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NUMERICAL SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND

BY

ABRAHAM A. TAL

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ABRAHAM A. TAL

Visiting Research Assistant Professor
Computer Science Center
University of Maryland

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ABSTRACT

The equation $\int_0^1 K(s,t)\, x(t)\, dt = y(s)$, with $K(s,t)$ continuous in both variables, has a solution for only those functions $y(s)$ which have a Fourier expansion $\sum \eta_i \varphi_i$ such that $\sum (\eta_i/\lambda_i)^2 < \infty$. Approximate solutions obtained by finite difference discretization are highly oscillatory even when the order of the corresponding matrix is only moderately large. We prove that the method of steepest descent does provide smooth approximations. A number of numerical examples are given (sec. 4-5).

In the applications, it occurrs sometimes that the righthand side is known at a small number of points only. For this case we propose a choice for the solution and establish a relationship between this solution and that of the integral equation (sec. 2-3).

1. Introduction

This report concerns the numerical solution of Fredholm equations of the first kind.

$$(1.1) \quad Ax(s) = \int_0^1 K(x,t)\ x\ (t)\ dt = y(s)$$

where $K(s,t) = K(t,s)$ is a real and continuous function in $0 \leq s,\ t \leq 1$.

It is well known (see e.g. [1] p. 135) that (1.1) possesses a solution in $L^2(o.1)$ only for those functions Y on the righthand side for which the series $\sum_i [(y,\varphi_i) / \lambda_i]^2$ converges; here $\{\lambda_i,\varphi_i\}$ is the eigensystem of. $A\varphi = \lambda\varphi$. On the other hand, in many applications the function y on the righthand side is usually only known at finitely many points. Accordingly, in Sec. 2 we introduce a moment type discretization of (1.1) which possesses an infinity of solutions. After selecting a "minimal solution" of this system, we establish a relationship between this function and the solution of (1.1) - if the latter exists.

A further problem connected with the equation (1.1) is the noncontinuous dependence of the solution on the righthand side.

Indeed, addition of a highly oscillatory bounded function - e.g. sin(nt)-to x(t) will not appreciable affect the righthand side for n large enough since $\int_0^1 K(s,t)\sin(nt)dt \to 0$ as $n \to \infty$. The numerical manifestation of this instability have been described repeatedly ([2], [3], [4]). As a consequence, when a finite-difference discretization of the type

$$(1.2) \quad h\sum_i w_i K(s_j,t_i)\ x\ (t_i)\ dt = Y(s_j)$$

is applied to (1.1), the approximate solutions of (1.2) become oscillatory when the mesh size of the subdivision tend to zero. This is due to the near singularity of the system (1.2) which in turn follows from the continuity of K(s,t).

In order to eliminate these oscillations, various smoothing procedures have been proposed ([2],[3],[4]). Smoothing is essentially equivalent with the elimination of those components of the solution which belong to the smaller eigenvalues and corresponding eigenfunctions. Accordingly, it is desirable to estimate the closeness of approximation as a function of the eignevalues. In sec. 4 we show that the method of steepest descent does generate a smooth sequence of approximations which admits an estimate of this kind.

Finally we demonstrate on a number of examples the numerical procedures involved.

2. The Minimal Solution

In this section we utilize the method of optimal approximation

[5] in dealing with a discretized version of (1.1).

In many applications the righthand side $y(s)$ of (1.1) is only known at finitely many points $s_i \epsilon [o,1]$ rather than as a function defined at each point of the interval. Accordingly, instead of equation (1.1) we consider the problem of solving the set of linear relations

$$\int_o^1 K(s_i,t)\ x\ (t)\ dt = y(s_i)$$

or

$$(2.1)\qquad (K_i,x) = y_i \qquad i = 1,\ldots,\ n\ ,\quad y_i = y(s_i),$$

for short.

For the sake of simplicity let us assume that $s_{i+1} - s_i =$ const, $i = 0,\ldots,\ N-1$. We can also assume that the N functions $K_i = K(s_i,t)$ are linearly independent i.e., that they span an N-dimensional linear subspace K_n of $L^2(o,1)$. In this subspace (2.1) has a unique solution.

$$(2.2)\qquad x^*(t) = \alpha_1 K_1(t) +\ldots+ \alpha_n K_n(t).$$

In fact, substitution of (2.2) into (2.1) yields for the (α_i) the equations

$$(2.3)\qquad \sum_i (K_i,\ K_j)\ \alpha_i = y_j \qquad j = 1,\ldots,N.$$

The matrix of this system is the Gramian $((K_i,\ K_j))$ which is nonsingular due to the independence of the K_i.

Any function x^*+x_2 with $x_2 \in K_n^{\perp}$ is a solution of (2.1), and conversly, any solution $x(t)$ of (2.1) can be uniquely represented in the form $x = x_1+x_2$ where $x_1 \in K_n$ and $x_2 \in K_n^{\perp}$. Evidently, $x_1 = x^*$ and since $(x^*,x_2) = 0$ it follows that

(2.4) $\|x^*\| \leq \|x\|$

for any solution x of (2.1).

In summary, we assumed that the physical system satisfies the set of linear relations (2.1); and we found that then any solution of (2.1) can be represented in the form $x = x^*+x_2$ where x^* is uniquely determined by (2.1) and $x_2 \in K_n^{\perp}$. Since x_2 is quite arbitrary, there is no reason to accept any particular one of these solutions without further information. Moreover, it should be clear from the outset that this additional information should concern x_2 only.

Let us consider two possibilities for the additional information.

Suppose that physical considerations indicate that the particular solution to be determined is contained in the neighborhood of a given function w, and that $w = w_1+w_2$ with $w_1 \in K_n$ and $w_2 \in K_n^{1}$. The, either $w_1 = x^*$ or else w is incompatible with (2.1). In either case, the solution x^*+w_2 should evidently be the one to be adopted.

If additional information concerning the solution x(t) has the form of a bound $\|x\|^2 \leq r^2$, then r has to satisfy $\|x^*\|^2 \leq r^2$. If $\|x^*\|^2 < r^2$ then, any properly normalized element $x \in K_n^{\perp}$ can be added to x^*.

For reasons to be explained presently, we choose from the totality of the solutions of (2.1) the "minimal solution" x^* as our reference function.

Consider the case when (1.1) does possess a solution and (2.1) is a discretization of the integral equation on a finite set $0 = s_o < s_1 < \ldots < s_n = 1$. It is not necessarily true that the sequence of minimal solutions of (2.1) converges to the solution of (1.1) as $n \to \infty$. However, as the following discussion shows, under certain conditions a modified procedure does generate a convergent sequence.

Assume that (1.1) has a continuously differentiable solution $z(t)$ which takes on the boundary values $z(o) = a$ and $z(1) = b$. Then, we obtain from (1.1) by partial integration

(2.4) $\int_o^1 J(s,t)\ \zeta(t)\ dt = \eta(s)$

where

$J(s,t) = \int_o^t K(s,t)\ dt$

$\eta(s) = -\ y(s) + bJ(s,1) - aJ(s,0)$ and

$\zeta(t) = \frac{d}{dt}(z(t))$.

For the sake of simplicity suppose that $a=o$ and consider the problem of minimizing the quadratic form (ζ,ζ) under the conditions

(2.5) $(J_i,\zeta) = \eta_i$ and $\int_o^1 \zeta\ (t)dt = b$

where

$J_i = J(s_i,t)\ ;\ \eta_i = \eta(s_i)\ \ i=1,\ldots,n.$

This problem has a solution for every partition $0 = s_1 < s_2 < \ldots < s_n = 1$ provided that $\det(b_{ij}) \neq 0$ where $b_{ij} = b_{ji} = (J_i, J_j)$ for $i,j = 1,\ldots,N$ and $b_{n+1,j} = b_{j,n+1} = \int_0^1 J_j(t)\,dt$ $\quad j=1,\ldots,N$ and $b_{n+1,n+1} = 1$

This latter condition is satisfied if the functions $J_1(t), \ldots, J_n(t), 1$, are linearly independent which may always be assumed. It is now easy to prove the following theorem:

__Theorem__ If (1.1) has a continuously differentiable solution $z(t)$ which assumes the boundary values $z(0) = 0$, $z(1) = b$, and $\zeta_n(t)$ is the minimal solution of (2.5), then the sequence

$$(2.6) \quad x_n(t) = \int_0^t \zeta_n(t)\,dt$$

converges to $z(t)$ in the L^2 norm if

$\|(K,z) - (K,x_n)\| \to 0$ as $n \to \infty$.

__Proof__ Since $z(t)$ satisfies the conditions (2.5) for all N, and $\zeta_n(t)$ is the minimal solution under the same conditions, we have

$$\left\|\frac{d}{dt}(x_n(t))\right\| = \|\zeta_n(t)\| \leq \left\|\frac{d}{dt}(z(t))\right\|$$

Thus, the sequence $(x_n(t))$ is uniformly bounded $(x_n(0) = 0)$ and equicontinuous. Moreover, it satisfies the set of linear relations

$$(K_i, x_n) = y_i \qquad i=1,\ldots,n.$$

Consider now the linear transformation A of (1.1) restricted to the set (x_n). Since A is continuous and $\{Ax_n\}$. Hence, from

$\|\int_0^1 K(s,t)\ x_n(t)dt-y(s)\| \to 0$ as $N \to \infty$ it follows that

$$\|x_n - z\| \to 0 \text{ as } n \to \infty.$$

Remark: (1.1) has a continuously differentiable solution for example, if $\frac{\partial}{\partial s}(K(s,t)$ exists and is continuous in $0 \leq s,\ t \leq 1$, and if the righthand side has a finite Fourier expansion in terms of the eigenfunctions of $K(s,t)$.

Uniqueness of the solution is not required. (x_n) converges to that solution of (1.1) for which $\int_0^1 [\dot{x}(t)]^2 dt$ is minimum.

3. The Hypercircle Inequality

Since the inverse of our integral operator is not continuous, there seems to be no way to obtain a priori estimates for the approximation. Nevertheless, we can obtain a posteriori estimates if a bound for the solution $z(t)$ of (1.1) is known.

Let $L=L(x)$ be a linear functional in $L^2(o,1)$ represented by the function $h(t)$, i.e., $L(x) = (h,x)$ for all $x \in L^2$. Furthermore, let $x=x(t)$ be any solution of the system (2.1) and $k_1(t),\ldots, k_n(t)$ an orthonormal basis for the subspace K_n spanned by $K_1(t),\ldots,K_n(t)$. Then, the hypercircle inequality [5] states, that

$$(3.1)\quad |L(x) - L(x^*_n)|^2 \leq (\|x\|^2-\|x^*_n\|^2(\|h\|^2-\sum_{i=1}^{n} |L(k_i)|^2)$$

provided x^*_n is the minimal solution of (2.1). Since $z(t)$ is also a solution of (2.1), the inequality (3.1) applies with $x = z$.

Among the various possible choices of the functional h the most obvious one is the following delta type function

$$h = \begin{cases} m & \\ & \text{for} \\ o & \end{cases} \begin{matrix} t - 1/2m < + \leq t + 1/2m \\ 1\, t - t1 > 1/2m. \end{matrix}$$

In this case $\|h\|^2 = 1$ and by Bessel's inequality $0 \leq (\|h\|^2 - \sum_{i=1}^{n} |(h,k_i)|^2) < 1$. ubstituting this into (3.1) we obtain

(3.2) $\quad |(h,z) - (h,x^*_n)|^2 \leq \|z\|^2 - \|x^*_n\|^2$

For m large we may interpret (h,x) as the value of x(t) at $t = \tau$, and (3.2) becomes a uniform estimate for the approximation of z(t) in terms of the minimal solution $x^*_n(t)$.

Applying the Schwartz inequality to any one of the relations (2.1) - which are of course satisfied by x^*_n - we find that

$$- \|x^*_n\|^2 \leq - y_i^2 \;/\; \|K_i\|^2$$

or

(3.3) $\quad |(h,z) - (h,x^*_n)|^2 \leq \zeta^2 - \mu^2$

with $\mu^2 = \max_i y_i^2 / \|K_i\|^2$ and ζ^2 an upper bound for $\|z\|^2$.

The inequality (3.3) is only of theoretical value since we are usually unable to compute the exact solution of (2.3). Let $\alpha^* = (\alpha^*_1, \ldots, \alpha^*_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ denote the exact and and approximate solution of (2.3), respectively, and set $x^*_n = \Sigma\alpha^*_i K_i(t)$ and $x_n = \Sigma\alpha_i K_i(t)$. From (3.3) we obtain than

$$|(h,z)-(h,x_n)|^2 \le |(h,z)-(h,x_n^*)|^2+|(L,x_n^*)-(h,x_n)|^2$$

$$\le \zeta^2-\mu^2 + |(h,\sum_{i=1}^{n} (\alpha_i^*-\alpha_i)K_i)|^2$$

$$\le \zeta^2-\mu^2 + \max_{\substack{0\le t\le 1\\ i=1,\dots,n}} |K_i(t)|^2 \|K^{-1}\| \, \|\varepsilon\|^2 \tag{3.4}$$

where $\|K^{-1}\|$ is the (A_2) norm of the inverse of the Gramian $((K_i,K_j))$ and $\|\varepsilon\|$ the corresponding norm of the vector $\alpha^*-\alpha$.

For large n the Gramian tends to be rather illconditioned and the estimate (3.4) accordingly becomes meaningless. For small n the term $\zeta^2-\mu^2$ will be more significant. By definition we have $\mu^2 = \max y_i/\|K_i\|^2$, and this indicates that the discretization points should be chosen in such a manner that μ is as large as possible.

4. The Method of Steepest Descent

According to the last remark, the "minimal solution" of the system (2.3) admits a meaningful estimate only when the number N of discretization points is small. On the other hand, when the righthand side of (1.1) is known at a large number of points we will not discard part of the available information in order to keep the order of the corresponding algebraic system small. Another difficulty connected with the solution of (large order) nearly singular algebraic systems was mentioned in the introduction namely, the oscillation of the approximate solutions obtained by direct methods.

In this section we return to the finite difference discretization (1.2) of the integral equation. However, instead of devising a smoothing procedure imposed on the system (1.2) by additional constraints, we show that the method of steepest descent [6] automatically screens out components corresponding to smaller eigenvalues, thereby generating smooth approximate solutions.

We consider first a simplified version of the steepest descent method. Let x_o and y be vectors in R_n, A a positive definite N xN matrix and (x_n) the sequence generated by the algorithm

(4.1) $x_{n+1} = x_n - \mu r_n$

$r_n = Ax_n - y$.

It is easy to see that x_n converges to the solution x* of the equation Ax = y if $0<\mu<2/\lambda_1$ and λ_1 is the largest eigenvalue of A. Indeed, let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > o$ be the eigenbalues and $\varphi_1,\ldots,\varphi_n$ the corresponding eigenfunctions of A with the property that $(\varphi_i,\varphi_j) = \delta_{ij}$. Then, $r_o = Ax_o - y = \Sigma\, \rho_{oi}\varphi_i$ and $r_{n+1} = (I-\mu A)r_n = \ldots = (I-\mu A)^n r_o$

or

$$r_{n+1} = \Sigma_i (1-\lambda_i\mu)^n \rho_{oi}\varphi_i$$

Since $0< \mu< 2/\lambda_1$ we have $|1-\lambda_i\mu|<1$ i=1,...,N.

Thus, $r_n \to o$ and $x_n \to x^*$.

On the other hand, it is clear that the rate of convergence is different in the different directions φ_i . Let $\mu=2\theta/\lambda_1$, $0<\theta<1$. If $\theta \leq 1/2$ the convergence is fastest in the direction of φ_1, since in this case

$$0 \leq 1 - \mu\lambda_1 \leq 1 - \mu\lambda_i \quad i = 1, \ldots, N.$$

We can improve the convergence in the direction of φ_i - by choosing $\theta = \lambda_1/2\lambda_i$ - if $\lambda_i > \lambda_1/2$.

If k is the first index such that $\lambda_k \leq \lambda_1/2$ then

$$1 - \mu\lambda_i > 1 - \lambda_i/\lambda_k \qquad i = k+1, \ldots, N.$$

This last inequality shows that for any choice of $0<\theta<1$ the convergence factor $(1-\mu\lambda_i)$ will be very close to 1 for the small eigenvalues.

From (4.1) we obtain

$x_{n+1}-x^* = (I-\mu A)(x_n-x^*) = \ldots = (I-\mu A)^{n+1}(x_o-x^*)$. hoosing $x_o = 0$ and setting $x_n = \Sigma\zeta_{ni}\varphi_i$, $x^* = \Sigma\zeta_i^*\varphi_i$ we obtain

$$(4.2) \qquad \zeta_{ni} = [1-(1-\mu\lambda_i)^n]\zeta_i^* ,$$

which is an estimate for the coefficient of φ_i in the n-th iterate.

If the matrix A stems from the discretization of a continuous kernel K(s,t), most of the eigenvalues will cluster around zero, so that the contribution of the oscillating eigenfunctions to the n-th iterate will remain small, for n moderately large. (In actual computations oscillation did not take place even after 200 iterations.)

We consider now the standard steepest descent method, i.e., the method where the successive iterates are generated by the algorithm

(4.3) $x_{n+1} = x_n - \alpha_n r_n$

$r_n = Ax_n - y$

$\alpha_n = (r_n, r_n)/(Ar_n, r_n)$

The rate of convergence can be estimated by the inequality ([6] p. 608)

$$\|x_n - x^*\| \leq \frac{\|Ax_0 - y\|}{\lambda_N} \left(\frac{\lambda_1 - \lambda_N}{\lambda_1 + \lambda_N} \right)^n$$

where λ_1 is again the largest and λ_N the smallest eigenvalue of A. Since in our case λ_N is very small, this a priori estimate is rather pessimistic. As before, we can show that convergence is faster in the direction corresponding to larger eigenvalues.

Retaining the previous notation we obtain from (4.3)

(4.4) $\epsilon_{n+1,i} = \zeta_{n+1,i} - \zeta_i^* = (\zeta_{oi} - \zeta_i^*)(1-\alpha_o\lambda_i)\ldots(1-\alpha_n\lambda_i)\ (i=1,\ldots,N)$

and

(4.5) $\alpha_n = \dfrac{\Sigma_i \lambda_i^2 \epsilon_{ni}^2}{\Sigma_i \lambda_i^3 \epsilon_{ni}^2}$

Evidently, $\lambda_1 \geq \alpha_n^{-1} \geq \lambda_N$. As long as $\alpha_n^{-1} > \lambda_1/2$ convergence takes place in every direction φ_i like in the case of the simplified steepest descent method. Also, since the rate of convergence is slow in the directions corresponding to small eigenvalues, the iteration produces smooth approximations for n moderately large.

Let us consider now the case of $\alpha_n^{-1} \leq \lambda_1/2$. There exists integers $k=k(n)$ and $L = L(n)$ with the property that

(4.6) $\lambda_k \geq 2\alpha_n^{-1} \geq \lambda_{k+1} \geq \cdots \geq \lambda_L \geq \alpha_n^{-1} \geq \lambda_{L+1} \geq \cdots \geq \lambda_n$

From (4.5) and (4.6) it follows that

(4.7) $$\sum_{i=1}^{L} \lambda_i^2 \epsilon_{in}^2 (\lambda_i - \alpha_n^{-1}) = \sum_{L+1}^{n} \lambda_i^2 \epsilon_{in}^2 (\alpha_n^{-1} - \lambda_i)$$

Setting $\max_{L+1 \leq i < N} \epsilon_{in}^2 = \beta^2$ we find that

$$\epsilon_{ni}^2 \lambda_i^2 \alpha_n^{-1} \leq \epsilon_{in}^2 \lambda_i^2 (\lambda_i - \alpha_n^{-1}) \leq N \beta^2 \lambda_{L+1}^2 \alpha_n^{-1}$$
$$\leq N \beta^2 (\alpha_n^{-1})^3$$

or

$$\epsilon_{ni}^2 \leq N \beta^2 \left(\frac{\alpha_n^{-1}}{\lambda_i}\right)^2 \quad \text{for all } i=1,\ldots,k.$$

This inequality shows that α_n^{-1} can be small only when the error in the direction of large eigenvalues is sufficiently small. Thus, in this case too, the first iterations are in the direction of the larger eigenvalues.

(It may be of interest to note that α_n^{-1} is the center of mass of the error squares ϵ_{ni}^2 weighted by λ_i^2 and situated at the points λ_i respectively. From this fact the following qualitative picture of the iteration process emerges: At the n-th step of iteration the largest decrease in ϵ_{ni}^2 takes place at points closest to α_n^{-1}. If $\alpha_n^{-1} > \lambda_p/2$ (where λ_p is the largest eigenvalue for which $\epsilon_{ni}^2 > o$), then the overall decrease in ϵ_{ni}^2 is larger on the right of α_n^{-1} and the center of mass of the quantities $\epsilon_{n+1,i}^2 \lambda_i^2$ moves to the left: $\alpha_{n+1}^{-1} < \alpha_n^{-1}$. Similarly, if $\alpha_n^{-1} < \lambda_p/2$ then $\alpha_{n+1}^{-1} > \alpha_n^{-1}$. Therefore, we can expect that the sequence $\{\alpha_n^{-1}\}$ will osillate around $\lambda_p/2$. Also, since iteration with $\alpha_n^{-1} < \lambda_p/2$ causes divergence in the direction of eigenvalues $\lambda_i > 2\alpha_n^{-1}$, the sequence of residuals $r_n = \|Ax_n - y\|$ will also show an oscillatory pattern.)

Making again the choice $x_o = 0$, we obtain from (4.4) an a posteriori estimate for ζ_{ni}

$$|\zeta_{ni}| \leq \quad |1-(i-\alpha_o\lambda_i)\dots(1 - \alpha_{n-1}\lambda_i)|$$

in terms of a bound $b^2 > \| x^* \|^2$ and the eigenvalue λ_i. In particular, if $\lambda_i < \epsilon$, $\max_k \alpha_k = \alpha$ and $\epsilon\ \alpha < 1$ we find that

$$|\zeta_{ni}| \leq b\ |n\ \alpha\ \epsilon\ |$$

5. Numerical Solutions

The convergence of the simplified method of steepest descent (4.1) is very slow. Although, it is relatively easy to approximate the first eigenvalue λ_1 and execute a large number of iterations $x_{n+1} = x_n - \mu r_n$ with $\mu = 2\theta/\lambda_1$ $0 < \theta < 1$ however, this advantage does not compensate for the low rate of convergence.

A better approach is to execute a small number of iterations (usually one or two) with each element of a sequence

$\mu_i = 2\Theta_i/\lambda_1$ e.g., $\Theta_i = (1 + i/N)/2$. The results obtained by this procedure are generally not inferior to those obtained by the standard steepest descent method; moreover we do not run the risk of oscillating approximations. Tables 1a and 1b exhibit the solutions obtained by these two methods. Table 1b also shows the oscillation of the coefficient α_n as described in the previous section.

A more effective version of the steepest descent method ([6] p. 608) is based on the formula

(5.1) $x_{n+1} = x_n + \alpha_1 r_n + \alpha_2 A r_n + \ldots + \alpha_p A^{p-1} r_n$

with

$$r_n = Ax_n - y$$

The coefficients α_i are determined from the condition that the functional $(Ax,x) - (x,x)$ be a minimum i.e., the α_i satisfy the system of equations

(5.2) $(A^{j-1} r_n, r_n) + \sum_{k=1}^{p} \alpha_k (A^{j+k-1} r_n, r_n) = 0 \qquad j = 1,\ldots,p.$

This method is more then p times faster then the standard steepest descent method. In actual computation the sequence of iterates is somewhat similar to that of an asymptotic series: while the first few iterates are approaching the solution, the iterates of higher order usually tend to infinity. The residual norm $r_n = \| Ax_n - y \|$ shows a similar pattern: at first r_n diminishes, but later it increases indefinitely. (In contrast to the standard steepest descent method - we cannot expect here oscillation since x_{n+1} is a linear combination of several independent vectors $x_n, r_n, \ldots, A^{p-1} r_n$ and it does not point in the direction of a single eigenvector.) However, it is only approximately true that the best approximation (in the sense of uniform norm) corresponds to the smallest value of $\| r_n \|$. As a rule we have several approximations whose residuals are of the same order. From these we choose according to a predetermined criterion. E.g., in our numerical experiments the solutions are polynomials, analytic functions and rational functions with no singularities. Invariably, the smoothest iterate ($\int_0^1 (\ddot{x}_n)^2 = \min$) furnished the best (uniform) approximation.

Except for tables 1a and 1b, the term "steepest descent method" refers to the iteration defined by formulas (5.1) and (5.2). We found experimentally the value $p = 3$ yielding the best results, solving (5.2) by a simple Gaussian elimination. A larger value for p would speed up "convergence" so much that no smooth approximation would be obtained at all.

It is known that the proper choice of the initial approximation is essential for success with the steepest descent method. The choice $x_o = y$ yields $r_o = Ax_o - y = \Sigma \lambda_i(\lambda_i - 1)\zeta_i^* \varphi_i$. Appearently, the components of r_o in the direction of the smaller eigenvalues are relatively small and the best approximation at the beginning (see sec. 4) is achieved in the direction of the larger eigenvalues.

The choice $x_o = Ay$ (or even $x_o = A^2y$) is still better.

The method of conjugate gradients has some of the characteristics of the steepest descent method, but it is essentially a direct (finite) method. Consequently, the iterates of higher order are oscillatory. Nevertheless, the first iterates which are nonoscillatory do provide in some cases better (uniform) approximations than those produced by the steepest descent method. Another advantage of the former is in the relevance of double precision procedures which yield results twice as good as those obtained by single precision procedures. However, the iterates are generally less smooth then the ones obtained by the steepest descent method.

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References

[1] Courant, R., and Hilbert, D., Methods of Mathematical Physics, I. Intersc. Pub., New York, 1953.

[2] D. L. Phillips, A Technique for the Numerical Solution of Certain Integral Equations of the First Kind, J. A. C. M. 9(1962), pp. 84-97.

[3] A. N. Tihonov, Solution of Incorrectly Formulated Problems and the Regularization Method, Soviet Math. Dokl. 4(1963), 1035.

[4] Baker, Fox, Mayers and Wright, Numerical Solution of Fredholm Integral Equation of the First Kind, The Computer's Journal, July 1964.

[5] M. Golomb and H. F. Weinberger, Optimal Approximation and Error Bounds, Symp. on Numerical Approximation, (Ed. Langer), Univ. of Wisconsin 1958.

[6] L. V. Kantorovich and G. P. Akilov, Functional Analysis in Normed Spaces, MacMillan, N. Y. 1964.

THETA	ITER. NO.	ERROR NORM	RESIDUAL NORM	ITER. SOL. NORM
0.500	2	0.4908E-01	0.2976E-03	0.8001E 00
0.550	4	0.4848E-01	0.2822E-03	0.8001E 00
0.600	6	0.4789E-01	0.2662E-03	0.8001E 00
0.650	8	0.4731E-01	0.2499E-03	0.8001E 00
0.700	10	0.4674E-01	0.2334E-03	0.8002E 00
0.750	12	0.4621E-01	0.2170E-03	0.8002E 00
0.800	14	0.4570E-01	0.2007E-03	0.8002E 00
0.850	16	0.4522E-01	0.1847E-03	0.8002E 00
0.900	18	0.4479E-01	0.1692E-03	0.8002E 00
0.950	20	0.4439E-01	0.1542E-03	0.8002E 00
0.960	22	0.4403E-01	0.1404E-03	0.8003E 00
0.970	24	0.4373E-01	0.1277E-03	0.8003E 00
0.980	26	0.4345E-01	0.1161E-03	0.8003E 00
0.990	28	0.4321E-01	0.1054E-03	0.8003E 00
1.000	30	0.4300E-01	0.9565E-04	0.8003E 00
1.010	32	0.4281E-01	0.8673E-04	0.8003E 00

EXACT SOLUTION NORM = 0.8017E 00

$$\int_o^1 \frac{1}{1+s+t}\, x(t)dt = \frac{1}{1+(1+s)^2}\left[\ln\left(\frac{2+s}{1+s}\right) + \frac{\pi(1+s)}{4} - \frac{1}{2}\ln 2\right] = Y(s),$$

$$x(t) = (1+t^2)^{-1}, \qquad x_o(t) = Y(t)$$

$$x_{n+1} = x_n - \mu_i r_n \qquad \mu_i = 2\theta_i/\lambda_1$$

Table 1a

ALPHA	ITER. NO.	ERROR NORM	RESIDUAL NORM	ITER. SOL. NORM
1.8649	1	0.4952E-01	0.3[illegible]88E-03	0.8001E 00
77.9108	2	0.4247E-01	0.1653E-04	0.8005E 00
3.1030	3	0.4244E-01	0.1346E-04	0.8005E 00
4.6699	4	0.4239E-01	0.1650E-04	0.8005E 00
3.1023	5	0.4236E-01	0.1343E-04	0.8005E 00
4.6711	6	0.4231E-01	0.1647E-04	0.8005E 00
3.1015	7	0.4228E-01	0.1340E-04	0.8005E 00
4.6741	8	0.4223E-01	0.1644E-04	0.8005E 00
3.0998	9	0.4220E-01	0.1337E-04	0.8005E 00
4.6774	10	0.4215E-01	0.1641E-04	0.8005E 00
3.0986	11	0.4212E-01	0.1334E-04	0.8005E 00
4.6790	12	0.4207E-01	0.1638E-04	0.8005E 00
3.0980	13	0.4204E-01	0.1332E-04	0.8005E 00
4.6826	14	0.4199E-01	0.1636E-04	0.8005E 00
3.0965	15	0.4196E-01	0.1328E-04	0.8005E 00
4.6858	16	0.4191E-01	0.1633E-04	0.8005E 00
3.0948	17	0.4188E-01	0.1326E-04	0.8005E 00
4.6881	18	0.4183E-01	0.1630E-04	0.8005E 00
3.0939	19	0.4180E-01	0.1323E-04	0.8005E 00
4.6913	20	0.4175E-01	0.1628E-04	0.8005E 00

EXACT SOLUTION NORM = 0.8017E 00

$$\int_0^1 \frac{1}{1+s+t}\, x(t)dt = \frac{1}{1+(1+s)^2}\left[\ln\left(\frac{2+s}{1+s}\right) + \frac{\pi(1+s)}{4} - \frac{1}{2}\ln 2\right] = Y(s),$$

$$x(t) = (1+t^2)^{-1} \quad , \quad x_o(t) = AY(t)$$

$$x_{n+1} = x_n - \alpha_n r_n$$

Table 1b

AFTER 5 ITERATIONS BY STEEPEST DESCEND

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	1.000000	0.998785	0.
0.025	0.999375	0.996229	0.7451E-08
0.050	0.997506	0.997087	0.
0.075	0.994406	0.996405	0.
0.100	0.990099	0.992952	-0.7451E-08
0.125	0.984615	0.986818	0.
0.150	0.977995	0.979027	0.
0.175	0.970285	0.972952	-0.7451E-08
0.200	0.961538	0.962398	0.
0.225	0.951814	0.951675	0.
0.250	0.941176	0.940872	-0.7451E-08
0.275	0.929692	0.929018	-0.7451E-08
0.300	0.917431	0.915332	0.
0.325	0.904466	0.902102	0.
0.350	0.890869	0.888402	0.
0.375	0.876712	0.875151	-0.7451E-08
0.400	0.862069	0.859754	-0.3725E-08
0.425	0.847009	0.845917	-0.7451E-08
0.450	0.831601	0.830984	-0.7451E-08
0.475	0.815910	0.814174	-0.3725E-08
0.500	0.800000	0.798944	0.
0.525	0.783929	0.783867	-0.3725E-08
0.550	0.767754	0.766907	0.3725E-08
0.575	0.751527	0.751931	-0.7451E-08
0.600	0.735294	0.737017	-0.7451E-08
0.625	0.719101	0.720431	-0.3725E-08
0.650	0.702988	0.704745	0.
0.675	0.686990	0.687492	0.3725E-08
0.700	0.671141	0.672216	0.
0.725	0.655469	0.657082	0.
0.750	0.640000	0.642243	-0.3725E-08
0.775	0.624756	0.627540	-0.3725E-08
0.800	0.609756	0.612152	0.
0.825	0.595017	0.596963	0.
0.850	0.580552	0.581173	0.
0.875	0.566372	0.566429	0.3725E-08
0.900	0.552486	0.551069	0.3725E-08
0.925	0.538902	0.537735	0.
0.950	0.525624	0.523682	0.3725E-08
0.975	0.512656	0.509953	0.
1.000	0.500000	0.495563	0.

ERROR NORM = 0.1780E-02 RESIDUAL NORM = 0.4328E-08

EXACT SOL. NORM = 0.8017E 00 ITERATED SOL. NORM = 0.8017E 00

$$\int_0^1 (1+s+t)^{-1}\, x(t)dt = \left[\ln\left(\frac{2+s}{\sqrt{2}(1+s)}\right) + \frac{\pi}{4}(1+s)\right]\left[1+(1+s)^2\right]^{-1} = Y(s)$$

$$x(t) = (1+t^2)^{-1} \quad ; \quad x_o(t) = Y(t)$$

Table 1c

AFTER 5 ITERATIONS BY CONJUGATE GRADIENT

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	1.000000	0.992505	-0.9313E-06
0.025	0.999375	0.994811	-0.8196E-06
0.050	0.997506	0.996255	-0.7302E-06
0.075	0.994406	0.996716	-0.6557E-06
0.100	0.990099	0.993244	-0.5737E-06
0.125	0.984615	0.987464	-0.4917E-06
0.150	0.977995	0.981413	-0.4321E-06
0.175	0.970285	0.974358	-0.3725E-06
0.200	0.961538	0.963774	-0.3092E-06
0.225	0.951814	0.953225	-0.2533E-06
0.250	0.941176	0.942757	-0.2086E-06
0.275	0.929692	0.929871	-0.1602E-06
0.300	0.917431	0.915698	-0.1080E-06
0.325	0.904466	0.902701	-0.6706E-07
0.350	0.890869	0.888899	-0.2980E-07
0.375	0.876712	0.874962	0.3725E-08
0.400	0.862069	0.859820	0.3725E-07
0.425	0.847009	0.845242	0.7078E-07
0.450	0.831601	0.829928	0.1006E-06
0.475	0.815910	0.813841	0.1341E-06
0.500	0.800000	0.797694	0.1639E-06
0.525	0.783929	0.782136	0.1900E-06
0.550	0.767754	0.765811	0.2198E-06
0.575	0.751527	0.750807	0.2384E-06
0.600	0.735294	0.734724	0.2608E-06
0.625	0.719101	0.719250	0.2831E-06
0.650	0.702988	0.703346	0.3055E-06
0.675	0.686990	0.687578	0.3316E-06
0.700	0.671141	0.671547	0.3465E-06
0.725	0.655469	0.656228	0.3651E-06
0.750	0.640000	0.641953	0.3763E-06
0.775	0.624756	0.626491	0.3949E-06
0.800	0.609756	0.611295	0.4135E-06
0.825	0.595017	0.596695	0.4284E-06
0.850	0.580552	0.581926	0.4433E-06
0.875	0.566372	0.567111	0.4619E-06
0.900	0.552486	0.552728	0.4731E-06
0.925	0.538902	0.539037	0.4843E-06
0.950	0.525624	0.525260	0.4955E-06
0.975	0.512656	0.511852	0.5066E-06
1.000	0.500000	0.498532	0.5178E-06

ERROR NORM = 0.2037E-02 RESIDUAL NORM = 0.3923E-06

EXACT SOL. NORM = 0.8017E 00 ITERATED SOL. NORM = 0.8017E 00

$$\int_0^1 (1+s+t)^{-1} x(t)dt = [\ln(\frac{2+s}{\sqrt{2}(1+s)}) + \frac{\pi}{4}(1+s)][1+(1+s)^2]^{-1} = Y(s)$$

$$x(t) = (1+t^2)^{-1} \quad ; \quad x_o(t) = Y(t)$$

Table 1d

AFTER 10 ITERATIONS BY STEEPEST DESCEND

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	C.	-0.034562	-0.8941E-07
0.025	C.078459	0.059576	-0.5215E-07
0.050	0.156434	0.150119	-0.3725E-07
0.075	C.233445	0.236676	-0.1490E-07
0.100	C.309017	0.317344	0.7451E-08
0.125	C.382683	0.395378	0.1490E-07
0.150	C.453990	0.469754	0.2235E-07
0.175	0.522499	0.538346	0.2235E-07
0.200	C.587785	0.603001	0.2235E-07
0.225	0.649448	0.662769	0.2235E-07
0.250	0.707107	0.717418	0.2235E-07
0.275	C.760406	0.767022	0.2235E-07
0.300	C.809017	0.812638	0.2235E-07
0.325	C.852640	0.851630	0.1490E-07
0.350	C.891007	0.886640	0.7451E-08
0.375	C.923880	0.915953	0.7451E-08
0.400	C.951057	0.940386	0.
0.425	0.972370	0.959309	-0.7451E-08
0.450	0.987688	0.973052	-0.7451E-08
0.475	0.996917	0.981342	-0.1490E-07
0.500	1.000000	0.983729	-0.1490E-07
0.525	C.996917	0.981887	-0.1863E-07
0.550	C.987688	0.974080	-0.1863E-07
0.575	0.972370	0.961068	-0.1863E-07
0.600	0.951057	0.942667	-0.1863E-07
0.625	C.923880	0.918254	-0.1118E-07
0.650	0.891007	0.890382	-0.2235E-07
0.675	0.852640	0.855517	-0.1118E-07
0.700	0.809017	0.816625	-0.2235E-07
0.725	C.760406	0.771932	-0.1118E-07
0.750	0.707107	0.721187	-0.3725E-08
0.775	0.649448	0.666912	-0.3725E-08
0.800	0.587785	0.606960	-0.3725E-08
0.825	0.522499	0.542521	0.
0.850	0.453991	0.472187	0.3725E-08
0.875	0.382683	0.396927	0.1118E-07
0.900	0.309017	0.318741	0.
0.925	0.233445	0.234155	0.3725E-08
0.950	0.156434	0.145129	0.3725E-08
0.975	0.078459	0.050840	0.1118E-07
1.000	0.000000	-0.047206	0.

ERRCR NORM = 0.1403E-01 RESIDUAL NORM = 0.1943E-07

EXACT SOL. NORM = 0.7071E 00 ITERATED SOL. NORM = 0.7070E 00

$$\int_0^1 \exp(-st)x(t)dt = \pi(1+\exp(-s))(s^2+\pi^2)^{-1}$$

$x(t) = \sin(\pi t)$; $x_o(t) = t$

Table 2a

AFTER 15 ITERATIONS BY CONJUGATE GRADIENT

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	0.	0.008187	-0.3576E-06
0.025	0.078459	0.082931	-0.3576E-06
0.050	0.156434	0.164748	-0.3651E-06
0.075	0.233445	0.232887	-0.3651E-06
0.100	0.309017	0.300737	-0.3576E-06
0.125	0.382683	0.374083	-0.3576E-06
0.150	0.453990	0.452294	-0.3651E-06
0.175	0.522499	0.516930	-0.3651E-06
0.200	0.587785	0.589576	-0.3725E-06
0.225	0.649448	0.649442	-0.3725E-06
0.250	0.707107	0.703779	-0.3725E-06
0.275	0.760406	0.759872	-0.3725E-06
0.300	0.809017	0.807247	-0.3800E-06
0.325	0.852640	0.857706	-0.3874E-06
0.350	0.891007	0.891204	-0.3800E-06
0.375	0.923880	0.929571	-0.3874E-06
0.400	0.951057	0.953038	-0.3874E-06
0.425	0.972370	0.971098	-0.3874E-06
0.450	0.987688	0.990510	-0.3949E-06
0.475	0.996917	1.000132	-0.4023E-06
0.500	1.000000	1.002294	-0.3986E-06
0.525	0.996917	1.000346	-0.4023E-06
0.550	0.987688	0.991933	-0.4023E-06
0.575	0.972370	0.970315	-0.4023E-06
0.600	0.951057	0.953460	-0.4061E-06
0.625	0.923880	0.917000	-0.4023E-06
0.650	0.891007	0.894712	-0.4098E-06
0.675	0.852640	0.850099	-0.4023E-06
0.700	0.809017	0.813357	-0.4098E-06
0.725	0.760406	0.758530	-0.3986E-06
0.750	0.707107	0.700925	-0.3949E-06
0.775	0.649448	0.645235	-0.3949E-06
0.800	0.587785	0.587179	-0.3912E-06
0.825	0.522499	0.525390	-0.3874E-06
0.850	0.453991	0.452302	-0.3800E-06
0.875	0.382683	0.378100	-0.3688E-06
0.900	0.309017	0.317954	-0.3688E-06
0.925	0.233445	0.235955	-0.3614E-06
0.950	0.156434	0.161085	-0.3502E-06
0.975	0.078459	0.077297	-0.3316E-06
1.000	0.000000	-0.002204	-0.3241E-06

ERROR NORM = 0.4199E-02　　　RESIDUAL NORM = 0.3812E-06

EXACT SOL. NORM = 0.7071E 00　　　ITERATED SOL. NORM = 0.7072E 00

$$\int_0^1 \exp(-st)x(t)dt = \pi(1+\exp(-s))(s^2+\pi^2)^{-1}$$

$$x(t) = \sin(\pi t) \quad ; \quad x_o(t) = t$$

Table 2b

AFTER 15 ITERATIONS BY CONJUGATE GRADIENT

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	0.	0.008553	0.4120D-08
0.025	0.078459	0.080145	0.5592D-08
0.050	0.156434	0.156977	0.3740D-09
0.075	0.233445	0.232295	0.2623D-08
0.100	0.309017	0.303284	0.7652D-09
0.125	0.382683	0.378566	-0.4640D-08
0.150	0.453990	0.452110	-0.3049D-09
0.175	0.522499	0.521397	-0.4328D-08
0.200	0.587785	0.587266	-0.6018D-08
0.225	0.649448	0.649913	-0.2582D-08
0.250	0.707107	0.706623	-0.6019D-08
0.275	0.760406	0.761418	-0.5332D-08
0.300	0.809017	0.814617	-0.3242D-08
0.325	0.852640	0.856054	-0.7669D-09
0.350	0.891007	0.891905	-0.4162D-08
0.375	0.923880	0.927474	-0.2061D-08
0.400	0.951057	0.954152	-0.2241D-08
0.425	0.972370	0.973202	-0.1331D-08
0.450	0.987688	0.988328	-0.6695D-08
0.475	0.996917	0.997249	-0.6157D-08
0.500	1.000000	1.000159	-0.1081D-08
0.525	0.996917	0.994917	-0.2186D-08
0.550	0.987688	0.985244	0.6840D-09
0.575	0.972370	0.969738	0.1749D-08
0.600	0.951057	0.948517	0.2061D-08
0.625	0.923880	0.920113	-0.1252D-08
0.650	0.891007	0.887163	0.3179D-09
0.675	0.852640	0.849501	0.9876D-09
0.700	0.809017	0.806526	-0.3036D-09
0.725	0.760406	0.760934	0.4937D-09
0.750	0.707107	0.709346	0.1740D-08
0.775	0.649448	0.650499	0.2039D-09
0.800	0.587785	0.589588	0.2600D-08
0.825	0.522499	0.526582	0.3596D-08
0.850	0.453990	0.457236	0.1062D-08
0.875	0.382683	0.387114	0.1418D-09
0.900	0.309017	0.308361	0.2413D-08
0.925	0.233445	0.233071	0.2491D-08
0.950	0.156434	0.155436	0.3261D-08
0.975	0.078459	0.076327	0.1285D-08
1.000	0.000000	-0.005883	0.2028D-08

ERROR NORM = 0.2706D-02 RESIDUAL NORM = 0.3093D-08

EXACT SOL. NORM = 0.7071D 00 ITERATED SOL. NORM = 0.7071D 00

$$\int_0^1 \exp(-st)x(t)dt = \pi(1+\exp(-s))(s^2+\pi^2)^{-1}$$

$x(t) = \sin(\pi t)$; $x_o(t) = t$

Double precision

Table 2c

AFTER 5 ITERATIONS BY STEEPEST DESCEND

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	0.	-0.037162	-0.3576E-06
0.025	0.078459	0.057958	-0.2757E-06
0.050	0.156434	0.149357	-0.2086E-06
0.075	0.233445	0.236392	-0.1565E-06
0.100	0.309017	0.318276	-0.8941E-07
0.125	0.382683	0.396485	-0.5215E-07
0.150	0.453990	0.470393	-0.7451E-08
0.175	0.522499	0.539462	0.2235E-07
0.200	0.587785	0.604108	0.5215E-07
0.225	0.649448	0.663498	0.8196E-07
0.250	0.707107	0.718189	0.1043E-06
0.275	0.760406	0.767612	0.1192E-06
0.300	0.809017	0.812775	0.1341E-06
0.325	0.852640	0.852529	0.1416E-06
0.350	0.891007	0.886467	0.1416E-06
0.375	0.923880	0.915553	0.1490E-06
0.400	0.951057	0.939994	0.1565E-06
0.425	0.972370	0.958605	0.1565E-06
0.450	0.987688	0.972040	0.1490E-06
0.475	0.996917	0.980629	0.1416E-06
0.500	1.000000	0.983187	0.1378E-06
0.525	0.996917	0.981113	0.1304E-06
0.550	0.987688	0.973264	0.1229E-06
0.575	0.972370	0.960433	0.1080E-06
0.600	0.951057	0.942091	0.1006E-06
0.625	0.923880	0.918159	0.8941E-07
0.650	0.891007	0.890101	0.6706E-07
0.675	0.852640	0.855454	0.5588E-07
0.700	0.809017	0.816669	0.2608E-07
0.725	0.760406	0.772041	0.1118E-07
0.750	0.707107	0.722253	-0.3725E-08
0.775	0.649448	0.667505	-0.3353E-07
0.800	0.587785	0.607560	-0.5960E-07
0.825	0.522499	0.543141	-0.8941E-07
0.850	0.453991	0.472847	-0.1155E-06
0.875	0.382683	0.397946	-0.1453E-06
0.900	0.309017	0.318624	-0.1937E-06
0.925	0.233445	0.234015	-0.2272E-06
0.950	0.156434	0.144598	-0.2682E-06
0.975	0.078459	0.050096	-0.3092E-06
1.000	0.000000	-0.048607	-0.3576E-06

ERROR NORM = 0.1471E-01   RESIDUAL NORM = 0.1478E-06

EXACT SOL. NORM = 0.7071E 00   ITERATED SOL. NORM = 0.7070E 00

$$\int_0^1 e^{-st} x(t)dt = \pi(1+e^{-s})(s^2+\pi^2)^{-1} = Y(s)$$

$x(t) = \sin(\pi t)$; $x_o(t) = Y(t)$

Table 2d

AFTER 1C ITERATIONS BY CONJUGATE GRACIENT

T	EXACT SCLUTICN	ITERATED SCL.	RESIDUAL
0.	C.	C.C13626	C.4470E-07
0.C25	C.C78459	C.C8C566	C.447CE-07
0.C50	C.156434	C.159344	C.447CE-07
0.C75	C.233445	C.238CC8	0.5215E-07
0.1C0	C.309017	C.296455	0.5215E-07
0.125	C.382683	C.378358	0.5215E-07
0.150	C.453990	C.451654	C.5215E-07
0.175	C.522499	C.514459	0.6706E-07
0.200	C.587785	C.590059	0.5960E-07
0.225	C.649448	C.646266	0.6706E-07
0.250	C.7071C7	C.704859	0.6706E-07
0.275	C.7604C6	C.752866	0.6706E-07
0.3C0	C.809017	C.812383	0.6706E-07
0.325	C.852640	C.859355	C.6706E-07
0.350	C.8910C7	C.891141	0.7451E-07
0.375	C.923880	C.933534	0.7451E-07
0.4C0	C.951057	C.953662	0.7451E-07
0.425	C.972370	C.9783C1	C.7451E-07
0.450	C.987688	C.994732	0.7451E-07
0.475	C.996917	C.996277	0.7451E-07
0.5C0	1.C00000	1.000988	0.7823E-07
0.525	C.996917	C.999638	0.7823E-07
0.550	C.987688	C.988261	0.8196E-07
0.575	C.972370	0.9718C9	0.8196E-07
0.6C0	C.951057	C.9490C3	0.7823E-07
0.625	C.923880	0.917479	0.8568E-07
0.650	C.8910C7	C.891732	0.7823E-07
0.675	C.852640	C.8471CC	0.8196E-07
0.7C0	C.809017	0.814362	0.7823E-07
0.725	C.76C4C6	C.756838	0.8568E-07
0.750	C.7071C7	C.703183	0.8568E-07
0.775	C.649448	C.645690	0.8568E-07
0.8C0	C.587785	0.592754	0.8568E-07
0.825	C.522499	C.523171	C.8568E-07
0.850	C.453991	C.453886	0.8941E-07
0.875	C.382683	C.376291	0.9313E-07
0.9C0	C.309017	0.320231	0.8941E-07
0.925	C.233445	0.239423	0.9313E-07
0.950	C.156434	C.159544	0.9686E-07
0.975	C.078459	0.075493	0.1043E-06
1.0C0	C.0C0000	0.000439	0.1043E-06

ERRCR NCRM = C.5220E-C2 RESIDUAL NORM = 0.7668E-07

EXACT SCL. NCRM = 0.7C71E 00 ITERATED SOL. NORM = C.7072E 00

$$\int_0^1 e^{-st}x(t)dt = \pi(1+e^{-s})(s^2+\pi^2)^{-1} = Y(s)$$

$x(t) = \sin(\pi t)$; $x_o(t) = Y(t)$

Table 2e

AFTER 5 ITERATIONS BY STEEPEST DESCEND

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	1.000000	0.987169	0.5588E-07
0.025	0.952381	0.950110	0.3725E-07
0.050	0.909091	0.912321	0.2235E-07
0.075	0.869565	0.871024	0.2608E-07
0.100	0.833333	0.837341	0.7451E-08
0.125	0.800000	0.804248	0.
0.150	0.769231	0.773260	-0.7451E-08
0.175	0.740741	0.744298	-0.1118E-07
0.200	0.714286	0.713739	-0.7451E-08
0.225	0.689655	0.690370	-0.1863E-07
0.250	0.666667	0.667022	-0.2235E-07
0.275	0.645161	0.644453	-0.2235E-07
0.300	0.625000	0.623165	-0.2608E-07
0.325	0.606061	0.601213	-0.1863E-07
0.350	0.588235	0.583111	-0.1863E-07
0.375	0.571429	0.567248	-0.2235E-07
0.400	0.555556	0.551971	-0.2235E-07
0.425	0.540541	0.537749	-0.2235E-07
0.450	0.526316	0.520483	-0.1118E-07
0.475	0.512821	0.510403	-0.1863E-07
0.500	0.500000	0.499006	-0.1863E-07
0.525	0.487805	0.487280	-0.1490E-07
0.550	0.476190	0.476035	-0.1118E-07
0.575	0.465116	0.465418	-0.7451E-08
0.600	0.454545	0.455686	-0.7451E-08
0.625	0.444444	0.447332	-0.7451E-08
0.650	0.434783	0.439149	-0.7451E-08
0.675	0.425532	0.430060	-0.3725E-08
0.700	0.416667	0.417984	0.7451E-08
0.725	0.408163	0.411600	0.3725E-08
0.750	0.400000	0.404986	0.3725E-08
0.775	0.392157	0.394788	0.1118E-07
0.800	0.384615	0.389086	0.7451E-08
0.825	0.377358	0.380049	0.1304E-07
0.850	0.370370	0.371947	0.1863E-07
0.875	0.363636	0.364966	0.1676E-07
0.900	0.357143	0.358678	0.1490E-07
0.925	0.350877	0.348698	0.2235E-07
0.950	0.344828	0.340073	0.2608E-07
0.975	0.338983	0.332395	0.2794E-07
1.000	0.333333	0.324663	0.2794E-07

ERROR NORM = 0.3494E-02 RESIDUAL NORM = 0.1854E-07

EXACT SOL. NORM = 0.5774E 00 ITERATED SOL. NORM = 0.5774E 00

$$\int_0^1 (1+s+t)^{-1}\, x(t)dt = (\ln 3+\ln(1+s) - \ln(2+s)) \cdot (2s+1)^{-1}$$

$$x(t) = (1+2t)^{-1} \quad ; \quad x_o(t) = -t$$

Table 3a

AFTER 10 ITERATIONS BY STEEPEST DESCENT

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	1.000000	0.996266	0.3725E-08
0.025	0.952381	0.953670	-0.3725E-08
0.050	0.909091	0.914420	-0.7451E-08
0.075	0.869565	0.864887	0.3725E-08
0.100	0.833333	0.832835	-0.3725E-08
0.125	0.800000	0.800218	-0.3725E-08
0.150	0.769231	0.769975	-0.3725E-08
0.175	0.740741	0.744678	-0.7451E-08
0.200	0.714286	0.707735	0.3725E-08
0.225	0.689655	0.690517	-0.3725E-08
0.250	0.666667	0.670728	-0.7451E-08
0.275	0.645161	0.648722	-0.7451E-08
0.300	0.625000	0.625326	-0.3725E-08
0.325	0.606061	0.600637	0.3725E-08
0.350	0.588235	0.585875	0.
0.375	0.571429	0.569477	0.
0.400	0.555556	0.556730	-0.3725E-08
0.425	0.540541	0.541525	-0.3725E-08
0.450	0.526316	0.520222	0.3725E-08
0.475	0.512821	0.511638	0.
0.500	0.500000	0.502190	-0.3725E-08
0.525	0.487805	0.489625	-0.3725E-08
0.550	0.476190	0.477166	-0.3725E-08
0.575	0.465116	0.463661	0.
0.600	0.454545	0.456566	-0.3725E-08
0.625	0.444444	0.446570	-0.3725E-08
0.650	0.434783	0.437450	-0.3725E-08
0.675	0.425532	0.428332	-0.3725E-08
0.700	0.416667	0.412706	0.3725E-08
0.725	0.408163	0.407438	0.
0.750	0.400000	0.401885	-0.3725E-08
0.775	0.392157	0.390411	0.
0.800	0.384615	0.386392	-0.3725E-08
0.825	0.377358	0.376725	0.
0.850	0.370370	0.369274	0.
0.875	0.363636	0.364267	-0.1863E-08
0.900	0.357143	0.361566	-0.5588E-08
0.925	0.350877	0.350170	-0.1863E-08
0.950	0.344828	0.342902	0.
0.975	0.338983	0.337571	0.
1.000	0.333333	0.333571	-0.1863E-08

ERROR NORM = 0.2674E-02 RESIDUAL NORM = 0.3690E-08

EXACT SOL. NORM = 0.5774E 00 ITERATED SOL. NORM = 0.5774E 00

$$\int_0^1 (1+s+t)^{-1}\, x(t)dt = [\ln 3+\ln(\frac{s+1}{s+2})](2s+1)^{-1} = Y(s)$$

$$x(t) = (1+2t)^{-1} \quad ; \quad x_o(t) = Y(t)$$

Table 3b

AFTER 5 ITERATIONS BY CONJUGATE GRADIENT

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	1.000000	0.993548	-0.3017E-06
0.025	0.952381	0.949321	-0.2682E-06
0.050	0.909091	0.908376	-0.2384E-06
0.075	0.869565	0.870046	-0.1974E-06
0.100	0.833333	0.834898	-0.1751E-06
0.125	0.800000	0.802044	-0.1490E-06
0.150	0.769231	0.771582	-0.1304E-06
0.175	0.740741	0.743018	-0.1155E-06
0.200	0.714286	0.716200	-0.8568E-07
0.225	0.689655	0.691458	-0.7823E-07
0.250	0.666667	0.668153	-0.6333E-07
0.275	0.645161	0.646255	-0.4843E-07
0.300	0.625000	0.625718	-0.3725E-07
0.325	0.606061	0.606339	-0.2235E-07
0.350	0.588235	0.588126	-0.1118E-07
0.375	0.571429	0.571032	-0.3725E-08
0.400	0.555556	0.554966	0.7451E-08
0.425	0.540541	0.539696	0.1118E-07
0.450	0.526316	0.525083	0.2980E-07
0.475	0.512821	0.511551	0.3353E-07
0.500	0.500000	0.498643	0.3725E-07
0.525	0.487805	0.486308	0.4843E-07
0.550	0.476190	0.474629	0.5588E-07
0.575	0.465116	0.463510	0.6706E-07
0.600	0.454545	0.453053	0.7451E-07
0.625	0.444444	0.443089	0.7451E-07
0.650	0.434783	0.433534	0.8568E-07
0.675	0.425532	0.424423	0.8568E-07
0.700	0.416667	0.415592	0.1043E-06
0.725	0.408163	0.407293	0.1043E-06
0.750	0.400000	0.399454	0.1118E-06
0.775	0.392157	0.391779	0.1192E-06
0.800	0.384615	0.384527	0.1229E-06
0.825	0.377358	0.377594	0.1285E-06
0.850	0.370370	0.370837	0.1378E-06
0.875	0.363636	0.364480	0.1397E-06
0.900	0.357143	0.358440	0.1416E-06
0.925	0.350877	0.352365	0.1527E-06
0.950	0.344828	0.346606	0.1602E-06
0.975	0.338983	0.341185	0.1621E-06
1.000	0.333333	0.335885	0.1695E-06

ERROR NORM = 0.1531E-02 RESIDUAL NORM = 0.1206E-06

EXACT SOL. NORM = 0.5774E 00 ITERATED SOL. NORM = 0.5773E 00

$$\int_0^1 (1+s+t)^{-1} x(t)dt = (\ln 3+\ln(1+s) - \ln(2+s)) \cdot (2s+1)^{-1}$$

$$x(t) = (1+2t)^{-1} \quad ; \quad x_0(t) = 0$$

Table 3c

AFTEP   5 ITERATIONS BY CONJUGATE GRACIENT

T	EXACT SCLLTICN	ITERATED SCL.	RESICUAL
0.	1.COCOCO	C.998360	C.339CE-06
0.C25	C.952381	C.952777	0.2943E-06
0.C50	C.909091	C.91C4C1	C.2608E-06
0.075	C.869565	C.868322	C.2384E-06
0.1CC	C.833333	C.833365	0.2086E-06
0.125	C.8CCCCC	C.80C486	C.1788E-06
0.150	C.769231	C.77C071	0.149CE-06
0.175	C.74C741	C.741735	C.1267E-06
0.2CC	C.714286	C.712612	C.1155E-06
0.225	C.689655	C.689886	0.8941E-07
0.250	C.666667	C.667132	C.6333E-07
0.275	C.645161	C.645413	C.4843E-07
0.3C0	C.6250CC	0.624861	0.298CE-07
0.325	C.606061	C.604932	0.2235E-07
0.350	C.588235	C.587143	0.3725E-08
0.375	C.571429	C.571319	-0.1118E-07
0.4C0	C.555556	C.555175	-C.2608E-07
0.425	C.54C541	0.540925	-0.3725E-07
0.450	C.526316	C.524972	-0.4470E-07
0.475	C.512821	C.512882	-0.6333E-07
0.5C0	C.50C0C0	0.50C824	-0.7823E-07
0.525	C.4878C5	0.487953	-0.8196E-07
0.550	C.476190	C.476377	-0.9686E-07
0.575	C.465116	C.464327	-0.9686E-07
0.6C0	C.454545	C.4547C8	-0.1080E-06
0.625	C.444444	C.444981	-0.1229E-06
0.650	C.434783	C.435728	-0.1304E-06
0.675	C.425532	C.426395	-0.1378E-06
0.7C0	C.416667	C.416293	-0.1378E-06
0.725	C.408163	C.407763	-0.1490E-06
0.750	C.40C0C0	0.40C641	-0.1565E-06
0.775	C.392157	C.391959	-0.1602E-06
0.8C0	C.384615	C.384942	-0.1676E-06
0.825	C.377358	C.377732	-0.1714E-06
0.850	C.37C37C	0.369956	-0.1751E-06
0.875	C.363636	C.363891	-0.1825E-06
0.9C0	C.357143	C.358544	-0.1919E-06
0.925	C.350877	0.3506C4	-0.1900E-06
0.950	C.344828	C.343871	-0.1919E-06
0.975	C.338983	C.338263	-0.1993E-06
1.0C0	C.333333	C.332672	-0.2030E-06

EPRCR NORM = C.7044E-C3   RESIDUAL NORM = 0.1495E-06

EXACT SCL. NCRM = 0.5774E 00   ITERATED SOL. NORM = C.5774E 00

$$\int_0^1 (1+s+t)^{-1}x(t)dt = [\ln 3+\ln(\frac{s+1}{s+2})](2s+1)^{-1} = Y(s)$$

$$x(t) = (1+2t)^{-1} \quad ; \quad x_o(t) = Y(t)$$

Table 3d

AFTER 5 ITERATIONS BY STEEPEST DESCEND

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	1.000000	0.950989	0.9835E-06
0.025	1.051271	1.023810	0.7749E-06
0.050	1.105171	1.090256	0.6557E-06
0.075	1.161834	1.170301	0.5066E-06
0.100	1.221403	1.231767	0.3874E-06
0.125	1.284025	1.304048	0.2086E-06
0.150	1.349859	1.373659	0.1192E-06
0.175	1.419068	1.438848	0.
0.200	1.491825	1.513541	-0.1192E-06
0.225	1.568312	1.586752	-0.1788E-06
0.250	1.648721	1.662407	-0.2682E-06
0.275	1.733253	1.741382	-0.3278E-06
0.300	1.822119	1.824536	-0.3874E-06
0.325	1.915541	1.916452	-0.4172E-06
0.350	2.013753	2.005602	-0.4470E-06
0.375	2.117000	2.107799	-0.5066E-06
0.400	2.225541	2.209036	-0.4768E-06
0.425	2.339647	2.323191	-0.5066E-06
0.450	2.459603	2.444460	-0.5364E-06
0.475	2.585710	2.565025	-0.5066E-06
0.500	2.718282	2.701992	-0.5066E-06
0.525	2.857651	2.833919	-0.4470E-06
0.550	3.004166	2.987274	-0.4172E-06
0.575	3.158193	3.144352	-0.3874E-06
0.600	3.320117	3.306120	-0.3278E-06
0.625	3.490343	3.481065	-0.2980E-06
0.650	3.669297	3.665736	-0.2384E-06
0.675	3.857426	3.859337	-0.1788E-06
0.700	4.055200	4.063477	-0.8941E-07
0.725	4.263114	4.276669	-0.5960E-07
0.750	4.481689	4.499338	0.2980E-07
0.775	4.711470	4.734928	0.1043E-06
0.800	4.953032	4.976761	0.1639E-06
0.825	5.206980	5.231727	0.2533E-06
0.850	5.473947	5.499043	0.3278E-06
0.875	5.754603	5.773236	0.4023E-06
0.900	6.049647	6.064326	0.4768E-06
0.925	6.359819	6.363456	0.5662E-06
0.950	6.685894	6.672333	0.6557E-06
0.975	7.028687	6.993561	0.7153E-06
1.000	7.389056	7.329735	0.7749E-06

ERROR NORM = 0.1871E-01 RESIDUAL NORM = 0.4239E-06

EXACT SOL. NORM = 0.3661E 01 ITERATED SOL. NORM = 0.3661E 01

$$\int_0^1 \exp(-st)x(t)dt = [\exp(2-s)-1](2-s)^{-1} = Y(s)$$

$$x(t) = \exp(2t) \quad ; \quad x_0(t) = Y(t)$$

Table 4a

AFTER 15 ITERATIONS BY CONJUGATE GRADIENT

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	1.000000	1.000020	-0.5960E-07
0.025	1.051271	1.111297	-0.8941E-07
0.050	1.105171	1.014121	-0.5960E-07
0.075	1.161834	1.210845	-0.1490E-06
0.100	1.221403	1.124544	-0.8941E-07
0.125	1.284025	1.292979	-0.1788E-06
0.150	1.349859	1.325212	-0.1490E-06
0.175	1.419068	1.386173	-0.1788E-06
0.200	1.491825	1.481047	-0.2086E-06
0.225	1.568312	1.557698	-0.1788E-06
0.250	1.648721	1.599505	-0.1788E-06
0.275	1.733253	1.744027	-0.2086E-06
0.300	1.822119	1.821549	-0.1788E-06
0.325	1.915541	1.913860	-0.2086E-06
0.350	2.013753	1.946898	-0.1788E-06
0.375	2.117000	2.178568	-0.2384E-06
0.400	2.225541	2.140462	-0.2086E-06
0.425	2.339647	2.333626	-0.2086E-06
0.450	2.459603	2.508983	-0.2384E-06
0.475	2.585710	2.662900	-0.2384E-06
0.500	2.718282	2.797174	-0.2384E-06
0.525	2.857651	2.866102	-0.1788E-06
0.550	3.004166	3.013337	-0.2086E-06
0.575	3.158193	3.159742	-0.2086E-06
0.600	3.320117	3.276696	-0.1788E-06
0.625	3.490343	3.485498	-0.2086E-06
0.650	3.669297	3.644044	-0.1788E-06
0.675	3.857426	3.848406	-0.1490E-06
0.700	4.055200	4.008420	-0.1490E-06
0.725	4.263114	4.295977	-0.1490E-06
0.750	4.481689	4.431120	-0.1043E-06
0.775	4.711470	4.701271	-0.8941E-07
0.800	4.953032	4.946289	-0.7451E-07
0.825	5.206980	5.179943	-0.2980E-07
0.850	5.473947	5.460063	-0.1490E-07
0.875	5.754603	5.742625	0.2980E-07
0.900	6.049647	6.057471	0.7451E-07
0.925	6.359819	6.375849	0.1043E-06
0.950	6.685894	6.664310	0.1639E-06
0.975	7.028687	7.038487	0.2235E-06
1.000	7.389056	7.477353	0.2682E-06

ERROR NORM = 0.3932E-01   RESIDUAL NORM = 0.1711E-06

EXACT SOL. NORM = 0.3661E 01   ITERATED SOL. NORM = 0.3661E 01

$$\int_0^1 \exp(-st)x(t)dt = [\exp(2-s)-1](2-s)^{-1} = Y(s)$$

$$x(t) = \exp(2t) \quad ; \quad x_o(t) = Y(t)$$

Table 4b

AFTER 25 ITERATIONS BY CONJUGATE GRADIENT

T	EXACT SOLUTION	ITERATED SOL.	RESIDUAL
0.	1.000000	1.054028	-0.1729E-05
0.025	1.051271	1.070571	-0.1401E-05
0.050	1.105171	1.125671	-0.1103E-05
0.075	1.161834	1.150167	-0.8941E-06
0.100	1.221403	1.220460	-0.6855E-06
0.125	1.284025	1.263295	-0.5066E-06
0.150	1.349859	1.332290	-0.3278E-06
0.175	1.419068	1.403121	-0.2086E-06
0.200	1.491825	1.475439	-0.8941E-07
0.225	1.568312	1.555024	0.
0.250	1.648721	1.644672	0.8941E-07
0.275	1.733253	1.725375	0.1192E-06
0.300	1.822119	1.820458	0.1788E-06
0.325	1.915541	1.917790	0.2086E-06
0.350	2.013753	2.029152	0.2384E-06
0.375	2.117000	2.117999	0.2086E-06
0.400	2.225541	2.248912	0.2384E-06
0.425	2.339647	2.353317	0.2086E-06
0.450	2.459603	2.467240	0.1788E-06
0.475	2.585710	2.589263	0.1490E-06
0.500	2.718282	2.721318	0.1192E-06
0.525	2.857651	2.868656	0.8941E-07
0.550	3.004166	3.013575	0.2980E-07
0.575	3.158193	3.166223	-0.2980E-07
0.600	3.320117	3.331997	-0.5960E-07
0.625	3.490343	3.493902	-0.1490E-06
0.650	3.669297	3.672653	-0.1788E-06
0.675	3.857426	3.855883	-0.2384E-06
0.700	4.055200	4.056486	-0.2980E-06
0.725	4.263114	4.251178	-0.3576E-06
0.750	4.481689	4.479546	-0.3874E-06
0.775	4.711470	4.702878	-0.4619E-06
0.800	4.953032	4.943784	-0.5215E-06
0.825	5.206980	5.200781	-0.5364E-06
0.850	5.473947	5.467219	-0.5811E-06
0.875	5.754603	5.749951	-0.6109E-06
0.900	6.049647	6.044646	-0.6258E-06
0.925	6.359819	6.358051	-0.6706E-06
0.950	6.685894	6.692924	-0.6706E-06
0.975	7.028687	7.036841	-0.6706E-06
1.000	7.389056	7.391184	-0.6706E-06

ERROR NORM = 0.1165E-01 RESIDUAL NORM = 0.5145E-06

EXACT SOL. NORM = 0.3661E 01 ITERATED SOL. NORM = 0.3660E 01

$$\int_0^1 \exp(-st)x(t)dt = [\exp(2-s)-1](2-s)^{-1} = Y(s)$$

$x(t) = \exp(2t)$; $x_o(t) = Y(t)$

Table 4c

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