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Lambert's Problem and Cross-Product Steering

William E. Moore

Analytical Mechanics Associates, Inc.

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Analytical Mechanics Associates, Inc.

Westbury, Long Island, New York

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Lambert's Problem and Cross Steering

Introduction

The construction of a free-fall conic trajectory requiring a given transit time between two fixed position vectors is called Lambert's Problem. In reference 1, S. Pines presented a solution for this problem which has several attractive features, including uniform representation for all types of conic orbits, reasonable speed, and the ability to handle cases where multiple revolutions around the central body are required. When this method is incorporated into a scheme of guidance logic, three facts become apparent. First, the possibility of multiple orbit solutions may be discarded, with a resulting simplification of the formulas. Second, the guidance logic needs the time-rate-of-change of the solution velocity vector for obtaining the direction of the thrust vector. This logic (cross-product steering) is described in many places. See, for example, reference 2. Third, the situation with respect to first guesses is much improved since the guidance logic involves repeated solution of Lambert's Problem with only slowly varying initial position and time.

This note consists of two parts. One gives the formulas for solving Lambert's Problem in the case of only single orbits; the other gives formulas for the derivatives of several parameters when the initial positions change in accordance with a given velocity vector and a fixed terminal position and time.

Simplified Equations for Single Orbits

Let the motion take place in a three dimensional cartesian coordinate system with a point (= body) having μ for a gravitational constant at the origin. Let $R(t)$ be the vector representing the position of the second body at time t. We state Lambert's Problem, then, as follows. Let the moving body be at position $R(t_o) = R_o$ at time t_o, and at position $R(t_1) = R_1$ at time $t_1 = t_o + \Delta t$, Δt being the transit time. The solution is that trajectory along which the body will travel from R_o to R_1 in elapsed time Δt. It can be described by the position and velocity vectors at time t_o. Thus, the solution is found when the initial velocity vector $V_R = V_R(t_o)$ is obtained. Let:

$$r_o = (R_o \cdot R_o)^{\frac{1}{2}} \qquad (1)$$

$$\text{and } r_1 = (R_1 \cdot R_1)^{\frac{1}{2}}$$

Then the angle δ between the position vectors satisfies the relation:

$$\cos \delta = \frac{(R_o \cdot R_1)}{r_o r_1} \qquad (2)$$

Unfortunately, equation (2) is not sufficient to define δ; in the plane determined by R_o and R_1, the motion may be through either of two angles, one less than, the other more than, 180^o. The case where R_o and R_1 are parallel, and do not determine a plane, is not considered here. Suppose, temporarily, that the motion takes place through the smaller of the two possible angles. Let H_1 be the unit vector parallel to the angular momentum vector if the motion were to take place in the smaller of the two angles:

$$H_1 = \frac{R_o \times R_1}{|R_o \times R_1|}$$

The current unit angular vector $H_o = \dfrac{R_o \times \dot{R}_o}{|R_o \times \dot{R}_o|}$, where $\dot{R}_o = \dot{R}\,(t_o)$ is the current velocity vector. Remembering that the guidance logic determines a thrust vector so as to change the motion from that described by H_o to that described by H_1, we pick the value of δ to get the smaller change Φ in unit angular momentum vector. The angle Φ is defined by:

$$\text{Cos } \Phi = H_o \cdot H_1 = \frac{(R_o \times \dot{R}_o)}{|R_o \times \dot{R}_o|} \cdot \frac{(R_o \times R_1)}{|R_o \times R_1|} \qquad (3)$$

Φ will have its smallest value when cos Φ is closest to +1. In our case, there are only two possibilities for cos Φ, one positive and the other negative. Thus, if equation (3) should produce a negative value, the unit angular momentum vector H_o should be reversed; that is, the direction of motion should be reversed, or δ should be greater than 180^o. Since only the sign in (3) is important, for computational purposes, the positive denominators on the right hand sides may be discarded and the numerator replaced by:

$$e = \text{sgn}\,[r_o^2\,(\dot{R}_o \cdot R_1) - d_o\,(R_o \cdot R_1)\,] \qquad (4)$$

where $d_o = R_o \cdot \dot{R}_o$

Equation (11b) of [1] makes use of the function $F_o\,(\frac{\delta^2}{4})$. For single revolutions, it is possible to avoid an F - computation at this point, since

$$F_o\,(\frac{\delta^2}{4}) = \cos\frac{\delta}{2} = e\sqrt{\frac{\cos\,\delta + 1}{2}} \,,$$

where e is just the value defined in (4). Thus,

$$\eta = \sqrt{2r_o \, r_1} \; F_o \left(\frac{\delta^2}{2}\right) = e \sqrt{r_o r_1} \sqrt{\cos\delta + 1} =$$

$$e \sqrt{R_o \cdot R_1 + r_o \, r_1} \qquad (5)$$

The solution consists of an iteration procedure, using $\alpha^2 = -\theta^2$ as an independent variable, when θ is the difference in eccentric anomalies of the two positions. For any value of α^2, define $F_i(\alpha^2) = \sum_{j=o} \frac{(\alpha^2)^j}{(i+2j)!}$

$$(i = 0, 1, \ldots 5) \qquad (6)$$

Note that α^2 as here defined, is the negative of the α^2 implicitly defined by equation (4b) of [1]. In equation (11e) of [1], F_o occurs again, this time as $F_o\left(\frac{\alpha^2}{4}\right)$, which can be replaced by:

$$k \sqrt{\frac{F_o(\alpha^2) + 1}{2}} \quad \text{where}$$

$$k = +1 \quad \text{when } \alpha^2 > -\pi^2$$

$$-1 \quad \text{when } \alpha^2 < -\pi^2$$

The iteration process is as follows: Guess the initial value α_o^2 of α^2. Compute $F_i(\alpha_n^2)$. For fixed n, the F_i have only the one argument just shown, so we drop it for simplicity.

$$\omega = r_o + r_1 - k\eta \sqrt{F_o + 1} \qquad (7)$$

$$G = \frac{\omega^{3/2} F_3}{F_2^{3/2}} + \eta \, \omega^{\frac{1}{2}} \qquad (8)$$

$$\phi = \frac{d(F_3 F_2^{-3/2})}{d\alpha^2} = F_2^{-5/2} \left[3/4 \left(F_5 + F_3^2\right) - \tfrac{1}{4} F_4 - \tfrac{1}{2} F_2 F_4 \right] \qquad (9)$$

$$\frac{dG}{d\alpha^2} = 3/8 \, \frac{\omega^{\frac{1}{2}} \eta F_3}{F_2} + 1/8 \, \frac{\eta^2 F_2^{\frac{1}{2}}}{\omega^{\frac{1}{2}}} + \omega^{3/2} \phi \qquad (10)$$

$$\alpha_{n+1}^2 = \alpha_n^2 + \frac{G - \sqrt{\mu}\,\Delta t}{\frac{dG}{d\alpha^2}} \tag{11}$$

Test to see if α^2_{n+1} is sufficiently close to α_n^2 or if G is sufficiently close to $\sqrt{\mu}\,\Delta t$. If either is true, we are through; if not, compute $F_i(\alpha^2_{n+1})$ and return to (7)[1].

Finally, after the iteration process has converged, the required velocity vector V_R is given by:

$$V_R = \frac{1}{g}\,(R_1 + f\,R_o) \tag{12}$$

When $$\frac{1}{g} = \frac{\sqrt{\mu}}{\eta\,\omega^{\frac{1}{2}}} \quad \text{and} \quad f = \frac{r_1 - k\eta\sqrt{1 + F_o}}{r_o} \tag{13}$$

unless R_1 and R_o are separated by 180°. (Note, nevertheless, that the iteration process in equations (7)-(11) will converge even in this exceptional case).

Time derivatives

Next, we apply this solution in a guidance scheme. The standard procedure is (1) to sense the current position and velocity R_o and $\dot{R}_o$; (2) to solve Lambert's Problem, using these vectors and a given target position R_1 at time t_1; (3) to use the resulting velocity vector $V_R = V_R\,(t)$ to determine a desired thrust vector; (4) to apply the needed thrust, or to simulate it in the integrator ; and (5) to return to (1) after some time under the influence of the acceleration due to thrust and other accelerations, repeating the whole procedure until the actual and desired velocity vectors agree.

The cross-product steering law for determining the thrust in (3) requires the time derivative of V_R. To compute this requires knowing $\frac{d\eta}{dt}$, $\frac{d\omega}{dt}$, and $\frac{d\alpha^2}{dt}$. The variables involved--η, ω, α^2, and V_R itself--are all functions of the current position only (the target position and time are constant in time during thrusting changes), so the derivatives will be functions of the current position and velocity vectors only.

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[1] Formula (10) is obtained from the derivative formulas $\frac{d\,F_i}{d\,\alpha^2} = \frac{1}{2}\,(i\,F_{i+2} - F_{i+1})$ and the recursion formulas-- $F_i = \frac{1}{i!} + \alpha^2 F_{i+2}$

Making use of:

$$\frac{d\, r_o}{dt} = \frac{R_o \cdot \dot{R}_o}{r_o} = \frac{d_o}{r_o} \;,$$

we have that:

$$d\eta = \frac{e\left(\dot{R}_o \cdot R_1 + \frac{d_o\, r_1}{r_o}\right)}{2\, e\, \eta} = \frac{(r_o R_1 + r_1 R_o)\cdot \dot{R}_o}{2\, \eta\, r_o} \qquad (14)$$

The derivation of $\frac{d\omega}{dt}$ and $\frac{d\alpha^2}{dt}$ must be carried out simultaneously. From equation (7) we have

$$\frac{d\omega}{dt} = \frac{d_o}{r_o} - \frac{(1 + F_o)^{\frac{1}{2}}\,(r_o\, R_1 + r_1\, R_o)\cdot \dot{R}_o}{2\, \eta\, r_o} + \qquad (15)$$

$$\frac{k\, \eta\, F_1}{4\,(1 + F_o)^{\frac{1}{2}}}\; \frac{d\,\alpha^2}{dt}$$

At the end of the iteration, we have:

$$\sqrt{\mu}\,(t_1 - t_o) = G = \frac{\omega^{3/2}\, F_3}{F_2^{\,3/2}} + \eta\, \omega^{\frac{1}{2}}$$

Differentiating this equation, remembering that ϕ in equation (10) is $\frac{dG}{d\alpha^2}$, we get:

$$-\sqrt{\mu} = \omega^{3/2}\, \phi\, \frac{d\alpha^2}{dt} + \omega^{\frac{1}{2}}\, \frac{d\eta}{dt} + \left(3/2\; \frac{\omega^{\frac{1}{2}}\, F_3}{F_2^{\,3/2}} + \frac{\eta}{2\omega^{\frac{1}{2}}}\right) \frac{d\omega}{dt} \qquad (16)$$

Combining (15) and (16), we have:

$$\frac{d\omega}{dt} = \frac{4\, \phi\, \omega^{3/2}\, (1 + F_o)^{\frac{1}{2}} \left(\frac{d_o}{r_o} - \frac{(1 + F_o)^{\frac{1}{2}}}{\sqrt{2}}\, \frac{d\eta}{dt}\right) - \eta\, F_1\, \left(\sqrt{\mu} + \omega^{\frac{1}{2}}\, \frac{d\eta}{dt}\right)}{4\, \phi\, \omega^{3/2}\, (1 + F_o)^{\frac{1}{2}} + \eta\, F_1\, \left(3/2\; \frac{\omega^{\frac{1}{2}}\, F_3}{F_2^{\,3/2}} + \frac{\eta}{2\sqrt{\omega}}\right)} \qquad (17)$$

and

$$\frac{\dfrac{d\alpha^2}{dt} = -\sqrt{\mu} - \omega^{\frac{1}{2}} \dfrac{d\eta}{dt} - \left(3/2\, \dfrac{\omega^{\frac{1}{2}} F_3}{F_2^{\,3/2}} + \dfrac{\eta}{2\omega^{\frac{1}{2}}} \right) \dfrac{d\omega}{dt}}{\phi\, \omega^{3/2}} \tag{18}$$

Finally, noting that (see equation 13):

$$\frac{d \log g}{dt} = \frac{1}{g} \frac{dg}{dt} = \frac{1}{\eta} \frac{d\eta}{dt} + \frac{1}{2\omega} \frac{d\omega}{dt} , \tag{19}$$

$$\frac{dV_R}{dt} = \frac{1}{g} \left\{ - \frac{d \log g}{dt} R_1 + \left[f \frac{d \log g}{dt} - \frac{1}{r_o} \left(\frac{\omega d_o}{r_o^2} - \frac{d\omega}{dt} \right) \right] R_o - f \dot{R}_o \right\} \tag{20}$$

For step-by-step integration methods, equation (18) may be used to predict a first guess for α^2 at the next time step. Indeed, if Δt is the integration step, and if α^2 (t) is the value of α^2 at the current time,

$$\alpha^2 (t + \Delta t) = \alpha^2 (t) + \frac{d\alpha^2}{dt} \Delta t.$$

<u>References</u>:

[1] Pines, S.: "A Uniform Solution of Lambert's Problem", NASA IN 65-FM-166, dated December 1965.

[2] "Guidance System Operation Plan AS-278", MIT Instrumentation Laboratory, R-547, Volume I, pp. 5-51, 5-53, dated October 1966.