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CASCADE THEORY OF PLASMA TURBULENCE IN A STRONG MAGNETIC FIELD

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ABSTRACT

For a quasi-neutral plasma consisting of cold ions embedded in hot electrons, the turbulent motion of the ions can be described by a system of dynamical equations determining the velocity and the electric potential. The system is submitted to a cascade approximation and yields two integral equations determining the two spectral functions. Three transport functions are involved: a transfer function across the velocity spectrum, a transfer function across the potential spectrum, and a production function representing the production of potential energy from kinetic energy and causing an exchange between the two spectra. Solutions are found for the following cases: (a) for a collisionless plasma, $F \sim k^{-3}$ (inertial subrange), $G \sim k^{-5}$ (dissipative subrange), (b) for a collisional plasma, $F \sim k^{-2}$ (inertial subrange), $G \sim k^{-3/2}$ (inertial subrange), $G \sim k^{-4.5}$ (dissipative subrange), (c) for a dilute suspension of charged particles, $F \sim k^{-5/3}$ (Kolmogoroff law of inertial subrange), $G \sim k^{-13/3}$ (dissipative subrange). Here F = velocity spectrum, G = potential spectrum, and k = wave-number. From the spectral results, the anomalous diffusion λ_k is investigated: (a) for a weak turbulence, $\lambda = \varepsilon/\omega_c^2$ (collisional plasma), $\lambda_k = a^2/\omega_c$ (collisionless plasma), (b) for a strong turbulence, the above values of diffusion have to be multiplied by a factor $t_c(\omega_c, \omega_0) R^{\frac{1}{2}}$, where $t_c = \omega_c^{-1}$ (collisional plasma), and $t_c = \omega_0^{-1}$ (collisionless plasma). Here ω_c = cyclotron frequency, $a^2 = kT_e/M$ (T_e = electron temperature,

M = ion mass), $\mathcal{E}$ = rate of collisional dissipation of turbulent energy, R = velocity vorticity function = square of rate of strain, J = potential vorticity function, $\omega_{0} \doteq (J\omega_{c})^{\frac{1}{3}}$. The presence of a strong turbulence is to weaken the ω_c dependence. The Bohm diffusion is found to be valid for a weak turbulence in a collisionless plasma.

1. MECHANISM OF ENERGY EXCHANGES IN HYDRODYNAMIC TURBULENCE AND PLASMA TURBULENCE.

It has long been recognized since Boussinesq[1] that the statistical effect of the fluctuations upon a mean, or background motion, is to produce an eddy viscosity, according to a mixing-length concept, that, in analogy with the molecular motions, will introduce an energy dissipation. The latter is found proportional to the product of the eddy viscosity with the vorticity function, which is defined as the square of the rate of strain. For a homogeneous and isotropic turbulence in the absence of a mean motion, we may draw a spectrum of eddies, and categorize the eddies into two groups: The big eddies of wave number smaller than k and the smaller eddies of wave number greater than k, k being a variable. Then by analogy with the Boussinesq's mixing-length concept, the big eddies may assume the role of a background motion creating a vorticity function

$$R_o(k) = 2\int_0^k dk'\, k'^2 F(k') \qquad (1)$$

where F is the velocity spectrum, and the smaller eddies produce an eddy viscosity ν_k, so that the rate of turbulent dissipation, due to the nonlinear mode coupling, or energy transfer across the spectrum, is given by

$$\nu_k R_o$$

the transfer function, while the rate of collisional dissipation, due to the kinematic viscosity ν, is

$$\nu R_o$$

[1] J.J. Boussinesq, Théorie de l'écoulement tourbillonant et tumultueux des liquides, Gautier-Villars (1897).

Hence the total rate of dissipation $\mathcal{E}$ yields the equation

$$(\nu + \nu_k) R_o = \varepsilon \qquad (2)$$

called the equation of energy balance in the framework of hydrodynamic turbulence.

On a dimensional basis Heisenberg[2] proposed a formula for ν_k

$$\nu_k = \int_k^\infty dk \, (F/k^3)^{\frac{1}{2}} \qquad (3)$$

and subsequently was able to solve (2).

A spectrum is divided into a non-universal range and a universal range. The former range often depends on particular experimental conditions covering very low wave-numbers. The universal range is subdivided into an inertial subrange and a dissipative subrange. The inertial subrange is governed by the nonlinear transfer alone. The dissipative subrange is characterized by a sudden drop of the spectrum. Heisenberg[2] found the spectrum

$$F = \text{const}\ \varepsilon^{2/3} k^{-5/3} \qquad (4)$$

in the inertial subrange, in agreement with Kolmogoroff's similitude theory[3].

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[2] W. Heisenberg, Z. f. Phys. 124, 628 (1948).

[3] A. N. Kolmogaroff, C. R. Acad. Sci. URSS, 30, 301(1941).

We would like to extend the above picture of the mechanism of energy exchanges to a plasma turbulence. Here a simple physical argument can be presented first. Since the random velocity dissipates and couples with other modes, both the collisional dissipation function νR_o and the transfer function $\nu_k R_o$ should subsist in the plasma turbulence. But the plasma has the added feature of correlating the velocity with the electric potential, so that a new transport function Φ_k, called production function, will appear, producing the potential energy from the kinetic energy. Hence, the equation of energy balance (2) should be generalized to

$$(\nu + \nu_k) R_o + \Phi_k = \varepsilon \tag{5}$$

for a plasma. The potential of the self-consistent electric field, as originated from density fluctuations, also couples to other modes in its turn, due to its nonlinearity, and therefore yields a new transfer function $\lambda_k J_o$; the latter is proportional to the product of a new eddy viscosity λ_k by the vorticity function

$$J_o(k) = 2 \int_0^k dk' \, k'^2 \, G(k') \tag{6}$$

relating to the spectrum G of the potential. There is a balance between the transfer function and the production function expressed by

$$\lambda_k J_o - \Phi_k = 0 \tag{7}$$

As will be shown later, the production function is also a transport function, to be expressed again as a product of a certain diffusion by the vorticity function (6). The system of equations (5) and (7) will be called equations of energy balance for a plasma.

Obviously the arguments given above to the equations of energy balance (5) and (7) are oversimplified, and should be given a more dynamical foundation. But this poses a very difficult task of solving a system of nonlinear dynamical equations. The dimensional method of Heisenberg and Kolmogoroff are too arbitrary in the present program of having to deal with many functions of similar dimensions. The newer methods, e.g. stochastic methods[4] and diagram techniques[5], which are already laborious in the derivation of the simplest Kolmogoroff law, will prove even more difficult for plasma turbulence. We feel that at the present time it is first necessary and important to determine the dominant physical features. Therefore, we resort to an approximate method called "cascade approximation". It is based on the categorization into groups of eddies as mentioned earlier, and studies the coupling between the groups.

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[4] R. H. Kraichnan, J. Fluid Mech. 5, 497 (1959).

[5] A. V. Shut'ko, Dokl. Akad. Nank USSR 158, 1058 (1964) [English transl: Soviet Phys., Doklady 9, 857 (1965)].

Following the derivation of the energy balance, the transport functions ν_k, λ_k and Φ_k will be determined. The solutions will be sought for a collisional and collisionless plasma, as well as for a dilute suspension of charged particles. The anamalous diffusions, including the Bohm diffusion and the turbulent diffusion, will be calculated from the spectral functions.

2. DYNAMICAL EQUATIONS OF AN ION PLASMA

Consider a quasi-neutral plasma consisting of cold ions and hot electrons. The hot electrons are in equilibrium with their density n_e equal to the ion density n following the distribution

$$n = n_e = n_0 \exp(\psi/a)$$

where n_0 is the average density, ψ is the electric potential and a is the phase velocity with electron temperature T_e and ion mass M :

$$a^2 = kT_e/M$$

The low ion temperature T_i , as originated from the effect of pressure, offers a negligible modification to the phase velocity. Let $\omega_c = eB_0/M$ be the cyclotron frequency from an external and constant magnetic field B_0 in the x_3 - direction, assumed larger than any frequency scale entering in the problem. Further, u_i are the velocity components of the ions in the directions perpendicular to the magnetic field. The dynamical equations of the plasma are

$$\frac{du_i}{dt} - \epsilon_{ijk}\, u_j\, \gamma_k\, \omega_c = -a \frac{\partial \psi}{\partial x_i} \tag{8a}$$

$$\frac{d\psi}{dt} = -a\, \nabla \cdot \underset{\sim}{u} \tag{8b}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \frac{\partial}{\partial \underset{\sim}{x}}$$

Here ϵ is an antisymmetric unit tensor, and $\gamma = 0, 0, 1$. The system (8) forms the dynamical equations upon which we shall apply the cascade approximation and investigate the turbulent spectra.

We shall introduce some general assumptions:

(i) We consider a homogeneous and isotropic turbulence in the directions perpendicular to the magnetic field.

(ii) The compressibility effects may manifest itself in the following instances: (a) The compressibility of the fluid, by inducing density changes, may excite fluctuations of the electric potential, as represented by the right hand sides of (8); (b) the compressibility may influence the pressure; (c) the compressibility may enter in some convective process as in d/dt. We shall assume that the effect (a) is much larger than the effects (b) and (c) in view of the high electron temperature. This assumption is analogous to the Boussinesq assumption in gravity waves, where the compressibility effect is neglected everywhere except in the buoyency force.

3. CASCADE APPROXIMATION

In the spectral analysis we seldom need the full information about each individual Fourier component. Therefore we divide the components into two groups, e.g. for the velocity we have

$$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{u}_o(\underset{\sim}{x}) + \underset{\sim}{u}'(\underset{\sim}{x})$$

where $\underset{\sim}{u}_o$ represents the group of big eddies with wave-numbers up to the value k, and $\underset{\sim}{u}'$ represents the group of smaller eddies with wave-numbers greater than k, k being a variable. $\underset{\sim}{u}_o$ is quasi-stationary, while $\underset{\sim}{u}'$ is a rapidly varying variable. A similar decomposition is applied to ψ. The two groups can be screened by an average

$$\langle \cdots \rangle_k$$

over a length scale k^{-1}, so that the dynamical equations for $\underset{\sim}{u}_o$, ψ_o, $\underset{\sim}{u}'$ and ψ' can be generated from (8) by such a procedure, as follows :

$$\begin{aligned} \frac{Du_{oi}}{Dt} - \epsilon_{ijk}\, u_j\, \gamma_k\, \omega_\ell &= -a\frac{\partial \psi_o}{\partial x_i} - \left\langle u_j' \frac{\partial u_i'}{\partial x_j}\right\rangle_k \\ \frac{D\psi_o}{Dt} &= -a\frac{\partial \underset{\sim}{u}_o}{\partial \underset{\sim}{x}} - \left\langle \underset{\sim}{u}' \cdot \frac{\partial \psi'}{\partial \underset{\sim}{x}}\right\rangle_k \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{Du_i'}{Dt} - \epsilon_{ijk}\, u_j'\, \gamma_k\, \omega_\ell &= -a\frac{\partial \psi'}{\partial x_i} - u_j' \frac{\partial u_{oi}}{\partial x_j} \\ \frac{D\psi'}{Dt} &= -\underset{\sim}{u}' \cdot \frac{\partial \psi_o}{\partial \underset{\sim}{x}} \end{aligned} \tag{10}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{u}_o \cdot \frac{\partial}{\partial \underset{\sim}{x}}$$

We apply the cascade system (9) to the analysis of the spectra in an isotropic and homogeneous plasma turbulence in the direction perpendicular to the magnetic field.

The system of cascade equations (9) include a coupling between the big and smaller eddies, involving two transport coefficients, which are calculated from (10). To this end, we make two assumptions applied especially to smaller eddies:

(a) As the smaller eddies contain little energy, the equations (10) are linearized.

(b) The smaller eddies are responsible for the transport properties, created by stretching the big eddies. During this process there is a secondary loss of energy by compressibility which is negligible.

After solving (10), the coupling terms in (9) become nonlinear. Besides, there are the nonlinear convective terms on the left hand sides of (8) and (9). The ones in (8) will generate correlation functions of ever increasing higher orders, but we disregard (8) and use (9) instead. The nonlinear convective terms in (9) are not serious as they will drop out upon forming energy equations and using the assumption (i) and (ii) about homogeneity and compressibility. The energy equations based upon (9) have the advantage of describing the development of the spectra directly.

We may stipulate the fundamental difference between the more familiar quasilinear method and the cascade method for the solution of a nonlinear differential equation. The quasilinear method is a perturbation method based upon an expansion of a function of small magnitude into a series of decreasing order of magnitudes. The method is therefore only applicable to a weak turbulence. In the cascade method, the function is not restricted to a small magnitude, but when it is decomposed into a cascade of two groups of eddies, that group representing smaller eddies is assumed of small magnitude, while the group of big eddies remains of finite amplitude. Therefore the method is adaptable to a strong turbulence.

4. COLLISIONAL AND COLLISIONLESS DISSIPATIONS

Upon multiplying (8) by $\underset{\sim}{u}_0$ and ψ_0 and taking an average over an infinitely large interval, denoted by $\langle \cdots \rangle$, we obtain the equations of energy balance

$$\frac{1}{2}\left\langle \frac{D}{Dt} u_0^2 \right\rangle = -(\nu + \nu_k) R_0 - \Phi_k \tag{11}$$

$$\frac{1}{2}\left\langle \frac{D}{Dt} \psi_0^2 \right\rangle = -\lambda_k J_0 + \Phi_k \tag{12}$$

In the energy equation (11) we have included a kinematic viscosity ν.

In the following we shall consider an equilibrium state of turbulence and apply the general assumptions (i) and (ii) introduced earlier so that the time and space derivatives of any mean value will vanish. Further, there must exist external energy sources or sinks maintaining the conservation of energy flux at all k, so that (11) and (12) may reduce to (5) and (7) respectively.

The calculations of the stresses from (10) gives

$$\nu_k = \frac{1}{2}\int_0^\infty d\tau \left\langle \underset{\sim}{u}'(0) \cdot \underset{\sim}{u}'(\tau) \right\rangle_k \cos \omega_c \tau$$

$$\lambda_k = \frac{1}{2}\int_0^\infty d\tau \left\langle \underset{\sim}{u}'(0) \cdot \underset{\sim}{u}'(\tau) \right\rangle_k$$

$$\Phi_k = \left\langle u' \cdot \Xi' \right\rangle_k$$

$$= \int_0^\infty d\tau \left\langle \underset{\sim}{\Xi}'(0) \cdot \underset{\sim}{\Xi}'(\tau) \right\rangle_k \cos \omega_c \tau$$

where $\underset{\sim}{E}'$ is the self-consistent electric field related to ψ' by

$$\underset{\sim}{E}' = - a \nabla \psi'$$

We shall omit the detailed calculations of the functions Φ_k and ν_k, but write their results in the following simple approximate expressions:

$$\Phi_k \cong \langle E'^2 \rangle_k \, t_c(\omega_c) = a^2 \left\langle \left(\frac{\partial \psi'}{\partial x} \right)^2 \right\rangle_k t_c(\omega_c)$$

$$\nu_k \cong \langle u'^2 \rangle_k \, t_c(\omega_c)$$

where the time scale is simply

$$t_c(\omega_c) = \omega_c^{-1} \tag{13}$$

in the case of a strong magnetic field. As the magnetic field is not effective in λ_k, the time scale is found to have a more complicated form there:

$$\lambda_k = \int_k^\infty dk' \, F(k') \, t_c[k', F(k')] \tag{14a}$$

with

$$t_c(k, F) = [R_o(k)]^{-\frac{1}{2}} \tag{14b}$$

and consequently

$$\lambda_k = \int_k^\infty dk' \, F(k) \, [R_o(k')]^{-\frac{1}{2}} \tag{14c}$$

The formula (14c) differs from the eddy viscosity (3), proposed by Heisenberg on a dimensional basis.

The production function Φ_k and the eddy viscosity ν_k can also be written in terms of the spectral functions F and G for the velocity and the potential respectively.

$$\Phi_k = \lambda(J - J_o), \qquad J = J_o(k=\infty) \tag{15}$$

$$\nu_k = \frac{1}{\omega_c}\int_k^\infty dk' F(k') \tag{16}$$

We note that in the expression for Φ_k a coefficient

$$\lambda = a^2/\omega_c \tag{17}$$

comes out that has the structure of the Bohm diffusion.

It is obvious that the production function Φ_k, written above as the product of the Bohm diffusion λ with the vorticity functions $J - J_o$, may play the role of a negative dissipation as in (7). This is made more explicit if we rewrite (7) in the form

$$(\lambda + \lambda_k) J_o = \eta \tag{18}$$

analogous to the energy balance (2) in the framework of a hydrodynamic turbulence. In this form, the transfer, or mode coupling $\lambda_k J_o$, is drained by a dissipation λJ_o which is most effective at large k. This dissipation is not of the molecular origin as was in (2), but comes from the fluctuations giving rise to a Bohm diffusion λ. The dissipations λJ_o and $\lambda J \equiv \eta$ are called collisionless dissipations.

The equations (5) and (7), with their transport expressions (14), (15) and (16), form the fundamental system of energy balance and coupling between the two spectral functions. In conclusion, the mechanism of energy balance takes the following picture: The energy in the F-spectrum is transferred across the spectrum by the amount $\nu_k R_o$ to be drained ultimately by molecular, or collisional dissipations νR_o and $\nu R \doteq \varepsilon$ at large k. Simultaneously there is an exchange of energy, or a coupling between the two spectra F and G, forming a new drain. The energy in the G-spectrum is also transferred across its spectrum by the amount $\lambda_k J_o$, to be drained by collisionless dissipations λJ_o and $\lambda J \doteq \eta$ at large k. Now we distinguish a collisionless plasma with $\lambda \gg \nu$, and a collisional plasma with $\lambda \ll \nu$. In the former case, the nonlinear transfer across the spectrum is drained mainly by the coupling Φ_k, rather than by collisional dissipations, as $\nu R \ll \Phi_k$, and in the latter case, the drain is controlled by the collisional dissipations with a negligible coupling, as $\Phi_k \ll \nu R_o$.

5. SPECTRAL FUNCTIONS IN A COLLISIONLESS PLASMA

Consider the case of a F-spectrum, which is likely to prevail in the inertial subrange in view of the negligible collision, and under such a regime, a G-spectrum will develop in the dissipative subrange on account of the early drop of the spectrum by the dominant Bohm diffusion.

It is simpler to consider the differential form of (5) and (18):

$$\begin{aligned} \lambda J_0' - \nu_k R_0' &= 0 \\ \lambda J_0' + \lambda_k' J &= 0 \end{aligned} \tag{19}$$

obtained by neglecting collisions and by making the approximations

$$R_0 \cong 0, \quad J_0 \cong J, \qquad \lambda_k \ll \lambda$$

on account of the mixed inertial and dissipative subranges.

A substraction between the two equations (19) gives

$$\nu_k R_0' = -\lambda_k' J$$

or

$$-\frac{u'^2 R_0'}{\lambda_k'} = J\omega_c \equiv \omega_0^3 \tag{20}$$

with a left hand side depending on the F-spectrum alone, suggesting that the time scale ω_0^{-1} is the sole parameter entering into the functional structure of the F-spectrum. Since the spectrum has the dimension

$$F \sim u'^2 \ell$$

with the variables u' and ℓ as the velocity and the diameter of the smallest eddy, we need two independent parameters to determine the functional structure of F, for which we choose

ω_o and λ ; whence

$$u' \sim (\omega_o \lambda)^{\frac{1}{2}} \tag{21a}$$

$$\ell \sim (\lambda/\omega_o)^{\frac{1}{2}} \tag{21b}$$

Consequently

$$F = \omega_o \lambda (\lambda/\omega_o)^{\frac{1}{2}} [k(\lambda/\omega_o)^{\frac{1}{2}}]^{-m}$$

where the term within the brackets is a dimensionless quantity whose exponent m should be so chosen as to make F independent of λ, as required by the condition (20). To this end, we find $m = 3$, giving

$$F = \text{const } \omega_o^2 k^{-3} \tag{22}$$

and it entails from (19) and (22)

$$G = \text{const } (J \omega_o/\lambda) k^{-5} \tag{23}$$

The constants have a numerical value close to unity.

The k^{-5} power law of the spectrum (23) has been measured at Harwell[6] in the frequency range of 5-15 Mc/sec, and at Princeton[7,8] in the frequency range of 10-100 kc/sec (see Fig. 1 and 2).

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[6] D. J. H. Wort and M. A. Heald, Plasma Phys. (J. Nuclear Energy Pt. C) 7, 79-81 (1965).

[7] F. F. Chen, Phys. Rev. Letters 15, 381 (1965).

[8] N. D'Angelo and L. Enriques, Phys. Fluids 9, 2290 (1966).

From the dimensional considerations, ψ' obeys the dimension

$$a\frac{\psi'}{\ell} \sim \frac{u'}{t}$$

or

$$\psi'^2 \sim \frac{1}{a^2}\left(\frac{\ell}{t}\right)^4$$

so that the 5th power law follows immediately:

$$G \sim (at^2)^{-2}\, k^{-5} \tag{24}$$

If the time scale (20) is chosen, (24) confirms the formula (23). However, if we had chosen a time scale ω_c^{-1}, (24) would become

$$G \sim \frac{\omega_c^4}{a^2}\, k^{-5} \tag{25}$$

a formula suggested sometimes for experimental usage[7]. Unfortunately the formula (25) does not account for any dependence on the strength of turbulence, a condition surely unacceptable for strong turbulence. We can resolve analytically the system (19), yielding the same results (22) and (23).

In conclusion, in a collisionless plasma, the F-spectrum has its nonlinear transfer $\nu_k R_o$ drained by a conversion into a potential energy, bypassing the molecular dissipation. The potential perpetuates its own transfer $\lambda_k J_o$ in the G-spectrum. The spectra obtained follow the inertial law $F \sim k^{-3}$ and the dissipative law $G \sim k^{-5}$.

6. SPECTRAL FUNCTIONS IN A COLLISIONAL PLASMA

In a collisional plasma, the evolution of the inertial spectrum of velocity is governed by (2), reduced to

$$\nu_k R_o = \varepsilon \tag{26}$$

It is noted that the coupling is negligible as compared to the drain by collision ε . The two equations of energy balance (18) and (26) are then decoupled and may be solved separately. On account of (16), equation (26) can be rewritten as

$$u'^2 R_o = \varepsilon \omega_c \equiv \alpha^2 \tag{27}$$

indicating that α is the sole parameter characterizing the inertial subrange.

As the spectrum has the dimension

$$F = u'^2 \ell \tag{28}$$

involving two scales u' and ℓ , we need another parameter, e.g. ω_c , for the dimensional determination of the spectrum. We assume that the velocity and the diameter of the smallest eddy can be characterized by the rate of collisional dissipation ε and the life time ω_c^{-1}. For this purpose we write u' and ℓ in terms of α and ω_c as follows:

$$u' \sim \alpha/\omega_c , \qquad \ell \sim \alpha/\omega_c^2 \tag{29}$$

and thus reduce (28) to

$$F \sim \alpha^3 \omega_c^{-4} \left(k \alpha \omega_c^{-2}\right)^{-m}$$

The condition (27) requires that F depends only on α and not on ω_c , yielding m = 2 and consequently

$$F = const\ \alpha\ k^{-2} \tag{30}$$

This procedure is similar to that of Kolomogoroff who uses ε and ν as parameters. The reason to choose the parameters ε and ω_c instead is to relate formula (29) to a diffusion $u'\ell \sim \varepsilon\, \omega_c^{-2}$, as originated from the smallest eddies, instead of the molecular viscosity ν .

The G-spectrum follows

$$J_o = \frac{\eta}{\lambda + \lambda_k}$$

on account of (5), yielding, upon substituting F from (30):

(i) for the inertial subrange ($k<k_B$)

$$G = const\ \eta\ \alpha^{-\frac{1}{2}}\ k^{-3/2} \tag{31a}$$

(ii) for the dissipation subrange ($k>k_B$)

$$G = const\ \eta\ \alpha^{\frac{1}{2}}\ \lambda^{-2} k^{-9/2} \tag{31b}$$

with the critical wave number k_B as a criterion of the spectral drop

$$k_B = (\alpha/\lambda^2)^{1/3}$$

The numerical constants in (30) and (31) are of the order of unity.

The power laws k^{-5} and $k^{-4.5}$ of the dissipative subranges (25) and (31b) for the collisionless and collisional plasmas respectively are very close, and are difficult to be distinguished experimentally. However, in a plasma with a strong collisional dissipation and a strong magnetic field, the wave number of the spectral drop is pushed to a very high value, then the inertial subrange is unequivocally developed, and the $k^{-3/2}$ law of G-spectrum can be ascertained. Some measurements at Saclay seem to indicate such a subrange.

7. DILUTE SUSPENSION OF CHARGED PARTICLES IN A NEUTRAL FLUID

The spectral study made in Sections 5 and 6 refers to the diffusion of particles as a Bohm diffusion, to the exclusion of molecular diffusion. This is true with a fully ionized gas. However, in many applications in ionosphere the charged particles may be a dilute suspension and diffuse by molecular motions in an ambient neutral atmosphere. Since the production function is so weak as not to influence the motion of the atmosphere, the inertial subrange of the F-spectra is determined by the equation

$$\lambda_k R_o = \varepsilon , \qquad \text{with} \qquad \varepsilon = \nu R$$

and the G spectrum is determined by

$$(D + \lambda_k) J_o = \eta , \qquad \text{with} \quad \eta = DJ$$

where D is the molecular diffusion. The system can be integrated, giving the solutions

$$F = \text{Const}\ \varepsilon^{2/3} k^{-5/3}$$

in the inertial subrange, in agreement with the Kolmogoroff law (4), and

$$G = \text{const}\ \eta\, \varepsilon^{-1/3} k^{-5/3} \left[1 + (k/k_D)^{4/3} \right]^{-2} \tag{32}$$

with

$$k_D = \left(\varepsilon / D^3 \right)^{1/4}$$

characterizing the drop of G-spectral due a dissipation by molecular diffusion.

When $\nu / D \ll 1$, k_D becomes a small quantity, reducing (32) to

$$G = \text{Const}\ D^{-2} \eta\, \varepsilon^{-1/3} k^{-13/3} \tag{33a}$$

in the dissipative range ($k \ll k_D$).

When $\nu/D \gg 1$, or $k/k \ll 1$ the charged particles become frozen to the fluid and the G-spectrum follows the inertial subrange of the F-spectrum to assume the formula

$$G = \mathrm{Const}\ \eta\ \varepsilon^{-1/3}\ k^{-5/3} \tag{33b}$$

The power law $k^{-5/3}$ agrees with the Kolmogoroff law of turbulence.

The discrepancy between the two spectral laws (33a) and (33b) is interesting, as it points to a paradox in the theory of radar scattering from ionosphere. According to that theory, the spectrum of the scattered power is proportional to the density spectrum of the electrons suspended in the ionosphere, invoking tacitly the Kolmogoroff law for the latter spectrum. It is known that the theory of scattering considers a randomly stratified medium without motions, and the Kolmogoroff theory excludes a random density. Hence it is paradoxal to invoke a density spectrum following the Kolmogoroff law. The present consideration shows that the paradox can be lifted under the circumstance $\nu/D \gg 1$, and that the theory fails in the case $\nu/D \ll 1$. When we deal with a dilute suspension of electrons with their mass much smaller than the mass of the neutral atoms, as in the actual ionosphere the latter case prevails and the density spectrum should not follow the Kolmogoroff law (33b).

8. DIFFUSION IN A TURBULENT PLASMA

We begin with a weakly turbulent plasma, where the mode coupling is weak, so that the duration of correlation can be taken as the life time of the smallest eddies $t_c(\omega_c, \omega_o)$ introduced in the spectral scales (21) and (29). Under such a circumstance we can write the diffusion (14a) in the form

$$(\lambda_k)_{weak} \sim u'^2 t_c(\omega_c, \omega_o)$$

If we use such scales (21) and (29), we obtain easily

$$(\lambda_k)_{weak} = \varepsilon \omega_c^{-2} \tag{34a}$$

and

$$(\lambda_k)_{weak} = \lambda \tag{34b}$$

for a collisional and collisionless plasma respectively.

We remark that the dissipation ε is defined as

$$\varepsilon = \nu \langle (\partial u'_i / \partial x_j)^2 \rangle_k$$
$$\cong \tau_c^{-1} \langle u'^2 \rangle_k$$

where τ_c^{-1} is the collision frequency, and $\frac{1}{2}\langle u'^2 \rangle_k$ is the kinetic energy of the smallest eddy. If we replace the turbulent energy by the thermal energy in the dissipation

$$\varepsilon \cong \tau_c^{-1} a^2$$

transforming (34a) to

$$(\lambda_k)_{laminar} = \tau_c^{-1} \omega_c^{-2} a^2$$

we degenerate to a well known formula of classical diffusion.

The diffusion by strong turbulence can be **calculated from** the spectra (22) and (30), using (14a) rewritten in the form

$$(\lambda_k)_{strong} = u'^2 \; t_c(k,F)$$

It turns out that the results of the calculations are simply

$$\frac{(\lambda_k)_{strong}}{(\lambda_k)_{weak}} = \frac{t_c(k,F)}{t_c(\omega_c,\omega_o)} \equiv A(\omega_c,\omega_o)$$

If we take (14b) for $t_c(k,F)$ for a strong turbulence, and

$$t_c(\omega_c) = \omega_c^{-1}, \qquad t_c(\omega_o) = \omega_o^{-1}$$

for a weak turbulence in a collisional and collisionless plasma respectively, we find correspondingly

$$A(\omega_c) = \omega_c R_o^{-\frac{1}{2}}, \qquad A(\omega_o) = \omega_o R_o^{-\frac{1}{2}}$$

These ratios are larger than unity and weakens the dependence of the diffusion with ω_c. More explicitly we have

$$(\lambda_k)_{strong} = \frac{\varepsilon}{R^{\frac{1}{2}}} \frac{1}{\omega_c}, \quad \text{collisional plasma} \tag{35a}$$

$$(\lambda_k)_{strong} = \frac{a^2}{\omega_c} \frac{\omega_o}{R^{\frac{1}{2}}}, \quad \text{collisionless plasma} \tag{35b}$$

Thus the diffusion by strong turbulence varies proportionately to ω_c^{-1} and $\omega_c^{-2/3}$ respectively, as compared to the diffusion by weak turbulence proportional to ω_c^{-2} and ω_c^{-1}.

Unlike the Bohm diffusion (17) and (34b), the turbulent diffusions (35) depend on the amplitude of turbulence. Unfortunately very few experiments coordinate the anomolous diffusion to the state of turbulence, e.g. spectrum, so that it is difficult to find experimental verification of the theoretical results. However, there are indications that the diffusion should depend on the strength of turbulence, pointing to the inadequacy of the Bohm formula. In the past years it has always been recognized that wherever turbulence is found, the Bohm formula of diffusion should be applied. The present calculations indicate that the Bohm formula (34b) is not valid for a strong turbulence. It is valid for a weak turbulence in a collisionless plasma, but not in a collisional plasma, for which the formula (34a) should apply. In this connection we may note that it is not surprising that several experiments in weakly turbulent plasmas (e.g. by Buchelnikova et al.[10]) found a diffusion varying with ω_c^{-2} and depending on the turbulent energy, in contradiction to the Bohm formula. For a strong turbulence in collisional and collisionless plasmas, the formulas (35a) and (35b) should apply, they again differ significantly from the Bohm formula.

[10] N.S. Buchel'nikova, R.A. Salimov and Yu. I. Eidel'man, Soviet Phys. JETP 25, 548-556 (1967).

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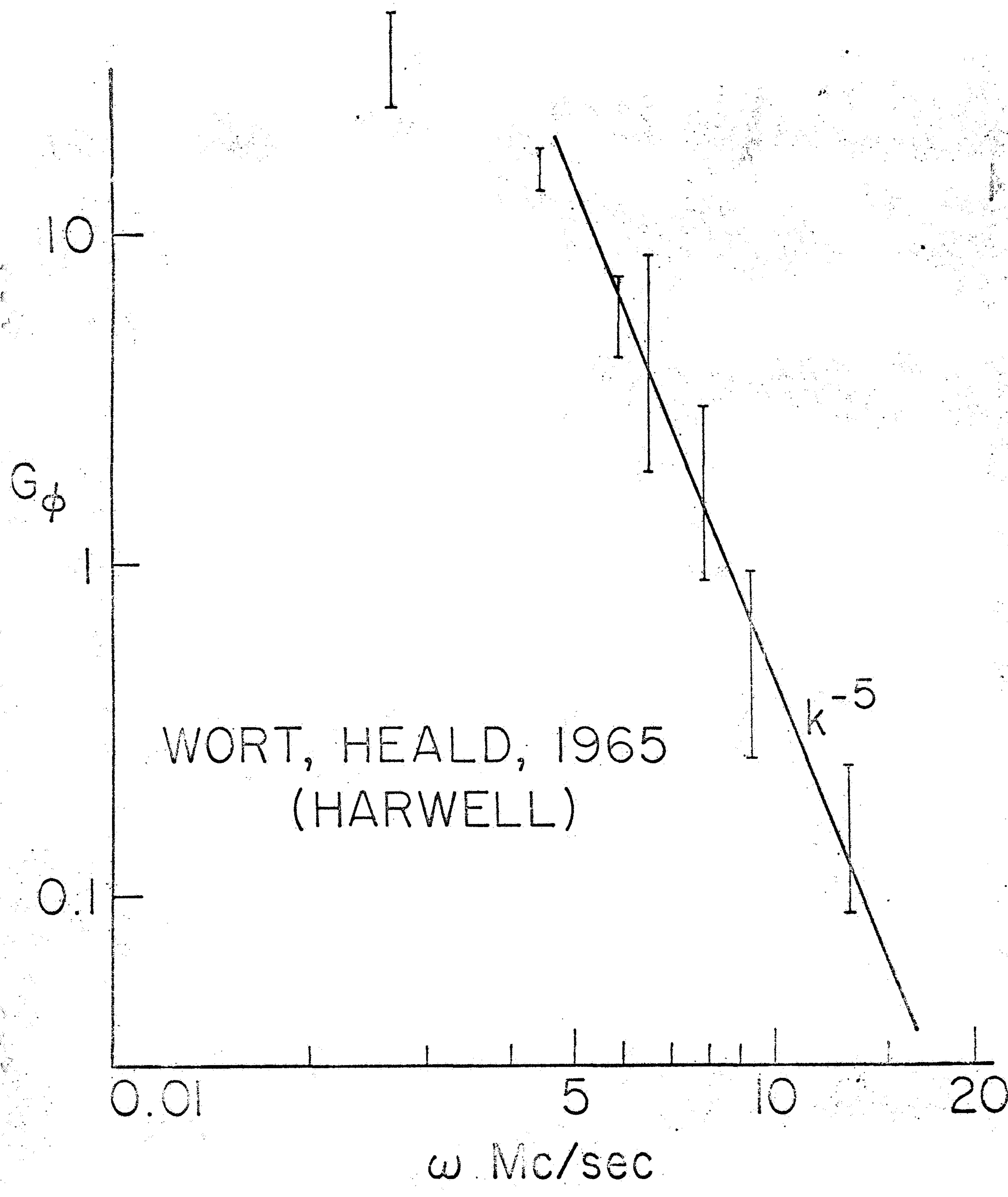

Fig. 1. Spectrum of the Electric potential, Ref.6.

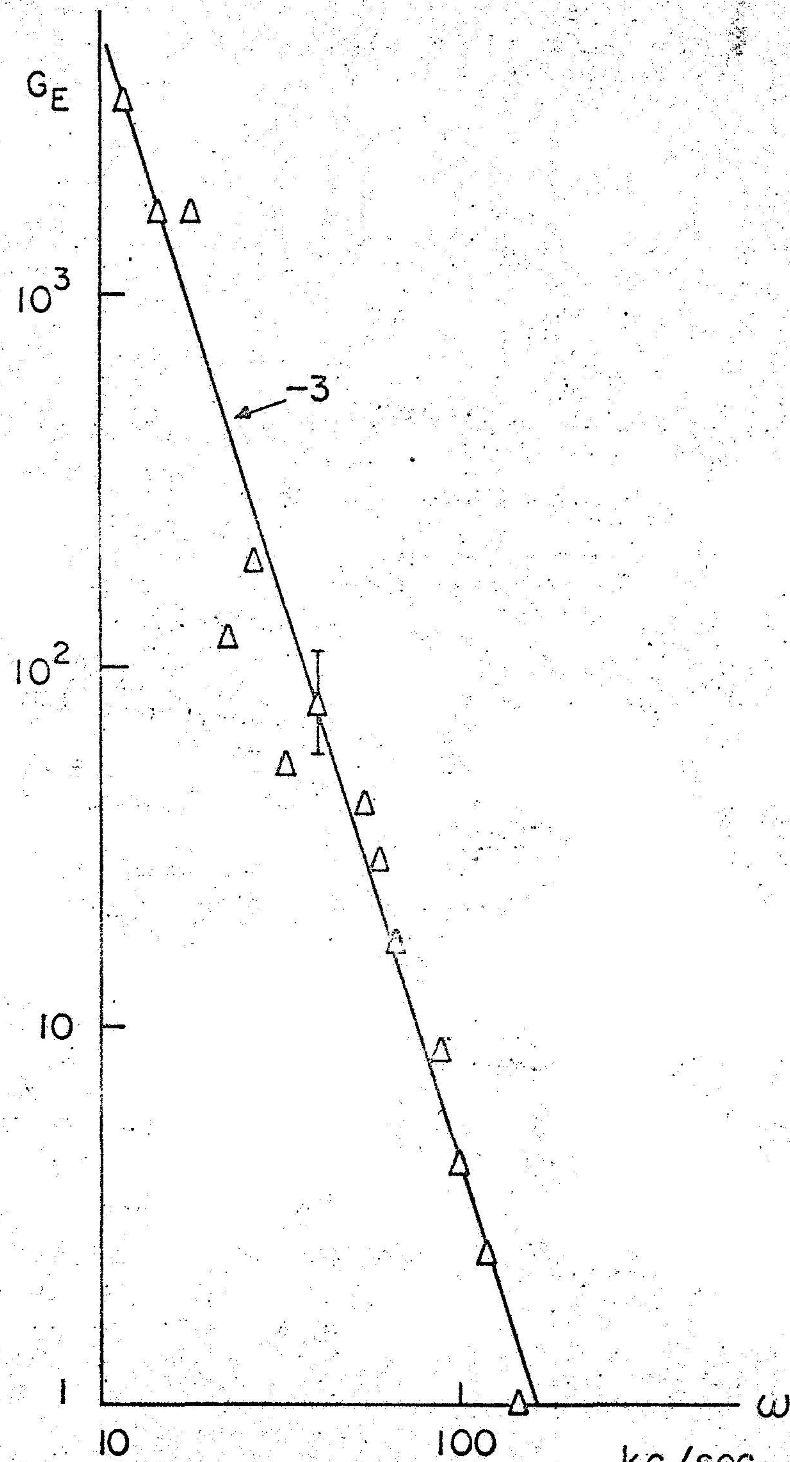

Fig. 2. Spectrum of the Electric Field Fluctuations.
Ref. 9.