NGR-32-003-037

THE NON-LINEAR TRANSIENT BEHAVIOR OF SECOND, THIRD AND FOURTH ORDER PHASE-LOCKED LOOPS

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Abstract. The non-linear transient behavior of second, third and fourth order phase-locked loops is obtained from the projections of the respective state spaces onto the various hyper-planes. For various driving functions, regions of the parameter space are specified such that the transients may be determined by the linear model.

INTRODUCTION

The object of this paper is to determine parameter regions of linear operation with respect to certain input signals for second, third, and fourth order phase-locked loops (PLL's). That is, certain bounds on the input phase driving functions will be established such that the PLL operates in much the manner predicted by linear analysis when subjected to these inputs.

These bounds are determined by comparing the peak transient error as calculated from the linear model with that calculated from the more exact non-linear differential equation for the various PLL configurations. When the peak transient errors so calculated are within 10% of each other, it is concluded that a linear analysis is valid. Thus, for a given system the magnitude of a particular phase input is varied until the above condition just fails to hold. This sets a bound for a given input upon a given PLL configuration. Thus, the signal and system parameter region is partitioned into regions of linear and nonlinear operation. With this knowledge of the bounds available, easily predictable, stable operation of given loops are assured for input signals not exceeding the specified limits.

The input signals of concern are the frequency step, ramp, and parabola. these being the aperiodic signals for which it is possible to reduce the steady-state error to zero in a second, third, and fourth order PLL respectively. For the attainment of zero steady-state errors as just mentioned it is necessary that the system type at least equal the system order, and these are the cases that will be considered. For example, when the second order system is investigated, it is a second order loop with two pure integrators. Therefore, the loop filter will contain an integrator. The other integration being contributed by the voltage controlled oscillator.

For each order PLL, restricted regions of the parameters space is studied in order to partition the parameter space into regions of linear and nonlinear operation. The restricted regions investigated are those neighborhoods containing the points in the parameter space determined by the two optimization criterians:

- CASE I Minimum integral square error
- CASE II Minimum error for steady state sinusoidal input as frequency approaches zero.[1]

Each of these optimizations require a system type number equal to the system order.

STATE VARIABLE MODEL

In order to determine the nonlinear response of the PLL, the describing nonlinear differential equation is formulated in terms of state variables and the projections of the state space onto the various hyperplanes is determined with the aid of a digital computer. Plots of state variable versus time are also obtained from analog simulation.

1. S. Gupta and R. J. Solen, "Optimum Filters for Second and Third Order-Phase Locked Loops by an Error-Function Criterion" IEE Trans. on Space Electronics and Telemetry, Vol. Set-11 pp. 54-62, June 1965.

For the nonlinear fourth order loop the differential equation relating the error and input phase is

$$\ddddot{\Theta}_e = \ddddot{\Theta}_i + C\omega_o \cos \Theta_e \dot{\Theta}_e^3 + 3\,C\,\omega_o \sin(\Theta_e)\dot{\Theta}_e \ddot{\Theta}_e - C\,\omega_o \cos \Theta_e \dddot{\Theta}_e + b\,\omega_o^2 \sin(\Theta_e)\dot{\Theta}_e^2 - b\,\omega_o^2 \cos(\Theta_e)\ddot{\Theta}_e \quad (1)$$

as obtained from the general expression[2]

$$\Theta_e = \Theta_i - K\frac{F(p)}{p} \sin \Theta_e \quad (2)$$

Similarily, the third and second order equations may be developed:

$$\dddot{\Theta}_e = \dddot{\Theta}_i + b\omega_o \sin \Theta_e \dot{\Theta}_e^2 - b\omega_o \cos \Theta_e \ddot{\Theta}_e - a\,\omega_o^2 \cos \Theta_e \dot{\Theta}_e - \omega_o^3 \sin \Theta_e \quad (3)$$

$$\ddot{\Theta}_e = \ddot{\Theta}_i - a\omega_o \cos \Theta_e \Theta_e + \omega_o \sin \Theta_e \quad (4)$$

The fourth order loop is formulated in terms of state variables via the following substitutions:

$$\begin{aligned} Y_1 &= \Theta_e \\ \dot{Y}_1 &= \dot{\Theta}_e = Y_2 \\ \dot{Y}_2 &= \ddot{\Theta}_e = Y_3 \\ \dot{Y}_3 &= \dddot{\Theta}_e = Y_4 \end{aligned} \quad (5)$$

The resulting state variable expression is:

$$\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \\ \dot{Y}_3 \\ \dot{Y}_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \omega_o^4 \sin Y_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ddddot{\Theta}_i \end{bmatrix} \quad (6)$$

2. A.J. Viterb: Principles of Coherent Communications, New York: McGraw Hill, 1966, p.45.

where

$a_{11} = 0, \quad a_{12} = 1, \quad a_{13} = 0, \quad a_{14} = 0$

$a_{21} = 0, \quad a_{22} = 0, \quad a_{23} = 1, \quad a_{24} = 0$

$a_{31} = 0, \quad a_{32} = 0, \quad a_{33} = 0, \quad a_{34} = 1$

$a_{41} = 0$

$a_{42} = (Y_2^2 C\omega_o - a\omega_o^3) \cos Y_1 + Y_2 b\omega_o^2 \sin Y_1$

$a_{43} = Y_2^3 C\omega_o \sin Y_1 - b\omega_o^2 \cos Y_1$

$a_{44} = C\omega_o \cos Y_1$

Similarily, the third and second order equations in state space are:

$$\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \\ \dot{Y}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin Y_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dddot{\Theta}_i \end{bmatrix} \quad (7)$$

where

$a_{11} = 0, \quad a_{12} = 1, \quad a_{13} = 0$

$a_{21} = 0, \quad a_{22} = 0, \quad a_{23} = 1$

$a_{31} = 0$

$a_{32} = a \cos Y_1 + bY_2 \sin Y_1$

$a_{33} = b \cos Y_1$

and

$$\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2\xi\omega \cos X_1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_o^{\,2} \sin Y_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \ddot{\theta}_i \end{bmatrix} \tag{8}$$

Digital computer algorithims applicable to the solution of n first order differential equations are utilized to obtain numberical solutions of the state variable equations for various driving functions. The projections of the n-dimensional state space onto the various hyper-planes are obtained. Projections for the linear and nonlinear second, third, and fourth order loops are obtained and compared while conducting a parameter sweep. The corresponding roots of the linearized characteristic equation for the parameter sweeps are tabulted below. Figures 1, 2, and 3 depict the data of the tables. Also shown are the maximum values of $\Delta\omega$ for which the 10% limit on peak overshoot holds.

Second order system

TABLE 1

Roots of Linearized Characteristic Equation	$\Delta\omega$ for Agreement Within 10%
1. $P_1 = -.707 + .707$ (Case I) $P_2 = -.707 - j\ .707$	$\theta_i(t) = \Delta\omega\ t$ $\Delta\omega = 1.2$
2. $P_1 = -.5 + j\ .866$ (Case II) $P_2 = -.5 - j\ .866$	$\theta_i(t) = \Delta\omega$ $\Delta\omega = 0.8$
3. $P_1 = -1$ $P_2 = -1$	$\theta_i(t) = \Delta\omega t$ $\Delta\omega = 1.2$

Third order system

TABLE 2

Roots of Linearized Characteristic Equation	$\Delta\omega$ for Agreement within 10%	
1. $P_1 = -1.755$ $P_2 = 1.23 + j.745$ (Case II) $P_3 = -.123 - j.745$	$\theta_i(t) = \Delta\omega t$ $\Delta\omega = 2.175$	$\theta_i(t) = \frac{\Delta\omega}{2} t^2$ $\Delta\omega = 1.15$

Table 2 (Continued)

2. $P_1 = -1.609$ $P_2 = -.195 + j.764$ $P_3 = -.195 - j.764$	$\theta_i(t) = \Delta\omega t$ $\Delta\omega = 2.1$	$\theta_i(t) = \frac{\Delta\omega}{2} t^2$ $\Delta\omega = 1.15$
3. $P_1 = -1.44$ $P_2 = -.280 + j.785$ $P_3 = .280 - j.785$	$\theta_i(t) = \Delta\omega t$ $\Delta\omega = 2.1$	$\theta_i(t) = \frac{\Delta\omega}{2} t^2$ $\Delta\omega = 1.15$
4. $P_1 = -1.24$ $P_2 = -.381 + j.814$ $P_3 = -.381 - j.814$	$\theta_i(t) = \Delta\omega t$ $\Delta\omega = 2.1$	$\theta_i(t) = \frac{\Delta\omega}{2} t^2$ $\Delta\omega = 1.167$
5. $P_1 = -1.00$ $P_2 = -.500 + j.866$ (Case I) $P_3 = -.500 - j.866$	$\theta_i(t) = \Delta\omega t$ $\Delta\omega = 2.178$	$\theta_i(t) = \frac{\Delta\omega}{2} t^2$ $\Delta\omega = 1.217$

Fourth order system

TABLE 3

Roots of Linearized Characteristic Equation	$\Delta\omega$ for Agreement within 10%	
1. $P_1 = -.957 + j\ 1.23$ $P_2 = -.957 - j\ 1.23$ $P_3 = -.043 + j.641$ (Case II) $P_4 = -.043 - j.641$	$\theta_i(t) = \Delta\omega t$ $\Delta\omega = 2.038$	$\theta_i(t) = \frac{\Delta\omega}{6} t^3$ $\Delta\omega = 1.003$
2. $P_1 = -.082 + j.657$ $P_2 = -.082 - j.657$ $P_3 = -1.175 + j.949$ $P_4 = -1.175 - j.949$	$\theta_i(t) = \Delta\omega t$ $\Delta\omega = 2.085$	$\theta_i(t) = \frac{\Delta\omega}{6} t^3$ $\Delta\omega = 1.033$
3. $P_1 = -1.108 + j\ 1.108$ $P_2 = -1.108 - j\ 1.108$ $P_3 = -.0459 + j.616$ $P_4 = -.0459 - j.616$	$\theta_i(t) = \Delta\omega t$ $\Delta\omega = 2.203$	$\theta_i(t) = \frac{\Delta\omega}{6} t^3$ $\Delta\omega = 1.052$

Table 3 (Continued)

4. $P_1 = -.901 + j.774$	$\Theta_i(t)=\Delta\omega t$	$\Theta_i(t)=\frac{\Delta\omega}{6}t^3$
$P_2 = -.901 - j.774$	$\Delta\omega=2.5$	$\Delta\omega=1.226$
$P_3 = -.330 + j.774$		
$P_4 = -.330 - j.774$		
5. $P_1 = -.383 + j.924$	$\Theta_i(t)=\Delta\omega t$	$\Theta_i(t)=\frac{\Delta\omega}{6}t^3$
$P_2 = -.383 - j.924$	$\Delta\omega=2.58$	$\Delta\omega=1.254$
$P_3 = -.924 + j.383$ (Case I)		
$P_4 = -.924 - j.383$		

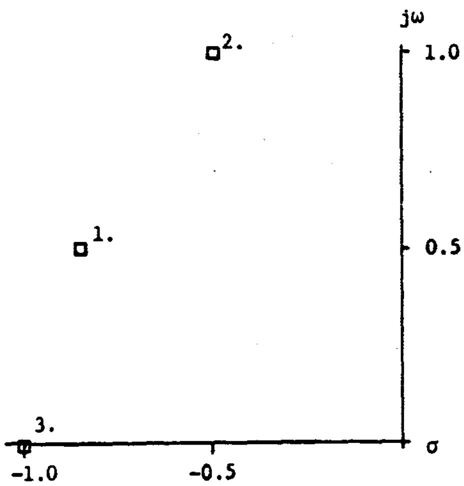

Figure 1. Roots of linearized characteristic equation for second order system. Number by roots indicates corresponding data from Table 1.

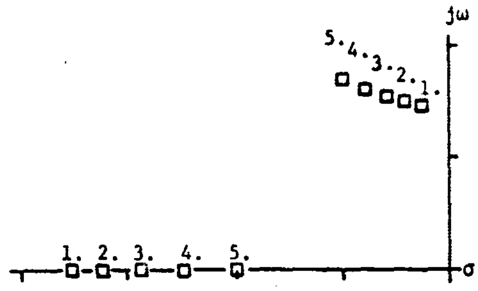

Figure 2. Roots of linearized characteristic equation for third order system. Number by roots indicates corresponding data from Table 2.

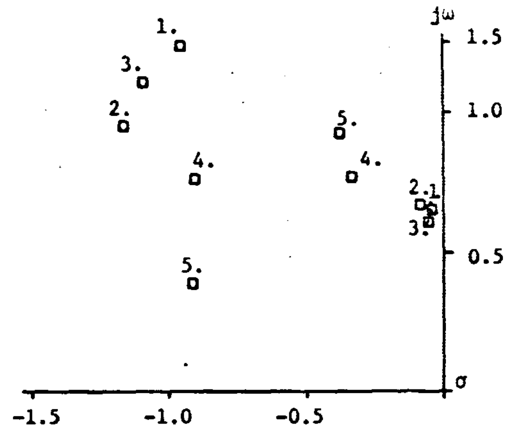

Figure 3. Roots of linearized characteristic equation for fourth order system. Number by roots indicates corresponding data from Table 3.

CONCLUSION

The linearization of equation 1, 2, and 4 for the analysis of the inputs considered yield valid solutions in the regions of parameter space specified. That is, with the loop gain constant and filter parameter adjusted to the region specified by the roots of the linearized equation, bounds for which a linear analysis is applicable are determined.

This work was supported by NASA grant #NGR-32-003-037