}{x_1}\right)$$

But, $\sqrt{X_1^2 - x_1^2} = \pm x_2$, so

$$X_2 = \tan^{-1}\left(\frac{x_2}{x_1}\right) \quad . \tag{57}$$

The new state variables $\{X_1, \ldots, X_6\}$ are actually cylindrical coordinates and velocities, and the new multiplier $\Lambda_2 = \sum_{j=1}^{6} \lambda_j \frac{\partial x_j}{\partial X_2}$ is the integral A_3. The remaining multipliers are defined by

$$\Lambda_i = \sum_{j=1}^{6} \lambda_j \frac{\partial x_j}{\partial X_i}, \quad (i = 1, \ldots, 6) \tag{58}$$

where the inverse transformation, $x = \phi(X)$, is defined by

$$\begin{aligned} x_1 &= X_1 \cos X_2 & x_4 &= X_4 \cos X_2 - X_5 \sin X_2 \\ x_2 &= X_2 \sin X_2 & x_5 &= X_4 \sin X_2 + X_5 \cos X_2 \\ x_3 &= X_3 & x_6 &= X_6 \ . \end{aligned} \tag{59}$$

In the preceding development there were innumerable possibilities for defining five of the six X_i's, and in fact, we did not make the most natural choice. Our choice was strictly motivated by familiarity with the new state, i.e., cylindrical coordinates and velocities.

Since time is not involved in the transformation $\{x(X), \lambda(X, \Lambda)\}$, the variational Hamiltonian in the $\{X, \Lambda\}$-system can be obtained by a straight substitution into Eq. (13), i.e.,

$$K(X, \Lambda\ t) \equiv H[x(X), \lambda(X, \Lambda), t] = X_4 \Lambda_1 + \frac{X_5}{X_1} \Lambda_2 + X_6 \Lambda_3 + \left[\frac{X_5^2}{X_1} - kX_1(X_1^2 + X_3^2)^{-\frac{3}{2}}\right]\Lambda_4$$

$$- \frac{X_4 X_5}{X_1} \Lambda_5 - kX_3(X_1^2 + X_3^2)^{-\frac{3}{2}} \Lambda_6 + \frac{T}{m} \sqrt{\Lambda_4^2 + \Lambda_5^2 + \Lambda_6^2} \quad . \tag{60}$$

Since X_2 is an ignorable coordinate, a system of only ten differential equations defines the problem if the time-history of X_2 is unimportant.

By making use of the remaining integrals, A_1 and A_2, the system can be reduced to nine differential equations. In this case, only the differential equations for $X_i (i = 1, \ldots, 6)$, Λ_1, Λ_4, and Λ_5 need to be integrated, since

$$
\begin{aligned}
\Lambda_2 &= A_3 \\
\Lambda_6 &= \frac{1}{X_5}[-(A_1 \cos X_2 + A_2 \sin X_2) + X_6 \Lambda_5 \\
&\quad + \frac{X_3}{X_1} (\Lambda_2 + X_5 \Lambda_4 - X_4 \Lambda_5)] \\
\Lambda_3 &= \frac{1}{X_1}[(A_2 \cos X_2 - A_1 \sin X_2) + X_3 \Lambda_1 + X_6 \Lambda_4 - X_4 \Lambda_6].
\end{aligned}
\tag{61}
$$

The reason for solving for Λ_6 and Λ_3 (instead of two other multipliers or state variables) is that these equations are undefined only when $X_1 \equiv r$, $X_5 \equiv r\dot{\theta}$ are equal to zero. If the coordinate system is chosen in such a way that the motion is in or near the $x_3 \equiv z = 0$ - plane, then X_1 and X_5 should be nonzero for most missions.

Finally, it should be cautioned that Eqs. (61) cannot be used in an iteration scheme (for converging optimal trajectories) which utilizes approximate differential equations. For in such an analysis, A_1 and A_2 are not constants of the motion on the iterates leading to the optimum. Thus, Eqs. (61) are useful when an indirect iteration scheme (as opposed to a direct scheme) is being used.

IV. CLOSURE

In the previous sections, the relationship between the known linear integrals of the optimal trajectory problem and the classical extended point-transformation was fully exploited. It was shown that there does not exist a canonical transformation which allows two or more of the integrals to be new canonic variables. With regard to a single integral, a method of Whittaker was utilized to generate an extended point-transformation which included one of the integrals as a new multiplier. We observed that there exist innumerable point-transformations which allow a given integral to be a new multiplier, and that the point-transformation generated by cylindrical coordinates was just one of many possibilities. Also, the remaining constants of the motion were used to reduce the number of differential equations from eleven to nine.

Finally, since the value of $\Lambda_2 = A_3$ is known for certain classes of missions (i.e., missions in which the terminal boundary conditions do not involve X_2 possess the transversality condition $\Lambda_2(t_f) = 0$), only five initial multipliers need to be estimated for an indirect iteration scheme. Similarly, if the numerical values of A_1 and/or A_2 can be obtained from transversality conditions or other means, the order of the iteration scheme will be reduced accordingly.

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